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Abstract :

We analyze a stack protocol of the Capetanakis-Tsybakov-Mikhaïlov type for resolving collisions in a multiaccess broadcast channel and study in detail the mean collision resolution intervals (CRI) . We obtain a functional equation for the generating function of the mean CRI's which is non local with a non-commutative iteration semi-group. Using Mellin transform techniques and properties of the iteration semigroup we show that for arrival rates smaller than a fixed threshold, the mean CRI for n colliders is asymptotically proportional to n. Ergodicity conditions are also given.

Résumé :

Cet article présente l'analyse d'un protocole de type Capetanakis-Tsybakov-Mikhaïlov pour la résolution de collisions dans un réseau à accès multiple. On y étudie en détail les temps moyens de résolution de collisions. La série génératrice de ces quantités vérifie une équation fonctionnelle non locale dont le groupe d'itération est non commutatif. L'utilisation de la transformation de Mellin en conjonction avec des propriétés stochastiques du groupe d'itération permet de montrer que le temps moyen de résolution de n collisions est essentiellement linéaire en n. Nous concluons par des conditions d'ergodicité.

Introduction

For more than ten years, a huge literature has been devoted to the analysis of the performance of a multiaccess satellite or ground broadcast channel used as a single communication medium in packet-switching networks (ALOHA type systems). We briefly recall the basic operation involved.

a) A single broadcast channel is shared among many independent users (sources, stations, ...) which emit packets. Channel time is discrete and measured in "transmission slots" - packet duration being equal to one slot. Sources are synchronized so that transmissions are initiated at the beginning of a slot, i.e at time 0,1,2,... .

b) Each transmission is broadcast to every user-including the emitter. When several stations transmit simultaneously, packets will collide (interfere) and it is assumed that none of them is received correctly: these collisions are handled as transmission errors.

The resolution of the contention is clearly the Gordian knot of that multiple access scheme.

Many protocols for conflict resolution have been suggested and sometimes modelled. Most of them are based on the original ALOHA system and on several cognate varieties: when conflicts occur, each station will retransmit randomly in the next slot with some given probability. The main drawback of these protocols is that the system is inherently unstable in the absence of external control, when the number of stations is large (say, an infinite source) - see [FAY75].

1. The C.T.M. Algorithm

In this paper we consider the Capetanakis-Tsybakov-Mikhailov (C.T.M.) collision resultion algorithm (CRA), which proves ergodic if the mean input rate of new packets is not too large. It requires no external control. Moreover, this CRA allows transmitters to broadcast a message for the first time immediately in the slot when it is available. Thus, they do not have to monitor the channel continuously, but only when they have an active message.

1.1 Description of the CTM algorithm with continuing input

(for a complete survey see [MAS81])

Points a) and b) of the basic operation hold.

α) Each user monitors his own stack consisting of cells k = 0,1,2,... . A new packet entering the stack at time t is entered in cell 0.

β) From a global point of view, the set of all user stacks will be considered as a conceptual "super stack" made of frames. The frame i is the union of all packets occupying cell i in their respective stack and a packet can leave the system only when it is in frame 0 (successful transmission).

The procedure RESOLVE written below, describes the operation of the super stack. This can be summarized as follows:

i) At the end of each transmission slot, colliding packets (if any) are divided into two groups according to some Bernoulli trial, each collider flipping a coin.

ii) One group is allowed to retransmit in the next slot. The other groups will retransmit later - namely after collisions of the first group have been solved.

```
PROCEDURE RESOLVE
    COMMENT: "N(i) is the number of packets in frame i;
              DEPTH is the depth of the stack."
    DEPTH ← 1
    IF N(0) = 0 or 1 THEN collision:=false; N(i-1) ← N(i), i≥1 ;
              DEPTH ← DEPTH+1
    ELSE BEGIN collision:=true; N(i+1) ← N(i), i≥1 ;
              DEPTH ← DEPTH+1
    COMMENT: "each of the N(0) users flips a coin.  The result is
              saved in the variable RANDOM which takes the value
              P (resp. Q) with probability p (resp. q), p+q=1."
      k=1;  NEW N(0) ← 0;
    BEGIN FOR k=1,..N(0)
      IF RANDOM=P, THEN COMMENT "STAY in frame 0"
         NEW N(0) ← NEW N(0)+1
     ELSE COMMENT "GOTO FRAME 1"
     N(1) ← N(1)+1;
     END;
     N(0) ← NEW N(0);
   END;
END RESOLVE
```

Assume that n packets initially collide in frame 0, i.e. the super stack is in the state [N(0)=n, N(1), N(2) ..., N(DEPTH)]. The time necessary to clear frame 0 - hence getting the state [N(0)=0, N(1), N(2) ..., N(DEPTH)] - will be called the mean resolution interval, and denoted by L_n.

In the next section, a stochastic model is studied to derive the expected length $\alpha_n \stackrel{def}{=} E(L_n)$ of the CRI.

Let us remark at once that some variations of the preceding algorithm can be presented to save slots which would otherwise be doomed (see [Massey]; [FAY,HOF82]). This happens when the first group (resulting from the tossing of packets in frame 0) is empty, in which case the following slot will surely produce collisions. We refer the reader to the concluding section 5 at the end of the paper.

1.2 A Functional Equation for the Generating function of mean CRI

Assumptions:

H1: There is an infinite number of identical sources.

H2: The number of new packets which appear in the system in one slot is a random variable independent of t and of the history of the channel up to time t - having a Poisson distribution of parameter λ.

The definition of the channel protocol provides the following recursive relation for the random variables L_n:

$$L_0 = L_1 = 1 \qquad (1)$$

$$L_n = 1 + L_{I+X} + L_{n-I+Y}, \quad n\geq 2 \qquad (2)$$

where:

I: the number of messages immediately retransmitted (Bernoulli trial with parameter p).

X: the number of new arrivals in that immediate slot.

Y: the number of new arrivals in the slot following the resolution of the I+X messages.

X and Y are Poisson random variables with mean λ.

Taking expectation in (1) and (2) yields

$$\alpha_0 = \alpha_1 = 1 \qquad (3)$$

$$\alpha_n = 1 + \sum_{k\geq 0} \sum_{l\geq 0} \sum_{i=0}^{n} p^i q^{n-i} e^{-2\lambda} \frac{\lambda^k}{k!} \frac{\lambda^\ell}{\ell!} (\alpha_{i+k} + \alpha_{n-i+l}) \qquad (4)$$

Relations of type (4) have been studied in [TSY-VVED80] only when $p=q=\frac{1}{2}$, although it was in the case of more general input. The arguments were different and, as will appear later, the mathematical problem is completely different when $p\neq q$.

Introducing

$$\alpha(z) \overset{\text{def}}{\equiv} \sum_{n=v}^{\infty} \alpha_n \frac{z^n}{n!}$$

$$\varphi(z) \overset{\text{def}}{\equiv} e^{-z} \alpha(z) \qquad , \qquad (5)$$

we get from (3) and (4)

$$\varphi(z) - \varphi(\lambda+pz) - \varphi(\lambda+qz) = 1 - e^{-z}[2\,\varphi(\lambda)(1+z)+z\varphi'(\lambda)] \qquad (6)$$

where $\varphi'(\lambda) = \left.\frac{d\varphi(z)}{dz}\right|_{z=\lambda}$

Putting successively in (6) $z = \frac{\lambda}{p}$ and $z = \frac{\lambda}{q}$, eliminating $\varphi(2\lambda)$, one obtains $\varphi'(\lambda)$ in terms of $\varphi(\lambda)$:

$$\varphi'(\lambda) = 2(K-1)\,\varphi(\lambda).$$

Where

$$K = \frac{e^{\frac{\lambda}{p}} - e^{\frac{\lambda}{q}}}{\frac{\lambda}{q} e^{\frac{\lambda}{q}} - \frac{\lambda}{p} e^{\frac{\lambda}{p}}} \qquad (7)$$

Finally (6) can be written as:

$$\varphi(z) - \varphi(\lambda+pz) - \varphi(\lambda+qz) = 1-2\varphi(\lambda)e^{-z}(1+Kz) \quad .$$

2. An Iteration Scheme for the Functional Equation

In this section we develop an iterative scheme for solving the basic functional equation:

$$\varphi(z) - \varphi(\lambda+pz) - \varphi(\lambda+pz) = 1 - 2\varphi(\lambda)e^{-z}\ (1+Kz) \tag{1}$$

where

$$K = \frac{e^{\frac{\lambda}{p}} - e^{\frac{\lambda}{q}}}{\frac{\lambda}{q}e^{\frac{\lambda}{q}} - \frac{\lambda}{p}e^{\frac{\lambda}{p}}} \quad , \qquad \varphi(0) = 1 \quad , \quad \varphi'(0) = 0 \tag{2}$$

and for expressing the Taylor coefficients of

$$\alpha(z) = e^{z}\varphi(z) \tag{3}$$

This requires the introduction of a non-commutative iteration semigroups whose properties are also of use in the later asymptotic analysis. The solutions appear as sums indexed on this iteration semigroup.

