On the control of strongly nonlinear systems

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Résumé : Ce rapport contient trois travaux sur le contrôle de systèmes distribués singuliers à forte nonlinéarité, dans un cas elliptique, parabolique et hyperbolique, respectivement. On montre que pour une fonction coût naturelle, on peut obtenir le système d'optimalité singulier du même type que pour les systèmes à une nonlinéarité moins forte.

Abstract : This report contains three papers on the control of strongly nonlinear singular distributed systems, in an elliptic, a parabolic and a hyperbolic case, respectively. We show that for a natural cost function the singular optimality system of the same type may be derived as for the less strongly nonlinear systems.

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On the control of strongly nonlinear systems I.

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Throughout this paper let Ω denote a bounded open set in $\mathbb{R}^n$ $(n \in \mathbb{N})$ with the boundary Γ of class C^∞ and let $f: \mathbb{R} \to \mathbb{R}$ be an arbitrary function of class C^1. Fix the numbers $1 < \alpha < \infty$, $1 < \beta < \infty$, $N > 0$ arbitrarily and put for brevity $\gamma = \min\{\alpha, \beta\}$. Let $z_d \in L^\alpha(\Omega)$ be arbitrarily given and let U_{ad} be a non-void convex, closed subset in $L^\beta(\Omega)$. Furthermore we put

$$(1) \qquad J(v,z) = \frac{1}{\alpha} \| f(z) - z_d \|^\alpha_{L^\alpha(\Omega)} + \frac{N}{\beta} \| v \|^\beta_{L^\beta(\Omega)} .$$

A pair (v,z) is said to be <u>admissible</u> if

$$(2) \qquad v \in U_{ad},\ z \in W_0^{1,\gamma}(\Omega),\ f(z) \in L^\alpha(\Omega) \text{ and } \Delta z + f(z) + v = 0.$$

We shall assume that

(3) there exists at least one admissible pair.

This is satisfied for example if $U_{ad} = L^\beta(\Omega)$.

A pair (u,y) is said to be <u>optimal</u> if it is admissible and if

$$(4) \qquad J(u,y) = \inf \{ J(v,z) : (v,z) \text{ is admissible} \}.$$

The first result of this paper will be the following

T h e o r e m 1. There exists at least one optimal pair. □

Given an open subset ω of Ω we denote by $\widetilde{\mathcal{D}(\omega)}$ the set of extensions by 0 of the functions from $\mathcal{D}(\omega) = C_0^\infty(\omega)$ to Ω.

Assume that

(5) $$\gamma > \frac{n}{2}$$

and

(6) there exists $v_0 \in U_{ad}$ and a non-void open subset ω of Ω such that $v_0 + \widetilde{\mathcal{D}(\omega)} \subset U_{ad}$.

Under these assumptions we shall prove the

T h e o r e m 2 . To any optimal pair (u,y) there exists a triplet (u,y,p) such that

(7a) $u \in \mathcal{U}_{ad}$

(7) $y \in W^{2,\gamma}(\Omega) \cap W_0^{1,\gamma}(\Omega)$, $p \in W^{2,\alpha'}(\Omega) \cap W_0^{1,\alpha'}(\Omega)$,

(8) $\Delta y + f(y) + u = 0$,

(9) $\Delta p + f'(y)\, p + |f(y) - z_d|^{\alpha - 1} \operatorname{sgn}(f(y) - z_d)\, f'(y) = 0$,

(10) $\int_\Omega \left(p + N |u|^{\beta - 1} \operatorname{sgn} u \right) (v-u)\, dx \geqslant 0 \quad \forall\, v \in U_{ad}$.

R e m a r k s . Theorem 2 solves a problem raised by J. L. Lions in his book [6] / Chapitre 3 , Paragraph 16 , Problem 25 / for the case $f(x) = e^x$. In view of this theorem, to find the optimal pairs it is worth while to seek first the triplets (u,y,p) satisfying (7a)-(10).

The author is grateful to J. L. Lions for the fruitful

discussions.

Before turning to the proof of the theorems, for the reader's convenience we recall an important proposition on the weak solutions of the Dirichlet problem.

P r o p o s i t i o n . Let us given two functions $f, g \in L^{\mu}(\Omega)$ $(1 < \mu < \infty)$ and assume that

$$(11) \qquad \int_{\Omega} f \, \Delta\xi \, dx = \int_{\Omega} g \, \xi \, dx \, , \ \forall \xi \in W^{2,\mu'}(\Omega) \cap W_0^{1,\mu'}(\Omega) \ .$$

Then $f \in W^{2,\mu}(\Omega) \cap W_0^{1,\mu}(\Omega)$ and

$$(12) \qquad \| f \|_{W^{2,\mu}(\Omega)} \leq C \, \| g \|_{L^{\mu}(\Omega)}$$

where C is a constant.

P r o o f . It follows from the (deep) regularity results of S. Agmon, A. Douglis and L. Nirenberg [2] that there exists $f^* \in W^{2,\mu}(\Omega) \cap W_0^{1,\mu}(\Omega)$ such that (11) and (12) hold true for f^* instead of f. Then $f - f^* \in L^{\mu}(\Omega)$ and

$$\int_{\Omega} (f - f^*) \Delta\xi \, dx = 0 \qquad \forall \xi \in W^{2,\mu'}(\Omega) \cap W_0^{1,\mu'}(\Omega) \ .$$

Furthermore, applying again the above regularity theorem, there exists $\xi \in W^{2,\mu'}(\Omega) \cap W_0^{1,\mu'}(\Omega)$ such that

$$\Delta\xi = |f - f^*|^{\mu - 1} \operatorname{sgn}(f - f^*)$$

Therefore $\int_{\Omega} |f - f^*|^{\mu} dx = 0$ whence $f = f^*$ and the proposition is proved. □

P r o o f   o f   T h e o r e m   1 . Let (v_k, z_k) be a minimizing sequence of admissible pairs i.e. such that

$$J(v_k, z_k) \longrightarrow \inf \{ J(v,z) : (v,z) \text{ is admissible} \}.$$

Then by (1),(2) and (12) the sequences

$$\|v_k\|_{L^\beta(\Omega)}, \quad \|f(z_k)\|_{L^\alpha(\Omega)}, \quad \|z_k\|_{W^{2,\gamma}(\Omega)}$$

are bounded. Passing, if it is needed, to subsequences, we may assume that

(13) $\quad v_k \longrightarrow u \qquad$ weakly in $L^\beta(\Omega)$,

(14) $\quad z_k \longrightarrow y \qquad$ weakly in $W^{2,\gamma}(\Omega)$.

It follows from the Rellich - Kondrasov theorem that the imbedding $W^{2,\gamma}(\Omega) \subset L^1(\Omega)$ is compact; therefore (14) implies that

(15) $\quad z_k \longrightarrow y \qquad$ strongly in $L^1(\Omega)$

and then by the Riesz lemma we may also suppose that

(16) $\quad z_k \longrightarrow y \qquad$ almost everywhere in Ω.

Now $f(z_k)$ is bounded in $L^\alpha(\Omega)$ and $f(z_k) \longrightarrow f(y)$ almost everywhere in Ω, therefore by Lemma 1.3 in [4], Chapitre 1

(17) $\quad f(z_k) \longrightarrow f(y) \qquad$ weakly in $L^\alpha(\Omega)$.

