Ambiguity and transcendence

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Abstract: *We establish that several classical context free languages are inherently ambiguous by proving that their counting generating functions, when considered as analytic functions, exhibit some characteristic form of transcendental behaviour.*

Résumé: On établit le caractère inhéremment ambigu de plusieurs langages algébriques (*context-free*) classiques en montrant que leurs séries génératrices de dénombrements, lorsqu'elles sont considérées en tant que fonctions analytiques, ont des comportements qui sont caractéristiques de fonctions transcendantes.

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AMBIGUITY AND TRANSCENDENCE

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ABSTRACT

We establish that several classical context free languages are inherently ambiguous by proving that their counting generating functions, when considered as analytic functions, exhibit some characteristic form of transcendental behaviour.

1. INTRODUCTION

We propose here an analytic method for approaching the problem of determining whether a ***context-free*** language is ***inherently ambiguous***. This method (which cannot be universal since the problem is highly undecidable) is applied to several context-free languages that had resisted previous attacks by purely combinatorial arguments. In particular, we solve here a conjecture of Autebert, Beauquier, Boasson, Nivat [1] by establishing that the "Goldstine language" is inherently ambiguous. Our technique is also applied to a number of context-free languages of rather diverse structural types.

There are relatively few types of languages that have been proved to be inherently ambiguous. This situation owes mostly to the fact that classical proofs of inherent ambiguity have to be based on a combinatorial argument of some sort considering *all possible grammars* for the language. Such proofs are therefore scarce and relatively lengthy.

At an abstract level, our methodology is related to a more general principle, namely the construction of ***analytical models for combinatorial problems***. Informally, the idea is as follows:

> ***To determine if a problem P belongs to a class $\mathbf{C}$, associate to elements ω of $\mathbf{C}$ adequately chosen analytic objects $\vartheta(\omega)$ so that a (possibly partial) characterization of $\vartheta(\mathbf{C})$ can be obtained. If $\vartheta(P) \notin \vartheta(\mathbf{C})$, then P does not belong to $\mathbf{C}$.***

At such a level of generality, this principle is of course of little use. However it has been successfully applied in the past in the derivation of non trivial lower bounds in complexity theory:

1. Shamos and Yuval [18] have obtained interesting lower bounds for the complexity of computing the mean distance of points in a Euclidian space by considering ***the Riemann surface associated to the complex multivalued function*** (especially its ***branch points***) that continues the function defined by the original problem. They obtain in this way an $\Omega(n^2)$ lower bound on the complexity of the problem. The fact that the proof of this particular result was subsequently "algebracized" by Pippenger [13] does not limit the interest of their approach.

2. More recently Ben-Or [3] has obtained a number of lower bounds for membership problems, including for instance the distinctness problem, set equality and inclusion .... His method consists in considering ***the topological structure of the real algebraic variety*** (the number of connected components) associated to a particular problem and relate it to the inherent complexity of that problem.

Our approach here is to examine properties of ***generating functions of context-free languages*** especially when these functions are considered as ***analytic functions*** instead of plain formal power series. The situation in this case is greatly helped by the fact that, from an old theorem of Chomsky and Schutzenberger, the ordinary generating function of an ***unambiguous*** context-free language is ***algebraic*** as a ***series***, and thus also as an analytic ***function***. Therefore, we can simply prove that a context-free language is inherently ambiguous provided we establish that its generating function is a ***transcendental function***.

Proofs of transcendence for analytic functions appear to be fortunately appreciably simpler than for real numbers. A method of choice consists in establishing the transcendence of a function by investigating its ***singularities***, in particular showing that it has a ***non-algebraic singularity*** (the way algebraic functions may become singular is well characterized), or infinitely many singularities or even a ***natural boundary***.

In the sequel, we shall state some useful transcendence criteria for establishing inherent ambiguity of context-free languages, and present a number of applications to specific languages.

A note about our presentation: it should be clear in what follows that we have made no attempt at deriving the simplest or most elementary proofs of inherent ambiguities of languages. We have instead tried to demonstrate the variety of techniques that may be employed here as they should also prove useful in future applications. It should also be clear that a very large number of languages are amenable to these techniques and some random sampling has been exercised to keep this paper within reasonable size limits.

2. SOME INHERENTLY AMBIGUOUS LANGUAGES

A context free grammar G is ***ambiguous*** iff there exists a least one word in the language generated by G that can be parsed according to G in two different ways. A context-free language L is ***inherently ambiguous*** iff ***any*** grammar that generates L is ambiguous.

