Covering radius in weakly metric association schemes

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RAYON DE RECOUVREMENT DANS LES SCHEMAS FAIBLEMENT METRIQUES

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Abstract :

We introduce the concept of a weakly metric association scheme, a generalization of metric schemes. Applications are vertex transitive graphs, codes over Z_q for the Lee metric, "permutation codes" for the Hamming metric, arithmetic codes for the modular distance.

In this context, we define the notion of covering radius of a subset.

We give a general version of the Stanton-Kalbfleisch inequalities, leading to a linear programming bound on the minimal cardinality of a subset of given covering radius. Equality in the constraints characterize perfect sets, and determines uniquely their inner distribution if the scheme is metric. This yields a Lloyd theorem for vertex transitive graphs , which specializes to Bassalygo's results in Lee scheme, and Biggs' results in distance-transitive graphs.

The outer distribution matrix of a subset allows us to generalize Delsarte upper bound on covering radius, and Mac-Williams lower bound on the external distance, yielding a strong necessary condition on completely regular perfect subsets.

Résumé:

Les schémas d'association faiblement métriques, sont une généralisation des schémas métriques. Les principales applications sont les graphes sommets-transitifs, les codes sur Z/qZ pour la métrique de Lee, les codes de permutation pour la métrique de Hamming, les codes arithmétiques pour la distance modulaire.

Dans ce contexte, nous définissons la notion de rayon de recouvrement d' un sous-ensemble.
Nous généralisons les inégalités de Stanton et Kalbfleisch , ce qui conduit à une borne de programmation linéaire sur la cardinalité minimale d' un sous ensemble de rayon de recouvrement donné.
L' égalité dans les contraintes du programme caractérise les sous-ensembles parfaits, et détermine uniquement leur distribution interne si le schéma est métrique . Cela mène à un "théorème de Lloyd " pour les graphes sommets-transitifs, qui admet pour cas particuliers le résultat de Bassalygo pour le schéma de Lee, et le résultat de Biggs pour les graphes distance-transitifs.

La matrice de distribution externe d'un sous-ensemble permet de généraliser la borne supérieure de Delsarte sur le rayon de recouvrement et la borne inférieure de Mac-Williams sur la distance extérieure. Nous en déduisons des conditions nécessaires d' existence sur les sous-ensembles parfaits complètement réguliers.

1. Introduction:

In his thesis [12] Delsarte shown that the theory of association schemes - an algebraic setting issued from Statistics and Group theory [1]- was a natural framework for coding theory. By use of linear programming , he recovered many classical bounds on the size of codes with given packing radius. At about the same time, N.Biggs proved the Lloyd theorem in distance regular graphis [5], which are exactly metric schemes.
In this work , we generalize these ideas in two ways:
- First, we introduce the class of weakly metric schemes, which includes strictly metric schemes . Our motivation is the study of Lee distance over Z_q^n and modular distance over Z_M, which behave less 'properly' than hamming metric. We give analogues of classical results from McWilliams and Delsarte.
- Second, we apply linear programming to covering problems rather then packing problems. For this we need a general version of Stanton Kalbfleisch inequalities, which are derived from the very axioms of association schemes by simple counting. The case of equality in the contraints yields both classical and new results on perfect sets, by linear algebra, without any use of orthogonal polynomials.
We refer to [21], [22] for any undefined term from group theory, and to [1], [18], [12] for any undefined term concerning association schemes.

