On the equiconvergence of eigenfunction expansions associated with ordinary linear differential operators

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Résumé

On considère les séries de Fourier d'une fonction de carré sommable, correspondant aux systèmes orthonormaux complets constitués des fonctions propres des opérateurs différentiels linéaires ordinaires. Toutes ces séries se comportent localement de la même manière. Le résultat reste valable pour les bases de Riesz.

Abstract

We consider the Fourier series of a square-integrable function, corresponding to the complete orthonormal systems consisting of eigenfunctions of ordinary linear differential operators. All these series behave locally in the same way. The result remain valid for Riesz bases, too.

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The equiconvergence theorems are very useful in the spectral investigation of differential operators because many results known for the most special operators may be transferred by their application to more general ones. For the case of orthonormal bases consisting of eigenfunctions of second-order operators several results have been obtained since the beginning of this century, see e.g. [1],[2],[3],[5],[7],[8],[14],[19],[22],[23]. All these results are contained in a result of I. Joó and the author in [10]. It concerns also the non-selfadjoint case i.e. when eigenfunctions of higher order are also used and when the system is not orthonormal but only a Riesz basis. (On the existence of such Riesz bases see [4],[11],[20],[21].) The proof was based on an efficient method due to V. A. Il'in (see e.g. [2]) of the constant application of some mean value formulas.

The aim of the present paper is to extend this result for differential operators of higher order. In some special cases this was already done in [15]. Our main tool will be a generalized Titchmarsh type formula derived in [12]. We note that it is not a mean value formula if the order of the differential operator is odd. In some cases a simpler expression was found for its coefficients by I. Joó in [9]; these expressions are important because they make possible to obtain sharp estimates for the coefficients. We shall need several results proved in the papers [10],[17] and [18], too.

By and large the following result will be proved: all Riesz bases (and in particular all orthonormal bases) consisting of eigenfunctions (maybe also of higher order) of some n-order linear differential operator locally behave in the same way. Here we stress two circumstances:

- there are no assumptions on the distribution of the eigenvalues: they can be arbitrary complex numbers;
- there are no boundary conditions.

As an immediate consequence of this result we note that (for example) Carleson's theorem remains valid for "all" eigenfunction expansions.

All the preliminary results used in this paper are contained in [10], [12], [17] and [18]. All the results of the papers [12] and [17] are needed. From [18] we need the case $i = 0$ of Theorem 3; for its proof it is not necessary to apply the results of the paper [16]. From [10] we use only a result of technical character (Lemma 6). For the reader's convenience we collect in section 2 all preliminary results used in this paper.

1. Formulation of the main result

Let G be an open interval on the real line, n a natural number, $q_s \in H^{n-s}_{loc}(G)$ a complex function $(s = 2,\dots,n)$ and consider the differential operator

$$L u = u^{(n)} + q_2 u^{(n-2)} + q_3 u^{(n-3)} + \dots + q_n u$$

defined on $H^n_{loc}(G)$. (Recall that, by definition, $H^k_{loc}(G)$ is the set of all complex functions $v \in L^2_{loc}(G)$ having distributional derivatives in $L^2_{loc}(G)$ of order up to k.)

As usual, a function $u \not\equiv 0$ is called an eigenfunction of order 0 (of the operator L) with some eigenvalue $\lambda \in \mathbb{C}$ if

$$L u = \lambda u .$$

Furthermore, a function u is called an eigenfunction of order k (of the operator L) with some eigenvalue $\lambda \in \mathbb{C}$ $(k = 1,2,\dots)$ if the function

$$u^* := L u - \lambda u$$

is an eigenfunction of order $k-1$ with the same eigenvalue λ.

Let ~~us~~ be now given a system $(u_r)_{r=1}^{\infty}$ of eigenfunctions and denote o_r (resp. λ_r) the order (resp. the eigenvalue) of u_r. Assume that the following three conditions are satisfied:

(C 1) (u_r) is a Riesz basis i.e. (u_r) is the image of an orthonormal basis under a linear topological isomorphism of $L^2(G)$;

(C 2) $\sup o_r < \infty$;

(C 3) if $o_r \geqslant 1$ then $u_r^* = u_s$ for some index s .

By (C 1) there exists a unique system (v_r) in $L^2(G)$ such that

$$\langle u_r, v_s \rangle = \delta_{rs} \quad \text{(the Kronecker symbol)} .$$

Introduce the following notations:

$$(1) \quad |\nu_r| = \max \left\{ |\mathrm{Im}\, \mu| : \mu \in \mathbb{C} \text{ and } \mu^n = \lambda_r \right\} ,$$

$$\sigma_\nu(f,x) = \sum_{|\nu_r| < \nu} \langle f, v_r \rangle \, u_r(x)$$

$$\left(\nu > 0, \ f \in L^2(G), \ x \in G \right) ,$$

$$S_\nu(f,x,R) = \int_{x-R}^{x+R} \frac{\sin \nu(y-x)}{\pi(y-x)} f(y)\, dy$$

$$\left(\nu > 0, \ f \in L^2(G), \ x \pm R \in G \right) .$$

The aim of this paper is to prove the following result:

T h e o r e m . To any compact subinterval K of G there exists a number $R_0 > 0$ such that

$$\lim_{\nu \to \infty} \ \sup_{x \in K} \left| S_\nu(f,x,R) - \sigma_\nu(f,x) \right| = 0$$

whenever $f \in L^2(G)$ and $0 < R < R_0$. $\square$

Remarks.

(i) We note that $S_\gamma(f,x,R)$ does not depend on the system (u_r).

(ii) The conditions of the theorem are very weak. The assumptions (C 1) - (C 3) are practically satisfied for ~~all known~~ many Riesz bases of eigenfunctions. We emphasize that there are no assumptions on the distribution of the eigenvalues λ_r. Several sufficient conditions are known for the existence of orthonormal bases of eigenfunctions (see e.g. [21]), and more generally, for the existence of Riesz bases of eigenfunctions (see [4], [11], [20]).

(iii) For second-order operators several equiconvergence theorems were proved from the beginning of this century, see e.g. [1], [2], [3], [5], [7], [8], [14], [19], [22]. These results are contained in a theorem of Joó and Komornik, proved in [10] by developing an important method of V. A. Il'in [2]. This result is also a slightly stronger than the case $n = 2$ of the above theorem: instead of $q_2 \in L^2_{loc}(G)$ it was sufficient to assume that $q_2 \in L^1_{loc}(G)$.

(iv) The proof of the just mentioned result of Joó and Komornik is not applicable for the general case. However, by integrating by parts we obtain the desired estimates also in this case. On the other hand, in [12] a new method (based on a suitable generalization of the well-known Titchmarsh formula) was developed for the spectral investigation of n-order differential operators. Using this method, several results were proved, see e.g. [12] - [18]. The present paper represents a new evidence for the efficiency of this method.

Some special cases of the theorem of the present paper were proved in [15].

2. Preliminary results

A./ We shall need the following estimate, being a consequence of some results of the papers [17], [18] : putting

$$(2) \qquad |\varrho_r| = \min \{ |\mathrm{Re}\,\mu| : \mu \in \mathbb{C} \text{ and } \mu^n = \lambda_r \},$$

to any compact intervals $K_1 \subset G$, $K_2 \subset \operatorname{int} K_1$ there exists a positive constant ε_o such that

$$(3) \qquad \|u_r\|_{L^\infty(K_2)} e^{|\varrho_r| \cdot \varepsilon_o} \leq \frac{1}{\varepsilon_o} \|u_r\|_{L^2(K_1)} \qquad (r = 1,2,\dots).$$

B./ In [12] we derived a generalization of the well-known Titchmarsh formula for n-order operators; the results cited in A./ were proved by the use of this formula. Now we need another generalization of the Titchmarsh formula.

Denote $\omega_1, \dots, \omega_n$ the n-th roots of unity and set $m = \left[\frac{n+1}{2}\right]$. For $0 \neq \mu \in \mathbb{C}$, $t > 0$ and $-mt < y < (n-m)t$ we denote by $f_k(\mu, t)$ the elementary symmetric polynomial of degree $m-k$ of the variables $e^{\mu\omega_1 t}, \dots, e^{\mu\omega_n t}$ with the main coefficient $(-1)^k$ $(k = m-n, \dots, m)$ and we put

$$F(\mu, t, y) = \sum_{\ell = 1 + \left[\frac{-y}{t}\right]}^{m} f_k(\mu, t) \sum_{p=1}^{n} \frac{\omega_p}{n\mu^{n-1}} e^{\mu\omega_p(y+kt)}.$$

One can easily see that f_k and F can be continuously extended for all $\mu \in \mathbb{C}$, $t \geqslant 0$ and $-mt \leq y \leq (n-m)t$. Furthermore, the extended function F has the following properties for any fixed $\mu \in \mathbb{C}$ and $t > 0$:

(4) $F(\mu,t,\cdot)$ is $n-2$ times continuously differentiable in $(-mt, (n-m)t)$ and

$$D_3^i F(\mu,t,-mt+0) = D_3^i F(\mu,t,(n-m)t-0) = 0 \quad (i = 0,\dots,n-2).$$

(5) $F(\mu,t,\cdot)$ is n times continuously differentiable in $(kt, (k+1)t)$ and $D_3^n F = \mu^n F \quad (-m \leq k \leq n-m-1)$.