A prototype of an inherently ambiguous language is

$$L = \{ a^m b^n c^p \mid n=m \text{ or } n=p \} \tag{1}$$

and the proof of its inherent ambiguity proceeds by showing, by means of some iteration theorem, that ***any*** grammar for L needs to generate words of the form $a^n b^n c^n$ at least twice for large enough n. (See *e.g.* Harrison's book [10] for similar classical proofs).

In this paper, we propose to prove the inherent ambiguity of a number of languages of various types that are structurally more complex than the above example:

Theorem 1: [Languages with constraints on the number of occurrences of letters] ***The languages O_3, O_4 are inherently ambiguous, where:***

$$O_3 = \{ w \in \{a,b,c\}^* \mid |w|_a = |w|_b \text{ or } |w|_a = |w|_c \}$$

$$O_4 = \{ w \in \{x, \bar{x}, y, \bar{y}\}^* \mid |w|_x = |w|_{\bar{x}} \text{ or } |w|_y = |w|_{\bar{y}} \} .$$

Theorem 2: [Crestin's language formed with products of palindromes] *The language C is inherently ambiguous, where:*

$$C = \{ w_1 w_2 \mid w_1, w_2 \in \{a,b\}^* ; w_1 = w_1^t, w_2 = w_2^t \}$$

with w^t denoting the mirror image of w.

Theorem 3: [A simple linear language] *The language S is inherently ambiguous, where:*

$$S = \{ a^n b v_1 a^n v_2 \mid v_1, v_2 \in \{a,b\}^* \}$$

Theorem 4: [Languages with a comb-like structure] *The languages P_1, P_2 are inherently ambiguous, where:*

$$P_1 = \{ \underline{n}_1 \underline{n}_2 n \cdots \underline{n}_{2k} \mid [\text{for all } j \; n_{2j} = n_{2j-1}] \text{ or } [\text{for all } j \; n_{2j} = n_{2j+1}, n_{2k} = n_1] \}$$

$$P_2 = \{ \underline{n}_1 \underline{n}_2 \cdots \underline{n}k \mid [n_1 = 1, \text{for all } j \; n_{2j} = 2n_{2j-1}] \text{ or } [\text{for all } j \; n_{2j} = 2n_{2j+1}] \}$$

where for integer n, $\underline{n}$ denotes the unary representation of n in the form of $a^n b$.

Theorem 5: [Languages deriving from the Goldstine language] *The languages $G_{\neq}$, $G_<$, $G_>$, $H_{\neq}$ are inherently ambiguous, where:*

$$G_{\neq} = \{ \underline{n}_1 \underline{n}_2 \cdots \underline{n}_p \mid \text{for some } j \; n_j \neq j \}$$

$$G_< = \{ \underline{n}_1 \underline{n}_2 \cdots \underline{n}_p \mid \text{for some } j \; n_j < j \}$$

$$G_> = \{ \underline{n}_1 \underline{n}_2 \cdots \underline{n}_p \mid \text{for some } j \; n_j > j \}$$

$$H_{\neq} = \{ \underline{n}_1 \underline{n}_2 \cdots \underline{n}_p \mid \text{for some } j \; n_j \neq p \}$$

Theorem 6: [Languages obeying local constraints] *The languages K_1, K_2 are inherently ambiguous, where:*

$$K_1 = \{ \underline{n}_1 \underline{n}_2 \cdots \underline{n}_k \mid \text{for some } j \; n_{j+1} \neq n_j \}$$

$$K_2 = \{ \underline{n}_1 \underline{n}_2 \cdots \underline{n}_k \mid \text{for some } j \; n_{j+1} \neq 2n_j \}$$

Theorem 7: [A language based on binary representations of integers] *The language B is inherently ambiguous, where:*

$$B = \{ \underset{\sim}{n}_1 \underset{\sim}{n}_2 \cdots \underset{\sim}{n}_k \mid n_1 \neq 1 \text{ or for some } j \; n_{j+1} \neq n_j + 1 \}$$

in which $\underset{\sim}{n}$ denotes the binary representation of integer n followed by a marker.

Many of these results are actually known but have been included here for the sake of illustrating the power of the methods we employ: the case of languages like O_2, O_3 is easily reduced to the ambiguity of languages like L defined in (1). The language C has been studied combinatorially by Crestin [7] who proved that it is of *inherent unbounded ambiguity*. We establish here the transcendence of its generating function, which proves a conjecture of Kemp. The result concerning language S is a weaker form of a result due to Shamir [17] by which

$$\{ u c v_1 u^t v_2 \mid u, v_1, v_2 \in \{a,b\}^* \}$$

is infinitely ambiguous. The language P_2 has been studied by Kemp [12] who proved that the asymptotic density of a closely related language is transcendental, thereby establishing its ambiguity.