With as before $\sigma_1(z) = \lambda+pz$ and $\sigma_2(z) = \lambda+qz$, we introduce the following definitions:

(i) We let H be the semigroup of linear substitutions generated by σ_1,σ_2, where the semigroup operation is the composition of functions. The identity of H, denoted by ϵ, is thus the function $\epsilon(z) = z$ for all $z \in G$.

(ii) Any substitution in H can be written under the form

$\sigma = \sigma_{i_1}\ \sigma_{i_2}\ \ldots\ \sigma_{i_n}$ where $n \geq 0$ and $i_j \in \{1,2\}$,

and we set

$$|\sigma|_1 = \text{card}\{j/i_j=1\} \quad ; \quad |\sigma|_2 = \text{card}\ \{j/i_j=2\};$$

$$|\sigma| = |\sigma_1| + |\sigma_2| \ ,$$

this last quantity being called the <u>length</u> of substitution σ.

(iii) The subset of H formed with substitutions of length n is denoted by H_n, so that

$$H_n = \{\sigma\in H/\ |\sigma|=n\} \ .$$

The semigroup H satisfies the obvious decompositions:

$$H = \{\epsilon\} \cup \sigma_1 H \cup \sigma_2 H \qquad (4)$$

$$H = \{\epsilon\} \cup H\sigma_1 \cup H\sigma_2 . \qquad (5)$$

and correspondingly for H_n:

$$H_n = \sigma_1 H_{n-1} \cup \sigma_2 H_{n-2} \qquad (6)$$

$$H_n = H_{n-1}\sigma_1 \cup H_{n-2}\sigma_2 \ . \qquad (7)$$

<u>Notation</u>: If α, β are complex numbers, we define

$$(\alpha;\beta)^\sigma = \alpha^{|\sigma|_1} \beta^{|\sigma|_2} \ .$$

We can now state:

<u>Proposition 1</u>: If α, β are complex numbers satisfying the contraction condition:

$$|\alpha| + |\beta| < 1$$

and if $t(z)$ is an entire function, then the functional equation

$$f(z) - \alpha\, f(\sigma_1(z)) - \beta\, f(\sigma_2(z)) = t(z) \qquad (8)$$

has a unique entire solution given by

$$f(z) = \sum_{\sigma\in H} (\alpha;\beta)^\sigma\, t(\sigma(z)) \ . \qquad (9)$$

Proof:

(i) Existence: For z in C and any $\sigma \in H$, we have

$$|\sigma(z)| \leq \max\left(|z|, \frac{\lambda}{q}\right) \quad .$$

Let $\mu(z) = \{|t(x)| / x \in C, \quad |x| \leq \max(|z|, \frac{\lambda}{q})\}$

then the sum in (8) is absolutely convergent and its modulus is bounded by

$$\mu(z) \sum_{\sigma \in H} (|\alpha|; |\beta|)^{\sigma} = \frac{\mu(z)}{1-|\alpha|-|\beta|} \quad .$$

Thus for z in any bounded domain, f(z) as given by (9), is a uniformly convergent sum of analytic functions and therefore is itself analytic. Using decomposition (5), we have:

$$f(z) = t(z) + \sum_{\sigma \in H\sigma_1} (\alpha;\beta)^{\sigma}\, t(\sigma(z)) + \sum_{\sigma \in H\sigma_2} (\alpha;\beta)^{\sigma}\, t(\sigma(z))$$

$$= t(z) + \alpha \sum_{\tau \in H} (\alpha;\beta)^{\tau}\, t(\tau\sigma_1(z)) + \beta \sum_{\tau \in H} (\alpha;\beta)^{\tau}\, t(\tau\sigma_2(z))$$

$$= t(z) + \alpha\, f(\sigma_1(z)) + \beta\, f(\sigma_2(z)),$$

so that f(z) satisfies equation (8).

(ii) Unicity: To discuss the solutions of (8), we need to consider domains D such that

$$\sigma_1 D \subset D \qquad \sigma_2 D \subset D \quad .$$

Such domains necessarily contain the real interval $[\frac{\lambda}{p};\frac{\lambda}{q}]$ since for any z, the set

$$H[z] = \{\sigma(z) \;/\; \sigma \in H\}$$

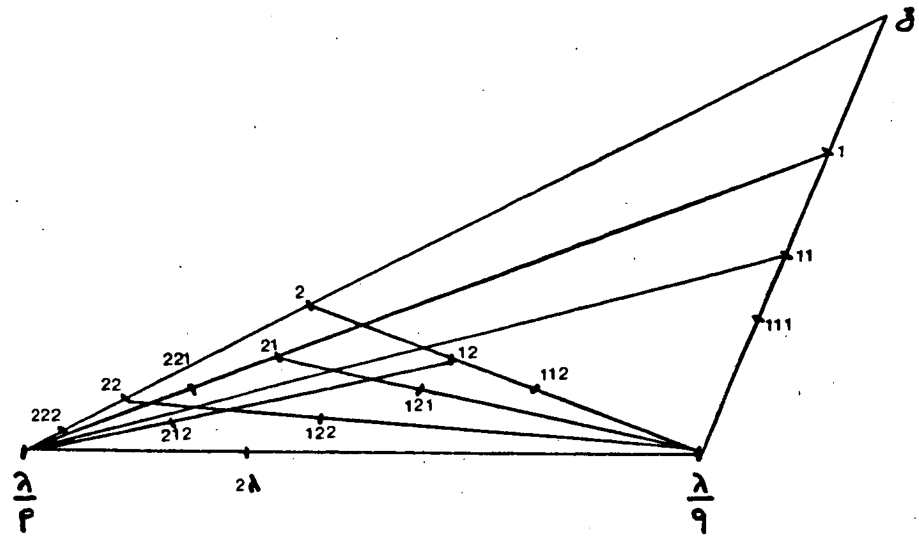

Figure 1: The successive transforms of a point z, in the case $p = \frac{2}{3}$, $q = \frac{1}{3}$. The point $\sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_m}(z)$ is labelled $i_1 i_2 \ldots i_m$.

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admits this interval at the set of its accumulation points (see Figure 1). Let f_1 and f_2 be two analytic solutions of equation (8) in such a domain D. By iteration of the functional relation, f_1 and f_2 are necessarily entire. Setting

$$\delta(z) = f_1(z) - f_2(z) \quad ,$$

we see that $\delta(z)$ satisfies the relation

$$\delta(z) - \alpha\delta(\sigma_1(z)) - \beta\delta(\sigma_2(z)) = 0 \quad ,$$

whence by iteration for all $n>0$:

$$\delta(z) = \sum_{\sigma\in H_n} (\alpha; \beta)^{\sigma} \, \delta(\sigma(z)) \quad . \qquad (10)$$

We prove that $\delta(z)$ is of bounded modulus in the whole of the complex plane, hence constant by Liouville's theorem: let

$$M = \max \{|\delta(z)| \; / \; |z| \leq \frac{2\lambda}{q}\} \quad .$$

For any $z \in C$, there exists some $n \in N$ such that

$$|\sigma(z)| \leq \frac{2\lambda}{q} \quad \text{for all } \sigma \in H_n \quad .$$

Thus with (10) and this value of n

$$|\delta(z)| \leq M \sum_{\sigma \in H_n} (|\alpha|;|\beta|)^{\sigma} = M(|\alpha|+|\beta|)^n < M \quad ,$$

which therefore establishes the uniform boundedness of δ. Obviously we can only have $\delta(z) = 0$ which proves the two solutions f_1 and f_2 to coincide. □

<u>Lemma 1</u>: For all $n \geq 2$, let

$$g_n = (-1)^n \sum_{\sigma \in H} e^{-\sigma(0)} (p^n, q^n)^{\sigma}$$

$$k_n = (-1)^n \sum_{\sigma \in H} e^{-\sigma(0)} \sigma(0)(p^n, q^n)^{\sigma}$$

and define

$$D(\lambda) = \sum_{n \geq 2} [(1-Kn)\, g_n + Kk_n] \frac{\lambda^n}{n!} \quad .$$

A necessary and sufficient condition for equation (1) to have an entire solution is that

$$D(\lambda) \neq -\tfrac{1}{2} \; , \quad \text{and} \quad K \neq \infty \quad .$$

If this condition is satisfied, one has

$$\alpha_n = 1 - \frac{2}{1+2D(\lambda)} [T_n + KU_n + Kn\, V_n] + 0(n^{1-\eta}) \quad , \; \eta > 0$$

where

$$T_n = \sum_{\sigma \in H} e^{-\sigma(0)} [(1-(p;q)^{\sigma})^n - 1 + n(p;q)^{\sigma}]$$

$$U_n = \sum_{\sigma \in H} e^{-\sigma(0)} (p;q)^{\sigma} [(1-(p;q)^{\sigma})^n - 1 + n(p;q)^{\sigma}]$$

$$V_n = \sum_{\sigma \in H} e^{-\sigma(0)} (p;q)^{\sigma} [(1-(p;q)^{\sigma})^{n-1} - 1] \quad .$$

