Non convex methods for computing free boundary equilibria of axially symmetric plasmas

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NON CONVEX METHODS FOR COMPUTING FREE BOUNDARY EQUILIBRIA OF AXIALLY SYMMETRIC PLASMAS

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ABSTRACT. - This paper analyses the convergence of an iteration scheme adapted to a class of nonconvex problems where the functional to be minimized is the difference between two convex functions. These problems include a mathematical description of free boundary equilibria for axially symmetric plasmas governed by equations containing *discontinuous nonlinear terms*. A Galerkin method is used to discretize these partial differential equations in the space-variables. Numerical computations are performed on examples via a finite element method.

RESUME. - Cet article analyse la convergence d'un schéma itératif adapté à une classe de problèmes non convexes où la fonctionnelle à minimiser est la différence de deux fonctions convexes. Ces problèmes incluent la description mathématique d'un plasma à frontière libre en équilibre axisymétrique gouverné par des équations comportant des *termes non linéaires discontinus*. Ces équations aux dérivées partielles sont discrétisées en variable d'espace par une méthode de Galerkin. Des calculs numériques sont effectués sur des exemples par la méthode des éléments finis.

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0. - INTRODUCTION.

As it was pointed out independently in two papers (H. BREZIS-H. BERESTYCKI [3] and one of the authors (M.S.) [17]), there exists a very efficient iteration scheme for computing general free boundary equilibria of axially symmetric plasmas. In [3], it is proved that for any initialization some subsequence converges to a solution, whilst in [17],[18] local convergence around variational solutions is proved and numerical computations are performed (including a bifurcation case).

The aim of this paper is to improve the mathematical background of this scheme and to show that it can be generalized to other problems.

The paper begins with the study of the abstract subdifferential problem of finding u such that

$$(0.1) \qquad \partial F(u) \cap \partial G(u) \neq \emptyset ,$$

where F and G are both convex functions and $\partial F(u)$ denotes the subdifferential of F at u.

We are interested in questions of existence and approximation of solutions of (0.1). The duality theory (non convex duality) for these problems was first studied by J.F. TOLAND (cf. [23],[24]).

In Section 1, we prove the existence of at least one solution for (0.1) without assuming $J = G-F$ to be bounded from below ; then we generalize the iteration scheme quoted before and prove its convergence to a solution of (0.1) for all initializations. As far as we know, these results are new (a more restrictive framework was considered by the second author in [18]).

In Section 2, using an internal approximation of the space (which includes Galerkin method and finite elements), we define an approximation of the abstract subdifferential problem (0.1) and prove its convergence. We also give the discrete analogue of the continuous iteration scheme.

In Section 3, we study the general set-valued equation

$$(0.2) \qquad \begin{cases} \mathcal{L}u \in \partial_s \phi(\cdot,u(\cdot)) \text{ in } \Omega, \\ u = \text{const. (unknown) on } \Gamma = \partial\Omega, \\ -\int_\Gamma \frac{\partial u}{\partial \nu_{\mathcal{L}}} d\sigma = I, \end{cases}$$

where I is a prescribed constant, $\mathcal{L}$ an elliptic operator and $\phi(x,s)$ a function which is convex with respect to s. Sec. 1 and 2 yield the existence of at least one solution and the convergence of an iteration scheme for Eq. (0.2). This problem (resp. the associated iteration scheme) differs from the set-valued equations studied in other papers (see for instance K.C. CHANG [4]-[6], I. MASSABO [12], I.M. and C.A. STUART [13]) mainly by the nature of the prescribed data (the integral of the conormal derivative is given instead of the value of u on Γ).