The other cases seem to be new. In particular, the case of the language $G_{\neq}$, which is exactly the Goldstine language, solves the conjecture of Autebert *et al.*. Although it seems quite plausible at first sight that such languages must be inherently ambiguous, the difficulty owes to the fact that when attempting to apply iteration theorems, (like Ogden's lemma), some of them (most notably the Goldstine language) behave almost like regular languages.

3. AN OVERVIEW OF TRANSCENDENCE CRITERIA USED FOR ESTABLISHING INHERENT AMBIGUITY

To any language $L \subset A^*$ (A a finite alphabet) we associate its ***enumeration sequence*** defined by:

$$l_n = card\{ w \in L \mid |w| = n \} .$$

This sequence is characterized by its generating function, called the ***generating function*** of language L:

$$l(z) = \sum_{n \geq 0} l_n z^n .$$

This function is an analytic function in a neighbourhood of the origin, and its radius of convergence ρ satisfies:

$$\frac{1}{card\ A} \leq \rho \leq 1 .$$

Consideration of analytical properties of function $l(z)$ or, in an often equivalent manner, of asymptotic properties of the sequence $\{l_n\}$ permits in a number of cases to establish inherent ambiguity of the context-free language L by means of the following classical theorem of Chomsky and Schutzenberger [5]:

Theorem: *Let $l(z)$ be the generating function of a context-free language L. If L is unambiguous, then $l(z)$ is an algebraic series (function) in one variable.*

This classical theorem is established in a constructive manner by transforming an unambiguous definition of the language into a set of polynomial equations. It will be used in the sequel under the trivially equivalent form:

Corollary: *If the generating function $l(z)$ of a context-free language L is transcendental, then L is inherently ambiguous.*

The above corollary (see [14] for general information on languages and formal power series) therefore permits to conclude as to the inherent ambiguity of a language provided the following two conditions are met:

(i) [The counting condition]: One has at one's disposal a combinatorial decomposition of the language, in a way that gives access to the sequence l_n and permits to "express" $l(z)$.

(ii) [Transcendence condition]: A transcendence criterion is available to establish the non-algebraic character of $l(z)$.

We now proceed with the statement of a few simple transcendence criteria of which applications will be given in the following sections. The first one is obvious:

Theorem A: *Let $l(z)$ be an algebraic series of $Q[[z]]$, ω an algebraic number. Then $l(\omega)$ is algebraic.*

Criterion A: [Transcendence of values at an algebraic point] *If $l(z)$ is a series of $Q[[z]]$, and if $l(\omega)$ is transcendental for some algebraic ω, then $l(z)$ is transcendental*

Theorem B: *An algebraic function $l(z)$ defined by an equation*

$$P(z, l(z)) = 0$$

has a finite number of singularities that are algebraic numbers satisfying the equation:

$$\text{Resultant}_y [P(z,y), \frac{\partial P(z,y)}{\partial y}] = 0 .$$

This last result is of course a very classical one (see for instance Seidenberg's book [16]) and comes from the implicit function theorem for analytic functions.

Criterion B: *A function having infinitely many singularities (for instance a natural boundary) is transcendental.*

In the sequel, this result is used to establish the ambiguity of Crestin's language C taking advantage of Kemp's determination of its generating function which appears to have infinitely many singularities. Other applications stem from the existence of natural boundaries for ***lacunary series*** (as an application of theorems of Hadamard, Borel and Fabry).

A more refined way of establishing the transcendence of a series consists in observing the appearance of transcendental elements in local expansions around a singularity. Indeed, for an algebraic function, one has:

Theorem C: *If $l(z)$ is algebraic it admits in the vicinity of a singularity a fractional power series expansion of the type*

$$l(z) = \sum_{k \geq -m} a_k (1-\frac{z}{\alpha})^{kr} , \quad m \in \mathbf{N}, r \in \mathbf{Q}^+ ,$$

where the a_k are algebraic.

The above expansion is nothing but the familiar Puiseux expansion of an algebraic function.

Criterion C: *If $l(z)$ has in the vicinity of a singularity an asymptotic equivalent that is not of the form:*

$$\omega (1-\frac{z}{\alpha})^r$$

with ω and α algebraic and r rational, then $l(z)$ is transcendental.