(6) $$-D_3^{n-1} F(\mu,t,(n-m)t-0) = f_{m-n}(\mu,t),$$

$$D_3^{n-1} F(\mu,t,-kt+0) - D_3^{n-1} F(\mu,t,-kt-0) = f_k(\mu,t)$$

$$(m-n < k < m),$$

$$D_3^{n-1} F(\mu,t,-mt+0) = f_m(\mu,t).$$

Using these properties and integrating by parts we obtain for any $u \in H^n_{loc}(G)$ the formula

(7) $$\sum_{\ell=m-n}^{m} f_k(\mu,t)\, u(x+kt) + \int_{x+(m-n)t}^{x+mt} F(\mu,t,x-\tau)\left[\mu^n u(\tau) - u^{(n)}(\tau)\right] d\tau =$$

whenever $t > 0$, $x+(m-n)t \in G$ and $x+mt \in G$.

C./ Apply the formula (7) to the eigenfunction u_r. Denoting by μ_r an arbitrary n-th root of λ_r, we obtain

$$(8)\quad \sum_{k=m-n}^{m} f_k(\mu_r,t)\, u_r(x+kt) + \int_{x+(m-n)t}^{x+mt} F(\mu_r,t,x-\tau)\left[\sum_{s=2}^{n} q_s(\tau)u_r^{(n-s)}(\tau) - u_r^{*}(\tau)\right] d\tau = 0$$

whenever $t > 0$, $x+(m-n)t \in G$ and $x+mt \in G$.

For $n = 2$ and $o_r = 0$ this reduces to the Titchmarsh formula. For $n = 2$ and $o_r \neq 0$ it was found by Joó [6]. For $o_r = 0$, n arbitrary (then $u_r^* \equiv 0$) the formula (8) is a special case of a more general formula derived in [12]. We note that the above simple form of the coefficients f_k (which has great importance to obtain some estimates in the sequel) was proved by Joó [9].

We shall frequently use two equivalent forms of the formula (8). Index the n-th roots of λ_r such that

$$\mathrm{Re}\,\mu_{r,1} \geq \dots \geq \mathrm{Re}\,\mu_{r,n}$$

and put $\mu_r = \mu_{r,m}$, $\varrho_r = \mathrm{Re}\,\mu_r$, $\nu_r = \mathrm{Im}\,\mu_r$. These notations are in keeping with the former ones used in (1), (2), (3) and (8). Denote $g_k(\mu_r,t)$ and $G(\mu_r,t,y)$ (resp. $h_k(\mu_r,t)$ and $H(\mu_r,t,y)$) the functions obtained from $f_k(\mu_r,t)$ and $F(\mu_r,t,y)$ by dividing by $e^{(\mu_{r,1}+\dots+\mu_{r,m-1})t}$ (resp. $e^{(\mu_{r,1}+\dots+\mu_{r,m})t}$).

Then from (8) we obtain the following two formulas:

$$(9)\quad \sum_{k=m-n}^{m} g_k(\mu_r, t)\; u_r(x+kt) + \int_{x+(m-n)t}^{x+mt} G(\mu_r, t, x-\tau)\left[\sum_{s=2}^{n} q_s(\tau) u_r^{(n-s)}(\tau) - u_r^{*}(\tau)\right] d\tau = 0 ,$$

$$(10)\quad \sum_{k=m-n}^{m} h_k(\mu_r, t)\; u_r(x+kt) + \int_{x+(m-n)t}^{x+mt} H(\mu_r, t, x-\tau)\left[\sum_{s=2}^{n} q_s(\tau) u_r^{(n-s)}(\tau) - u_r^{*}(\tau)\right] d\tau = 0 .$$

D./ It follows obviously from (4) that

$$(11)\quad D_3^i\, G(\mu_r, t, -mt+0) = D_3^i\, G(\mu_r, t, (n-m)t-0) = 0 ,$$

$$D_3^i\, H(\mu_r, t, -mt+0) = D_3^i\, H(\mu_r, t, (n-m)t-0) = 0$$

$$\left(0 \leq i \leq n-2 ,\quad r = 1,2,\ldots\right) .$$

The following estimates follow directly from the definition of the coefficients in the formulas (8), (9), (10), we refer to [17] for some details . In all these estimates we assume that $\varrho_r \geqslant 0$ and $t > 0$.

5] and

$$(12)\quad g_k(\mu_r, t) ,\; h_k(\mu_r, t) ,\; D_2 g_k(\mu_r, t) \text{ and } D_2 h_k(\mu_r, t)$$

tend to 0 if $|k| \geqslant 2$ and $|\mu_r t| \longrightarrow \infty$.

(13) $g_1(\mu_r, t)$ and $g_{-1}(\mu_r, t)$ remain bounded,

$D_2 g_1(\mu_r, t)$, $D_2 g_{-1}(\mu_r, t)$ and $g_0(\mu_r, t) - e^{\mu_r t}$

tend to 0 if $\varrho_r t \longrightarrow \infty$ and $\dfrac{|\nu_r|}{\varrho_r} \longrightarrow \infty$.

(14) $h_1(\mu_r, t)$, $h_{-1}(\mu_r, t)$, $D_2 h_1(\mu_r, t)$, $D_2 h_{-1}(\mu_r, t)$ and

$h_0(\mu_r, t) - 1$ tend to 0 if $\varrho_r t \longrightarrow \infty$ and

$\dfrac{|\nu_r|}{\varrho_r}$ remains bounded.

(15) $g_{-1}(\mu_r, t)$, $g_1(\mu_r, t)$ and $h_{-1}(\mu_r, t)$ remain bounded,

$D_2 g_{-1}(\mu_r, t)$, $D_2 g_1(\mu_r, t)$, $g_1(\mu_r, t) + 1$,

for n odd $D_2 h_{-1}(\mu_r, t)$ and $h_0(\mu_r, t) - 1$,

for n even $g_{-1}(\mu_r, t) + 1$

tend to 0 if $|\nu_r t| \longrightarrow \infty$ and $\varrho_r t$ remains

bounded.

(16) For any real number ν the fractions

$$\frac{g_0(\mu_r, t) - g_0(i\nu, t)}{t} \quad \text{and} \quad \frac{h_1(\mu_r, t) - h_1(i\nu, t)}{t}$$

remain bounded (uniformly in ν) if $\varrho_r t$ and $|\mu_r - i\nu|$ remain bounded.

$$(17) \qquad \frac{D_3^i\, G(\mu_r, t, y)}{|\mu_r|^{i+1-n}\, e^{\varrho_r (t-|y|)}} \quad \text{and} \quad \frac{D_3^i\, H(\mu_r, t, y)}{|\mu_r|^{i+1-n}\, e^{-\varrho_r |y|}}$$

are uniformly bounded $(i = 0,\dots,n-1,\ r = 1,2,\dots)$.

E./ It follows from Theorem 2 in [12] that for any compact subinterval K of G there exists a constant $C > 0$ such th

$$\|u_r'\|_{L^\infty(K)} \leq C\,(1+|\mu_r|)\cdot\|u_r\|_{L^\infty(K)} \qquad (r = 1,2,\dots).$$

F./ Finally we recall two important properties of the Riesz bases: the generalized Bessel inequality and the generalized Parseval identity. First, there exists a constant C such that

$$(18) \qquad \sum_{r=1}^{\infty} |\langle u_r, w\rangle|^2 \leq C\, \|w\|^2_{L^2(G)} \qquad \forall\ w \in L^2(G).$$

secondly,

$$(19) \qquad \langle f,w\rangle = \sum_{r=1}^{\infty} \langle f,v_r\rangle \langle u_r,w\rangle \qquad \forall\ f,w \in L^2(G).$$

3. Estimation of the sum of squares of the eigenfunctions

In this section, under the conditions of the Theorem we shall prove the following strong estimate:

P r o p o s i t i o n . To any compact subinterval K of G there exists a positive number ε such that

$$\sup_{\nu \geq 1} \sum_{|\nu - |\nu_r|| \leq 1} \|u_r\|^2_{L^\infty(K)} \, e^{|\varrho_r| \cdot \varepsilon} < \infty \quad . \square$$

This result will follow from several lemmas.