The remaining part of the paper is devoted to physical applications. The main one (Sec. 4) is the determination of free boundary plasma axisymmetric equilibria by solving STOKES-GRAD-SHAFRANOV equation, which is a special case of Eq. (0.2). Even in the case of set-valued nonlinear terms $\partial_s\phi$, we obtain the existence of solutions and the convergence of an iteration scheme ; that way, we improve the existence results of R. TEMAM [21], H. BERESTYCKI-H. BREZIS [2], J.P. PUEL [14] and approximation results of H. BERESTYCKI-H. BREZIS [3] which were all proved assuming the continuity of ϕ'_s. Let us mention that R. CIPOLATTI also uses convexity methods in [8] to study $-\Delta u = \lambda P_K(u)$ (where K is some convex set of functions) which generalizes the plasma model-problem $-\Delta u = \lambda u_+$. As other application, we present in Sec. 5 a convergent iteration scheme for computing the shape of a heavy rotating string (cf. I.I. KOLODNER [10], J.F. TOLAND [24]). Both applications are illustrated by numerical results. Many other applications are expected.

1. - AN ABSTRACT SUBDIFFERENTIAL PROBLEM.

1.1. The abstract framework.

This section is devoted to the non-standard problem of finding points where the subdifferentials of two convex functions have a non-empty intersection. Our most interesting result is an algorithm for computing solutions of such a problem. We first prove the existence of solutions through the closely related problem of minimizing the difference between the two convex functionals on a non-convex set. We then propose an algorithm and study it.

The abstract framework of this study is the following.

Given a reflexive Banach space $(V,\|\cdot\|)$, Z a closed subspace of V (which can be $\{0\}$) and F,G two functionals on V, we assume that

(1.1) F is convex, weakly continuous on V, (1.2) G is convex, continuous on V,

(1.3) G is invariant by translation with respect to Z,

(1.4) there exists $\alpha > 0$ such that for all $u, v \in V$ and all $L \in \partial G(u)$

$$G(u+v) - G(u) \geq L(v) + \alpha \|\dot{v}\|^2_{V/Z} \quad (^1).$$

Then, the problem we consider is that of finding $u \in V$ such that

(1.5) $\partial F(u) \cap \partial G(u) \neq \emptyset$.

This problem will be often referred to as problem (1.5) for simplicity. The following remark helps us to locate the solutions of this problem if there are any.

Remark 1.1 : It is easy to check that assumption (1.3) is equivalent to

(1.3)' for all $v \in V$, $\partial G(v) \subset Z^o$. (2)

Moreover this condition (that disappears if $z = \{0\}$) allows us to consider G as a continuous convex functional on V/Z which is even strictly convex due to (1.4). ∎

If us is any solution of (1.5), it is now clear that $\partial F(u) \cap Z^o$ is not empty. As a consequence, u necessarily belongs to the set

(1.6) $K = \{v \in V : F(v) \leq F(v+z) \text{ for all } z \in Z\}$.

As we shall see problem (1.5) is closely related to the minimization problem

(1.7) $\displaystyle\inf_{v \in K} \{G(v) - F(v)\}$

when a few additional coercivity conditions are met.

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(1) The quotient space V/Z is a Banach for the norm $\|\dot{v}\|_{V/Z} = \displaystyle\inf_{z \in Z} \|v+z\|$

(2) The polar set Z^o of the subspace Z is given by

$$Z^o = \{L \in V^* : L(z) = 0 \text{ for all } z \in Z\}$$

where V^* denotes the dual space of V.

These assumptions are

(1.8) for each $v \in V$, $F(v+z) \to +\infty$ when $z \in Z$, $||z|| \to +\infty$,

(1.9) the functional $J = G-F$ is bounded from below on K,

(1.10) $\begin{cases} \text{the set } K \text{ is bounded in } V \\ \text{or } J(v) \to +\infty \text{ when } v \in K, \ ||v|| \to +\infty . \end{cases}$

Remark 1.2 : Concerning the terminology, we shall refrain from calling the solutions u of $\partial F(u) \cap \partial G(u) \neq \emptyset$ the critical points of $J = G-F$ (as did J.F. TOLAND in [24]) because this property depends on the decomposition $J = G-F$, as it was pointed out by K.C. CHANG [6].

Quadratic case for G.