It is also well known that the local behaviour of a function in the vicinity of its singularities is closely reflected by the asymptotic behaviour of its Taylor coefficients. Corresponding "transfer" lemmas rely on contour integration techniques, like the Darboux method.

Theorem D: *If $l(z)$ is an algebraic function that is analytic at the origin, then its n Taylor coefficient l_n has an asymptotic equivalent of the form:*

$$l_n = \frac{\alpha^n n^s}{\Gamma(s+1)} \sum_{i \geq 0}^{m} C_i \omega_i^n + O(\alpha^n n^t) , \qquad (\Delta)$$

where $s \in \mathbf{Q} / \{-1,-2,-3, \cdots\}$, $t < s$, α is an algebraic number and the C_i, ω_i are algebraic with $|\omega_i| = 1$.

Criterion D: *Let $l(z)$ be a function analytic at the origin; if its Taylor coefficients l_n do not satisfy an asymptotic expansion of type (Δ), then $l(z)$ is transcendental.*

In passing, Criterion D generalizes a result of Berstel [4] who observed that if there exists an integer α such that the limit

$$\lambda = \lim \frac{l_n}{\alpha^n}$$

exists and is a transcendental number, then $l(z)$ is a transcendental function, so that L cannot be an unmabiguous context-free language. Theorem D does provide a ***generalized density*** characterization for unambiguous context-free languages that extends Berstel's results.

A particularly useful set of applications of Theorem D is for coefficients with asymptotic equivalents of the form:

$$l_n \sim \gamma \alpha^n n^r .$$

If either r is irrational, α transcendental or $\gamma\Gamma(r+1)$ is transcendental, then $l(z)$ is a transcendental function. Therefore the following asymptotic behaviours are characteristic of transcendental functions:

$$O(e^n n^r) \; ; \; O(\alpha^n n^{\sqrt{2}}) \; ; \; O(\frac{\alpha^n}{n}) \; ; \; O(\frac{\alpha^n}{n^2}) \; ; \; \pi^{\frac{1}{2}} 4^n n^{-\frac{3}{2}} \; \cdots$$

(Notice however for the last example that $\pi^{-\frac{1}{2}} 4^n n^{-\frac{3}{2}}$ *does* occur in the expansion of algebraic functions.)

The last batch of methods is based on a theorem by Comtet [6] that any algebraic function satisfies a linear differential equation with polynomial coefficients, a fact which is reflected on its Taylor coefficients by:

Theorem E: *If $l(z)$ is algebraic, then there exist a set of polynomials $p_0(z), \cdots, p_m(z)$ such that:*

$$\sum_{j=0}^{m} p_j(z) \frac{d^j l(z)}{dz^j} = 0 .$$

Thus, there exist a set of polynomials $q_0(u), \cdots, q_m(u)$ such that for all $n \geq n_0$:

$$\sum_{j=0}^{m} q_j(n) l_{n-j} = 0 .$$

Criterion E: *Let $l(z)$ be an analytic function. If there does not exist a finite sequence of polynomials $q_0, q_1, \cdots, q_m$ such that for n large enough:*

$$\sum_{j=0}^{m} q_j(n) l_{n-j} = 0 ,$$

then $l(z)$ is transcendental.

This criterion comes as a useful complement (or as an alternative) to transcendence proofs based on lacunary series mentioned in relation to Criterion B since it applies obviously to any series

$$\sum_{n\geq 0} a_n x^{c_n}$$

such that $\sup(c_{n+1}-c_n) = +\infty$.

4. TRANSCENDENCE OF VALUES OF GENERATING FUNCTIONS

This method is in principle the most direct. However, in practice, it turns out to be rather hard to apply because of the relative scarcity of transcendence results for real numbers. It is applied here to the following languages:

1. The language O_4 is by definition the union of two context-free languages whose intersection has a generating function that is an ***elliptic integral***. Using a classical result in transcendence theory concerning values of such integrals at algebraic points [15,II,Sect. 4], we establish the transcendence of the generating function of O_4.
2. Language P_2: it has a generating function whose expression involves the ***Fredholm series***:

   $$F(x) = \sum_{n\geq 0} x^{2^n}$$

   and the approximation theorem of Thue-Siegel-Roth [15,I,Sect. 6, Th. 8] shows the value of the series to be transcendental at any point $x=\frac{1}{q}$ for integral q.

This last example is inspired by the construction due to Kemp [12] of a context-free language with a transcendental density.

5. FUNCTIONS WITH INFINITELY MANY SINGULARITIES

Criterion B expresses that any function with infinitely many singularities is transcendental. Such a property may either be apparent on the very expression of the function or it may result from theorems on ***lacunary series***.