L e m m a 1 .

$$\sup_{\nu \geq 1} \sum_{\substack{|\nu - \nu_r| \leq 1 \\ \nu_r \geq A \cdot \varrho_r \\ \varrho_r \geq A}} \left(\frac{\|u_r\|_{L^2(K)}}{\varrho_r} \right)^2 = \underline{\underline{o}}(1) \quad (A \longrightarrow \infty) .$$

P r o o f . Fix $R > 0$ such that $K_{2nR} \subset G$ where $K_\delta := \{x : \operatorname{dist}(x,K) \leq \delta\}$. For any $\nu \geq 1$, $x \in K$, $R \leq t \leq 2R$ and $r \in I_\nu(A) := \{r : |\nu - \nu_r| \leq 1,\ \nu_r \geq A\,\varrho_r,\ \nu_r \geq A\}$, by the application of the formula (9) we obtain

$$- \int_R^{2R} e^{-i\nu t} g_0(\mu_r, t)\, dt \; u_r(x)$$

$$= \sum_{\substack{m-n \leq k \leq m \\ k \neq 0}} \int_R^{2R} e^{-i\nu t} g_k(\mu_r, t)\, u_r(x+kt)\, dt$$

$$- \int_R^{2R} e^{-i\nu t} \int_{x+(m-n)t}^{x+mt} G(\mu_r, t, x-\tau) \left[u_r^*(\tau) - \sum_{s=2}^{n} q_s(\tau)\, u_r^{(n-s)}(\tau) \right] d\tau\, dt .$$

Integrating by parts and using (11) hence we obtain

$$- \int_R^{2R} e^{-i\nu t} g_0(\mu_r, t)\, dt\, u_r(x)$$

$$= \sum_{\substack{m-n \le k \le m \\ k \ne 0}} g_k(\mu_r, 2R) \int_R^{2R} e^{-i\nu t} u_r(x+kt)\, dt$$

$$- \sum_{\substack{m-n \le k \le m \\ k \ne 0}} \int_R^{2R} D_2 g_k(\mu_r, t) \int_R^{t} e^{-i\nu\xi} u_r(x+k\xi)\, d\xi\, dt \quad .$$

$$+ \sum_{s=2}^{n} \sum_{i=0}^{n-s} (-1)^i \binom{n-s}{i} \int_R^{2R} e^{-i\nu t} \int_{x+(m-n)t}^{x+mt} D_3^{n-s-i+1} G(\mu_r, t, x-\tau)$$

$$\cdot \int_{x+(m-n)t}^{\tau} q_s^{(i)}(\xi) u_r(\xi)\, d\xi\, d\tau\, dt$$

$$- \int_R^{2R} e^{-i\nu t} \int_{x+(m-n)t}^{x+mt} D_3\, G(\mu_r, t, x-\tau) \int_{x+(m-n)t}^{\tau} u_r^*(\xi)\, d\xi\, d\tau\, dt \quad .$$

The following estimates will be uniform in ν , x , r , t , τ when $A \longrightarrow \infty$. Using the estimates (12), (13), (17), with suitably defined functions w_1, w_2, w_3, $w_4 \in L^2(G)$ (which depend also on ν, x, t, τ, k , s , i) we obtain

$$(1-\bar{\bar{o}}(1)) \left| \frac{u_r(x)}{\varrho_r} \right| \le \sum_{\substack{m-n \le k \le m \\ k \ne 0}} O(1) \left| \langle w_1, u_r \rangle \right|$$

$$+ \sum_{\substack{m-n \le k \le m \\ k \ne 0}} \int_R^{2R} O(1) \left| \langle w_2, u_r \rangle \right| dt$$

$$+ \sum_{s=2}^{n} \sum_{i=0}^{n-s} \int_R^{2R} \int_{x+(m-n)t}^{x+mt} O(1) \left| \langle w_3, u_r \rangle \right| d\tau\, dt$$

$$+ \int_R^{2R} \int_{x+(m-n)t}^{x+mt} O(1) \left| \langle w_4, u_r^* \rangle \right| d\tau\, dt \quad .$$

Taking the square of this inequality, summarizing for $r \in I_\nu(A)$ using (18) and (C3) we obtain

$$\sum_{r \in I_\nu(A)} \left| \frac{u_r(x)}{\varrho_r} \right|^2 \leq \sum_{\substack{m-n \leq k \leq m \\ k \neq 0}} O(1) \; \|w_1\|^2_{L^2(G)}$$

$$+ \sum_{\substack{m-n \leq k \leq m \\ k \neq 0}} \int_R^{2R} O(1) \; \|w_2\|^2_{L^2(G)} \; dt$$

$$+ \sum_{s=2}^{n} \sum_{i=0}^{n-s} \int_R^{2R} \int_{x+(m-n)t}^{x+mt} O(1) \; \|w_3\|^2_{L^2(G)} \; d\tau dt$$

$$+ \int_R^{2R} \int_{x+(m-n)t}^{x+mt} O(1) \; \|w_4\|^2_{L^2(G)} \; d\tau dt \quad .$$

Furthermore, one can easily see that

$$\|w_i\|^2_{L^2(G)} = O(1) \quad , \; i = 1,2,3,4 \quad ,$$

therefore

$$\sum_{r \in I_\nu(A)} \left| \frac{u_r(x)}{\varrho_r} \right|^2 = O(1) \quad .$$

Integrating on K, we obtain the required estimate. □

L e m m a 2 .

$$\sup_{\nu \geq 1} \sum_{\substack{|\nu+\nu_r| \leq 1 \\ \nu_r \leq -A\varrho_r \\ \varrho_r \geq A}} \left(\frac{\|u_r\|_{L^2(K)}}{\varrho_r} \right)^2 = O(1) \qquad (A \longrightarrow \infty) .$$

P r o o f . Quite similar to that of Lemma 1, replacing $e^{-i\nu t}$ by $e^{i\nu t}$. □

L e m m a 3 . For any fixed $A > 0$ we have

$$\sum_{\substack{|\nu_r| < A\cdot\varrho_r \\ \varrho_r \geq B}} \|u_r\|^2_{L^2(K)} = \underline{\underline{O}}(1) \ (B \longrightarrow \infty).$$

P r o o f . Fix $R > 0$ such that $K_{2nR} \subset G$. For any $x \in K$, $R \leq t \leq 2R$, $r \in I(B) := \{ r: |\nu_r| < A\cdot\varrho_r \ , \ \varrho_r \geq B \}$, applying now the formula (10), integrating by parts and using (11), we obtain

$$- \int_R^{2R} h_0(\mu_r, t)\, dt\, u_r(x) = \sum_{\substack{m-n\leq k\leq m \\ k\neq 0}} h_k(\mu_r, 2R) \int_R^{2R} u_r(x+kt)\, dt$$

$$- \sum_{\substack{m-n\leq k\leq m \\ k\neq 0}} \int_R^{2R} D_2 h_k(\mu_r, t) \int_R^{t} u_r(x+k\xi)\, d\xi\, dt$$

$$+ \sum_{s=2}^{n} \sum_{i=0}^{n-s} (-1)^i \binom{n-s}{i} \int_R^{2R} \int_{x+(m-n)t}^{x+mt} D_3^{n-s-i+1} H(\mu_r, t, x-\tau)$$

$$\cdot \int_{x+(m-n)t}^{\tau} q_s^{(i)}(\xi)\, u_r(\xi)\, d\xi\, d\tau\, dt$$

$$- \int_R^{2R} \int_{x+(m-n)t}^{x+mt} D_3 H(\mu_r, t, x-\tau) \int_{x+(m-n)t}^{\tau} u_r^*(\xi)\, d\xi\, d\tau\, dt \ .$$

The following estimates will be uniform in x , r , t , τ when $B \longrightarrow \infty$. Using the estimates (12), (14), (17), with suitably defined functions $w_5, w_6, w_7, w_8 \in L^2(G)$ (having also the parameters x, t, τ, k, s, i) we obtain

$$\left(1-\bar{o}(1)\right)|u_r(x)| \leq \sum_{\substack{m-n\leq k\leq m \\ k\neq 0}} O(1)\,|\langle w_5, u_r\rangle|$$

$$+ \sum_{\substack{m-n\leq k\leq m \\ k\neq 0}} \int_R^{2R} O(1)\,|\langle w_6, u_r\rangle|\, dt$$

$$+ \sum_{s=2}^{n} \sum_{i=0}^{n-s} \int_R^{2R} \int_{x+(m-n)t}^{x+mt} O(1)\,|\langle w_7, u_r\rangle|\, d\tau\, dt$$

$$+ \int_R^{2R} \int_{x+(m-n)t}^{x+mt} O(1)\,|\langle w_8, u_r^*\rangle|\, d\tau\, dt\ .$$