Proof: In order to be able to use Lemma 1, we consider the functional equation

$$f(z) - p^2 f(\sigma_1(z)) - q^2 f(\sigma_2(z)) = (A+Bz)e^{-z} \quad (11)$$

where

$$A = -2\varphi(\lambda)(1-2K) \qquad B = -\varphi(\lambda)K \quad . \quad (12)$$

This equation is obtained by differentiating (8) formally twice. The coefficients p^2, q^2 satisfy the contraction condition and thus (11) admits the solution

$$f(z) = \sum_{\sigma\in H} (p^2;q^2)^\sigma (A + B\sigma(z))\, e^{-z}$$

under the sole condition that A and B - i.e. K and $\varphi(\lambda)$ - be defined. Assuming that condition to be fulfilled, and noticing that

$$\sigma(z) = \sigma(0) + (p;q)^\sigma z \;,$$

we find for the coefficients of f+:

$$f_n = n!\,[z^n]\; f(z)$$

the expression

$$f_n = (A-B_n)g_{n+2} + Bk_{n+2} \quad (13)$$

with g_n, k_n defined in the statement of Lemma 1. Since f is entire, it can be integrated twice. The result is the function

$$\varphi(z) = 1 + \sum_{n\geq 2} f_{n-2} \frac{z^n}{n!} \quad (14)$$

which is also entire and satisfies equation (1). Relations (12), (13), (14) give an explicit expression of f(z) for all z, assuming thus only K and $\varphi(\lambda)$ to be defined:

+ We denote by $[z^n]a(z)$ the n^{th} Taylor coefficient of a(z):

$$[z^n] \sum_m a_m z^m = a_n \quad .$$

$$\varphi(z) = 1 - 2\varphi(\lambda) \sum_{n\geq 2} [(1-Kn)g_n + Kk_n] \frac{z^n}{n!} \quad . \tag{15}$$

Instantiating (15) for $z = \lambda$, we thus find that a necessary condition for the existence of $\varphi(\lambda)$ is that $D(\lambda) \neq -\frac{1}{2}$, $K \neq \infty$ and the condition is also clearly sufficient. Thus when $D(\lambda) \neq -\frac{1}{2}$, solving (15), we find:

$$\varphi(\lambda) = [1 + 2D(\lambda)]^{-1} \quad . \tag{16}$$

The second part of the Lemma follows by taking the Taylor coefficients of $\alpha(z) = e^z\varphi(z)$ with $\varphi(z)$ given by (16) and grouping terms using standard binomial expansions. □

The next step is to use exponential approximations for the coefficients T_n, U_n, V_n (cf. [Kn73] p.131 for a similar situation). To that purpose, we introduce the quantities:

$$t(x) = \sum_{\sigma\in H} e^{-\sigma(0)}(e^{-a(\sigma)x} - 1 + a(\sigma)x) \tag{17}$$

$$u(x) = \sum_{\sigma\in H} e^{-\sigma(0)} \sigma(0)(e^{-a(\sigma)x} - 1 + a(\sigma)x) \tag{18}$$

$$v(x) = \sum_{\sigma\in H} e^{-\sigma(0)} a(\sigma)(e^{-a(\sigma)x} - 1) \tag{19}$$

where we have set $a(\sigma) = (p,q)^{\sigma}$. We prove:

<u>Lemma 2</u>: The collision resolution times satisfy the relation

$$\alpha_n = \frac{2}{1+2D(\lambda)} [t(n) + Ku(n) + Knv(n)] + O(n^{1-\eta})$$

for some real $\eta > 0$.

<u>Proof</u>: We use the expression of Lemma 1 and show that T_n, U_n, V_n are approximated for large n by resp. t(n), u(n), v(n). We only prove the result for T_n, the other cases being identical. From the definitions

$$T_n - t(n) = \sum_{\sigma \in H} e^{-\sigma(0)} \left[(1-a(\sigma))^n - e^{-a(\sigma)n}\right]$$

$$= \sum_{\sigma \in H} \delta(\sigma) \quad .$$

To evaluate this sum, we split it and define

$$S_1(\nu) = \sum_{|\sigma| \leq \nu} \delta(\sigma) \quad ; \quad S_2(\nu) = \sum_{|\sigma| > \nu} \delta(\sigma) \quad .$$

Since for positive a, $(1-a) < e^{-a}$, and since for $|\sigma| \leq \nu$, $a(\sigma) \geq q^{\nu}$, we find:

$$S_1(\nu) = O(2^{\nu} e^{-nq^{\nu}}) \quad . \tag{20}$$

Choosing ν such that $p^{\nu} < \frac{1}{2}$ ensures that for all σ: $|\sigma| > \nu$,

$$(1-a(\sigma))^n - e^{-a(\sigma)n} = e^{-a(\sigma)n} \, O(na^2(\sigma)),$$

uniformly in n and σ, so that:

$$S_2(\nu) = O(n \sum_{|\sigma| > \nu} a^2(\sigma))$$

$$= O(n(p^2+q^2)^{\nu}) \tag{21}$$

We can select

$$\nu = \log_q \left(\frac{\log^2 n}{n}\right) \quad ,$$

which ensures that $S_1(\nu)$ is exponentially small by (20):

$$S_1(\nu) = O(n^{c_1} e^{-\log^2 n}) \quad \text{for some } c_1;$$

The condition that $p^{\nu} = O(1)$ is also satisfied, so that by (21)

$$S_2(\nu) = O(n(p^2+q^2)^{\nu})$$

$$= O(n^{1-\eta}) \text{ for some strictly positive } \eta. \qquad \square$$

3. Asymptotic Analysis

We now propose to study the asymptotic behaviour of α_n as n gets large. Lemma 2 has reduced the problem to that of estimating the asymptotic equivalents of $t(x)$, $u(x)$ and $v(x)$ as $x \to \infty$. We first compute the Mellin transforms of t, u, v which appear to have factored forms in which both the gamma function and certain Dirichlet series related to the iteration group appear. We then use the classical correspondence between the singularities of Mellin transforms in a right half plane and terms in the asymptotic expansion of the original functions for large values of the arguments, a fact which comes from the inversion theorem for Mellin transforms.

Locating singularities of the Dirichlet series, and in particular estimating the dominant terms in their asymptotic expansions around their poles requires some deeper properties of the iteration group H. Once this is done, we can conclude with the asymptotic analysis of $t(x)$, $u(x)$, $v(x)$. The discussion distinguishes two cases based on certain arithmetical properties of the probabilities p, q, and we can conclude finally that α_n has a linear growth in n.

We start by introducing the two Dirichlet series:

$$\theta(s) = \sum_{\sigma \in H} e^{-\sigma(0)} (p^s, q^s)^\sigma \tag{1}$$

$$\xi(s) = \sum_{\sigma \in H} e^{-\sigma(0)} \sigma(0) (p^s, q^s)^\sigma \tag{2}$$

in which the sums are absolutely convergent for $Re(s)>1$. We also consider the function

$$\beta(x) = t(x) + Ku(x) + Kxv(x) \tag{3}$$

which appears in the approximation of α_n, and which is defined for all $x \geq 0$.

The Mellin transform of a function $f(x)$ defined on R^+ is denoted by $f*(s)$ or $M[f(x);s]$ and is given by

$$f*(s) = \int_0^\infty f(x)x^{s-1}dx \quad . \tag{4}$$

(See [Do55], [Da78] for basic properties and definitions of the Mellin transform.) We have:

<u>Lemma 3:</u> The Mellin transform of the function $\beta(x)$ of (3) is:

$$\beta*(s) = \theta(-s)(\Gamma(s)+K\Gamma(s+1)) + K\xi(-s)\Gamma(s)$$

and the integral (4) defining $\beta*$ is absolutely convergent for $s: -2<\mathrm{Re}(s)<-1$.

<u>Proof</u>: The Mellin transform satisfies the important functional property:

$$M[f(ax);s] = a^{-s}f*(s) \tag{5}$$

for any positive a. Applying (5) repeatedly we see that a function of the form

$$F(x) = \sum_{\sigma\in H} c_\sigma f(d_\sigma x) \tag{6}$$

has a transform of the form

$$F*(s) = f*(s) \sum_{\sigma\in H} c_\sigma (d_\sigma)^{-s} \tag{7}$$

which is valid provided s in the intersection of the domain of absolute convergence of $f*(s)$ and of the domain of absolute convergence of the sum that appears in (7).