It follows from (13) , (14) and (17) that (u,y) is admissible (we note that U_{ad} is weakly closed) and that

$$J(u,y) \leq \underline{\lim}\ J(v_k, z_k)$$

i.e. (u,y) is an optimal pair. □

P r o o f o f T h e o r e m 2 . The proof will be divided into several parts. Let (u,y) be an arbitrarily fixed optimal pait and put for each $\varepsilon > 0$

$$J_\varepsilon(v,z) = \frac{1}{\alpha}\|f(z) - z_d\|^\alpha_{L^\alpha(\Omega)} + \frac{N}{\beta}\|v\|^\beta_{L^\beta(\Omega)}$$

$$+\frac{1}{\varepsilon\gamma}\|\Delta z + f(z) + v\|^\gamma_{L^\gamma(\Omega)} + \frac{1}{\gamma}\|z - y\|^\gamma_{L^\gamma(\Omega)} + \frac{1}{\beta}\|v - u\|^\beta_{L^\beta(\Omega)}.$$

A pair (v,z) is called <u>ε - admissible</u> if

(18) $\quad v \in U_{ad}, \quad z \in W^{2,\gamma}(\Omega) \cap W^{1,\gamma}_0(\Omega) \quad$ and $\quad f(z) \in L^\alpha(\Omega)$.

It follows from the proposition that every admissible pair is also ε - admissible.

A pair $(u_\varepsilon, y_\varepsilon)$ is called <u>ε - optimal</u> if it is ε - admissible and if

(19) $\quad J_\varepsilon(u_\varepsilon, y_\varepsilon) = \inf\{J_\varepsilon(v,z) : (v,z) \text{ is } \varepsilon\text{ - admissible}\}$.

L e m m a 1 . For each $\varepsilon > 0$ there exists at least one ε - optimal pair $(u_\varepsilon, y_\varepsilon)$.

P r o o f . One can repeat the argument used in theorem 1 . □

Let us fix for each $\varepsilon > 0$ an ε - optimal pair $(u_\varepsilon, y_\varepsilon)$.

L e m m a 2 . The following relations hold true as ε tends to 0 :

(20) $u_\varepsilon \longrightarrow u$ strongly in $L^\beta(\Omega)$,

(21) $y_\varepsilon \longrightarrow y$ strongly in $W^{2,\gamma}(\Omega) \cap W_0^{1,\gamma}(\Omega)$,

(22) $f(y_\varepsilon) \longrightarrow f(y)$ strongly in $L^\alpha(\Omega)$.

P r o o f . It follows from the obvious relation

(23) $J_\varepsilon(u_\varepsilon, y_\varepsilon) \leq J_\varepsilon(u, y) = J(u, y)$

that the sequences

$\|u_\varepsilon\|_{L^\beta(\Omega)}$, $\|f(y_\varepsilon)\|_{L^\alpha(\Omega)}$, $\|y_\varepsilon\|_{W^{2,\gamma}(\Omega)}$

are bounded and that

(24) $\Delta y_\varepsilon + f(y_\varepsilon) + u_\varepsilon \longrightarrow 0$ strongly in $L^\gamma(\Omega)$.

Therefore every subsequence of $(u_\varepsilon, y_\varepsilon)$ has another subsequence such that

(25) $u_\varepsilon \longrightarrow \hat{u}$ weakly in $L^\beta(\Omega)$,

(26) $y_\varepsilon \longrightarrow \hat{y}$ weakly in $W^{2,\gamma}(\Omega) \cap W_0^{1,\gamma}(\Omega)$,

(27) $y_\varepsilon \longrightarrow \hat{y}$ almost everywhere in Ω ,

(28) $f(y_\varepsilon) \longrightarrow f(\hat{y})$ weakly in $L^\alpha(\Omega)$.

It follows from (23),(24),(25),(26),(28) that $(\hat{u}, \hat{y})$ is admissible and

$$J(\hat{u},\hat{y}) \leq \underline{\lim}\ J(u_\varepsilon, y_\varepsilon) \leq \overline{\lim}\ J_\varepsilon(u_\varepsilon, y_\varepsilon) \leq J(u,y) .$$

Being (u,y) optimal hence we conclude

$$J(\hat{u},\hat{y}) = \lim J(u_\varepsilon, y_\varepsilon) = \lim J_\varepsilon(u_\varepsilon, y_\varepsilon) = J(u,y)$$

whence

(29) $u_\varepsilon \longrightarrow u$ strongly in $L^\beta(\Omega)$,

(30) $y_\varepsilon \longrightarrow y$ strongly in $L^r(\Omega)$.

(25),(26),(29),(30) imply

(31) $\hat{u} = u , \hat{y} = y$.

Furthermore (28),(29) (or (25)) ,(31) and the relation $\lim J(u_\varepsilon, y_\varepsilon) = J(u,y)$ imply (22) .

Finally,(24) ,(29) (which is identical with (20)) and (22) imply

(32) $\Delta y_\varepsilon \longrightarrow - f(y) - u$ strongly in $L^r(\Omega)$.

Using the proposition ,(30) and (32) yield (21) . □

L e m m a 3 . The following relations hold true as ε tends to 0:

(33) $y_\varepsilon \to y$ strongly in $C(\bar{\Omega})$,

(34) $f(y_\varepsilon) \longrightarrow f(y)$ strongly in $C(\bar{\Omega})$,

(35) $f'(y_\varepsilon) \longrightarrow f'(y)$ strongly in $C(\bar{\Omega})$.

P r o o f . By the Sobolev imbedding theorem and by property (5) $W^{2,\gamma}(\Omega) \subset C(\bar{\Omega})$, therefore (21) implies (33). (34) and (35) hence follow because f and f' are locally uniformly continuous. □

Let us now set

$$p_\varepsilon = \frac{1}{\varepsilon} \left| \Delta y_\varepsilon + f(y_\varepsilon) + u_\varepsilon \right|^{\gamma - 1} \operatorname{sgn} \left(\Delta y_\varepsilon + f(y_\varepsilon) + u_\varepsilon \right) ,$$

then $p_\varepsilon \in L^{\gamma'}(\Omega)$ for all $\varepsilon > 0$.