2. Weakly metric schemes :

2.1. Definitions:

An association scheme with t classes consists of a finite set along with a partition $R = (R_0, R_1, ..., R_t)$ on $X \times X$, satisfying the following axioms:
$A_1 : R_0 = \{(x, x)/x \in X\}$
$A_2 : R_i^{-1} = \{(y, x)/\, (x, y) \in R_i\} = R_{i'}$ for some i'.
A_3 : the cardinality of $\{z/(x, z) \in R_i \; and \; (z, y) \in R_j\}$ is a function p_{ij}^k which depends on k , but not on $(x, y) \in R_k$.
$A_4 : \; p_{ij}^k = p_{ji}^k$
if we replace A_2 by the stronger condition $A_2{}' : \; R_i^{-1} = R_i$ the scheme is said of Bose-Mesner type (BM for short). We call weakly metric an association scheme equipped with a metric d constant on the classes of the scheme:

$$aR_kb \Rightarrow d(a, b) = d(k) \qquad (1)$$

When d is graphic [9] (definition in subsection 2.5) and $d(k) = k$, we recover exactly the definition of a metric scheme [12] In the following subsections we construct examples of both practical and theoretical interest.

2.2. An all-purpose construction

Let X be a finite set endowed with a metric d. We suppose that a subgroup of the group G of isometries of d acts transitively on X, and we consider the action of G on the cartesian product $X \times X$. Let $(R_0, R_1, ..., R_t)$ be the orbits with $R_0 = \{(x, x)/x \in X\}$.

Proposition 1: (X, R) is a weakly metric scheme.

Proof:
It is well-known that (X, R) is an association scheme [1],[6], the incidence algebra of which coincides with the centralizing algebra of the regular permutation representation of G.
Clearly d is constant on the R_i.

2.3. Vertex transitive graphs:

Let Γ be a simple, [3], undirected graph, with vertex set $V\Gamma$, and edge set $E\Gamma$. Taking $X = V\Gamma$, and $d = d_\Gamma$ the shortest chain distance on $V\Gamma$,we can apply paragraph 1.2. We construct the Lee sheme (denoted by $L(n, q)$), and Hamming scheme (denoted by $H(n, q)$), on Z_q^n in that way.
We denote by C_q (resp.K_q)the graph of the ordinary q-gon, (resp. the complete graph on q vertices) .Then the n-fold cartesian sum [3], $C(n, q)$ (resp. $K(n, q)$) admits as an automorphism group :

$$Aut(C_q) \int S_n = D_q \int S_n \qquad (2)$$

respectively:

$$Aut(K_q) \int S_n = S_q \int S_n \qquad (3)$$

where $\int$ denotes the wreath product [21], and D_q denotes the dihedral group [21].

The distance $d_{C(n,q)}$ (resp.$d_{K(n,q)}$) is simply the sum of the distances d_{C_q} (resp. d_{K_q}) on each summand graph [3]. We remark that the number of nontrivial orbits in $K(n, q) \times K(n, q)$ is n, which is equal to its diameter. This implies that $K(n, q)$ is distance-transitive [6], hence distance-regular. On the other hand, C(n,q) has diameter ns, with $s = [(q-1)/2]$, and $t = \binom{n+s}{s}$

orbits, and is not even distance regular, which is equivalent to say [12], that $L(n,q)$ is not P-polynomial [19].

2.4. Arithmetic codes for the modular distance:

We use $X = Z_M$, the integers modulo M, we let d be the modular distance in the sense of [8] with radix r, and for G the semi-direct product of T_M by G_r. T_M is the group of translations of Z_M, and G_r the group of permutations of Z_M generated by multiplications by r, and -1. We assume that r is prime to M, so that G_r is indeed a group and $d(rx, ry) = d(x, y)$ (see [8] for a proof in the case of $M = r^m - 1$). Roughly speaking the orbits of G_r on Z_M are the cyclotomic classes merged with their opposite. For instance $M = 17, r = 2$, the nontrivial orbits of G_r on Z_M are