We use the classical transforms

$$\int_0^\infty (e^{-x} - 1 + x)\, x^{s-1}dx = \Gamma(s) \quad \text{for} \quad s: -2<\mathrm{Re}(s)<1 \tag{8}$$

$$\int_0^\infty (e^{-x} - 1)\, x^{s-1}dx = \Gamma(s) \quad \text{for} \quad s: -1<\mathrm{Re}(s)<0 \quad . \tag{9}$$

Applying (7) to the sum giving t(x), we thus find that

$$t^*(s) = \Gamma(s) \sum_{\sigma \in H} e^{-\sigma(0)} \{(p,q)^{\sigma}\}^{-s} \quad ,$$

where the condition on (8) is $-2<\mathrm{Re}(s)<-1$ and the condition on the sum is $\mathrm{Re}(s)< -1$. The transforms of u and v are dealt with in a similar way, whence the result by linearity of the transform. □

By the inversion theorem $\beta(x)$ is expressible in terms of $\beta^*(s)$ as the integral

$$\beta(x) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \beta^*(s) x^{-s} ds \;, \quad \text{for any c: } -2<c<-1. \qquad (10)$$

We propose to evaluate (10) by shifting the line of integration to the right taking residues of the integrand into account. The first residues give the dominant terms in the asymptotic expansion of $\beta(x)$ as $x \to \infty$. This however requires some more detailed analytic information on $\theta(s)$ and $\xi(s)$.

To treat $\theta(s)$ and $\xi(s)$ simultaneously, we thus consider Dirichlet series given by a sum of the form

$$\omega(s) = \sum_{\sigma \in H} r(\sigma(0)) \; (p^s,p^s)^{\sigma} \qquad (11)$$

where r(u) is any continuously differentiable function on $[0;\frac{\lambda}{q}]$. We have:

Proposition 1: The function $\omega(s)$ is meromorphic for $\mathrm{Re}(s)>0$. It has a simple pole at s=1, and around that point admits the expansion

$$\omega(s) = \frac{1}{(s-1)} \quad \frac{1}{h(p,q)(\frac{\lambda}{q} - \frac{\lambda}{p})} \qquad \int_{\frac{\lambda}{p}}^{\frac{\lambda}{q}} r(u)du + \int_{0}^{\frac{\lambda}{q}} r'(u)w(u)du + O(s-1)$$

where $h(p,q)$ is the entropy function:

$$h(p,q) = p \log p^{-1} + q \log q^{-1}$$

and $w(u)$ is a weight function independent of $r(u)$.

We first prove that $\omega(s)$ has, in the half plane $Re(s)>0$, the same singularities as $(1-p^s-q^s)^{-1}$.

<u>Lemma 4</u>: The function

$$\pi(s) = \omega(s)(1-p^s-q^s)$$

is analytic for $Re(s)>0$ and uniformly bounded in any half plane $Re(s)\geq\eta>0$.

<u>Proof</u>: For $Re(s)>1$, the sum expressing $\omega(s)$ is absolutely convergent, so that we can regroup terms in the expression of $\pi(s)$. We have:

$$\pi(s) = \sum_{\sigma\in H} r(\sigma(0))(p^s;q^s)^{\sigma} - \sum_{\sigma\in H} r(\sigma(0))\, p^s(p^s,q^s)^{\sigma} - \sum_{\sigma\in H} r(\sigma(0))q^s(p^s,q^s)^{\sigma} \qquad (12)$$

We first transform the second and third sums in (12)

$$\sum_{\sigma\in H} r(\sigma(0))p^s(p^s;q^s)^{\sigma} = \sum_{\tau\in H\sigma_1} r(\tau\sigma_1^{-1}(0))(p^s;q^s)^{\tau} \qquad (13)$$

$$\sum_{\sigma\in H} r(\sigma(0))\, q^s(p^s;q^s)^{\sigma} = \sum_{\tau\in H\sigma_2} r(\tau\sigma_2^{-1}(0))\, (p^s;q^s)^{\tau}\ ; \qquad (14)$$

We then use the decomposition $H = \{\epsilon\} \cup H\sigma_1 \cup H\sigma_2$ in the first sum in (12), and group terms with those of (13), (14), so that

$$\pi(s) = r(0) - \sum_{\tau\in H\sigma_1} (r(\tau\sigma_1^{-1}(0)) - r(\tau(0)))\ (p^s,q^s)^{\tau}$$

$$- \sum_{\tau\in H\sigma_2} (r(\tau\sigma_2^{-1}(0)) - r(\tau(0))\ (p^s,q^s)^{\tau} \quad . \qquad (15)$$

Equation (15) is valid for $Re(s)>1$; now using the observations:

$$\tau(a) - \tau(b) = (p,q)^{\tau} (a-b)$$

$$r(\tau(a)) - r(\tau(b)) = O((p,q)^{\tau})$$

uniformly in τ for fixed a and b, we see that the sums in (15) are

$$O\left(\sum_{\tau \in H} |(p^{s+1},q^{s+1})^{\tau}| \right) = O([1-|p^{s+1}|-|q^{s+1}|]^{-1})$$

and therefore converge for $Re(s)>0$. Thus (15) provides the analytic continuation of $\pi(s)$ to the left of $s = 1$ and the Lemma is established. □

The next stage in the proof of Proposition 1 is to obtain the main terms in the expansion of $\omega(s)$ around $s = 1$. To that purpose, we decompose the sum expressing $\omega(s)$:

$$\omega(s) = \sum_{n\geq 0} \sum_{\sigma\in H_n} r(\sigma(0))(p^s,q^s)^{\sigma}$$

with, as before $H_n = \{\sigma\in H / \ |\sigma| = n\}$. We define the sequence of functions:

$$\phi_n^{(s)}(u) = \sum_{\substack{\sigma(0)<u \\ \sigma\in H_n}} (\bar{p}^s,\bar{q}^s)^{\sigma} \text{ with } \bar{p}^s = \frac{p^s}{p^s+q^s} \ ; \ \bar{q}^s = \frac{q^s}{p^s+q^s} \qquad (16)$$

Function $\phi_n^{(s)}$ is thus the cumulative distribution function of the discrete probability distribution which to point $\sigma(0)$ with $\sigma\in H_n$ associates the probability $(\bar{p}^s,\bar{q}^s)^{\sigma}$. The expression of $\omega(s)$ then becomes

$$\omega(s) = \sum_{n\geq 0} (p^s+q^s)^n \int r(u)d\phi_n^{(s)}(u) \qquad (17)$$

where the integral is a Riemann-Stieljes integral taken on R. As we shall see, when n tends to infinity, $\phi_n^{(s)}$ tends to a limit, and the value of this limit for $s = 1$ gives the main term in the local expansion of $\omega(s)$.

<u>Lemma 5</u>: For δ such that $p<\delta<1$, let $D(\delta)$ be the domain:

$$D(\delta) = \{s \in C \;/\; |\bar{p}^{(s)}|<\delta \text{ and } |\bar{q}^{s}|<\delta \;.$$

Then, for each s in $D(\delta)$, there exists a function $\phi_\infty^{(s)}(u)$ defined on R, such that

$$|\phi_n^{(s)}(u) - \phi_\infty^{(s)}(u)| \leq A\delta^n \quad \text{for all } u \text{ in } R \text{ and for some } A \in R.$$

In particular, $\phi_\infty^{(1)}$ has the explicit form

$$\phi_\infty^{(1)}(u) = \begin{cases} 0 & \text{if } u \leq \frac{\lambda}{p} \\ \dfrac{u - \frac{\lambda}{p}}{\frac{\lambda}{q} - \frac{\lambda}{p}} & \text{if } \frac{\lambda}{p} \leq u \leq \frac{\lambda}{q} \\ 1 & \text{if } \frac{\lambda}{q} \leq u. \end{cases}$$

<u>Proof</u>: From the definition of the $\phi_n^{(s)}$, using the decomposition $H_{n+1} = \sigma_1 H_n \cup \sigma_2 H_n$, we have:

$$\phi_{n+1}^{(s)}(u) = \sum_{\substack{\sigma \in H_n \\ \sigma_1\sigma(0)<u}} \bar{p}^{s}(\bar{p}^{s};\bar{q}^{s})^{\sigma} + \sum_{\substack{\sigma \in H_n \\ \sigma_2\sigma(0)<u}} \bar{q}^{s}(\bar{p}^{s};\bar{q}^{s})^{\sigma}$$

and since σ_1,σ_2 are monotone increasing:

$$\phi_{n+1}^{(s)}(u) = \bar{p}^{s}\phi_n^{(s)}(\sigma_1^{-1}(u)) + \bar{q}^{s}\phi_n^{(s)}(\sigma_2^{-1}(u)), \qquad (18)$$

with

$$\phi_0^{(s)}(u) = 0 \text{ if } u<0 \quad ; \quad \phi_0^{(s)}(u) = 1 \text{ if } 0 \leq u.$$