Let $G(v) = \frac{1}{2} a(v,v)$ be a functional on V satisfying the classical conditions

i) $(V, ||\cdot||)$ and $(H, |\cdot|)$ are Banach spaces ; V is reflexive and compactly embedded in H.

ii) $a(\cdot,\cdot)$ is a bilinear form on V which is continuous, symmetric, nonnegative and satisfies the coercivity condition

(1.11) $\begin{cases} \text{there exist real constants } \lambda \geq 0, \ C > 0 \text{ such that for all } v \in V \\ a(v,v) + \lambda |v|^2 \geq C \, ||v||^2 . \end{cases}$

We shall see below that i)-ii) yield

iii) the null space Z of $a(\cdot,\cdot)$ is $\{0\}$ or finite dimensional,

iv) there exists $\alpha > 0$ such that for all $v \in V$

$$a(v,v) \geq \alpha ||\dot{v}||^2_{V/Z} .$$

It is then clear that relations (1.2)-(1.4) are fulfilled since every quadratic functional G satisfies

$$G(u+v) - G(u) - G'(u)v = G(v).$$

In this framework relation (1.5) becomes

(1.5)' $\quad a(u,\cdot) \in \partial F(u)$

which is equivalent to the variational inequality

(1.5)" $\quad a(u,v) \leq F(u+v)-F(u)$ for all $v \in V$.

And in the particular case of a differentiable F, (1.5)' means $a(u,\cdot) = F'(u)$.[1]

Let us now prove iii)-iv). Because of the embedding $V \subset H$ and inequality (1.11), the norms on V and H are equivalent on Z. The identity operator on Z is therefore compact due to i), which yields iii) (cf. M. SCHECHTER [15]).

We argue by contradiction to prove iv). Assume that a sequence $\{\dot{v}_n\}$ satisfies

$$\|\dot{v}_h\| = 1 \ , \ a(\dot{v}_n,\dot{v}_n) < \frac{1}{n} \ .$$

In each class $\dot{v}_n$, we can select v_n such that

(1.12) $\quad 1 \leq \|v_n\| \leq 2 \ , \ a(v_n,v_n) < \frac{1}{n}$.

By compactness, some subsequence $\{v_{p(n)}\}$ converges weakly in V and strongly in H to some $u \in V$. The weak lower semi-continuity of convex continuous functionals yields

$$a(u,u) \leq 0 = \lim\inf a(v_{p(n)},v_{p(n)}) .$$

Since $a(\cdot,\cdot)$ is nonnegative, u belongs to Z. Then, applying relation (1.11) to $(v_{p(n)}-u)$ we get

$$a(v_{p(n)},v_{p(n)}) + \lambda|v_{p(n)}-u|^2 \geq C\,\|v_{p(n)}-u\|^2 \geq C\,\|\dot{v}_{p(n)}\|^2_{V/Z}$$

and letting $n \to +\infty$, we infer from this and (1.12) that

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([1]) This case was previously studied in [18] with $a(\cdot,\cdot)$ taken definite positiv

$$\liminf |v_{p(n)}-u|^2 \geq \frac{C}{\lambda} > 0$$

in contradiction with the strong convergence of $v_{p(n)}$ to u.

Remark 1.3 : It should be noted that hypotheses (1.9)-(1.10) only involve the behaviour of the functional J on K. There is indeed no global property of coercivity, since according to (1.3) and (1.8) we have $\lim\limits_{\substack{z\in Z \\ \|z\|\to+\infty}} J(v+z) = -\infty$ for all $v\in V$.

1.2. Two lemmas.

Hereinafter, a key-role is played by the following lemma

Lemma 1.1 : *Let* F, G *be two convex functionals on a Banach space* V. *We assume that two points* v_1, v_2 *of* V *satisfy*

$$\partial F(v_1) \cap \partial G(v_2) \neq \emptyset.$$

If we set J = G-F, *we then have*

$$J(v_1) \geq J(v_2).$$

Moreover if Z *is a closed subspace of* V *and if relation* (1.4) *is fulfilled, one has*

$$J(v_1) \geq J(v_2) + \alpha\|v_1-v_2\|^2_{V/Z} \; .$$

We shall also need a characterization of the set K (defined by (1.6)) given by

Lemma 1.2 : *Let* F *be a convex continuous functional on a Banach space* V, Z *a closed subspace of* V *and* Z^o *its polar set. For any* $u\in V$ *the conditions*

i) $F(u) \leq F(u+z)$ for all $z\in Z$ (i.e. $u\in K$)

ii) $\partial F(u) \cap Z^o \neq \emptyset$

are equivalent.