L e m m a 4 . For any $\xi \in W^{2,\alpha}(\Omega) \cap W_0^{1,\alpha}(\Omega)$ we have

$$\int_\Omega p_\varepsilon \left(\Delta \xi + f'(y_\varepsilon)\xi \right) + \left| f(y_\varepsilon) - z_d \right|^{\alpha -1} \operatorname{sgn}\left(f(y_\varepsilon) - z_d\right) f'(y_\varepsilon)\xi$$
$$+ \left| y_\varepsilon - y \right|^{\gamma - 1} \operatorname{sgn}(y_\varepsilon - y)\xi \, dx = 0 .$$

P r o o f . By the optimality of $(u_\varepsilon, y_\varepsilon)$ we have

$$\frac{d}{dt} J_\varepsilon \left(u_\varepsilon , y_\varepsilon + t\xi \right) \Big|_{t=0} = 0$$

provided that this derivative exists. And this is true, because, using Lebesgue's dominated convergence theorem we obtain

$$\frac{J_\varepsilon(u_\varepsilon, y_\varepsilon + t\xi) - J_\varepsilon(u_\varepsilon, y_\varepsilon)}{t} =$$

$$= \frac{1}{\alpha}\int_\Omega \frac{|f(y_\varepsilon + t\xi) - z_d|^\alpha - |f(y_\varepsilon) - z_d|^\alpha}{t}\,dx$$

$$+ \frac{1}{\gamma\varepsilon}\int_\Omega \frac{|\Delta(y_\varepsilon + t\xi) + f(y_\varepsilon + t\xi) + u_\varepsilon|^\gamma - |\Delta y_\varepsilon + f(y_\varepsilon) + u_\varepsilon|^\gamma}{t}\,dx$$

$$+ \frac{1}{\gamma}\int_\Omega \frac{|y_\varepsilon + t\xi - y|^\gamma - |y_\varepsilon - y|^\gamma}{t}\,dx$$

$$\longrightarrow \int_\Omega |f(y_\varepsilon) - z_d|^{\alpha-1}\,\mathrm{sgn}(f(y_\varepsilon) - z_d)\,f'(y_\varepsilon)\xi$$

$$+ \frac{1}{\varepsilon}|\Delta y_\varepsilon + f(y_\varepsilon) + u_\varepsilon|^{\gamma-1}\,\mathrm{sgn}(\Delta y_\varepsilon + f(y_\varepsilon) + u_\varepsilon)(\Delta\xi + f'(y_\varepsilon)\xi)$$

$$+ |y_\varepsilon - y|^{\gamma-1}\,\mathrm{sgn}(y_\varepsilon - y)\,\xi\,dx$$

and the lemma is proved. □

L e m m a 5 . For any $v \in U_{ad}$ we have

$$\int_\Omega (p_\varepsilon + N|u_\varepsilon|^{\beta-1}\,\mathrm{sgn}\,u_\varepsilon)(v - u_\varepsilon) + |u_\varepsilon - u|^{\beta-1}\,\mathrm{sgn}(u_\varepsilon - u)(v - u_\varepsilon)\,dx$$

$$\geqslant 0 .$$

P r o o f . By the optimality of $(u_\varepsilon, y_\varepsilon)$ we have

$$\frac{d}{dt} J_\varepsilon(u_\varepsilon + t(v - u_\varepsilon), y_\varepsilon)\Big|_{t=0} \geqslant 0$$

if this derivative exists. (We cannot say more in general because u_ε may be eventually a boundary point of U_{ad}). Taking into account that the sequence (u_ε) is bounded in $L^\beta(\Omega)$, by Lebesgue's dominated convergence theorem we obtain

$$\frac{J_\varepsilon(u_\varepsilon + t(v-u_\varepsilon), y_\varepsilon) - J_\varepsilon(u_\varepsilon, y_\varepsilon)}{t}$$

$$= \frac{N}{\beta}\int_\Omega \frac{|u_\varepsilon + t(v-u_\varepsilon)|^\beta - |u_\varepsilon|^\beta}{t}\,dx$$

$$+ \frac{1}{\gamma\varepsilon}\int_\Omega \frac{|\Delta y_\varepsilon + f(y_\varepsilon) + u_\varepsilon + t(v-u_\varepsilon)|^\gamma - |\Delta y_\varepsilon + f(y_\varepsilon) + u_\varepsilon|^\gamma}{t}\,dx$$

$$+ \frac{1}{\beta}\int_\Omega \frac{|u_\varepsilon - u + t(v-u_\varepsilon)|^\beta - |u_\varepsilon - u|^\beta}{t}\,dx$$

$$\longrightarrow \int_\Omega N|u_\varepsilon|^{\beta-1}(\operatorname{sgn} u_\varepsilon)(v-u_\varepsilon)$$

$$+ \frac{1}{\varepsilon}|\Delta y_\varepsilon + f(y_\varepsilon) + u_\varepsilon|^{\gamma-1}\operatorname{sgn}(\Delta y_\varepsilon + f(y_\varepsilon) + u_\varepsilon)(v - u_\varepsilon)$$

$$+ |u_\varepsilon - u|^{\beta-1}\operatorname{sgn}(u_\varepsilon - u)(v - u_\varepsilon)\,dx$$

and the lemma is proved. □ .

L e m m a 6 . The sequence p_ε is bounded in $L^{\gamma'}(\Omega)$.

P r o o f . Assume on the contrary that

$$(36) \qquad \|p_\varepsilon\|_{L^{\gamma'}(\Omega)} \longrightarrow \infty$$

for some subsequence. Being the sequence $\frac{p_\varepsilon}{\|p_\varepsilon\|_{L^{r'}(\Omega)}}$ bounded in $L^{r'}(\Omega)$, it is also bounded in $W^{2,\alpha'}(\Omega)$ by lemmas 3, 4 and by the proposition. Applying (5) and the Rellich-Kondrasov theorem there exists therefore another subsequence such that

$$(37) \qquad \frac{p_\varepsilon}{\|p_\varepsilon\|_{L^{r'}(\Omega)}} \longrightarrow q \quad \text{strongly in} \quad L^{r'}(\Omega) .$$

Passing to limit in lemmas 4, 5 and using (20), (33)-(37) we obtain

$$(38) \qquad \int_\Omega q\left(\Delta\xi + f'(y)\xi\right) dx = 0 \quad \forall \xi \in W^{2,\alpha}(\Omega) \cap W_0^{1,\alpha}(\Omega)$$

$$(39) \qquad \int_\Omega q\,(v-u)\,dx \geqslant 0 \qquad \forall \; v \in U_{ad} .$$

(6) and (39) imply $q \equiv 0$ in ω. Therefore, applying a unicity result of W. O. Amrein, A. M. Berthier and V. Georgescu [1], from (38) we conclude $q \equiv 0$ in Ω. But this contradicts to (37) whence $\|q\|_{L^{r'}(\Omega)} = 1$. □

Now we are ready to prove the theorem. Applying lemmas 2, 3, 4, 6 and the proposition, there exists a subsequence p_ε such that

$$(40) \qquad p_\varepsilon \longrightarrow p \quad \text{weakly in} \quad W^{2,\alpha'}(\Omega) \cap W_0^{1,\alpha'}(\Omega) .$$

Now (7) follows from (21) and (40) while (8) is an obvious because (u,y) is admissible. (9) and (10) follow from lemmas 4, 5 if we pass to limit and use the relations (20), (33) - (35) and (40).

The theorem is proved. □

R e m a r k s . (i) It is not necessary to apply the strong unicity result of W. O. Amrein, A. M. Berthier and V. Georgescu . Indeed, being $f'(y) \in C(\bar{\Omega})$ and $q \in W^{2,\alpha'}(\Omega) \cap W_0^{1,\alpha'}(\Omega)$ (by the proposition) a classical unicity result is also sufficient for our purposes.

(ii) The special case $\mu = 2$ of the proposition is much easier: instead of the results of S. Agmon, A. Douglis and L. Nirenberg it is then sufficient to use the regularity results only in the Hilbert space case. It is much more simple, see e.g. [3], [4] . However, this restricts our investigation to the case $\mu = 2$ i.e. (in view of condition (5)) to the case $n \leq 3$.