Let us consider the four regions:

$$R_1 = \{u \;/\; u < \tfrac{\lambda}{p}\} \quad ; \quad R_2 = \{u \;/\; \tfrac{\lambda}{p} \leq u < 2\lambda\}$$

$$R_3 = \{u \;/\; 2\lambda \leq u < \tfrac{\lambda}{q}\} \quad ; \quad R_4 = \{u \;/\; \tfrac{\lambda}{q} \leq u\} \quad ;$$

the substitutions σ_1^{-1} , σ_2^{-1} operate on these regions as follows:

$$\begin{aligned} &\sigma_1^{-1}(R_1) \subset R_1 && \sigma_2^{-1}(R_1) \subset R_1 \\ &\sigma_1^{-1}(R_2) \subset R_1 && \sigma_2^{-1}(R_2) \subset R_3 \\ &\sigma_1^{-1}(R_3) \subset R_1 \cup R_2 && \sigma_2^{-1}(R_3) \subset R_4 \\ &\sigma_1^{-1}(R_4) \subset R_4 && \sigma_2^{-1}(R_4) \subset R_4 \end{aligned} \tag{19}$$

In particular, each element of $[\frac{\lambda}{p},\frac{\lambda}{q}[$ has only one image by σ_1^{-1}, σ_2^{-1} in the interval. For any $I \subset R$, we define the norm

$$||f||_I = \sup \{|f(u)| \ / \ u \in I\} .$$

Let n_0 be such that

$$\sigma_1 \ \sigma_2^{n_0}(0) > \frac{\lambda}{p} \quad ;$$

then:

$$\forall n > n_0 \quad \forall \sigma \in H_n \quad : \quad \sigma_1\sigma(0) > \frac{\lambda}{p} . \tag{20}$$

Working on the four cases of (19), we first find that for all $n > n_0$, using (20):

$$||\phi_{n+1}^{(s)} - \phi_n^{(s)}||_{R_1} \leq \delta \ ||\phi_n^{(s)} - \phi_{n-1}^{(s)} \ ||_{R_1} \quad ;$$

thus

$$||\phi_{n+1}^{(s)} - \phi_n^{(s)}||_{R_1} \leq A_1 \ \delta^n \quad .$$

Similarly:

$$||\phi_{n+1}^{(s)} - \phi_n^{(s)}||_{R_2} \leq ||\phi_n^{(s)} - \phi_{n-1}^{(s)}||_{R_1} + \delta \ ||\phi_n^{(s)} - \phi_{n-1}^{(s)}||_{R_3} \quad ,$$

$$||\phi_{n+1}^{(s)} - \phi_n^{(s)}||_{R_3} = ||\phi_n^{(s)} - \phi_{n-1}^{(s)}||_{R_4} + \delta \ ||\phi_n^{(s)} - \phi_{n-1}^{(s)}||_{R_2} .$$

$$||\phi_{n+1}^{(s)} - \phi_n^{(s)}||_{R_4} = 0.$$

Therefore, for some A_2:

$$||\phi_n^{(s)} - \phi_{n-1}^{(s)}||_R \leq A_2\delta^n . \qquad (21)$$

Hence for each u, the sequence $\{\phi_n^{(s)}(u)\}$ is a Cauchy sequence, so that it converges to a limit $\phi_\infty^{(s)}(u)$. From (21) follows that, for some A:

$$||\phi_n^{(s)} - \phi_\infty^{(s)}|| < A\delta^n \qquad (22)$$

as was to be proved.

Using this result in conjunction with equation (18), one sees that $\phi_\infty^{(s)}$ satisfies the equation

$$\phi_\infty^{(s)}(u) = \bar{p}^s\phi_\infty^{(s)}(\sigma_1^{-1}(u)) + \bar{q}^s\phi_\infty^{(s)}(\sigma_2^{-1}(u)) \qquad (23)$$

with the boundary condition

$$\phi_\infty^{(s)}(u) = 0 \quad \text{if } u \leq \frac{\lambda}{p} \qquad \phi_\infty^{(s)}(u) = 1 \text{ if } \frac{\lambda}{q} \leq u , \qquad (24)$$

and it is easy to check that (23), (24) when s=1 are satisfied by the piecewise linear function of the statement of Lemma 5. □

Thus $\phi_\infty^{(1)}(u)$ is nothing but the cumulative distribution function associated to the uniform distribution on $[\frac{\lambda}{p};\frac{\lambda}{q}]$. We now proceed to use this result in conjunction with the expression (17) of $\omega(s)$. We define:

$$D(s) = \omega(s) - \frac{1}{1-p^s-q^s} \frac{1}{\frac{\lambda}{q}-\frac{\lambda}{p}} \int_{\frac{\lambda}{p}}^{\frac{\lambda}{q}} r(u)du \qquad (25)$$

which, a priori, exists only for Re(s) > 1. Using (17), (25) becomes

$$D(s) = \sum_{n\geq 0} (p^s+q^s)^n \left[\int r(u)\ d(\phi_n^{(s)}(u) - \phi_\infty^{(1)}(u))\right]$$

$$D(s) = \sum_{n\geq 0} (p^s+q^s)^n \left[\int r'(u)(\phi_\infty^{(1)}(u) - \phi_n^{(s)}(u))du\right] , \qquad (26)$$

as follows from summation by parts. Using the decomposition

$$\phi_{\infty}^{(1)}(u) - \phi_{n}^{(s)}(u) = \phi_{\infty}^{(1)}(u) - \phi_{n}^{(1)}(u) + \phi_{n}^{(1)}(u) - \phi_{n}^{(s)}(u) \quad ,$$

we have

$$D(s) = D_1(s) + D_2(s)$$

with

$$D_1(s) = \sum_{n\geq 0} (p^s+q^s)^n \int r'(u)\ (\phi_{\infty}^{(1)}(u) - \phi_{n}^{(1)}(u))\ du \qquad (27)$$

$$D_2(s) = \sum_{n\geq 0} (p^s+q^s)^n \int r'(u)\ (\phi_{n}^{(1)}(u) - \phi_{n}^{(s)}(u))\ du \qquad (28)$$

Using the geometric convergence (22) of Lemma 5, we see that $D_1(s)$ is actually analytic in a neighbourhood of s=1, and

$$D_1(1) = \sum_{n\geq 0} \int r'(u)(\phi_{\infty}^{(1)}(u) - \phi_{n}^{(1)}(u))\ du \quad .$$

$$D_1(1) = \int r'(u) w_1(u)\ du$$

with:

$$w_1(u) = \sum_{n\geq 0} (\phi_{\infty}^{(1)}(u) - \phi_{n}^{(1)}(u)) \quad . \qquad (29)$$

We prove

Lemma 6: Function $D_2(s)$ is analytic in a neighbourhood of s=1.

Proof: We use an indirect argument: since

$$D_2(s) = D(s) - D_1(s)$$

is the difference of two functions meromorphic at s=1, it is meromorphic there and has at most a simple pole at s=1 by Lemma 4. We propose to prove that as $s \to 1^+$

$$D_2(s) = o(\frac{1}{s-1})$$

which will establish that $D_2(s)$ is regular at s=1. We write first:

$$D_2(s) = \sum_{n\geq 0} d_n(s) \quad \text{with } d_n(s) = (p^s+q^s)^n \sum_{\upsilon\in H_n} r(\upsilon(0))[\bar{p}^s;\bar{q}^s)^{\upsilon}) .$$

We estimate the d_n's using the decomposition

$$H_n = H_k \times H_l \quad \text{with } k=k(n) = \lfloor\sqrt{n}\rfloor \;\; ; \;\; l=l(n) = n - \lfloor\sqrt{n}\rfloor . \qquad (30)$$

We have:

$$d_n(s) = (p^s+q^s)^n \sum_{\sigma\in H_k} \sum_{\tau\in H_l} r(\sigma\ \tau(0))[(\bar{p}^s;\bar{q}^s)^{\sigma\tau} - (p;q)^{\sigma\tau}]$$

Since r is assumed to be continuously differentiable

$$r(\sigma\tau(0)) = r(\sigma(0)) + 0((p;q)^{\sigma})$$

uniformly in σ,τ. Thus:

$$d_n(s) = (p^s+q^s)^n \sum_{\sigma\in H_k} \sum_{\tau\in H_l} r(\sigma(0))[(\bar{p}^s;\bar{q}^s)^{\sigma\tau} - (p;q)^{\sigma\tau}]$$

$$+ (p^s+q^s)^n\ 0(\sum_{\sigma\in H_k}\sum_{\tau\in H_l} ((\bar{p}^s;\bar{q}^s)^{\sigma\tau} + (p;q)^{\sigma\tau})(p;q)^{\sigma}). \qquad (30)$$

Using the fact that

$$\sum_{\tau\in H_l} (\bar{p}^s,\bar{q}^s)^{\tau} = 1,$$

we get

$$d_n(s)= (p^s+q^s)^n \sum_{\sigma\in H_k} r(\sigma(0))\ [(\bar{p}^s;\bar{q}^s)^{\sigma} - (p;q)^{\sigma}] + (p^s+q^s)^n\ 0(\sum_{\sigma\in H_k} (\bar{p}^{s+1};\bar{q}^{s+1})^{\sigma} +$$

$$+ (p^2;q^2)^{\sigma})$$

For s in a neighbourhood of 1, the second term is

$$0((p^{1-\epsilon}+q^{1-\epsilon})^n(p^{2-\epsilon}+q^{2-\epsilon})^n) \quad , \quad \epsilon>0$$

and thus is $0(M^n)$ for some M: $0<M<1$.