$$X_1 = \{1, 2, 4, 8, -1, -2, -4, -8\} \tag{4}$$

of modular weight 1, and

$$X_2 = \{3, 6, 12, 7, -3, -6, -12, -7\} \tag{5}$$

of modular weight 2, with the convention

$$x\, R_i\, y \;\Leftrightarrow\; x - y \in X_i \tag{6}$$

We obtain a two-classes association scheme in the BM sense, or equivalently a strongly regular graph [12]. (Here the cyclotomic classes coincide with their opposite). Its parameters in the notation of [7], are (17,8,2,4).
In the previous case the scheme was metric for the modular distance.This is not always the case.A counterexample is for $M = 31, r = 2$, with the orbits: $\{0\}, C_3 \bigcup C_7$, $C_5 \bigcup C_{11}$, $C_1 \bigcup C_{15}$ of weights 0,2,2,1 where C_i stands for the i^{th} cyclotomic class. This latter scheme is weakly metric, but not metric for the modular distance, since two classes of the scheme share the same modular weight.

2.5. "Permutation codes"

We construct an example where d is not graphic, that is, it is not a d_Γ. We let $X = S_n$, the symmetric group on n letters. Any permutation on n letters can be seen as a codeword of length n on an alphabet of size n [9]. We let d be the Hamming distance in $H(n, n)$. Let $\sigma \in S_n$, and $s(\sigma)$ denote the shape [17] of σ, we have

$$s(\sigma) = (s_1, s_2, ..., s_n) \tag{7}$$

where s_i is the number of i-cycles. We define the classes $R_{\mathbf{k}}$ by

$$\sigma\, R_{\mathbf{k}}\, \tau \;\Leftrightarrow\; s(\sigma\tau^{-1}) = \mathbf{k}$$

with **k** running over the partitions of n.

Proposition 2: (S_n, R) is a B.M scheme weakly metric for the distance d.

Proof: We use the construction of proposition 1 with X and d specified above. We note that:

$$d(\sigma, \tau) = n - s_1(\sigma\tau^{-1}) \tag{8}$$

Hence the group $S_n.S_n$ (direct product) acts as a group of isometries for d by left and right multiplication:

$$(\lambda, \mu).\sigma = \lambda\sigma\mu \tag{9}$$

for we have :

$$s_1(\lambda\sigma\mu(\mu^{-1}\tau^{-1}\lambda^{-1})) = s_1(\lambda\sigma\tau^{-1}\lambda^{-1}) = s_1(\sigma\tau^{-1}) \tag{10}$$

since $s(\sigma)$ is constant on the conjuguate classes of S_n. Conversely , these classes are exactly the sets with a given $s(\sigma)$ [17], so that the orbits of $S_n.S_n$ on $S_n \times S_n$ are exactly the $R_{\mathbf{k}}$.
It is clear that $S_n.S_n$ acts transitively on S_n.
Q.E.D
d is not graphic for , if $d(\sigma, \tau) = n$ it does not exist any ν with the property

$$\begin{cases} d(\sigma, \nu) & = n - 1 \\ d(\nu, \tau) & = 1 \end{cases} \tag{11}$$

since a permutation with $n - 1$ fixed points has to be the identity.

3. The covering radius of a set

3.1. Definitions:

Let Y denote a subset of X. Y is said to be a r-covering of X iff for any z in X :

$$d(z, Y) := min_{y \in Y} d(z, y) \leq r \tag{12}$$

The smallest r such that Y is a r-covering is called the covering radius of Y and is denoted by $\rho(Y)$.
Let $K(X, r)$ be the smallest cardinality of a subset of covering radius r.
In the following we shall be interested in lower bounding $K(X, r)$. First we give without proof some elementary properties which are shown for $H(n, 2)$ in [10].