(iii) The condition (6) (due to J. L. Lions) may be replaced by other conditions, for example by $U_{ad} = \{v \in L^{\beta}(\Omega) : v \geq 0$ almost everywhere in $\Omega\}$ (this is due to F. Murat) or by $0 \in U_{ad}$ and $\|z_d\|_{L^{\alpha}(\Omega)}$ is sufficiently small (this type of conditions is due to P. Rivera); see [6] , pp. 327-328. It is then natural to ask whether theorem 2 remains valid without any further assumption on U_{ad}. In view of some recent results of M. Ramaswamy [7] this does not seem to be true.

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On the control of strongly nonlinear systems II.

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Throughout this paper let Ω denote a bounded open set in $\mathbb{R}^n$ $(n \in \mathbb{N})$ with the boundary Γ of class C^∞, let T be a positive number and set $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$ and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be an arbitrary function of class C^1. Fix the numbers $1 < \alpha < \infty$, $1 < \beta < \infty$, $N > 0$ arbitrarily and put for brevity $\gamma = \min\{\alpha, \beta\}$. Let $z_d \in L^\alpha(Q)$ be arbitrarily given and let U_{ad} be a non-void convex, closed subset in $L^\beta(Q)$. Finally, put

$$(1) \qquad J(v,z) = \frac{1}{\alpha}\|f(z) - z_d\|^\alpha_{L^\alpha(Q)} + \frac{N}{\beta}\|v\|^\beta_{L^\beta(Q)} \ .$$

A pair (v,z) is said to be <u>admissible</u> if

$$(2) \qquad v \in U_{ad}\ ,\ z \in W^{2,1;\gamma}(Q)\ ,\ z = 0 \text{ on } \Sigma\ ,\ z(0) = 0\ ,$$

$$f(z) \in L^\alpha(Q) \quad \text{and} \quad z' - \Delta z - f(z) = v \ .$$

For the definition and properties of the spaces $W^{2,1;\gamma}(Q)$ we refer to [1] or [6]. / We shall assume that

(3) there exists at least one admissible pair.

/This is satisfied for example if $U_{ad} = L^\beta(Q)$ ./

An admissible pair is said to be <u>optimal</u> if

$$(4) \qquad J(u,y) = \inf\{J(v,z) : (v,z) \text{ is admissible}\} \ .$$

We shall prove the following results.

T h e o r e m 1 . There exists at least one optimal pair. □

T h e o r e m 2 . Assume that

(5) $$\gamma > \frac{n}{2} + 1 .$$

To any optimal pair (u,y) there exists then a function p such that the following conditions are satisfied:

(6a) $u \in U_{ad}$,

(6) $y \in W^{2,1;\gamma}(Q)$, $y = 0$ on Σ , $y(0) = 0$,

(7) $p \in W^{2,1;\alpha'}(Q)$, $p = 0$ on Σ , $p(T) = 0$,

(8) $y' - \Delta y - f(y) = u$ in Q ,

(9) $$-p' - \Delta p - f'(y)\, p = |f(y) - z_d|^{\alpha-1} \operatorname{sgn}(f(y) - z_d)\, f'(y)$$

in Q ,

(10) $$\int_Q \left(p + N\, |u|^{\beta-1} (\operatorname{sgn} u)\right)(v-u)\, dx\, dt \geq 0 \quad \forall\, v \in U_{ad} .$$ □

R e m a r k s . T h e o r e m 2 solves a problem raised by J. L. Lions in his book [9] / Chapitre 1 , Paragraph 20 , Problem 12 /. In view of this result, to seek the optimal pairs it is worth while to look for the triplets (u,y,p) satisfying (6a) - (10).

For the proof we shall adapt the method developed by J. L. Lions for such investigations with the following modification: in order to apply some compact imbedding theorems for anisotrop

Sobolev spaces, we have to use the L^p - theory of parabolic equations for $p \neq 2$ already in lower dimensions, too.

The author is indebted to J. L. Lions for stimulating discussions.

The following proposition is well-known. Nevertheless, for the reader's convenience we recall its proof.

P r o p o s i t i o n . Let us given two functions $f, g \in L^{\mu}(Q)$ $(1 < \mu < \infty)$ and assume that :

$$(11) \qquad \int_Q f\,(-\xi' - \Delta\xi)\,dx\,dt = \int_Q g\,\xi\,dx\,dt$$

$$\forall \xi \in W^{2,1;\mu'}(Q) \text{ such that } \xi = 0 \text{ on } \Sigma,\ \xi(T) = 0 .$$

Then $f \in W^{2,1;\mu}(Q)$, $f = 0$ on Σ, $f(0) = 0$ and

$$(12) \qquad \|f\|_{W^{2,1;\mu}(Q)} \leq C\,\|g\|_{L^{\mu}(Q)}$$

where C is an absolute constant.

P r o o f . It is known / see [3],[5] / that there exists $f^* \in W^{2,1;\mu}(Q)$ such that $f^* = 0$ on Σ, $f^*(0) = 0$ and (11),(12) hold true for f^* instead of f. Then $f - f^* \in L^{\mu}(Q)$ and

$$\int_Q (f - f^*)\,(-\xi' - \Delta\xi)\,dx\,dt = 0$$

$$\forall \xi \in W^{2,1;\mu'}(Q) \text{ such that } \xi = 0 \text{ on } \Sigma,\ \xi(T) = 0 .$$

By a second application of the above mentioned result, ξ may be chosen so as to satisfy also the condition

$$-\xi' - \Delta\xi = |f - f^*|^{\mu - 1} \operatorname{sgn}(f - f^*) .$$

Then we obtain $\int_Q |f - f^*|^{\mu} \, dx \, dt = 0$ whence $f = f^*$. □

P r o o f o f T h e o r e m 1 . Take a minimizing sequence

$$J(v_k, z_k) \longrightarrow \inf \{ J(v,z) ; (v,z) \text{ is admissible} \}$$

of admissible pairs, then the sequences

$$\|v_k\|_{L^{\beta}(Q)} , \quad \|f(z_k)\|_{L^{\alpha}(Q)} , \quad \|z_k\|_{W^{2,1;\gamma}(Q)}$$

are bounded by (1),(2) and (12). There exists therefore a subsequence such that

(13) $v_k \longrightarrow u$ weakly in $L^{\beta}(Q)$,

(14) $z_k \longrightarrow y$ weakly in $W^{2,1;\gamma}(Q)$,

(15) $z_k \longrightarrow y$ strongly in $L^1(Q)$,

(16) $z_k \longrightarrow y$ almost everywhere in Q ,

(17) $f(z_k) \longrightarrow f(y)$ weakly in $L^{\alpha}(Q)$;

here (15) follows from (14) by the Rellich-Kondrasov theorem, (16) follows from (15) by the Riesz lemma and (17) follows from (16) because the sequence $f(z_k)$ is bounded in $L^{\alpha}(Q)$ / see

8, Chapitre 1, Lemme 1.3 /. Being U_{ad} weakly closed, (u,y) is then admissible by (13), (14), (17), and

$$J(u,y) \leq \underline{\lim}\ J(v_k,z_k).$$

Therefore (u,y) is optimal. □

P r o o f o f T h e o r e m 2. Let us fix an optimal pair (u,y) arbitrarily and put for each $\varepsilon > 0$

$$J_\varepsilon(v,z) = \frac{1}{\alpha}\|f(z)-z_d\|^{\alpha}_{L^\alpha(Q)} + \frac{N}{\beta}\|v\|^{\beta}_{L^\beta(Q)}$$

$$+\frac{1}{\varepsilon\gamma}\|z'-\Delta z - f(z)-v\|_{L^\gamma(Q)} + \frac{1}{\gamma}\|z-y\|_{L^\gamma(Q)} + \frac{1}{\beta}\|v-u\|_{L^\beta(Q)}.$$

A pair (v,z) is said to be ε-admissible if

(18) $\quad v\in U_{ad}$, $z\in W^{2,1;\gamma}(Q)$, $z=0$ on Σ, $z(0)=0$ and $f(z)\in L^\alpha(Q)$.