Furthermore we have

$$\|w_i\|^2_{L^2(G)} = O(1)\ , \quad i = 5,6,7,8$$

and the proof can be finished by the same way as in Lemma 1 . □

L e m m a 4 . For any fixed $B > 0$ we have

$$\sup_{\nu\geq 1} \sum_{\substack{|\nu-\nu_r|\leq 1 \\ 0\leq \rho_r < B \\ \nu_r \geq D}} \|u_r\|^2_{L^2(K)} = \underline{\underline{o}}(1) \quad (D\longrightarrow\infty)\ .$$

P r o o f . Fix $0 < R_0 < \frac{|K|}{4}$ such that $K_{4nR_0} \subset G$ ($|K|$ denotes the length of K). We will show that for any fixed $0 < R < R_0$, we have the estimate

$$(20) \sum_{r \in I} |u_r(y)|^2 \leq C\left(1+\bar{\bar{o}}(1)\right) R \sum_{r \in I} \| u_r \|^2_{L^2(K)} + O(1)$$

$(D \longrightarrow \infty)$ uniformly in $\gamma \geqslant 1$, $y \in K$ and uniformly for any finite subset I of

$$J_y(D) := \left\{ r : |\gamma - \gamma_r| \leq 1,\ 0 \leq \rho_r < B,\ \gamma_r \geqslant D \right\}$$

(C is an absolut constant). Hence the lemma will follow easily. Indeed, integrating on K we obtain

$$\sum_{r \in I} \| u_r \|^2_{L^2(K)} \leq C\, |K| \left(1+\bar{\bar{o}}(1)\right) R \sum_{r \in I} \| u_r \|^2_{L^2(K)} + O(1) .$$

Choose at the beginning of the proof R so small that

$$C\, |K|\, R < \frac{1}{2} ,$$

then, being all the terms finite by the choice of I,

$$\sum_{r \in I} \| u_r \|^2_{L^2(K)} = O(1) ,$$

and, being $I \subset J_y(D)$ arbitrary,

$$\sum_{r \in \mathcal{F}_y(D)} \|u_r\|^2_{L^2(K)} = O(1)$$

as stated in the lemma.

Denote c the centre of K. We prove (20) differently in the following three cases:

a./ $y \geqslant c$,

b./ $y \leqslant c$ and n is even ,

c./ $y \leqslant c$ and n is odd .

a./ Applying the formula (10) with $x = y-t$ we obtain

$$- \int_R^{2R} g_1(\mu_r, t)\, dt\ u_r(y)$$

$$= \int_R^{2R} \left(g_0(\mu_r, t) - g_0(i\nu, t)\right) u_r(y-t)\, dt$$

$$+ \int_R^{2R} g_0(i\nu, t)\ u_r(y-t)\, dt$$

$$+ \sum_{\substack{m-n \le k \le m \\ k \ne 0,1}} g_k(\mu_r, 2R) \int_R^{2R} u_r(y-t+kt)\, dt$$

$$- \sum_{\substack{m-n \le k \le m \\ k \ne 0,1}} \int_R^{2R} D_2 g_k(\mu_r, t) \int_R^t u_r(y-\xi+k\xi)\, d\xi\, dt$$

$$+ \sum_{s=2}^{n} \sum_{i=0}^{n-s} (-1)^i \binom{n-s}{i} \int_R^{2R} \int_{y-t+(m-n)t}^{y-t+mt} D_3^{n-s-i+1} G(\mu_r, t, y-t-\tau)$$

$$\cdot \int_{y-t+(m-n)t}^{\tau} q_s^{(i)}(\xi)\, u_r(\xi)\, d\xi\, d\tau\, dt$$

$$- \int_R^{2R} \int_{y-t+(m-n)t}^{y-t+mt} D_3\, G(\mu_r, t, y-t-\tau) \int_{y-t+(m-n)t}^{\tau} u_r^*(\xi)\, d\xi\, d\tau\, dt \; .$$

The following estimates are uniform in γ, y , r , t , τ when D $\longrightarrow \infty$. Introducing the functions $w_9, \ldots, w_{13} \in L^2(G)$ (depending also on the parameters γ, y , τ, t , k , s , i) in a suitable way, by (12), (15), (16) and (17) we have

$$\left(R-\bar{\bar{o}}(1)\right) \left|u_r(y)\right| \leq C\, R^{3/2} \left\|u_r\right\|_{L^2(K)} + \left|\langle w_9, u_r \rangle\right|$$

$$+ \sum_{\substack{m-n \leq k \leq m \\ k \neq 0,1}} o(1) \left|\langle w_{10}, u_r \rangle\right| + \sum_{\substack{m-n \leq k \leq m \\ k \neq 0,1}} \int_R^{2R} o(1) \left|\langle w_{11}, u_r \rangle\right| dt$$

$$+ \sum_{s=2}^{n} \sum_{i=0}^{n-s} \int_R^{2R} \int_{y-t+(m-n)t}^{y-t+mt} o(1) \left|\langle w_{12}, u_r \rangle\right| \, d\tau\, dt$$

$$+ \int_R^{2R} \int_{y-t+(m-n)t}^{y-t+mt} o(1) \left|\langle w_{13}, u_r^* \rangle\right| \, d\tau\, dt \; .$$

Taking the square of both sides, summarizing for $r \in I$, taking into account that

$$\left\| w_i \right\|_{L^2(G)} = o(1) \quad , \; i = 9, \ldots, 13$$

and using (18) we obtain (20).

b./ Applying the formula (10) with $x = y+t$ we obtain almost the same formula as in the preceeding case, with the following changes:

- $y-t$ is replaced by $y+t$ everywhere ,
- instead of $g_1(\mu_r,t)$, $g_{-1}(\mu_r,t)$ is placed on the left hand side.

After it (20) may be derived exactly by the same manner as before because $g_{-1}(\mu_r,t) = -1 + \bar{\bar{\bar{o}}}(1)$ by (15) (this does not remain true if n is odd).

c./ We apply now the formula (11) :

$$- \int_R^{2R} h_o(\mu_r,t)\,dt\ u_r(y)$$

$$= \int_R^{2R} \Big(h_1(\mu_r,t) - h_1(i\nu,t)\Big)u_r(y+t)\,dt$$

$$+ \int_R^{2R} h_1(i\nu,t)\,u_r(y+t)\,dt$$

$$+ \sum_{\substack{m-n\le k\le m\\ k\ne 0,1}} h_k(\mu_r,2R)\int_R^{2R} u_r(y+kt)\,dt$$

$$- \sum_{\substack{m-n\le k\le m\\ k\ne 0,1}} \int_R^{2R} D_2h_k(\mu_r,t)\int_R^{t} u_r(y+k\xi)\,d\xi\,dt$$

$$+ \sum_{s=2}^{n}\sum_{i=0}^{n-s}(-1)^i\binom{n-s}{i}\int_R^{2R}\int_{y+(m-n)t}^{y+mt} D_3^{n-s-i+1}H(\mu_r,t,y-\tau)$$

$$\int_{y+(m-n)t}^{\tau} q_s^{(i)}(\xi)\, u_r(\xi)\, d\xi\, d\tau\, dt$$

$$- \int_{R}^{2R} \int_{y+(m-n)t}^{y+mt} D_3\, H(\mu_r, t, y-\tau) \int_{y+(m-n)t}^{\tau} u_r^*(\xi)\, d\xi\, d\tau\, dt.$$

Using (12), (15), (16) and (17), with obvious notation we obtain

$$\left(R-\bar{\bar{o}}(1)\right) |u_r(y)| \leq C\, R^{3/2} \|u_r\|_{L^2(K)} + |\langle w_{14}, u_r\rangle|$$

$$+ \sum_{\substack{m-n\leq k\leq m \\ k\neq 0,1}} o(1)\, |\langle w_{15}, u_r\rangle| + \sum_{\substack{m-n\leq k\leq m \\ k\neq 0,1}} \int_R^{2R} o(1)\, |\langle w_{16}, u_r\rangle|\, dt$$

$$+ \sum_{s=2}^{n} \sum_{i=0}^{n-s} \int_R^{2R} \int_{y+(m-n)t}^{y+mt} o(1)\, |\langle w_{17}, u_r\rangle|\, d\tau\, dt$$

$$+ \int_R^{2R} \int_{y+(m-n)t}^{y+mt} o(1)\, |\langle w_{18}, u_r^*\rangle|\, d\tau\, dt\, .$$

Furthermore

$$\|w_i\|_{L^2(G)} = O(1)\, , \quad i = 14, \ldots, 18$$

and (20) can be obtained as in part a./ . □

L e m m a 5 . For any fixed $B > 0$ we have

$$\sup_{\nu \geq 1} \sum_{\substack{|\nu + \nu_+| \leq 1 \\ 0 \leq \varrho_+ < B \\ \nu_+ \leq -D}} \|u_r\|^2_{L^2(K)} = \underline{\underline{O}}(1) \quad (D \longrightarrow \infty) .$$

P r o o f . This is quite similar to that of Lemma 4 , replacing in the formulas the term $g_0(i\nu, t)$ (resp. $h_1(i\nu, t)$) by $g_0(-i\nu, t)$ (resp. $h_1(-i\nu, t)$). □

L e m m a 6 . For any fixed $B, D > 0$ we have

$$\sum_{\substack{|\varrho_r| < B \\ |\gamma_r| < D}} \|u_r\|^2_{L^2(K)} < \infty .$$

P r o o f . We will show the existence of a constant C such that

$$(21) \qquad R^2 \sum_{r \in I} |u_r(y)|^2 \leq C R^3 \sum_{r \in I} \|u_r\|^2_{L^2(K)} + C$$

for any $y \in K$, $0 < R < \frac{|K|}{2n}$ and for any finite subset I of $J := \{ r: |\varrho_r| < B, \ |\gamma_r| < D \}$. Indeed, then choosing R such that

$$C R |K| \leq \frac{1}{2} ,$$

integrating on K and taking into account that I (and therefore $\sum_{r \in I} \|u_r\|^2_{L^2(K)}$) is finite, we obtain

$$\sum_{r \in I} \|u_r\|^2_{L^2(K)} \leq \frac{2 C |K|}{R^2}$$

uniformly in I ; hence the lemma follows.