3.2. The covering bound:

Let b_j be the volume of the ball of radius j for the distance d:

$$b_j = \sum_{d(i) \leq j} p^0_{i,i'} \tag{13}$$

Then, we have

$$| X | \leq | Y | \times b_{\rho(y)} \tag{14}$$

3.3. The superset lemma:

If W is a proper superset of Y, of minimal distance $d(W)$:

$$d(W) := min_{x \neq y} d(x, y) \tag{15}$$

we have

$$\rho(Y) \geq d(W) \tag{16}$$

3.4. Maximal sets:

Y is maximal for inclusion iff

$$\rho(Y) \leq (d(Y) - 1) \tag{17}$$

Application: We consider the metric scheme of alternating forms over F_{2^m} [13],[14], together with the distance

$$d(B, B') = (rank(B - B'))/2 \tag{18}$$

The optimal (m, r) sets introduced in [14] have maximum cardinality among the sets with minimum distance:

$$d((m, r)) = r \tag{19}$$

A fortiori, they are maximal for inclusion, and verify

$$\rho((m, r)) \leq r - 1 \tag{20}$$

Moreover they are strictly nested:

$$(m, r) \subset (m, r - 1) \tag{21}$$

so that we can apply the superset lemma:

$$\rho((m, r)) \geq d((m, r - 1)) \tag{22}$$

We conclude

$$\rho((m, r)) = r - 1 \tag{23}$$

4. A linear programming bound on K(X,r):

4.1. Linear constraints on the inner distribution of a set:

Let Y be a subset of X, x a point of X, and $A_k(x)$ the number of points of Y k-related to x. The inner distribution of Y is then defined as:

$$a_k := \left(\sum_{x \in Y} A_k(x)\right) / \mid Y \mid \tag{24}$$

Proposition 3: Necessary conditions for Y to be an r-covering are :

$$\forall x,\ \forall i \in [0..t]: \sum_{d(i)-r \leq d(k) \leq d(i)+r} A_k(x) \left(\sum_{d(j) \leq r} p_{ij}^k \right) \geq p_{ii'}^0 \tag{25}$$

Corollary 1: Necessary conditions for Y to be a r-covering are:

$$\forall i \in [0..t]: \sum_{d(i)-r \leq d(k) \leq d(i)+r} a_k \left(\sum_{d(j) \leq r} p_{ij}^k \right) \geq p_{ii'}^0 \tag{26}$$

Proof: Let us consider the set $X_i(x)$ defined by:

$$X_i(x) = \{z \in X / z\, R_i\, x\} \tag{27}$$

The definition of the intersection numbers yields:

$$\mid X_i(x) \mid = p_{ii'}^0 \tag{28}$$

Now any z in this set is at most r away from some y in Y, so that it exists an index j such that :

$$z\,R_j\,y \tag{29}$$

and:

$$d(j) \leq r \tag{30}$$

Such a y is k-related to the "origin" x, and according to the triangle inequality

$$|\,d(i) - r\,| \leq d(k) \leq d(i) + r \tag{31}$$

Then, by definition of the intersection numbers p_{ij}^{k} and of $A_k(x)$, the L.H.S of equation (25) counts the number of such y. Since the correspondence $y \leftrightarrow z$ is many-to-one, the L.H.S is superior to the R.H.S.
The corollary follows immediately by averaging equation (25) over x in Y. Q.E.D

Example 1: In $H(n,2)$ this specializes to the Stanton-Kalbfleisch inequalities [11], [20]. For r=1,2 this yields :

$$(n-i+1)\,A_{i-1} + A_i + (i+1)\,A_{i+1} \geq \binom{n}{i} \tag{32}$$

$$\binom{n-i+2}{2} A_{i-2} + (n-i+1)\,A_{i-1} + (1+in-i^2)\,A_i + (i+1)A_{i+1} + \binom{i+2}{2} A_{i+2} \geq \binom{n}{i} \tag{33}$$

A generating function for the intersection numbers of $H(n,q)$ will be given in the next subsection.

4.2. Equality in the constraints:

We recall that a set fulfilling the covering bound with equality is said to be perfect.

Proposition 4: The constraints on the inner distribution of Y are fulfilled with equality iff Y is perfect.