The first term can be estimated, computing derivatives, one finds that

$$(\bar{p}^{s};\bar{q}^{s})^{\sigma} - (p;q)^{\sigma} = O(k(n)(s-1))$$

Uniformly in σ. We have thus proved that

$$d_n(s) = O((p^s+q^s)^n k(n)(s-1) + O(M^n).$$

Since , as $x \to 1^-$

$$\Sigma \lfloor\sqrt{n}\rfloor x^n = O(1-x)^{-3/2} \quad ,$$

we thus obtain

$$D_2(s) = O(s-1)^{-\frac{1}{2}}$$

and $D_2(s)$ is analytic at s=1 by our preceding remarks. □

With Lemma 6 and observation (29), we can thus conclude that D(s) is analytic at s=1, so that

$$\omega(s) - \frac{1}{s-1} \quad \frac{}{h(p,q)(\frac{\lambda}{q} - \frac{\lambda}{p})} \int_{\frac{\lambda}{p}}^{\frac{\lambda}{q}} r(u)du \tag{31}$$

admits around 0 an expansion of the form

$$a_0 + a_1(s-1) + a_2(s-2)^2 + \ldots \quad .$$

To complete the proof of Proposition 1, we now look for an explicit expression of a_0. The problem is equivalent to that of determining D(1) whose existence is guaranteed by Lemma 6.

Using now the decomposition

$$\phi_\infty^{(1)}(u) - \phi_n^{(s)}(u) = \phi_\infty^{(s)}(u) - \phi_n^{(s)}(u) + \phi_\infty^{(1)}(u) - \phi_\infty^{(s)}(u)$$

we see that D(s) can also be rewritten under the form:

$$D(s) = \sum_{n\geq 0}(p^s+q^s)^n \int r'(u)(\phi_\infty^{(s)}(u) - \phi_n^{(s)}(u))\, du$$
$$+ (1-p^s-q^s)^{-1}\int r'(u)(\phi_\infty^{(1)}(u) - \phi_\infty^{(s)}(u))du \tag{32}$$

For Re(s)>1. When $s \to 1^+$, the first term converges to $D_1(1)$ given by expression (29). The second term is thus another form of $D_2(1)$, so that it has a limit when $s \to 1^+$ and is equal to:

$$D_2(1) = \int r'(u) w_2(u) du$$

with

$$w_2(u) = - \frac{1}{h(p,q)} \left[\frac{\partial}{\partial s} \phi_\infty^{(s)}(u)\right]_{s=1} .$$

To estimate $w_2(u)$, we differentiate the functional equation satisfied by $\phi_\infty^{(s)}$ and find:

$$\frac{\partial}{\partial s} \phi_\infty^{(s)}(u) = \bar{p}^s \frac{\partial}{\partial s} \phi^{(s)}(\sigma_1^{-1}(u)) + \bar{q}^s \frac{\partial}{\partial s} \phi^{(s)}(\sigma_2^{-1}(u))$$
$$+ L\, \phi^{(s)}(\sigma_1^{-1}(u)) - L\, \phi^{(s)}(\sigma_2^{-1}(u)) \qquad (33)$$

with $L = p\, q \log \frac{p}{q}$.

Let $\delta(x)$ be the triangular function

$$\delta(x) = L(\phi_\infty^{(1)}(\sigma_1^{-1}(x)) - \phi_\infty^{(1)}(\sigma_2^{-1}(x)) \quad . \qquad (34)$$

The graph of $\delta(x)$ is displayed on Figure 2. Instantiating (33) at s=1, and using the iterative scheme for solving this functional equation, we thus find that:

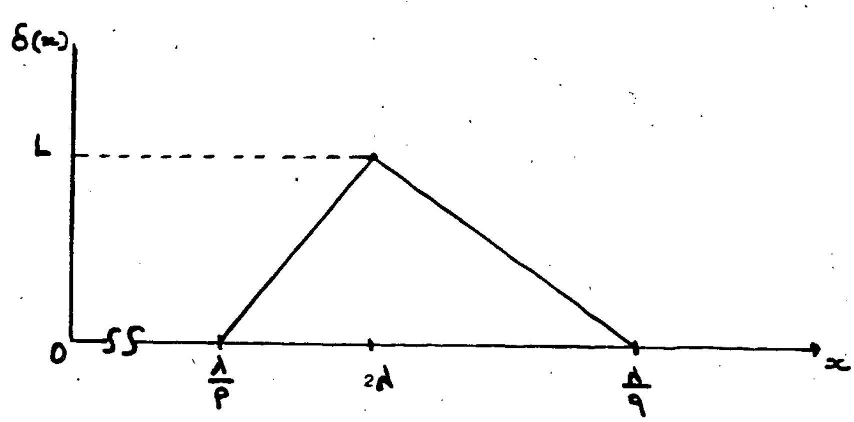

Figure 2: The graph of the triangular function $\delta(x)$

$$w_2(u) = \sum_{\sigma \in H} (p;q)^{\sigma} \, \delta(\sigma^{-1}(u)) \qquad (35)$$

Equation (35) defines a function which is nowhere differentiable and which is a superposition of triangular functions of smallest and smallest supports and amplitudes. Functions of a similar nature appear elsewhere in the analysis of algorithms

Function $w_1(u)$ in (29) can also be expressed in a similar way as:

$$w_1(u) = \sum_{\sigma \in H} (p;q)^{\sigma} \, \epsilon(\sigma^{-1}(u)) \qquad (36)$$

where $\epsilon(u)$ is the piecewise linear function

$$\epsilon(u) = \phi_{\infty}^{(1)}(u) - \phi_0^{(1)}(u)$$

whose graph is displayed on Figure 3.

These calculations thus complete the proof of Proposition 1 and $w(u) = w_1(u) + w_2(u)$ has an "explicit" expression through (35), (36).

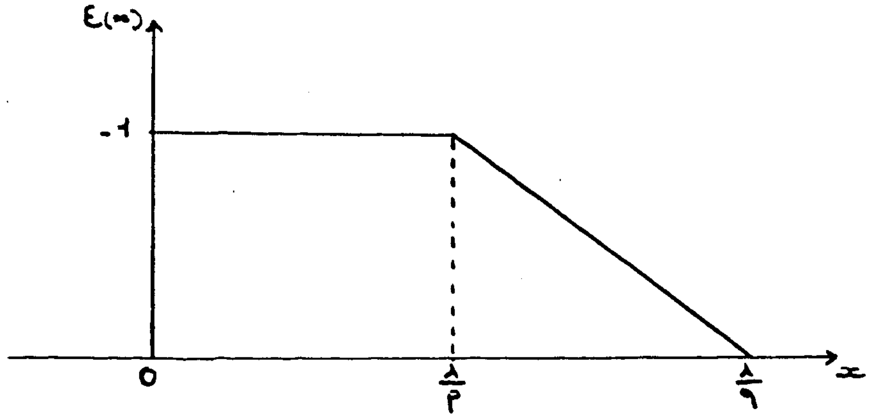

Figure 3: The graph of function ϵ.

Proposition 1 thus gives an expression for the first terms in the expansion of functions $\theta(-s)$, $\xi(-s)$ appearing in Lemma 3, around their singularity at $s=-1$. By the Mellin inversion theorem (10):

$$\beta(x) = \frac{1}{2i\pi} \int_{-\frac{3}{2}-i\infty}^{-3/2+i\infty} \beta*(s)x^{-s}ds \quad .$$

At last we establish a general property on the poles of $\omega(s)$ in the right half plane in order to move the integration contour in the above integral to the left, let

$$f(s) = p^s + q^s - 1,$$

and

$$Z(f) = \{s \;/\; Re(0) > 0; \quad f(s) = 0\} \quad .$$

The set $Z(f)$ coincides by Lemma 4 with the poles of $\omega(s)$ in the right half plane. We prove that $Z(f)$ is uniformly discrete in the following sense:

Lemma 7: There exists a real number $\delta > 0$ such that

$$\forall s, s' \in Z(f) \qquad |s - s'| > \delta \quad .$$

Proof: Assume a contrario that the Lemma is not satisfied. Then for all $\delta > 0$, there exists an s in $Z(f)$ and an a with $|a|<\delta$ such that:

$$p^s + q^s = 1 \quad \text{and } p^{s+a} + q^{s+a} = 1 \quad .$$

Eliminating q^s, we should have

$$p^s = \frac{1 - q^a}{p^a - q^a} \tag{38}$$

Now for small a, a local expansion shows that

$$\frac{1-q^a}{p^a-q^a} = -\frac{\log q}{\log \frac{p}{q}} + O(a) \quad .$$

The function

$$\lambda(x) = -\frac{\log(1-x)}{\log \frac{x}{1-x}}$$

satisfies:

$\lambda(x) > 1$ for all $x \in]0; \frac{1}{2}[$. Thus the right hand side of (38) is of modulus strictly larger than 1 for a small enough. There is thus a contradiction in (38) since for $Re(s) > 0$

$$|p^s| < 1 \quad ;$$

this establishes the Lemma. □

The argument used in Lemma 7 can actually be used to prove that all the elements of Z(f) are simple zeros of f and that for all $\delta > 0$ there exists an $\eta > 0$ such that:

$$\forall s', Re(s') > 0 \ , \ \forall s \in Z(f): |s - s'| > \delta \Rightarrow f(s') > \eta \quad . \qquad (39)$$

One then sees that for some fixed small enough $\epsilon > 0$ such that the minimal distance between points in Z(f) is larger than 4ϵ and for each integer n there exists a closed contour Γ_n with the following properties

(i) Γ_n consists of four curves:

$$\Gamma_n = \Gamma_n^{\,1} + \Gamma_n^{\,2} + \Gamma_n^{\,3} + \Gamma_n^{\,4}$$

with

$$\Gamma_n^{\,1} = \{-\tfrac{3}{2} + it \ / \ t \in [-n; +n]\}$$

$$\Gamma_n^{\,2} \subseteq \{z \ / \ Re(z) \in [-\tfrac{3}{2}; -\epsilon]; \ I_m(z) \in [n; n+2\epsilon]\}$$

$\Gamma_n^3 \subset \{z \;/\; Re(z) \in [-3\epsilon;-\epsilon];\quad Im(z) \in [-n-2\epsilon;\; n+2\epsilon]\}$

$\Gamma_n^4 \subset \{z \;/\; Re(z) \in [-\frac{3}{2};-\epsilon];\quad Im(z) \in [-n-2\epsilon;\; -n]\}$

(ii) Each point in Γ_n is at a distance at least ϵ from a zero of $f(-s)$.

Such a contour can be constructed by distorting a rectangular contour so as to avoid the zeros of $f(-s)$. We can thus assume that Γ_n is rectifiable and has length $O(n)$. Figure 4 displays the shape of such a contour. We now consider the integral

$$I_n = \frac{1}{2i\pi} \int_{\Gamma_n} \beta*(s)x^{-s}ds$$

with Γ_n oriented clockwise, and let n tend to infinity.

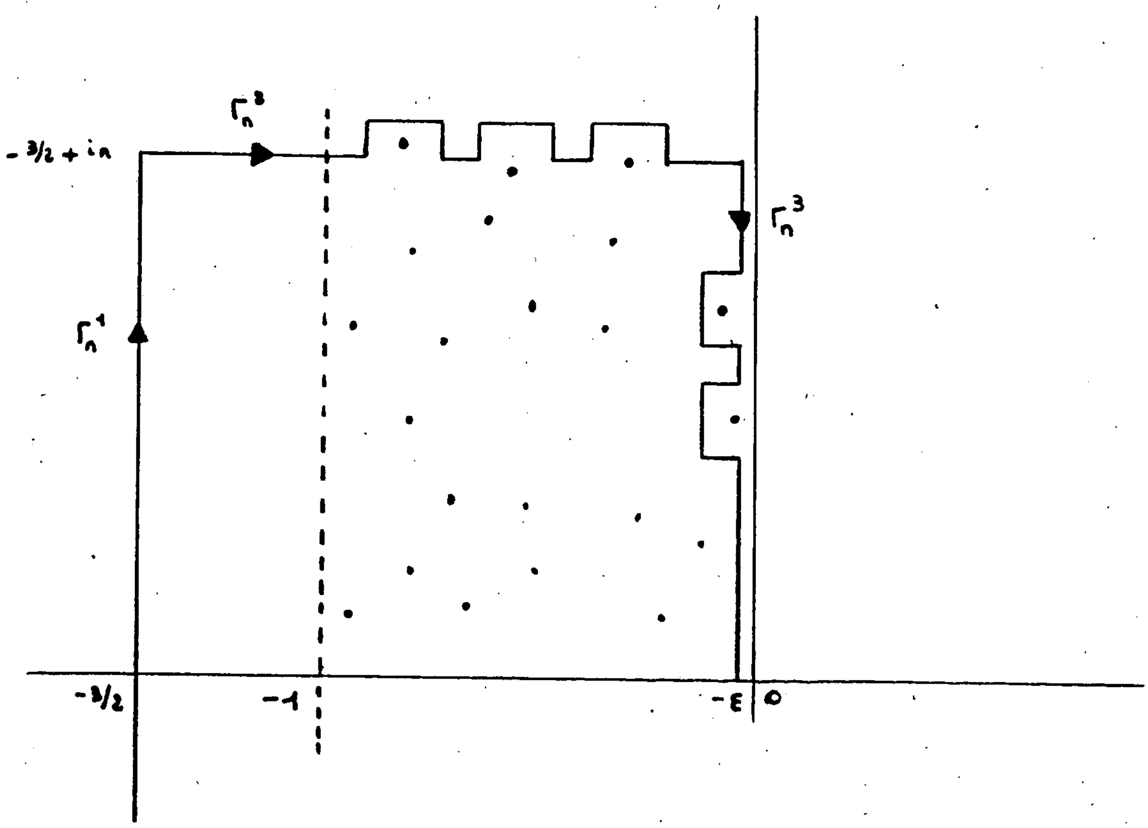

Figure 4: A schema showing the contour Γ_n in the upper half plane.

By the complement formula for the gamma function $\Gamma(z)$ has an exponential decrease at infinity along imaginary lines. On the other hand, along all of Γ_n, $\theta(-s)$ and $\xi(-s)$ are bounded by Lemma 4 and the remarks following it. Thus, the integral

$$\int \beta*(s)x^{-s}ds. \tag{40}$$

taken along Γ_n^2 and Γ_n^4 tends to zero exponentially fast. Integrals (40) along Γ_n^1 and Γ_n^3 tend to limits that are respectively $\beta(x)$ and a function of x which is $O(x^{3\epsilon})$ as x gets large. Taking residues into account by the Cauchy Theorem, we therefore get

$$\beta(x) = - \Sigma \operatorname{Res}(\beta*(s)x^{-s}) + O(x^{3\epsilon})$$

the sum being extended to all the poles that lie inside the Γ_n's. Because of the exponential decrease of $\Gamma(s)$ at $i\infty$ and the uniform discreteness of the set of zeros of f(s), the sum of the residues is absolutely convergent.

To conclude on the behaviour of $\beta(s)$, we estimate the residue of $\beta*(s)$ at $s=-1$. Introducing the notations:

$$\mu(r) = \frac{1}{h(p,q)} \frac{1}{\frac{\lambda}{q} - \frac{\lambda}{p}} \int_{\frac{\lambda}{p}}^{\frac{\lambda}{q}} r(u)du \tag{41}$$

$$\nu(r) = \int_0^{\frac{\lambda}{q}} r'(u)w(u)du, \tag{42}$$

we have around $s=-1$:

$$\theta(-s) = - \mu(e^{-u})(s+1)^{-1} + \nu(e^{-u}) + O(s+1)$$

$$\xi(-s) = - \mu(ue^{-u})(s+1)^{-1} + \nu(ue^{-u}) + (O(s+1)$$

$$\Gamma(s+1) = \frac{1}{s+1} - \gamma + O(s+1)$$

$$\Gamma(s) = \frac{-1}{s+1} + (\gamma-1) + O(s+1)$$

$$x^{-s} = x^{-1}(1 + (s+1)\log x + O(s+1)^2) \; .$$

With these expansions, we see that $\beta*(s)$ has only a simple pole at $s=1$. The residue there is found to be equal to $-A$ where

$$A = \frac{1}{h(p,q)} \frac{(e^{-\frac{\lambda}{p}} - e^{-\frac{\lambda}{q}})^2}{(\frac{\lambda}{q} - \frac{\lambda}{p})(\frac{\lambda}{q}e^{-\frac{\lambda}{q}} - \frac{\lambda}{p}e^{-\frac{\lambda}{p}})} + \int [Kue^{-u} - (2K+1)e^{-u}]\, w(u)du \; . \tag{43}$$

Similarly, for χ a pole of $(1-p^{-s}-q^{-s})^{-1}$, the residue of $\beta*(s)$ at χ is $-a(\chi)$ where:

$$a(\chi) = \frac{-1}{p^{\chi}\log p + q^{\chi}\log q} [\pi_1(\chi)(\Gamma(\chi) + K\,\Gamma(1+\chi)) + K\,\pi_2(\chi)\Gamma(\chi)] \; . \tag{44}$$

and π_1, π_2 are the π-functions of Lemma 4 associated to $r(u) = e^{-u}$, ue^{-u}.