Obviously every admissible pair is also ε - admissible.

A pair $(u_\varepsilon, y_\varepsilon)$ is said to be ε - optimal if it is ε - admissible and if

(19) $\quad J_\varepsilon(u_\varepsilon, y_\varepsilon) = \inf\{J_\varepsilon(v,z) : (v,z)$ is ε - admissible$\}$.

Repeating the proof of theorem 1 we obtain

L e m m a 1. For each $\varepsilon>0$ there exists at least one ε - optimal pair $(u_\varepsilon, y_\varepsilon)$. □

Let us fix an ε - optimal pair $(u_\varepsilon, y_\varepsilon)$ for each $\varepsilon > 0$.

L e m m a 2. If ε tends to 0 then

(20) $u_\varepsilon \longrightarrow u$ strongly in $L^\beta(Q)$,

(21) $y_\varepsilon \longrightarrow y$ strongly in $W^{2,1;\gamma}(Q)$,

(22) $f(y_\varepsilon) \longrightarrow f(y)$ strongly in $L^\alpha(Q)$.

P r o o f. It follows from the estimate

(23) $J_\varepsilon(u_\varepsilon, y_\varepsilon) \leq J_\varepsilon(u,y) = J(u,y)$

that the sequences

$$\|u_\varepsilon\|_{L^\beta(Q)}, \quad \|f(y_\varepsilon)\|_{L^\alpha(Q)}, \quad \|y_\varepsilon\|_{W^{2,1;\gamma}(Q)}$$

are bounded and that

(24) $y'_\varepsilon - \Delta y_\varepsilon + f(y_\varepsilon) - u_\varepsilon \longrightarrow 0$ strongly in $L^\gamma(Q)$.

Therefore every subsequence of $(u_\varepsilon, y_\varepsilon)$ has another subsequence such that

(25) $u_\varepsilon \longrightarrow \hat{u}$ weakly in $L^\beta(Q)$,

(26) $y_\varepsilon \longrightarrow \hat{y}$ weakly in $W^{2,1;\gamma}(Q)$,

$$y_\varepsilon \Rightarrow \hat{y} \quad \text{strongly in} \quad L^r(Q) ,$$

$$y_\varepsilon \to \hat{y} \quad \text{a. e. in} \quad Q ,$$

(27) $$f(y_\varepsilon) \to f(\hat{y}) \quad \text{weakly in} \quad L^\alpha(Q) .$$

Therefore $(\hat{u},\hat{y})$ is admissible and by the optimality of (u,y) hence we obtain also

$$J(u,y) \leq J(\hat{u},\hat{y}) \leq \underline{\lim}\, J(u_\varepsilon, y_\varepsilon) \leq \overline{\lim}\, J_\varepsilon(u_\varepsilon, y_\varepsilon) \leq J(u,y)$$

i.e.

$$J(u,y) = J(\hat{u},\hat{y}) = \lim J(u_\varepsilon, y_\varepsilon) = \lim J_\varepsilon(u_\varepsilon, y_\varepsilon) .$$

Hence (using also (27)) (20) follows and

(28) $$y_\varepsilon \Rightarrow y \quad \text{strongly in} \quad L^r(Q) ,$$

(29) $$f(y_\varepsilon) \to f(\hat{y}) \quad \text{strongly in} \quad L^\alpha(Q) ,$$

and, using also (25), (26), (24), (27),

(30) $$\hat{u} = u ,$$

(31) $$\hat{y} = y ,$$

(32) $$y'_\varepsilon - \Delta y_\varepsilon \to f(y) + u \quad \text{strongly in} \quad L^r(Q) .$$

(29) and (31) yield (22).

It follows from (28), (32) and from the proposition that (21) is also satisfied and the lemma is proved. □

L e m m a 3. If ε tends to 0 then

(33) $\quad y_\varepsilon \to y$ strongly in $C(\bar{Q})$,

(34) $\quad f(y_\varepsilon) \to f(y)$ strongly in $C(\bar{Q})$,

(35) $\quad f'(y_\varepsilon) \to f'(y)$ strongly in $C(\bar{Q})$.

P r o o f. It is well-known that under the hypothesis (5) the imbedding of $W^{2,1;\gamma}(Q)$ into $C(\bar{Q})$ is compact / see [1] or [6] /. Therefore (33) follows from (21). Furthermore, being f and f' locally uniformly continuous, (34) and (35) follow from (33). □

Let us now set

$$p_\varepsilon = -\frac{1}{\varepsilon}\left|y_\varepsilon' - \Delta y_\varepsilon - f(y_\varepsilon) - u_\varepsilon\right|^{\gamma-1} \operatorname{sgn}\left(y_\varepsilon' - \Delta y_\varepsilon - f(y_\varepsilon) - u_\varepsilon\right),$$

then $p_\varepsilon \in L^{\gamma'}(Q)$ for all $\varepsilon > 0$.

L e m m a 4. We have

$$\int_Q - p_\varepsilon\left(\xi' - \Delta\xi - f'(y_\varepsilon)\xi\right) + \left|f(y_\varepsilon) - z_d\right|^{\alpha-1} \operatorname{sgn}\left(f(y_\varepsilon) - z_d\right) f'(y_\varepsilon)\xi$$

$$+ \left|y_\varepsilon - y\right|^{\gamma-1} \operatorname{sgn}\left(y_\varepsilon - y\right)\xi \, dx\, dt = 0$$

whenever $\xi \in W^{2,1;\alpha}(Q)$, $\xi = 0$ on Σ, $\xi(0) = 0$.

P r o o f . Using the preceeding two lemmas and applying Lebesgue's dominated convergence theorem we can see that the above integral equals to

$$\frac{d}{dt} J_\varepsilon (u_\varepsilon , y_\varepsilon + t \xi)\Big|_{t=0} .$$

Being $(u_\varepsilon , y_\varepsilon)$ an optimal pair, hence the lemma follows. □

L e m m a 5 . We have

$$\int_Q \left(p_\varepsilon + N |u_\varepsilon|^{\beta -1} \operatorname{sgn} u_\varepsilon\right)(v-u_\varepsilon) + |u_\varepsilon - u|^{\beta-1} \operatorname{sgn}(u_\varepsilon - u)(v-u_\varepsilon)\, dx\, dt \geq 0$$

whenever $v \in U_{ad}$.