Denote again c the centre of K. To prove (21), we distinguish three cases:

a./ $y \geqslant c$,

b./ $y \leqslant c$ and $n \geqslant 2$,

c./ $y \leqslant c$ and $n = 1$.

a./ Apply the formula (8) with $x = y-mt$, $0 < t < R$, then we obtain

$$- \int_0^R f_m(\mu_r, t)\, dt\; u_r(y)$$

$$= \sum_{k=m-n}^{m-1} \int_0^R \Big(f_k(\mu_r,t) - f_k(0,0) \Big)\; u_r\big(y-mt+kt\big)\, dt$$

$$+ \sum_{k=m-n}^{m-1} \int_0^R f_k(0,0)\; u_r\big(y-mt+kt\big)\, dt$$

$$+ \sum_{s=2}^{n} \sum_{i=0}^{n-s} (-1)^i \binom{n-s}{i} \int_0^R \int_{y-nt}^{y} D_3^{n-s-i+1}\, F\big(\mu_r, t, y-\tau\big)$$

$$\cdot \int_{y-nt}^{\tau} q_s^{(i)}(\xi)\, u_r(\xi)\, d\xi\, d\tau\, dt$$

$$- \int_0^R \int_{y-nt}^{y} D_3\, F\big(\mu_r, t, y-\tau\big) \int_{y-nt}^{\tau} u_r^*(\xi)\, d\xi\, d\tau\, dt .$$

Being f_k, F smooth the functions

$$\frac{f_k(\mu_r,t) - f_k(0,0)}{t} \quad \text{and} \quad D_3^{n-s-i+1} F(\mu_r,t,y-\tau)$$

are bounded for $|\varrho_r| < B$, $|\gamma_r| < D$, $0 < t < R$ and $y-nt < \tau < y$. Furthermore, $|f_m(\mu_r,t)| \equiv 1$. Therefore, introducing the functions (depending on the parameters y, τ, k, s, i) $w_{19}, w_{20}, w_{21} \in L^2(G)$ in a suitable way, we obtain the estimates

$$R\,|u_r(y)| \leq C_1 R^{3/2} \|u_r\|_{L^2(K)} + \sum_{k=m-n}^{m-1} |\langle w_{19},u_r\rangle|$$

$$+ \sum_{s=2}^{n}\sum_{i=0}^{n-s}\int_0^R\int_{y-nt}^{y} C_1 |\langle w_{20},u_r\rangle|\, d\tau\, dt$$

$$+ \int_0^R\int_{y-nt}^{y} C_1 |\langle w_{21},u_r^*\rangle|\, d\tau\, dt$$

and

$$\|w_i\|_{L^2(G)} \leq C_1 \quad , i = 19, 20, 21$$

for some constant C_1. Hence (21) follows by the usual way.

b./ Apply now the formula (8) with $x = y + (n-m)t$, $0 < t < R$, then

$$- \int_0^R f_{m-n}(\mu_r, t)\, dt \; u_r(y)$$

$$= \sum_{k=m-n+1}^{m} \int_0^R \Big(f_k(\mu_r, t) - f_k(0,0) \Big) u_r\big(y+(n-m)t+kt\big)\, dt$$

$$+ \sum_{k=m-n+1}^{m} \int_0^R f_k(0,0)\, u_r\big(y+(n-m)t+kt\big)\, dt$$

$$+ \sum_{s=2}^{n} \sum_{i=0}^{n-s} (-1)^i \binom{n-s}{i} \int_0^R \int_y^{y+nt} D_3^{n-s-i+1} F(\mu_r, t, y+nt-\tau)$$

$$\cdot \int_y^{\tau} q_s^{(i)}(\xi)\, u_r(\xi)\, d\xi\, d\tau\, dt$$

$$- \int_0^R \int_y^{y+nt} D_3 F(\mu_r, t, y+nt-\tau) \int_y^{\tau} u_r^{\#}(\xi)\, d\xi\, d\tau\, dt \; .$$

Hence one can proceed as in the preceeding case because $n \geqslant 2$ implies also $\left| f_{m-n}(\mu_r, t) \right| \equiv 1$.

c./ This case can be treated quite similarly to the case b./, with a sole change: before the estimations we divide the formula written just above by $e^{\mu_r t}$. Then

$$\left| \frac{f_{m-n}(\mu_r, t)}{e^{\mu_r t}} \right| \equiv 1$$

and the usual procedure works. □

P r o o f o f t h e P r o p o s i t i o n . It follows from Lemmas 1 - 6 that

$$\sup_{\nu \geq 1} \sum_{\substack{|\nu - |\nu_r|| \leq 1 \\ \rho_r \geq 0}} \left(\frac{\| u_r \|_{L^2(K)}}{1 + \rho_r} \right)^2 < \infty .$$

If n is even then tha condition $\rho_r \geq 0$ is always satisfied. If n is odd then we have also

$$\sup_{\nu \geq 1} \sum_{\substack{|\nu - |\nu_r|| \leq 1 \\ \rho_r \leq 0}} \left(\frac{\| u_r \|_{L^2(K)}}{1 - \rho_r} \right)^2 < \infty$$

by a reflection principle described in the introduction of the paper [17]. Therefore we have in both cases

$$\sup_{\nu \geq 1} \sum_{|\nu - |\nu_r|| \leq 1} \left(\frac{\| u_r \|_{L^2(K_1)}}{1 + |\rho_r|} \right)^2 < \infty$$

for any compact subinterval K_1 of G . Applying the estimate (2) (choose $K_2 = K$) hence we obtain the proposition (with $\varepsilon = \frac{\varepsilon_0}{2}$) for example. □

4. Proof of the Theorem

The idea of the proof is the following. Putting

$$\delta(\gamma, |\gamma_r|) = \begin{cases} 1 & \text{if} \quad \gamma > |\gamma_r| , \\ \frac{1}{2} & \text{if} \quad \gamma = |\gamma_r| , \\ 0 & \text{if} \quad \gamma < |\gamma_r| . \end{cases}$$

and

$$w(x+t) = \begin{cases} \dfrac{\sin \gamma t}{\pi t} & \text{if} \quad |t| < R , \\ 0 & \text{otherwise} , \end{cases}$$

by the application of the proposition proved in the preceeding section we will show that for any compact subset K of G

$$(22) \quad \sup_{\gamma > 0} \ \sup_{x \in K} \sum_{r=1}^{\infty} \left| \langle u_r, w \rangle - \delta(\gamma, |\gamma_r|)\, u_r(x) \right|^2 < \infty$$

whenever R is sufficiently small $\left(w \in L^2(G) \right.$ depends on the parameters γ and $\left. R \right)$. Taking into account that

$$S_\gamma(f,x,R) = \langle f,w \rangle = \sum_{r=1}^{\infty} \langle f,v_r \rangle \langle u_r,w \rangle ,$$

$$\sigma_\gamma(f,x) = \sum_{r=1}^{\infty} \langle f,v_r \rangle \delta(\gamma, |\gamma_r|)\, u_r(x) - \frac{1}{2} \sum_{|\gamma_r| = \gamma} \langle f,v_r \rangle\, u_r(x)$$

applying the Cauchy - Schwarz inequality , (22) and the proposition again, we obtain

$$\sup_{\nu>0} \sup_{x\in K} \left| S_\nu(f,x,R) - \sigma_\nu(f,x) \right| \leq C \, \|f\|_{L^2(G)} \quad \left(\forall f \in L^2(G)\right),$$

with some constant C independent of f. Now it suffices to show that

$$\lim_{\nu\to\infty} \sup_{x\in K} \left| S_\nu(f,x,R) - \sigma_\nu(f,x) \right| = 0$$

for any f from a dense subset of $L^2(G)$. But this last property is satisfied for any finite linear combination f of the eigenfunctions u_r because then f is continuously differentiable and $\sigma_\nu(f,x) \equiv f(x)$ for ν sufficiently large, therefore one can apply a classical result of the theory of Fourier series (see [25], Volume 1, p. 55).