With these calculations, we have:

Theorem: The average time to resolve n collisions satisfies

$$\alpha_n = \frac{2A}{1+2D(\lambda)} n + \frac{2}{1+2D(\lambda)} \sum_{\chi} a(\chi) n^{-\chi} + O(n^{1-\eta})$$

the sum being extended to χ's satisfying:

$$1 - p^{-\chi} - q^{-\chi} = 0 \quad ; \quad -1 \le \mathrm{Re}(\chi) < -1+\eta \quad ; \quad \chi \neq -1$$

for any sufficiently small $\eta > 0$.

The sum in the expression of the theorem is a bounded fluctuating function whose amplitude is small compared to the value of A. Its asymptotic nature depends on very specific arithmetical properties of numbers p and q. In the sum

$$\Sigma\, a(\chi) u^{-\chi} \tag{45}$$

for large u, the $a(\chi)$ have an exponential decrease in $|\mathrm{Im}(\chi)|$ while the $u^{-\chi}$ increase with $|\mathrm{Re}(\chi)|$. Estimating the order of (45) thus necessitates to determine the relation between $\mathrm{Re}(\chi)$ and $\mathrm{Im}(\chi)$ for the leftmost χ's that are poles of $(1 - p^{-s} - q^{-s})^{-1}$. Setting

$$\mathrm{Re}(\chi) = -1 + \epsilon \quad ; \quad \mathrm{Im}(\chi) = t \quad ,$$

we look for the solutions of

$$p^{1-\epsilon}\, e^{it\log p} + q^{1-\epsilon}\, e^{it\log q} = 1 \tag{46}$$

where ϵ tends to 0. Simplifying the discussion, (46) decomposes into

$$\begin{cases} p^{1-\epsilon}\,\cos(t.\log p) + q^{1-\epsilon}\,\cos(t.\log q) = 1 & (47) \\ p^{1-\epsilon}\,\sin(t.\log p) + q^{1-\epsilon}\,\sin(t.\log q) = 0\ . & (48) \end{cases}$$

With α, β denoting the principal determinations of tlogp and tlogq in $]-\pi;\pi]$, (48) yields

$$\frac{\sin\alpha}{\sin\beta} = -\frac{q^{1-\epsilon}}{p^{1-\epsilon}} \quad , \tag{49}$$

so that α and β are of opposite signs. For ϵ small enough, from (47), (49) we see that α,β must be small and in the limit, when $\epsilon=0$, (47) can only be satisfied by $\alpha=\beta=0$.

Local expansions show that for some constants A,B:

$$|\alpha| \le A\epsilon^{\frac{1}{2}} \quad ; \quad |\beta| \le B\epsilon^{\frac{1}{2}} \quad . \tag{50}$$

Expressing the fact that α and β are determinations of tlogp, tlogq, then

$$t\log p = 2a\pi + \alpha$$

$$t\log q = 2b\pi + \beta$$

for integral a,b. Thus eliminating t and using (50), one must have

$$|\rho - \frac{a}{b}| < \frac{D\epsilon^{\frac{1}{2}}}{b} \quad \text{where } \rho = \frac{\log p}{\log q} \qquad (51)$$

Since ρ is linearly related to t, this represents a relation between ϵ and t, i.e. $Re(\chi)$ and $Im(\chi)$.

Corollary: If $\rho = \frac{\log p}{\log q}$ is rational; $\rho = \frac{d}{e}$ with $(d,e) = 1$, one has :

$$\alpha_n = \frac{2A}{1+2D(\lambda)} n + P(e.\log_p n) + O(n^{1-\eta})$$

for some $\eta > 0$, with P(u) a Fourier series of u with mean value 0.

The proof relies on the fact that (51) for small enough ϵ has the only trivial solutions $a = kd$, $b = ke$; thus the Dirichlet series in this case have a pole-free strip right of $Re(s) = -1$.

In general the fluctuating function (45) is O(n).

4. The Ergodicity Condition

Theorem: The necessary and sufficient condition to have a stable channel, i.e. $\alpha_n < \infty$ $\forall n$ finite is $\lambda < \lambda_{MAX}$ where λ_{MAX} is the first root of

$$1 + 2D(\lambda_{MAX}) = 0 \qquad (1)$$

(see equation 2.16)

Proof: The proof relies on results of section 3 together with standard results on Markov chains (see for example [ÇIN75] as follows.

The state of the system (super stack) at time t can be represented by a vector of variable length

$\widetilde{N}(t) = [N(0), N(1), \ldots N(DEPTH)]$

using notation of section 1, or setting $\widetilde{W} = [N(1), N(2) \ldots N(DEPTH)]$
$\widetilde{N}(t) = (N(0), \widetilde{W})$.

$\widetilde{N}(t)$ is clearly a Markov chain with a countable state space irreducible and aperiodic. In such a case it is well known that all states are of the same type (i.e. they are all transient or recurrent). Assuming the initial state is $[N(0) = n, \widetilde{W}]$, α_n is the mean time of first visit to state $[N(0) = 0, \widetilde{W}]$. Hence the α_n, $n \geq 2$, will be all either positive and finite or all infinite. Moreover there is at most one solution $\phi(z) = e^{-z}\alpha(z)$. (This last fact was proved by other arguments in section 2.)

From equation (2.15), (2.16), Lemma 1 of section 2 and the last theorem of section 3, it follows that there are only two possibilities:

i) $\phi(z)$ is an entire function, i.e. analytic in every finite region of the plane.

ii) $\phi(z)$ is degenerated : this occurs when either $k=\infty$, that is when $\lambda = \lambda_0, \lambda_1$ where

- λ_0 is the root of $\frac{\lambda}{q} e^{-\frac{\lambda}{q}} - \frac{\lambda}{p} e^{-\frac{\lambda}{p}} = 0$

- λ_1 is a root of $1+2D(\lambda_1) = 0$ see(2.16).

The above considerations show that the necessary and sufficient condition for the ergodicity of the system is to be in case (i).

Numerically, it can be seen that there exists a unique λ_1. Moreover,

$$0 < \lambda_1 < \lambda_0 \quad .$$

Hence, $\lambda_{MAX} = \lambda_1$ and the conclusion follows:

$$\begin{cases} \text{if } \lambda < \lambda_{MAX}, & \alpha(z) \text{ entire exists.} \\ \text{if } \lambda \geq \lambda_{MAX}, & \alpha_n = \infty \quad \forall n \geq 2 . \end{cases}$$

The proof is concluded. □

The ergodicity condition (1) is equivalent to

$$\lambda_{MAX} = \inf\{\lambda | 0 < \varphi(\lambda) < \infty\}$$

and the stochastic interpretation of $\varphi(\lambda)$ is easy.

Indeed $\varphi(\lambda) = \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n \alpha_n}{n!}$ represents the expected value of the mean collision resolution interval (in a wide sense, because α_0 and α_1 are taken into account).

Numerical Results

$$p = 0.5 + 10^{-7}$$

$$\lambda_{MAX} \simeq 0.3765$$

5. Conclusion

As said in section 1, other schemes can be proposed to get a higher λ_{MAX}. They essentially try to save "doomed" slots. This can lead for example to the following recursive relationships for the L_n:

$$L_n = 1 + \begin{cases} L_{I+X} + L_{n-I+y} & \text{if } I+X \neq 0 \\ L_n & \text{if } I+X = 0 \end{cases}$$

(see [MAS81], [FAHO82]).

The functional equation for the generating function of the $\alpha_n = E(L_n)$ is then non symmetrical with respect to p and q = 1-p. Moreover, there is a term involving $\alpha(qz)$. Nevertheless, the same methods of analysis could be applied.

Bibliography

[Kn73] Knuth, D.E.: "The Art of Computer Programming, Vol. 3: Sorting and Searching", Addison-Wesley, Reading (1973).

[Do55] Doetsch, G.: "Handbuch der Laplace Transformation", Birkhäuser, Basel (1955).

[Da78] Davies, B.: "Integral Transforms and Their Applications", Springer Verlag, New York (1978).

[FAHO82] Fayolle, G. Hofri, G.: "On the Capacity of a Collision Channel under Stack-based Collision Resolution Algorithms. Submitted to JACM. Internal Report, Technion University, January 1982.

[MAS81] Massey, J.L.: "Collision Resolution Algorithms and Random Access Communications". In "Multiuser Communications" (ed. G. Longo) CISM Courses and Lectures No. 265, pp.73-137, Springer Verlag, 1981.

[FAY75] Fayolle, G.: "Etude du Comportement d'un Canal Radio Partage entre Plusieurs Utilisateurs". Thése Docteur Ingenieur, Universite Paris 6, 1975.

[CIN75] Cinlar, E.: "Introduction to Stochastic Processes". Prentice Hall, 1975.

[TSY-VVED80] Tsybakov B.S., Vvedenskaya, N.D.: "Random Multiple-Access Stack Algorithm". Prob. of Inf. Trans. (English Translation) 16, No. 3, pp.230-243, 1980.

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