Proof of lemma 1.1 :

Let $L \in \partial F(v_1) \cap \partial G(v_2)$; by convexity we get

$$F(v_2) \geq F(v_1) + L(v_2 - v_1),$$

$$G(v_1) \geq G(v_2) + L(v_1 - v_2),$$

and adding these inequalities we obtain $J(v_1) \geq J(v_2)$. Furthermore, if G satisfies relation (1.4), replacing the second inequality by (1.4) we immediately get

$$J(v_1) \geq J(v_2) + \alpha \| \dot{v}_1 - \dot{v}_2 \|^2_{V/Z} . \blacksquare$$

Proof of lemma 1.2

It is a straightforward consequence of Hahn-Banach's theorem. Though a justification can be found in K.C. CHANG [4] we give here some more details.

Assume $\partial F(u) \cap Z^o$ is not empty, and let L be any element of this set. Since

$$F(u+v) \geq F(u) + L(v) \text{ for all } v \in V,$$

$$L(z) = 0 \text{ for all } z \in Z,$$

it is obvious that $F(u) \leq F(u+z)$ for all $z \in Z$.

Conversely, assume that i) is fulfilled. The convex functional $\tilde{F}$ defined on V by

$$\tilde{F}(v) = F(u+v) - F(u)$$

clearly satisfies[1]

$$Z \times \{0\} \cap \overset{\circ}{\widehat{\text{epi } \tilde{F}}} = \emptyset.$$

Hahn-Banach's theorem asserts that there is a closed hyperplane π of $V \times \mathbb{R}$ such that

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(1) epi $\tilde{F} = \{(v,t) \in V \times \mathbb{R} : \tilde{F}(v) \leq t\}$ and we denote by $\overset{\circ}{S}$ the interior of a set S.

(1.13) $\quad Z\times\{0\}\subset\pi$ and $\pi\cap\overset{\frown}{\overset{\circ}{\text{epi}\,\tilde{F}}}=\emptyset$.

Let $L\in V^*$, $\alpha\in\mathbb{R}$ such that

$$(v,t)\in\pi \Leftrightarrow L(v)-\alpha t = 0.$$

It follows from (1.13) that $L\in Z^o$ and $L(v)-\alpha t$ is never zero on epi $\tilde{F}$, thus keeps its sign unchanged, say the negative one. We easily get that $\alpha > 0$ and

$$L(v) \le \alpha \inf_{(v,t)\in \text{epi}\,\tilde{F}} t = \alpha\tilde{F}(v).$$

Therefore $\frac{1}{\alpha}$ L belongs to $\partial F(u)\cap Z^o$. The lemma is proved. ∎

1.3. Existence of solutions for problem (1.5).

In the framework specified in Sec. 1.1, we have the following existence result concerning problem (1.5).

Theorem 1 : *Assume that conditions* (1.1)-(1.4) *are fulfilled. Under assumption* (1.8) *the set* K *defined by* (1.6) *is not empty, and if problem* (1.7) *admits solutions, all of them satisfy* (1.5). *Since they have a minimim property, we shall call them variational solutions of* (1.5).

If we assume moreover that conditions (1.9)-(1.10) *are satisfied, there is at least one solution of the minimization problem* (1.7), *and therefore at least one (variational) solution of problem* (1.5).

Proof of theorem 1.1 :

Let us first check that the set K is not empty. Given $v\in V$, it is classical that the minimization problem

(1.14) $$\inf_{z\in Z} F(v+z)$$

has always a solution since Z (as well as V) is reflexive, and F is weakly continuous and satisfies (1.8). As a consequence, we have

(1.15) $\quad$ for each $v\in V$, there exists $z\in Z$ such that $v+z\in K$.

So, K is obviously not empty.