P r o o f . Using lemmas 2, 3 and applying the Lebesgue theorem we obtain that this integral equals to

$$\lim_{t \to 0+0} \frac{J_\varepsilon (u_\varepsilon + t(v-u_\varepsilon), y_\varepsilon) - J_\varepsilon (u_\varepsilon , y_\varepsilon)}{t} .$$

On the other hand the above difference quotient is nonnegative because $(u_\varepsilon , y_\varepsilon)$ is an ε-optimal pair. □

L e m m a 6 . The sequence p_ε is bounded in $L^{r'}(Q)$.

P r o o f . Assume, on the contrary, that

$$(36) \qquad \|p_\varepsilon\|_{L^{r'}(Q)} \longrightarrow \infty$$

for some subsequence. Then, applying the proposition, by lemmas 3, 4 we obtain that

$$\frac{p_\varepsilon}{\|p_\varepsilon\|_{L^{r'}(Q)}} \text{ is bounded in } W^{2,1;\alpha'}(Q) .$$

It follows from the condition (5) that the imbedding of $W^{2,1;\alpha'}(Q)$ into $L^{r'}(Q)$ is compact. There exists therefore another subsequence such that

$$(37) \qquad \frac{p_\varepsilon}{\|p_\varepsilon\|_{L^{r'}(Q)}} \longrightarrow q \quad \text{strongly in } L^{r'}(Q) .$$

Using this property and also (33)-(36), from lemma 4 we obtain

$$\int_Q - q\left(\xi' - \Delta\xi - f'(y)\,\xi\right) dx\, dt = 0$$

whenever $\xi \in W^{2,1;\alpha'}(Q)$, $\xi = 0$ on Σ, $\xi(0) = 0$.
Applying a unicity result of H. Brézis / see [9], Chapitre 1, Théoreme 2.1 ; here the theorem is proved only for $n \leqslant 3$ but this remains valid also for $n > 3$ /, hence we can conclude $q \equiv 0$. This contradicts to (37) whence $\|q\|_{L^{r'}(Q)} = 1$. □

Let us now turn to the proof of the theorem. Applying the lemmas 2, 3, 4, 6 and the proposition, there exists a subsequence p_ε such that

(38) $p_\varepsilon \longrightarrow p$ weakly in $W^{2,1;\alpha'}(Q)$ and then

also strongly in $L^{r'}(Q)$.

Using lemma 3 and (38), from lemma 4 we obtain

$$(39) \quad \int_Q - p\left(\xi' - \Delta\xi - f'(y)\xi\right) + \left|f(y) - z_d\right|^{\alpha-1} \operatorname{sgn}\left(f(y) - z_d\right) f'(y)\xi \, dxdt$$

$$= 0$$

whenever $\xi \in W^{2,1;\alpha}(Q)$, $\xi = 0$ on Σ and $\xi(0) = 0$. Now (6) and (8) are true because (u,y) is optimal, while (7) and (9) follow from (39) and from the proposition. Finally, (10) follows from lemma 5 by (20) and (38). The theorem is proved. □

R e m a r k . We used the proposition several times. For $\mu = 2$ it can be proved much more easily, using the Hilbert space theory of parabolic equations, see e.g. [7]. However, to treat the cases $n > 1$ i.e. the cases of practical interest, in view of the condition (5) it is necessary to apply more general results.

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ON THE CONTROL OF STRONGLY NONLINEAR HYPERBOLIC SYSTEMS

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The control theory of singular distributed systems was systematically studied by J.L. Lions in his book [1]. In general the nonlinear terms had polynomial growth and the author raised the problem of the control of strongly nonlinear systems i.e. when the nonlinear term is more general, e.g. of exponential type.

In [2] and [3], by developing the method used in [1] this problem was solved for some systems of elliptic and parabolic type. Using the $L^p (1 < p < \infty)$ regularity results known for these equations, it was not necessary to make any assumption on the nonlinear term. For the hyperbolic equations, however, we have no L^p-regularity results except p=2, therefore the above method cannot be applied directly.

The purpose of this paper is to overcome this difficulty. We will show that, making some new and very general assumptions on the growth of the nonlinear term (which are satisfied, for example for the functions having polynomial or exponential growth), a singular system of optimality can be derived in this case too.

Throughout this paper let Ω be a bounded open subset of $\mathbb{R}^n$ $(n \in \mathbb{N})$ with boundary Γ of class C^∞, T a positive number and set

$$Q = \Omega \times]0,T[, \qquad \Sigma = \Gamma \times]0,T[.$$

We are given a function $f \in C^1(\mathbb{R})$ such that

(1) $$\sup_{x\in\mathbb{R}} \frac{|f'(x)|}{1+|f(x)|} < \infty ,$$

(2) $$\sup_{\substack{x,y\in\mathbb{R}\\ |x-y|\leq 1}} \frac{|f(x)|}{1+|f(y)|} < \infty .$$

We remark that these conditions are satisfied for example if f is a polynomial or an exponential function. The assumption (1) is an improvement of another growth condition used by N. Barbu [6] and D. Tiba [7].

Let $\mathcal{U}_{ad}$ be a convex, closed subset of $L^2(Q)$, $y^0 \in H_0^1(\Omega)$, $y^1 \in L^2(\Omega)$ arbitrary given functions. We shall say that a pair (v,z) is admissible if

(3) $$\begin{aligned} &z'' - \Delta z - f(z) = v \text{ in } \mathcal{D}'(Q),\\ &z \in L^\infty(0,T\,;\,H_0^1(\Omega)),\ z' \in L^\infty(0,T\,;\,L^2(\Omega)),\\ &z(0) = y^0,\quad z'(0) = y^1,\\ &v \in \mathcal{U}_{ad},\ f(z) \in L^2(Q). \end{aligned}$$

Let us fix $z_d \in L^2(Q)$ and a number $N > 0$ arbitrarily and set

(4) $$\mathcal{J}(v,z) = \frac{1}{2}||f(z) - z_d||^2_{L^2(Q)} + \frac{N}{2}||v||^2_{L^2(Q)}.$$

An admissible pair (u,y) will be called optimal if

(5) $$\mathcal{J}(u,y) = \inf\ \mathcal{J}\{(v,z)\ |\ (v,z) \text{ is admissible}\}.$$

We shall always assume that

(6) there exists at least one admissible pair.

(This is satisfied for example if $\mathcal{U}_{ad} = L^2(Q)$ and $y^0 = y^1 = 0$). Then the following theorems will be proved :

Theorem 1

There exists at least one optimal pair. □

Theorem 2

Suppose that one of the following three assumptions is satisfied :

(i) $n = 1$,

(ii) $\mathcal{U}_{ad} = L^2(Q)$,

(iii) f and f' are bounded.