Proof: These conditions are realized iff the correspondence $y \leftrightarrow z$, defined in (29), is one-to-one, which means that the balls of radius r centered on the points of Y pack the space X. Q.E.D

Corollary 2: For a perfect subset Y in a metric scheme, the weight distribution $A_k(x)$ for an arbitrary origin x in Y, and the inner distribution a_k are uniquely determined by $\rho(Y)$ and are equal.

Proof:
For a perfect set Y it holds that:

$$d(Y) = 2\rho\,(Y) + 1 \tag{34}$$

yielding the "initial conditions"

$$a_0 = 1;\; a_i = 0 \;\; for\; 1 \leq i \leq 2\rho\,(Y) \tag{35}$$

The fact that the distance is graphic ensures that the coefficient of the unknown a_k in the system (26) (with equality) for $k = i + r$ does not vanish.
Then the systems (34) and (35) admit one and only one solution.
Q.E.D
This fact was already proved in [12], using the theory of orthogonal polynomials.

Proposition 5: In $H(n,q)$, for any q, perfect single error correcting codes if they exist have the weight distribution function:

$$A(Y) := \sum_{i=0}^{n} A_i y^i = (1+n(q-1))^{-1}\left[(1+(q-1)y)^n + n(q-1)(1+(q-1)y)^{\frac{n-1}{q}}(1-y)^{\frac{1+n(q-1)}{q}}\right] \quad (36)$$

Proof: In $H(n,q)$ the intersection numbers enjoy the generating function, [16], [4]:

$$\sum_{i,j} p_{ij}^k x^i y^j = [x+y+(q-2)xy]^k[1+(q-1)xy]^{n-k} \quad (37)$$

From this we obtain the recursion:

$$(q-1)(n-i+1)A_{i-1} + A_i + (i+1)A_{i+1} = \binom{n}{i}(q-1)^i \quad (38)$$

with limit conditions

$$A_0 = 1,\ A_{-1} = A_{n+1} = 0;\ A_1 = A_2 = 0 \quad (39)$$

which is equivalent to differential equation on $A(y)$ with conditions at the origin

$$A(0) = 1\ ,\ A'(0) = 0,\ A''(0) = 0 \quad (40)$$

a problem which admits one and only one solution. Tedious but direct computations show that the given expression is this solution. Q.E.D

Let $\psi_1(x)$ denote the 1^{st} order Lloyd polynomial in $H(n,q)$:

$$\psi_1(x) = 1 + n(q-1) - qx \quad (41)$$

Corollary 3: If $H(n,q)$ admits a perfect single error correcting code, then the root of ψ_1 is an integer.

$$q \mid 1 + n(q-1) \quad (42)$$

Remark:
The sphere packing condition does not imply the Lloyd condition for general q:
$n = 7\,; q = 10 \Rightarrow 1 + n(q-1) = 2^6$ which divides 10^6 but $q = 10$ does not divide 64.

4.3. A Lloyd theorem in vertex transitive graphs:

We recall that a vertex transitive graph can be embedded in an association scheme (section 2.2) that is weakly metric.
We introduce the incidence matrices of the scheme D_i of size $\mid V\Gamma \mid$ by $\mid V\Gamma \mid$, with (x,y) entries given by:

$$D_i(x,y) = \begin{cases} 1 & if\ xR_iy \\ 0 & otherwise \end{cases} \quad (43)$$

We consider the counting matrices $\hat{D}_i$ with integer entries, $\hat{D}_i(j,k) = p_{i,j}^k$. It is well known [6], [18], that these matrices form an algebra isomorphic to the incident algebra of the scheme, and we have the same eigenvalues , say $p_i(j)$, $i,j \in [0..t]$ and are simultaneously diagonalizable.
We define the Lloyd operator of order e to be :

$$L_e = \sum_{d(i)\le e} \hat{D}_i \quad (44)$$

and we denote by $\Pi_X(e)$ the cardinality of the set

$$\{x \in X / \, y R_i x, \; d(i) \leq e\} \tag{45}$$

where y is an arbitrary origin.