Suppose now that u is a solution of problem (1.7). We noticed that $u\in K$. Lemma 1.2 allows us to choose $L\in\partial F(u)\cap Z^o$; let us introduce a new minimization problem

$$\operatorname*{Inf}_{v\in V} \{G(v)-L(v)\}$$

Since the functional (G-L) is invariant under addition of an element of Z to the variable v, we may consider this problem as defined on V/Z. The reflexivity of V yields that of the quotient space V/Z ; the functional (G-L) is convex, continuous on V/Z and coercive i.e. $\lim_{\|\dot{v}\|\to+\infty} (G(\dot{v})-L(\dot{v})) = +\infty$. Concerning this last property we have indeed

$$G(v) \geq G(0) + L_o(v) + \alpha\|\dot{v}\|^2_{V/Z}$$

by application of (1.4) with $L_o \in \partial G(0)$, so that

$$G(\dot{v})-L(\dot{v}) \geq G(0) + \alpha\|\dot{v}\|^2_{V/Z} - \|L-L_o\|\,\|\dot{v}\|_{V/Z}$$

whence the coercivity follows.

The minimization problem

$$\text{(1.16)} \qquad \operatorname*{Inf}_{\mathcal{V}\in V/Z} (G(\mathcal{V})-L(\mathcal{V}))$$

is of the same classical kind as (1.14). It has a solution $\mathfrak{w}$, and this solution is unique by strict convexity (see Remark 1.1). We now prove that $\mathfrak{w} = \dot{u}$ and $L \in \partial G(u) \cap \partial F(u)$.

It follows from (1.16) that for any $w \in \mathfrak{w}$ we have $L \in \partial G(w)$. Thanks to property (1.15) we may suppose $w \in K \cap \mathfrak{w}$; since u is a solution of (1.7), we have $J(u) \leq J(w)$. On the other hand, since $L \in \partial F(u) \cap \partial G(w)$ we get by lemma 1.1 $J(w) + \alpha\|\dot{u}-\dot{w}\|^2_{V/Z} \leq J(u)$. It turns out that $\dot{u}=\dot{w}$, and as we saw before this yields $L \in \partial G(u)$; hence u satisfies (1.5).

Concerning the second part of the theorem, let $\{u_n\}$ be a minimizing sequence for J on the set K. Such a sequence is bounded in V due to (1.9)-(1.10). By reflexivity, there is a subsequence $\{u_{n_k}\}$ that weakly converges to some $u \in V$. Let us show that u is a solution of (1.7). First of all, we have $u \in K$. It is indeed clear from (1.1) and (1.6) that K is weakly closed. Moreover, it follows at once from (1.1)-(1.2) that J is weakly l.s.c. (lower semicontinuous) as the sum of two weakly l.s.c. functionals, so that $J(u) \leq \liminf J(u_{n_k})$. Since $u \in K$, we get $J(u) = \operatorname*{Inf}_{v\in K} J(v)$ as announced. ∎

1.4. Approximation of solutions of problem (1.5).

The method we used to prove theorem 1.1 ensures that a solution (even a variational solution) of problem (1.5) exists, but it does not enable us to compute it. We now study an algorithm that makes possible an effective computation of solutions. It also provides an alternative proof of theorem 1.1 as a by-result.

Iteration scheme.

The starting point is some $u_1 \in K$; once $u_1 \in K, \dots, u_n \in K$ have been obtained we get $u_{n+1} \in K$ in the following way.

Step 1 : let L_n be any element of the (non empty convex) set $\partial F(u_n) \cap Z^o$; we define $\mathcal{U}_{n+1} \in V/Z$ as the solution of the minimization problem

$$\operatorname*{Inf}_{\mathcal{U} \in V/Z} \{G(\mathcal{U}) - L_n(\mathcal{U})\}$$

Step 2 : then we take for u_{n+1} a solution of the minimization problem

$$\operatorname*{Inf}_{u \in \mathcal{U}_{n+1}} F(u).$$

Remark 1.4 : In the quadratic case previously described, $\mathcal{U}_{n+1} \in V/Z$ is defined by $a(\mathcal{U}_{n+1}, \mathcal{V}) = L_n(\mathcal{V})$ for all $\mathcal{V} \in V/Z$.