Then to any optimal pair (u,y) there exists a function p such that the following singular system of optimality holds :

(7) $y'' - \Delta y - f(y) = u$ in $\mathcal{D}'(Q)$,

(8) $p'' - \Delta p - f'(y)p = (f(y) - z_d)\, f'(y)$ in $\mathcal{D}'(Q)$,

(9) $\int_Q (p + Nu)(v - u)\,dxdt \geq 0 \quad \forall v \in \mathcal{U}_{ad}$,

(10) $y \in L^\infty(0,T\,;\, H_0^1(\Omega))$, $y' \in L^\infty(0,T\,;\, L^2(\Omega))$,

(11) $y(0) = y^0$, $y'(0) = y^1$,

(12) $p \in L^\infty(0,T\,;\, L^2(\Omega))$, $p' \in L^\infty(0,T\,;\, H^{-1}(\Omega))$,

(13) $p(T) = p'(T) = 0$, $\quad p = 0$ sur Σ ,

(14) $u \in \mathcal{U}_{ad}$. □

Proof of Theorem 1

Let (v_k, z_k) be a sequence of admissible pairs such that :

$$\mathcal{J}(v_k,z_k) \to \inf \{ \mathcal{J}(v,z) \mid (v,z) \text{ is admissible}\},$$

then the sequences

$$||v_k||_{L^2(Q)}, \; ||f(z_k)||_{L^2(Q)}, \; ||z_k'' - \Delta z_k||_{1^2(Q)}$$

are bounded by (3) and (4). Applying the usual a priori estimates for the wave equation and we conclude that

$$||z_k||_{L^\infty(0,T;H_0^1(\Omega))} \quad \text{and} \quad ||z_k'||_{L^\infty(0,T;L^2(\Omega))}$$

are also bounded whence by [4], Ch. 1, Th. 5.1.

$$z_k \text{ is precompact in } L^2(Q).$$

Using also the Riesz lemma, there exists therfore a subsequence of (v_k,z_k) such that

$$v_k \to u \text{ in } L^2(Q) \text{ weakly,}$$
$$z_k \to y \text{ in } L^\infty(0,T \,;\, H_0^1(\Omega)) \text{ weakly-star,}$$
$$z_k' \to y' \text{ in } L^\infty(0,T \,;\, L^2(\Omega)) \text{ weakly-star,}$$
$$z_k \to y \text{ in } Q \text{ almost everywhere,}$$
$$f(z_k) \to f(y) \text{ in } L^2(Q) \text{ weakly.}$$

(In the last step we applied [4], Ch. 1, Lemma 1.3). Hence it follows that (u,y) is admissible and

$$\mathcal{J}(u,y) \le \lim \mathcal{J}(v_k,z_k)$$

i.e. (u,y) is optimal. □

Proof of Theorem 2

Fixing an optimal pair (u,y) arbitrarily, for each $\varepsilon > 0$ set

$$\mathcal{J}_\varepsilon(v,z) = \mathcal{J}(v,z) + \frac{1}{2\varepsilon} ||z'' - \Delta z - f(z) - v||^2_{L^2(Q)} + $$
$$+ \frac{1}{2} ||z - y||^2_{L^2(Q)} + \frac{1}{2} ||v - u||^2_{L^2(Q)} .$$

A pair (v,z) will be called ε-admissible if

(15)
$$\begin{aligned} & z'' - \Delta z \in L^2(Q), \\ & z \in L^\infty(0,T ; H^1_0(\Omega)), \ z' \in L^\infty(0,T ; L^2(\Omega)), \\ & z(0) = y^0, \ z'(0) = y^1, \\ & v \in \mathcal{U}_{ad}, \ f(z) \in L^2(Q), \end{aligned}$$

and ε-optimal if (15) and

(16) $$\mathcal{J}_\varepsilon(v,z) = \inf \{\mathcal{J}_\varepsilon(w,x) \mid (w,x) \text{ is } \varepsilon\text{-admissible}\}$$

are satisfied.

Obviously, every admissible pair is also ε-admissible. Repeating the reasoning of Theorem 1 one can easily see that for each $\varepsilon > 0$ there exists at least one ε-optimal pair.

Let us fix an ε-optimal pair $(u_\varepsilon, y_\varepsilon)$ for each $\varepsilon > 0$.

Lemma 1

If $\varepsilon \to 0$ then

(17) $u_\varepsilon \to u$ in $L^2(Q)$ strongly,

(18) $y_\varepsilon \to y$ in $L^\infty(0,T ; H^1_0(\Omega))$ strongly,

(19) $y'_\varepsilon \to y'$ in $L^\infty(0,T\ ;\ L^2(\Omega))$ strongly,

(20) $f(y_\varepsilon) \to f(y)$ in $L^2(Q)$ strongly.

Proof

It follows from the estimate

(21) $\mathcal{J}_\varepsilon(u_\varepsilon,y_\varepsilon) \le \mathcal{J}_\varepsilon(u,y) = \mathcal{J}(u,y)$

that the sequences

$$||u_\varepsilon||_{L^2(Q)},$$

$$||f(y_\varepsilon)||_{L^2(Q)},$$

$$||y_\varepsilon||_{L^\infty(0,T;H^1_0(\Omega))},$$

$$||y'_\varepsilon||_{L^\infty(0,T;L^2(\Omega))}$$

are bounded and that

(22) $||y''_\varepsilon - \Delta y_\varepsilon - f(y_\varepsilon) - u_\varepsilon||_{L^2(Q)} \to 0.$

Therefore any subsequence of $(u_\varepsilon,y_\varepsilon)$ has another subsequence such that

(23) $u_\varepsilon \to \hat{u}$ in $L^2(Q)$ weakly,

(24) $y_\varepsilon \to \hat{y}$ in $L^\infty(0,T\ ;\ H^1_0(\Omega))$ weakly-star,

(25) $y'_\varepsilon \to \hat{y}'$ in $L^\infty(0,T\ ;\ L^2(\Omega))$ weakly-star,

(26) $y_\varepsilon \to \hat{y}$ in $L^2(Q)$ strongly,

(27) $y_\varepsilon \to \hat{y}$ almost everywhere in Q,

(28) $f(y_\varepsilon) \to f(\hat{y})$ in $L^2(Q)$ weakly.

It follows from (22), (23), (24), (25) and (28) that $(\hat{u}, \hat{y})$ is admissible. Using also (21) and the optimality of (u,y) we obtain

$$\mathcal{J}(u,y) \leq \mathcal{J}(\hat{u},\hat{y}) \leq \underline{\lim}\ \mathcal{J}(u_\varepsilon,y_\varepsilon) \leq \overline{\lim}\ \mathcal{J}(u_\varepsilon,y_\varepsilon) \leq \mathcal{J}(u,y)$$

where

(29) $(\hat{u},\hat{y}) = (u,y)$

and

(30) $\mathcal{J}(u_\varepsilon,y_\varepsilon) \to \mathcal{J}(u,y)$.

Now (17) and (20) follow from (23), (28), (29) and (30). Furthermore (26) and (29) yield

(31) $y_\varepsilon \to y$ in $L^2(Q)$ strongly.

Using again the a priori estimates for the wave equation, (18) and (19) follow from (17), (20), (22) and (31). □

Let us now set

(32) $p_\varepsilon = -\frac{1}{\varepsilon}(y''_\varepsilon - \Delta y_\varepsilon - f(y_\varepsilon) - u_\varepsilon) \quad (\in L^2(Q))$.

Lemma 2

For all $v \in \mathcal{U}_{ad}$ we have

(33) $\int_Q (p_\varepsilon + Nu_\varepsilon + u_\varepsilon - u)(v - u_\varepsilon)\ dxdt \geq 0$.

Proof

By the ε-optimality of $(u_\varepsilon, y_\varepsilon)$ we have

$$\frac{\mathcal{J}_\varepsilon(u_\varepsilon + t(v - u_\varepsilon), y_\varepsilon) - \mathcal{J}_\varepsilon(u_\varepsilon, y_\varepsilon)}{t} \geq 0$$

for all $0 < t < 1$.