Proposition 6: If there is a perfect set Y of Γ of covering radius e , then L_e has rank at most $t+2-\Pi_X(e)$.

Proof: We may suppose , w.l.o.g that y is in Y. Choose vertices u_i in number $\Pi_X(e)-1$ such that $y\, R_i\, u_i$ *and* $d(i) \leq e$. There are as many $g_i \in Aut\Gamma$ which map y on each u_i; then we obtain $\Pi_X(e)$ perfect sets Y_i :

$$Y_0 = Y; \; Y_1 = g_1(Y); .... Y_i = g_i(Y). \tag{46}$$

If we introduce the column vector $\mathbf{a_i}$ of inner distribution of a set Y_i and the column vector $\mathbf{k}$ such that

$$k_i = \frac{1}{|V\Gamma|} \mid X_i(y) \mid \tag{47}$$

Then we see that the condition of equality in (26) can be rewritten in matrix form

$$L_e \times \mathbf{a_i} = \mathbf{k} \tag{48}$$

From that we deduce

$$L_e(\mathbf{a_0} - \mathbf{a_i}) = 0 \tag{49}$$

By the choice of the sets Y_i the vectors $\mathbf{a_0} - \mathbf{a_i}$ are independent , nonzero and in the kernel of L_e:

$$dim(Ker\, L_e) \geq \Pi_X(e) - 1 \tag{50}$$

which is equivalent to : $rank(L_e) \leq t+1-\Pi_X(e)+1$. QED

As the incidence algebra of the scheme is finitely generated the Lloyd operator is a polynomial in the generating matrices of the algebra. Since the matrices can be diagonalized on the same basis, the rank condition can be expressed by the annulation of a multivariate polynomial. In the special case where Γ is distance transitive there is a unique generator , the Lloyd polynomial is monovariate and we recover Biggs' results [5].
By letting $\Gamma = C(n,q)$ we recover easily the Lloyd theorem for the Lee metric [2], [19].

4.4. The linear programming bound:

Let $p_k(i)$ denote the first eigenvalues of the scheme (X, R) . A well known result due to Delsarte [12] is that the dual inner distribution is non negative:

$$\sum_{i=0}^{t} a_i P_k(i) \geq 0 \tag{51}$$

Then, the problem of finding a good covering set leads to the following linear programming bound:

$$K(X,r) \geq min_{a_i}\{a_0 + a_1 + ... + a_t / a_i \geq 0, a_0 = 1, (26)(51)\} \tag{52}$$

5. Outer distribution of a subset:

We define the outer distribution matrix B of a subset Y by its (x, i) entry:

$$B_{x,i} = | \{y \in Y / x R_i y\} | \qquad (53)$$

We call e the packing radius of Y : $e = [\,(d(Y) - 1)/2\,]$
We call b the number of distinct rows of B for x not in Y.
We call s' the number of nonzero a'_k for k nonzero and we recall the inversion formula from [12]:

$$a_k = \frac{1}{|X|} \sum_{i=0}^{t} p_k(i) a'_i \qquad (54)$$

<u>Theorem:</u>

$$rank(B) = s' + 1 \qquad (55)$$

<u>Proof:</u>
It is a well known linear algebraic lemma that

$$rank(B) = rank(B^T B) \qquad (56)$$

We shall prove the following:

$$B^T B = \frac{|Y|}{|X|} P^T \Delta P \qquad (57)$$

where P is the matrice of size $(t+1) \times (t+1)$ of the first eigenvalues of the scheme

$$P_{i,k} = [p_k(i)] \qquad (58)$$

and where

$$\Delta = diag(a'_0, a'_1, ... a'_t) \qquad (59)$$

First an elementary counting argument shows that

$$\sum_{x \in X} B_{i,x} B_{j,x} = | Y | \sum_{k=0}^{t} p^k_{i,j} a_k \qquad (60)$$

Using the inversion formula and the following relation

$$p_i(u) p_j(u) = \sum_{k=0}^{t} p^k_{i,j} p_k(u) \qquad (61)$$

which stems from the same relation on adjacency matrices, yields:

$$\sum_{x \in X} B_{i,x} B_{j,x} = \frac{|Y|}{|X|} \sum_{u=0}^{t} p_i(u) a'_u p_j(u) \qquad (62)$$

which is the desired relation.