Assumptions (1.1)-(1.4) and (1.8)-(1.10) grant that for any choice of $u_o \in K$ we can construct a sequence $\{u_n\}$ with this algorithm. The arguments are the same as in the proof of theorem 1.1.

Concerning such a sequence we have the following convergence result

Theorem 1.2 : *Assume that hypotheses (1.1)-(1.4) and (1.8)-(1.10) are fulfilled. Then, every sequence $\{u_n\}$ obtained through the above mentioned algorithm is bounded and has at least one weakly convergent subsequence. The limit u of any weakly convergent subsequence $\{u_{n_k}\}$ is a solution of problem (1.5). Moreover, the sequence $\{\dot{u}_{n_k}\}$ converges strongly to $\dot{u}$ in V/Z and the corresponding sequence $\{L_{n_k}\}$ admits a subsequence that converges strongly in V^* to some $L \in \partial F(u) \cap \partial G(u)$.*

Proof : From the first step of the algorithm we infer that $L_n \in \partial G(u_{n+1})$ so that $L_n \in \partial F(u_n) \cap \partial G(u_{n+1})$. Lemma 1.1 then yields

$$(1.17) \qquad J(u_{n+1})+\alpha\|u_n-u_{n+1}\|^2_{V/Z} \le J(u_n).$$

It turns out that the sequence $\{J(u_n)\}$ decreases. But it is bounded from below according to (1.9) and therefore converges. Thanks to assumption (1.10) the boundedness of $\{J(u_n)\}$ yields that of $\{u_n\}$. By reflexivity of V, there is a subsequence $\{u_{n_k}\}$ that converges weakly and its limit u belongs to K since K is weakly closed. In order to show that u is a solution of (1.5), we mainly use two properties of the sequence $\{L_n\}$ which are

i) the sequence $\{L_n\}$ is bounded in V^*,
ii) $\lim L_n(v_n) = 0$ for any sequence $\{v_n\}$ weakly converging to 0.

Let us draw the consequences of i)-ii) before proving them. Thanks to i) we may suppose (modulo extraction of another subsequence) that L_{n_k} weakly converges to some $L \in V^*$. Since $L_{n_k} \in \partial G(u_{n_k+1}) \cap \partial F(u_{n_k})$ we have

$$(1.18) \qquad F(u_{n_k}+v) \ge F(u_{n_k}) + L_{n_k}(v) \text{ for all } v \in V,$$

$$(1.19) \qquad G(v+u_{n_k+1}-u_{n_k}) \ge G(u_{n_k+1})+L_{n_k}(v-u_{n_k}) \text{ for all } v \in V.$$

We want to pass to the limit in (1.18) and (1.19) in order to prove that $L \in \partial F(u) \cap \partial G(u)$. From i) and (1.18) we get

$$F(u+v) \ge F(u)+L(v) \text{ for all } v \in V$$

that is to say $L \in \partial F(u)$.

To deal with (1.19) we use ii) and consider G as a convex continuous functional on V/Z. Since the sequence $\{J(u_n)\}$ converges we infer from (1.17) that $\lim \|\dot{u}_n-\dot{u}_{n+1}\|_{V/Z} = 0$. Thus, in the space V/Z $\dot{u}_{n_k}-\dot{u}_{n_k+1} \to 0$ strongly and $\dot{u}_{n_k+1} \to \dot{u}$ weakly, which yields

$$(1.20) \qquad \begin{cases} \lim G(\dot{v}+\dot{u}_{n_k+1}-\dot{u}_{n_k}) = G(\dot{v}), \\ \liminf G(\dot{u}_{n_k+1}) \ge G(\dot{u}). \end{cases}$$

It is then clear from (1.19), (1.20) and ii) that

$$G(v) \geq G(u)+L(v-u)$$

or else $L \in \partial G(u)$. We have shown that $L \in \partial F(u) \cap \partial G(u)$ i.e. u is actually a solution of problem (1.5).