The rest of this section is devoted to the proof of the estimate (22). In the sequel we shall consider only the case $n > 1$ because the case $n = 1$ (then $Lu = u'$) can be easily led to the case $n = 2$ ($Lu = u''$).

L e m m a 7 . We have $\left|\frac{1}{R}\int_R^{2R} e^{\mu t}\, dt\right| \to 0$

if $\mu \in \mathbb{C}$, $\operatorname{Re}\mu \leq 0$, $R > 0$ and $|\mu R| \to \infty$.

P r o o f . It suffices to show that

$$\left|\frac{1}{R}\int_R^{2R} e^{\mu t}\, dt\right| \leq e^{\operatorname{Re}\mu R} \cdot \min\left\{1, \frac{4}{|\operatorname{Im}\mu R|}\right\} .$$

For this, first we note that obviously

$$\left|\frac{1}{R}\int_R^{2R} e^{\mu t}\, dt\right| \leq e^{\operatorname{Re}\mu R} .$$

On the other hand, applying the theorem of Bonnet, there exist $R \leq R_1, R_2 \leq 2R$ such that

$$\left|\frac{1}{R}\int_R^{2R} e^{\mu t}\, dt\right| \leq \left|\frac{1}{R}\int_R^{2R} e^{\operatorname{Re}\mu t} \cos \operatorname{Im}\mu t\, dt\right|$$

$$+ \left|\frac{1}{R}\int_R^{2R} e^{\operatorname{Re}\mu t} \sin \operatorname{Im}\mu t\, dt\right| = \left|\frac{1}{R} e^{\operatorname{Re}\mu R}\int_R^{R_1} \cos \operatorname{Im}\mu t\, dt\right|$$

$$+ \left|\frac{1}{R} e^{\operatorname{Re}\mu R}\int_R^{R_1} \sin \operatorname{Im}\mu t\, dt\right| \leq \frac{4}{|\operatorname{Im}\mu R|} e^{\operatorname{Re}\mu R} . \square$$

L e m m a 8 . To any compact intervals $K_1 \subset G$, $K \subset \operatorname{int} K_1$ there exists $R_0 > 0$ such that for any fixed $0 < R < R_0$

$$\sup_{x \in K} \int_0^R \left| \frac{u_r(x-t) + u_r(x+t) - 2u_r(x) \operatorname{ch} \mu_r t}{t} \right| dt \leq$$

$$\leq C \frac{\ln |\mu_r|}{|\mu_r|} \left(\| u_r \|_{L^\infty(K_1)} + \| u_r^* \|_{L^\infty(K_1)} \right)$$

whenever $|\mu_r|$ is sufficiently large. □

P r o o f . We shall use the notations of section 2 . We shall assume that $\varrho_r \geqslant 0$. The case $\varrho_r < 0$ hence can be obtained by the reflection principle mentioned at the end of section 3.

Putting

$$v_r(y) = u_r(y) + \int_x^y \sum_{p=1}^{n} \frac{\omega_p}{n \mu_r^{n-1}} e^{\mu_r \omega_p (y-\tau)}$$

$$\cdot \left(\sum_{s=2}^{n} q_s(\tau) u_r^{(n-s)}(\tau) - u_r^*(\tau) \right) d\tau$$

one can readily verify that $v_r \in H^n_{loc}(G)$ and $v_r^{(n)} = \mu_r^n v_r$. Consequently v_r is a linear combination of the functions

$$e^{\mu_r \omega_1 (y-x)}, \ldots, e^{\mu_r \omega_n (y-x)} .$$

By (3) we can fix $R_o > 0$ such that $K_{4nR_o} \subset K_1$ and

$$(23) \quad \| u_r \|_{L^\infty(K_{4nR_o})} \cdot e^{2 \varrho_r R_o} \leq C \, \| u_r \|_{L^\infty(K_1)} \quad (r = 1,2,\dots)$$

We shall distinguish two cases:

a./ n is odd and $n > 1$ i.e. $n = 2m-1$, $m \geq 2$.

b./ n is even i.e. $n = 2m$, $m \geq 1$.

a./ For any $x \in K$, $0 < S < 2 R_o$ and $0 < t < S$ the determinant

$$\begin{vmatrix} v_r(x-mS) & \dots & e^{-m\mu_{r,p}S} & \dots \\ \vdots & & \vdots & \\ v_r(x-2S) & \dots & e^{-2\mu_{r,p}S} & \dots \\ \left(v_r(x-t)+v_r(x+t)-2v_r(x)\operatorname{ch}\mu_r t\right) & \dots & \left(2\operatorname{ch}\mu_{r,p}t - 2\operatorname{ch}\mu_r t\right) & \dots \\ v_r(x+2S) & \dots & e^{2\mu_{r,p}S} & \dots \\ \vdots & & \vdots & \\ v_r(x+(m+1)S) & \dots & e^{(m+1)\mu_{r,p}S} & \dots \end{vmatrix}$$

$(p = 1,\dots,n)$ vanishes. Expanding it according to the first column, with obvious notation we obtain the formula

$$d(\mu_r, S) \left(u_r(x-t) + u_r(x+t) - 2\, u_r(x) \operatorname{ch} \mu_r t \right)$$

$$= \sum_{\substack{-m \leq k \leq m+1 \\ |k| \geq 2}} d_k(\mu_r, S, t) \; u_r(x+kS)$$

$$+ \int_{x-mS}^{x+(m+1)S} D(\mu_r, S, t, x-\tau) \left(\sum_{s=2}^{n} q_s(\tau)\, u_r^{(n-s)}(\tau) - u_r^{*}(\tau) \right) d\tau \quad .$$

One can see easily $\left(\text{cf. } (4) \right)$ that

$$D_4^j\, D(\mu_r, S, t, -(m+1)S) = D_4^j\, D(\mu_r, S, t, mS) = 0 \quad , \quad j = 0, \ldots, n-2 \quad .$$

Therefore the above formula implies

$$d(\mu_r, S) \left(u_r(x-t) + u_r(x+t) - 2\, u_r(x)\, \mathrm{ch}\, \mu_r t \right)$$

$$= \sum_{\substack{-m \le k \le m+1 \\ |k| \geq 2}} d_k(\mu_r, S, t)\; u_r(x+kS)$$

$$+ \sum_{s=2}^{n} \sum_{i=0}^{n-s} (-1)^i \binom{n-s}{i} \int_{x-mS}^{x+(m+1)S} D_4^{n-s-i}\, D(\mu_r, S, t, x-\tau)\; q_s^{(i)}(\tau)\, u_r(\tau)\, d\tau$$

$$- \int_{x-mS}^{x+(m+1)S} D(\mu_r, S, t, x-\tau)\; u_r^{*}(\tau)\, d\tau \quad .$$

Putting

$$Q(\mu_r, S) = e^{\left((m+1)\mu_{r,1} + \ldots + 2\mu_{r,m} - 2\mu_{r,m+1} - \ldots - m\mu_{r,n} \right) \cdot S} \quad ,$$

by the method of the paper [17] we obtain the estimates

$$\left|d_k(\mu_r,S,t)\right| \leqslant C_1 \left|Q(\mu_r,S)\right|\cdot\left|\mu_r t\right|\cdot\left(\left|e^{-\mu_{r,m-1}S}\right|+\left|e^{(\mu_{r,m}+\mu_{r,m+1})S}\right.\right.$$

$$\left|D_4^j\, D\left(\mu_r,S,t,x-\tau\right)\right| \leqslant C_1 \left|Q(\mu_r,S)\right|\cdot\left|\mu_r\right|^{j+1-n}\cdot$$

$$\cdot \min\left\{1,\ \left|\mu_r t\right|\right\}\cdot e^{\varrho_r S} \qquad ;$$

furthermore, being n odd there exists a constant $\alpha > 0$ with

$$\operatorname{Re}\mu_{r,m-1} \geqslant \alpha\cdot\left|\mu_r\right| \quad \text{and} \quad \operatorname{Re}\mu_{r,m+1} \leqslant -\alpha\left|\mu_r\right| \quad ,\ \forall r.$$

(In the above estimates C_1 denotes an absolute constant.) Using these estimates from the formula we obtain

$$\left|\frac{d(\mu_r,S)}{Q(\mu_r,S)}\right|\cdot\left|\frac{u_r(x-t)+u_r(x+t)-2\,u_r(x)\,\operatorname{ch}\mu_r t}{t}\right|$$

$$\leqslant C_2\left|\mu_r\right| e^{-\alpha\left|\mu_r\right|}\cdot\left\|u_r\right\|_{L^\infty(K_{2nS})}\ e^{\varrho_r S}$$

$$+\sum_{s=2}^{n}\sum_{i=0}^{n-s} C_2\left|\mu_r\right|^{2-s-i}\min\left\{\left|\mu_r t\right|^{-1},\ 1\right\}\cdot\left\|u_r\right\|_{L^\infty(K_{2nS})}\cdot e^{\varrho_r S}$$

$$+\ C_2\left|\mu_r\right|^{2-n}\ \min\left\{\left|\mu_r t\right|^{-1},\ 1\right\}\ \left\|u_r^{\#}\right\|_{L^\infty(K_{2nS})}\ e^{\varrho_r S}$$

with another absolute constant C_2.