Corollary 4: $s' \geq \Pi_X(e) - 1$

Proof:
Let y denote an arbitrary point of Y, fixed once and for all, and x such that xR_iy with $d(i) \leq e$. By the definition of $\Pi_X(e)$ there are $\Pi_X(e)$ such x with pairwise distinct i, yielding that many rows beginning with i zeros, hence that many independant rows. (i=0 is counted in this process). Q.E.D
In the case of a metric scheme this yields the MacWilliams inequality [12], [18] :

$$s' \geq e \tag{63}$$

Corollary 5: $\rho \leq s'$

Proof:
Same reasoning with x such that $d(x,y) = i$, $i = 0..\rho$ Q.E.D

Example : Consider an AN code [15] for the modular distance [8] with radix r and modulus M such that $M = AD$.

$$AN = \{ A.i \,/\, 1 \leq i \leq D\} \tag{64}$$

Then the additive dual of AN in the sense of [12 p 23-88] is DN and s' is the number of nontrivial orbits of G_r on DN. This is also the number of nontrivial orbits of G_r on Z_A , say n_A, since, for any integers a and $b \leq A$ we have that

$$a \equiv b(\pm p^i)[A] \Longleftrightarrow Da \equiv Db(\pm p^i)[M] \tag{65}$$

We obtain immediately the known [15] result:

$$\rho(AN) \leq n_A \tag{66}$$

As shown in [15] this bound is attained on examples and the number of modular weights of DN can be $< \rho(AN)$, so that the analogous statement of Delsarte bound in $H(n,q)$ is wrong in general.

Corollary 6: $b \geq s'$

Proof:
There are $b+1$ distinct rows in B, whose rank is $s'+1$.
Q.E.D
A set Y is said to be $r-regular$ if the row B(x) depends only on $d(x,Y)$ for $d(x,Y) \leq r$ and completely regular iff $\rho - regular$, i.e $\rho = b$.

Corollary 7: if Y is completely regular then $\rho = s'$

Proof:

Obvious from the two preceding corollaries.
Q.E.D

Proposition 7: If Y is a completely regular perfect code then:
$\Pi_X(e) - 1 = e$

Proof:
Y is perfect hence $\rho = e$.
Y is completely regular hence $\rho = s'$.
Since $s' \geq \Pi_X(e) - 1$ the result follows , for $e \leq \Pi_X(e) - 1$ always holds.
Q.E.D

We point out that there exists perfect codes in weakly metric schemes that are not completely regular. For instance we take $X = L(n, q)$ with $n = 2, q = 13$, and Y the negacyclic self dual perfect code [4] with generator polynomial $g(x) = x + 5$ and "negacycle representatives" (0,0), (1,5), (3,2), (6,4).
We have $\rho = e = 2$ and $s' = 3 \neq 2$. Therefore Y is not completely regular.

Proposition 8: For $q \geq 5$ *and* $e \geq 2$ there are no completely regular perfect codes in $L(n, q)$.

Proof:
Clearly $\Pi_X(e+1) - \Pi_X(e) \geq 1$
Since $q \geq 5,\ 1 \neq 2$ *in* Z_q *and* $\Pi_X(2) = 4$
then an easy recurrence shows that $\Pi_X(e) \geq e + 2$ for $e \geq 2$.
Q.E.D
This contrasts strongly with $H(n, q)$, where every perfect code is completely regular.It can be shown that, in $L(n, q)$, the negacyclic single-error correcting codes [4], for q a prime are completely regular.

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