Letting $t \to 0$ (33) follows without any difficulty. □

The following observation will play an important role in the sequel.

Lemma 3

If $\varepsilon \to 0$ then

(34) $f'(y_\varepsilon) \to f'(y)$ in $L^2(Q)$ strongly.

Proof

It follows from (20) that the sequence $|f(y_\varepsilon)|^2$ is uniformly absolutely integrable. Using property (1) then $|f'(y_\varepsilon)|^2$ and also $|f'(y_\varepsilon) - f'(y)|^2$ is uniformly absolutely integrable. On the other hand, it follows from (18) (or from (31)) that any subsequence of y_ε has another subsequence such taht

$$f'(y_\varepsilon) \to f'(y) \text{ almost everywhere in } Q.$$

We can therefore apply a theorem of Vitali [5] whence the lemma follows. □

REMARK

The application of Vitali's theorem is not a usual tool in such investigations. Nevertheless the above lemma is unnecessary in the case when $f(x) = e^x$. Indeed, in this case $f' = f$ and (34) coincides with (20). □

In the following lemma both assumptions (1) and (2) on the growth of f will be needed.

Lemma 4

We have

(35) $$\int_Q p_\varepsilon(\xi'' - \Delta\xi - f'(y_\varepsilon)\xi)\,dxdt = \int_Q (f(y_\varepsilon) - z_d)f'(y_\varepsilon)\xi + (y_\varepsilon - y)\xi\,dxdt$$

whenever $\xi \in C^2(\bar{Q})$, $\xi(0) = \xi'(0)$ in Ω and $\xi = 0$ on Σ.

Proof

It follows from (2) that $(u_\varepsilon, y_\varepsilon + s\xi)$ is ε-admissible for all $0 < s < 1$. Being $(u_\varepsilon, y_\varepsilon)$ ε-optimal we have therefore

(36) $$\frac{d}{ds}\,\mathcal{J}_\varepsilon(u_\varepsilon, y_\varepsilon + s\xi)\big|_{s=0} = 0$$

provided that this derivative exists. To show this, write

$$\frac{\mathcal{J}_\varepsilon(u_\varepsilon, y_\varepsilon + s\xi) - \mathcal{J}_\varepsilon(u_\varepsilon, y_\varepsilon)}{s} = \int_Q \frac{|f(y_\varepsilon + s\xi) - z_d|^2 - |f(y_\varepsilon) - z_d|^2}{2s}\,dxdt$$
$$+ \int_Q \frac{|y''_\varepsilon - \Delta y_\varepsilon - f(y_\varepsilon + s\xi) - u_\varepsilon + s\xi'' - s\Delta\xi|^2 - |y''_\varepsilon - \Delta y_\varepsilon - f(y_\varepsilon) - u_\varepsilon|^2}{2\varepsilon s}\,dxdt$$
$$+ \int_Q \frac{|y_\varepsilon - y + s\xi|^2 - |y_\varepsilon - y|^2}{2s}\,dxdt$$

and try to pass to the limit by the use of Lebesgue's dominated convergence theorem. One can readily see that the integrands tend to

$(f(y_\varepsilon) - z_d)\; f'_\varepsilon(y_\varepsilon)\xi$,

$- p_\varepsilon(\xi'' - \Delta\xi - f'(y_\varepsilon)\xi)$,

$(y_\varepsilon - y)\xi$

respectively, almost everywhere in Q. According to (36) this provides (35) if we show that the integrands are L^1-bounded in all the three cases.

In the first case, applying the Lagrange mean value theorem, for some $0 < \theta < 1$ we have

$$\left| \frac{|f(y_\varepsilon + s\xi) - z_d|^2 - |f(y_\varepsilon) - z_d|^2}{2s} \right| =$$

$$|(f(y_\varepsilon + \theta s\xi) - z_d)\ f'(y_\varepsilon + \theta s\xi)\xi| \leq$$

$$(|f(y_\varepsilon + \theta s\xi)| + |z_d|)\ .\ |f'(y_\varepsilon + \theta s\xi)|\ .\ |\xi| \leq$$

$$C\ .\ (1 + |f(y_\varepsilon)| + |z_d|)\ .\ (1 + |f(y_\varepsilon)|) \in L^1(Q)$$

with some constant C depending only on $||\xi||_{L^\infty(Q)}$; we used (1), (2) and the boundedness of ξ.

In the second case the integrand is equal to

$$\frac{1}{\varepsilon}\ (y_\varepsilon'' - \Delta y_\varepsilon - u_\varepsilon + \theta s\xi'' - \theta s\Delta\xi - f(y_\varepsilon + \theta s\xi))\ .\ (\xi'' - \Delta\xi - f_\varepsilon'(y + \theta s\xi)\xi)$$

and its absolute value may be estimated by

$$\frac{1}{\varepsilon}(|y_\varepsilon'' - \Delta y_\varepsilon| + |u_\varepsilon| + C + C\ .\ |f(y_\varepsilon)|)\ .\ (C + C\ .\ |f(y_\varepsilon)|) \in L^1(Q).$$

Finally, the third integrand is obviously L^1-bounded. □

<u>REMARK</u>

Up to this point none of the assumptions (i), (ii), (iii) was used. □

Lemma 5

$$||p_\varepsilon||_{L^2(Q)} \text{ is bounded.}$$

Proof

If condition (ii) is satisfied then (33) yields $p_\varepsilon = u - u_\varepsilon - Nu_\varepsilon$, and the lemma follows from (17).

In the remaining cases we can argue as in the proof of [1] ch. 2, Th. 2.1. Indeed, putting

$$m_\varepsilon = f'(y_\varepsilon),$$

$$F_\varepsilon = (f(y_\varepsilon) - z_d)\, f'(y_\varepsilon) + (y_\varepsilon - y),$$

in the case (i)

m_ε is bounded in $L^2(Q)$,

F_ε is bounded in $L^1(Q)$,

in the case (ii)

m_ε is bounded in $L^\infty(Q)$,

F_ε is bounded in $L^2(Q)$.

Applying Theorem 2.2 of [1] , ch. 2, in both cases we obtain that

p_ε is bounded in $L^\infty(0,T\,;\,L^2(\Omega))$,

p'_ε is bounded in $L^\infty(0,T\,;\,H^{-1}(\Omega))$

whence the lemma follows. □

Now we can finish the proof of Theorem 2. Indeed, passing to a subsequence we may assume by Lemma 5 that

(37) $\qquad p_\varepsilon \to p$ in $L^2(Q)$ weakly.

Using (17), (20), (34) and (37), by letting $\varepsilon \to 0$ from (33), (35) we obtain (9) and

(38) $$\int_Q p(\xi'' - \Delta\xi - f'(y)\xi)dxdt = \int_Q (f(y) - z_d)f'(y)\xi \, dxdt$$

whenever $\xi \in C^2(\bar{Q})$, $\xi(0) = \xi'(0) = 0$, $\xi = 0$ on Σ, which is equivalent to (8), (12) and (13). Finally, (7), (10), (11) and (14) follow from the admissibility of the pair (u,y). □

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