Let us now fix $0 < R < R_0$ arbitrarily. If $|\mu_r| > \frac{1}{R}$ then

$$\int_0^R \min\{|\mu_r t|^{-1}, 1\} dt \leq \int_0^{|\mu_r|^{-1}} 1 \, dt + \int_{|\mu_r|^{-1}}^R |\mu_r t|^{-1} dt$$

$$= |\mu_r|^{-1} \left(1 + \ln R + \ln |\mu_r|\right) ,$$

therefore if $R < S < 2R$ and $|\mu_r| > \max\{1, \frac{1}{R}\}$ then

$$\left|\frac{d(\mu_r, S)}{Q(\mu_r, S)}\right| \int_0^R \left|\frac{u_r(x-t) + u_r(x+t) - 2u_r(x) \operatorname{ch} \mu_r t}{t}\right| dt$$

$$\leq C_3 \frac{1 + \ln|\mu_r R|}{|\mu_r|} \left(\|u_r\|_{L^\infty(K_{4nR})} + \|u_r^*\|_{L^\infty(K_{4nR})}\right) e^{2 \varrho_r R}$$

If $|\mu_r|$ is sufficiently large then by Lemma 7 we have

$$\int_R^{2R} \left|\frac{d(\mu_r, S)}{Q(\mu_r, S)}\right| dS > \frac{R}{2}$$

whence

$$\int_0^R \left|\frac{u_r(x-t) + u_r(x+t) - 2u_r(x) \operatorname{ch} \mu_r t}{t}\right| dt$$

$$\leq C_4 \frac{\ln|\mu_r|}{|\mu_r|} \left(\|u_r\|_{L^\infty(K_{4nR})} + \|u_r^*\|_{L^\infty(K_{4nR})}\right) e^{2 \varrho_r R}$$

with some constant C_4 depending only on R. Taking into account (23) and the condition (C 3) hence the Lemma follows.

b./ For any $x \in K$, $0 < S < 2R_0$ and $0 < t < S$ the determinant

$$\begin{vmatrix} v_r(x) & \cdots & 1 & \cdots \\ \left(v_r(x-t)+v_r(x+t)-2v_r(x)\,\mathrm{ch}\,\mu_r t\right) & \cdots & \left(2\,\mathrm{ch}\,\mu_{r,p}t - 2\,\mathrm{ch}\,\mu_r t\right) & \cdots \\ \left(v_r(x-2S)+v_r(x+2S)\right) & \cdots & \left(2\,\mathrm{ch}\,2\mu_{r,p}S\right) & \cdots \\ \vdots & & \vdots & \\ \left(v_r(x-mS)+v_r(x+mS)\right) & \cdots & \left(2\,\mathrm{ch}\,m\mu_{r,p}S\right) & \cdots \end{vmatrix}$$

$(p = 1,\dots,m)$ vanishes. Expanding it according to the first column, with obvious notation we obtain the formula

$$d(\mu_r,S)\left(u_r(x-t)+u_r(x+t)-2u_r(x)\,\mathrm{ch}\,\mu_r t\right)$$

$$= \sum_{\substack{0\le k\le m\\ k\ne 1}} d_k(\mu_r,S,t)\left(u_r(x+kS)+u_r(x-kS)\right)$$

$$+ \int_{x-mS}^{x+mS} D(\mu_r,S,t,x-\tau)\left(\sum_{s=2}^{n} q_s(\tau)\,u_r^{(n-s)}(\tau) - u_r^{*}(\tau)\right)d\tau \ .$$

One can see easily that

$$D_4^j\, D\left(\mu_r, S, t, \pm mS\right) = 0 \quad , j = 0,\dots,n-2 \quad ,$$

therefore

$$d\left(\mu_r, S\right) \left(u_r(x-t) + u_r(x+t) - 2u_r(x)\, \mathrm{ch}\, \mu_r t\right)$$

$$= \sum_{\substack{0 \le k \le m \\ k \ne 1}} d_k\left(\mu_r, S, t\right) \left(u_r(x+kS) + u_r(x-kS)\right)$$

$$+ \sum_{s=2}^{n} \sum_{i=0}^{n-s} (-1)^i \binom{n-s}{i} \int_{x-mS}^{x+mS} D_4^{n-s-i}\, D\left(\mu_r, S, t, x-\tau\right) q_s^{(i)}(\tau) u_r(\tau)\, d\tau$$

$$- \int_{x-mS}^{x+mS} D\left(\mu_r, S, t, x-\tau\right) u_r^*(\tau)\, d\tau \quad .$$

Putting

$$Q\left(\mu_r, S\right) = e^{\left(m\mu_{r,1} + \dots + 2\mu_{r,m-1}\right) S}$$

we have the estimates

$$\left|d_k\left(\mu_r, S, t\right)\right| \le C_1 \left|Q\left(\mu_r, S\right)\right| \cdot \left|\mu_r t\right| \cdot \left| e^{\left(2\mu_{r,m} - \mu_{r,m-1}\right) S} \right|$$

$$\text{if} \quad m \geqslant 2 \; ,$$

$$\left|D_4^j\, D\left(\mu_r, S, t, x-\tau\right)\right| \le C_1 \left|Q\left(\mu_r, S\right)\right| \cdot \left|\mu_r\right|^{j+1-n} \cdot \min\left\{1, \left|\mu_r t\right|\right\} \cdot e^{\varrho_r S} \quad .$$

Furthermore,

$$d_0(\mu_r, S, t) \equiv 0 \quad \text{if} \quad m = 1$$

and there exists a constant $\alpha > 0$ such that

$$\operatorname{Re} \mu_{r,m-1} \geqslant \alpha \cdot |\mu_r| \qquad (r = 1, 2, \dots)$$

if $m \geqslant 2$. Therefore, fixing $0 < R < R_0$ arbitrarily, hence we obtain

$$\left|\frac{d(\mu_r, S)}{Q(\mu_r, S)}\right| \int_0^R \left|\frac{u_r(x-t) + u_r(x+t) - 2\, u_r(x)\operatorname{ch}\mu_r t}{t}\right| dt$$

$$\leqslant C_2 \cdot \frac{1 + \ln|\mu_r R|}{|\mu_r|} \left(\|u_r\|_{L^\infty(K_{4nR})} + \|u_r^*\|_{L^\infty(K_{4nR})} \right) e^{2 \varrho_r R}$$

if $R < S < 2R$ and $|\mu_r| > \max\left\{1, \frac{1}{R}\right\}$ and the proof can be finished as in part a./ . $\square$

L e m m a 9 . To any $R > 0$ there exists a constant $C > 0$ such that

$$\left| \frac{2}{\pi} \int_0^R \frac{\sin \nu t \ \operatorname{ch} \mu_r t}{t}\, dt - \delta(\nu, |\nu_r|) \right| \leqslant \frac{C\, e^{|\varrho_r| R}}{2 + |\nu - |\nu_r||}$$

for all $\nu > 0$ and $r = 1, 2, \dots$.

P r o o f . See in [10] . □

Let us now prove (22). Given a compact interval $K \subset G$ arbitrarily, fix another compact interval $K_1 \subset G$ such that $K \subset \operatorname{int} K_1$, and then a positive number $R_1 > 0$ such that

$$0 < R_1 < R_0 \qquad (R_0 \text{ is defined as in Lemma 8 }) ,$$

$$K_{4nR_1} \subset K_1 \quad ,$$

$$(24) \qquad \sup_{\nu \geq 1} \sum_{|\nu - |\nu_r|| \leq 1} \left(\|u_r\|_{L^\infty(K_1)} e^{|\varrho_r| R_1} \right)^2 < \infty \quad .$$

This choice is possible by the proposition of section 3 .

Fix $0 < R < R_1$ arbitrarily and fix a constant $A = A(R) > 2$ such that the assertion of Lemma 8 hold true whenever $|\mu_r| > A$. In the sequel C denotes diverse constants independent of $\nu \geq 1$, $x \in K$ and $r = 1,2,\ldots$.