1. The language S: a decomposition reveals that its generating function involves rationally the quantity:

   $$s(z) = z\sum_{k\geq 0} \frac{z^{2k}}{1-2z+z^{k+1}},$$

   a series which not too surprisingly is related to statistics on ***"runs"*** (repetitive sequences) [8], and accordingly occurs in an analysis by Knuth of carry propagation. It is easy to check that $s(z)$ has infinitely many poles around $\frac{1}{2}$ satisfying:

   $$z_k = \frac{1}{2}+2^{-k-1}+o(2^{-k})$$

   and it is therefore transcendent.

2. The language C has been introduced by Crestin [7] and Kemp [11] has shown that its generating function is:

$$l(z) = 1+2\sum_{m\geq 1}\psi(m)\frac{z^m(1+z^m)(1+2z^m)}{(1-2z^{2m})^2}$$

where $\psi(m)=\prod(1-p)$, the product being extended to all prime divisors of m. From that expressions easily follows that $l(z)$ has for $|z|\leq 1$ isolated singularities at points:

$$z_{m,j} = 2^{\frac{1}{2m}}e^{\frac{ij\pi}{m}} .$$

3. The Goldstine language $G_{\neq}$ has a generating function related to the ***theta functions***:

$$l(z) = \frac{1-z}{1-2z} - \sum_{j\geq 0} z^{j(j+1)}$$

which is clearly non algebraic as can be checked since, for instance, from the theory of lacunary series the sum has the circle $|z|=1$ as a natural boundary. The same treatment applies to $H_{\neq}$.

This last case is the one that initially motivated our study. The reader may consult [2] for some related enumeration issues.

6. LOCAL BEHAVIOUR AROUND SINGULARITIES

This is certainly the most comfortable method to apply. The mere appearance of logarithmic terms in the local expansion of a function around a singularity is sufficient to establish its transcendence. Such local analyses may often be treated by Mellin transform techniques, a fact not too surprising considering the arithmetical character of many of the languages we study.

1. The language K_1 has a generating function whose expression involves the ***Lambert series*** associated to the arithmetical ***divisor function***:

$$D(z) = \sum \frac{z^n}{1-z^n} .$$

The Mellin transform of $D(e^{-t})$ is

$$\int_0^\infty D(e^{-t})t^{s-1}dt = \zeta^2(s)\Gamma(s) ,$$

from which we obtain through a residue calculation:

$$D(z) \sim (z-1)\log(1-z) ,$$

as z tends to 1, a typically transcendent behaviour.

2. The proofs for K_2, P_1 follow by similar arguments.

7. GENERALIZED ASYMPTOTIC DENSITIES

These are based on the existence of generalized densities corresponding to expansion (Δ) of Theorem D.

1. The language $G_>$ has a ***bivariate*** generating function (recording the number of a's and b's in words) that involves the function:

$$b(x,y) = \sum_{j\geq 1} [y^j \prod_{k=1}^{j} \frac{1-x^j}{1-x}] .$$

The function $c(x,y)=b(x,y(1-x))$ has the same transcendence status as $b(x,y)$. The transcendence of c is in turn easily related to that of the function:

$$E(\overline{x}) = \prod_{j\geq 1} (1-x^j)^{-1}$$

which occurs in the theory of ***integer partitions***. Finally, the transcendence of $E(x)$ we may itself establish if we wish by considering the asymptotic form of its Taylor coefficients that is given by a celebrated theorem of Hardy and Ramanujan [9]:

$$E_n \sim \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4n\sqrt{3}} .$$

8. POLYNOMIAL RECURRENCES

This last part corresponds to Comtet's theorem:

1. The language B has a generating function where there intervenes the series

$$\Lambda(z) = \sum_{n\geq 1} z^{\lambda(n)} \qquad \lambda(n) = 2n + \sum_{p\leq n} [\log_2 p]$$

a series which because of large gaps cannot have coefficients that satisfy any fixed order polynomial recurrence.

2. Other cases involving theta series, like the Goldstine language, can be also treated in this way.

9. OPEN PROBLEMS

The following problems naturally suggest themselves: (1) Are there sufficient conditions on generating functions to ensure that a language is ***infinitely*** inherently ambiguous? (2) In which class of transcendental functions do generating functions of (ambiguous) context-free languages lie? For instance, in our work we came nowhere close to expressions involving the exponential function. (3) Accordingly are there results on generalized densities of (ambiguous) context-free languages? For instance, can the number of words of size n in a context-free language grow like $\exp(\sqrt{n})$?

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