Consider first the case when $|\mu_r| > A$. Applying Lemmas 8 and 9 we have

$$\left| \langle u_r, w \rangle - \delta(\nu, |\nu_r|) \; u_r(x) \right|$$

$$= \left| \int_0^R \frac{\sin \nu t}{\pi t} \left(u_r(x-t) + u_r(x+t) \right) dt - \delta(\nu, |\nu_r|) \; u_r(x) \right|$$

$$\leq \left| \int_0^R \frac{\sin \nu t}{\pi t} \left(u_r(x-t) + u_r(x+t) - 2u_r(x) \operatorname{ch} \mu_r t \right) dt \right|$$

$$+ \left| \frac{2}{\pi} \int_0^R \frac{\sin \nu t \operatorname{ch} \mu_r t}{t} \, dt - \delta(\nu, |\nu_r|) \right| \cdot |u_r(x)|$$

$$\leq c \left(\frac{\ln |\mu_r|}{|\mu_r|} + \left(2 + |\nu - |\nu_r||\right)^{-1} \right) \left(\|u_r\|_{L^\infty(K_1)} + \|u_r^*\|_{L^\infty(K_1)} \right) e^{|\varsigma_r| R}$$

for all $\nu \geq 1$ and $x \in K$. Using (24) and (C3) with any fixed $0 < \varepsilon < 1$ hence we obtain

$$\sum_{|\mu_r| > A} \left| \langle u_r, w \rangle - \delta(\nu, |\nu_r|) \, u_r(x) \right|^2$$

$$\leq c \sum_{|\mu_r| > A} \left(\left(1 + |\nu_r|\right)^{-2+\varepsilon} + \left(2 + |\nu - |\nu_r||\right)^{-2} \right) \cdot$$

$$\cdot \left(\|u_r\|_{L^\infty(K_1)} + \|u_r^*\|_{L^\infty(K_1)} \right)^2 e^{2 |\varsigma_r| R_1}$$

$$\leq c \sum_{i=1}^{\infty} \left(i^{-2+\varepsilon} + \left(1 + |\nu - i|\right)^{-2} \right) \sum_{i-1 \leq |\nu_r| < i} \left(\|u_r\|_{L^\infty(K_1)} \, e^{|\varsigma_r| R_1} \right)^2$$

$$\leq c \sum_{i=1}^{\infty} \left(i^{-2+\varepsilon} + \left(1 + |\nu - i|\right)^{-2} \right) \leq c$$

i.e.

$$(25) \quad \sum_{|\mu_r|>A} \left| \langle u_r, w\rangle - \delta(\nu, |\nu_r|) \; u_r(x) \right|^2 \leq C .$$

Consider now the case when $|\mu_r| \leq A$. For any $\nu \geqslant 1$ and $x \in K$, integrating by parts and taking into account that the improper integral $\int_0^\infty \frac{\sin x}{x} \, dx$ is convergent, we obtain

$$\left|\langle u_r, w \rangle\right| = \left| \int_0^R \frac{\sin \nu t}{\pi t} \, dt \left(u_r(x-R) + u_r(x+R) \right) \right.$$

$$\left. + \int_0^R \int_0^t \frac{\sin \nu \xi}{\pi \xi} \, d\xi \left(u_r'(x-t) - u_r'(x+t) \right) dt \right|$$

$$\leq C \left(\|u_r\|_{L^\infty(K_1)} + \|u_r'\|_{L^\infty(K_1)} \right) .$$

But $|\mu_r|$ is bounded, therefore by the result mentioned in section 2 , part E./ hence we can conclude that

$$\left|\langle u_r, w \rangle\right| \leq C \; \|u_r\|_{L^\infty(K_1)}$$

and

$$\left| \langle u_r, w\rangle - \delta(\nu, |\nu_r|) \; u_r(x) \right| \leq C \; \|u_r\|_{L^\infty(K_1)} \; .$$

Using (24) again hence we obtain

$$(26)\quad \sum_{|\mu_r| \leq A} \left| \langle u_r, w \rangle - \delta(\nu, |\nu_r|)\ u_r(x) \right|^2 \leq C .$$

(25) and (26) imply (22) and the proof of the Theorem is finished. □

R e m a r k . We note that in the proof of the Proposition in section 3 we did not use the full assumption (C 1) but only its consequence (18). Thus our result remains valid for all (not necessarily complete) orthonormal systems consisting of eigenfunctions of order 0 (for example). □

O p e n p r o b l e m s .

1. It would be interesting to know whether the assumption (C 3) is necessary for the validity of the Proposition.
2. From the viewpoint of applications the Theorem proved in this paper seems to be very general and satisfactory. However, from a pure mathematical viewpoint it would be useful to enlighten whether the result remains true for the more general differential operator

$$L u = u^{(n)} + q_1 u^{(n-1)} + \ldots + q_n u \quad , q_s \in L^1_{loc}(G), \ s = 1, \ldots,$$

References

[1] A. Haar, Reichenentwicklungen nach Legendreschen Polynomen. Math. Annalen, 78 (1917) , 121-136.

[2] В.А.Ильин, Необходимые и достаточные условия базисности и равносходимости с тригонометрическим рядом спектральных разложений 1-2. Дифференциальные уравнения, 16 (1980) , 771-794, 980-1009.

[3] В.А.Ильин, И.Йо, Точная оценка разности частичных сумм разложений, отвечающих двум произвольным неотрицательным самосопряжённым расширениям двух операторов типа Штурма - Лиувилля для абсолютной непрерывной функции. Дифференциальные уравнения, 15 (1979) , 1175-1193.

[4] Н. И. Ионкин, Решение одной краевой задачи теории теплопроводности с неклассическим краевым условием. Дифференциальные уравнения, 13 (1977) , 293-304.

[5] Ш.А.Исматуллаев, И.Йо, Теоремы о равносходимости. Acta Math. Acad. Sci. Hungar. (to appear) .

[6] I. Joó, Upper estimates for the eigenfunctions of the Schrödinger operator. Acta Sci. Math., 44 (1982), 87-93.

[7] И. Йо, Теорема типа равносходимости. ДАН УзССР, 4 (1983) , 6-8.

[8] I. Joó, On some spectral properties of the Schrödinger operator. Annales Univ. Sci. Budapest, Sect. Math. (to appear).

[9] I. Joó, Remarks to a paper of V. Komornik. Acta Sci. Math. (to appear) .

[10] I. Joó - V. Komornik, On the equiconvergence of expansions by Riesz bases formed by eigenfunctions of the Schrödinger operator. Acta Sci. Math. (to appear) .

[11] Г.М. Кесельман, О безусловной сходимости разложений по собственным функциям некоторых дифференциальных операторов. Изв. вузов СССР, Математика, № 2 (1964), 82-93.

[12] V. Komornik, Upper estimates for the eigenfunctions of higher order of a linear differential operator. Acta Sci. Math., 45 (1983), 261-271.

[13] V. Komornik, On the distribution of the eigenvalues of an orthonormal system consisting of eigenfunctions of higher order of a linear differential operator. Acta Math. Acad. Sci. Hungar., 42 (1983), 171-175.

[14] V. Komornik, An equiconvergence theorem for the Schrödinger operator. Acta Math. Acad. Sci. Hungar. (to appear).

[15] V. Komornik, On the equiconvergence of expansions by Riesz bases formed by eigenfunctions of a 2n-order linear differential operator. Acta Math. Acad. Sci. Hungar. (to appear

[16] V. Komornik, Some new estimates for the eigenfunctions of higher order of a linear differential operator. Acta Math. Acad. Sci. Hungar. (to appear).

[17] V. Komornik, Local upper estimates for the eigenfunctions of a linear differential operator. Acta Sci. Math. (to appear).

[18] V. Komornik, Lower estimates for the eigenfunctions of a linear differential operator. Acta Math. Acad. Sci. Hungar. (to appear).

[19] Б. М. Левитан, Разложение по собственным функциям. Москва, 1950.

[20] В. П. Михайлов, О базисах Рисса в $L^2(0,1)$. ДАН СССР, 144, 5 (1962), 981-984.

[21] М.А. Наймарк, Линейные дифференциальные ~~уравнения~~ операторы. Москва, 1969.

[22] V. A. Stekloff, Sur les expressions asymptotiques de certaines fonctions définies par des équations differentielles linéaires du deuxieme ordre, et leurs applications au probleme du développement d'une fonction arbitraire en series procédant suivant les dites fonctions.
Харьков. Сообщ. матем. об-ва, сер 2, т. 10, №№2-6, 1907-1909, с. 96-199.

[23] E. C. Titchmarsh, Eigenfunction expansions associated with second-order differential equations. Oxford, Clarendon Press, 1946.

[24] A. Zygmund, Trigonometric series. Cambridge University Press, 1959.

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