<u>Proof</u> of i). For each $v \in V$ we have

$$(1.21) \qquad |L_n(v)| \leq \max\ (F(u_n+v)-F(u_n), F(u_n-v)-F(u_n))$$

so that $\|L_n\| \leq \sup\limits_{\|v\| \leq 1} (F(u_n+v)-F(u_n))$. But the sequence $\{u_n\}$ is bounded in V, say by M, and the ball $\{v : \|v\| \leq M+1\}$ is weakly sequentially compact in V. The image of this ball by F is compact because F is weakly continuous. This proves i).

<u>Proof</u> of ii). Just take $v=v_n$ in (1.21) and apply the weak continuity of F.

Let us prove now that $\dot{u}_{n_k}$ converges strongly to $\dot{u}$ in the quotient space V/Z. We argue by contradiction, and suppose that some subsequence $\dot{u}_{n'_k}$ satisfies $\|\dot{u}-\dot{u}_{n'_k}\|_{V/Z} \geq \varepsilon > 0$. Then, according to (1.4) we have

$$G(u+u_{n'_k+1}-u_{n'_k}) \geq G(u_{n'_k+1})+L_{n'_k+1}(u-u_{n'_k})+\alpha\varepsilon^2.$$

Letting $n'_k \to +\infty$ and repeating previous arguments we get

$$G(u) \geq G(u)+\alpha\varepsilon^2$$

which contradicts $\varepsilon > 0$. The whole sequence $\{\dot{u}_{n_k}\}$ therefore converges strongly to u in the quotient space V/Z.

Finally, the last part of theorem 1.2 follows directly from the property

iii) given a reflexive Banach space V, if a sequence $\{L_n\}$ of V^* converges weakly to 0 and satisfies ii), then $\{L_n\}$ converges strongly to 0.

Proof of iii) : By the weak sequential compactness of the ball $\mathcal{B} = \{v \in V, \|v\| \leq 1\}$, for each L_n we get $v_n \in \mathcal{B}$ such that $\|L_n\| = L_n(v_n)$. If the sequence $\{\|L_n\|\}$ does not tend to 0, there exist $\varepsilon > 0$ and a subsequence $\{n_k\}$ such that $\|L_{n_k}\| \geq \varepsilon$ for all k and $\{v_{n_k}\}$ converges weakly to some $v \in V$. Since L_{n_k} converges weakly to 0 and satisfies ii), it follows from

$$\|L_{n_k}\| = L_{n_k}(v_{n_k}-v)+L_{n_k}(v)$$

that $\lim \|L_{n_k}\| = 0$ in contradiction with $\varepsilon > 0$. We thus have $\lim_{n\to+\infty} \|L_n\| = 0$. ∎

Remark 1.5 : If the space Z is finite dimensional the strong convergence of $\{\dot{u}_{n_k}\}$ in V/Z yields that of $\{u_{n_k}\}$ in V. In this case, the strong and weak topologies are indeed equivalent on Z which admits moreover a topological complement isomorphic to V/Z (cf. L. SCHWARTZ [16]).

Corollary 1.1 : *Assume that hypotheses* (1.1)-(1.4) *and* (1.8)-(1.10) *are fulfilled and that problem* (1.5) *admits only finitely many solutions. Let* $\{u_n\}$ *be a sequence obtained with the above mentioned algorithm. The whole sequence converges weakly to a solution* u *of problem* (1.5) *in* V, *whereas the sequence* $\{\dot{u}_n\}$ *converges strongly to* $\dot{u}$ *in the quotient space* V/Z. *Moreover, if the space* Z *is finite dimensional the sequence* $\{u_n\}$ *itself converges strongly.*

Proof : Let us prove first that the whole sequence $\{\dot{u}_n\}$ converges ; to do this, we shall use three properties of this sequence. We know by theorem 1.2 that cluster values of $\{u_n\}$ are solutions of problem (1.5). They form a finite set by assumption, and the cluster values of $\{\dot{u}_n\}$ are their projections in V/Z so that

i) the sequence $\{\dot{u}_n\}$ has only finitely many cluster values.

It also follows easily from theorem 1.2 that

ii) the sequence $\{\dot{u}_n\}$ forms a relatively compact set in V/Z (strong).

Moreover, it appears in the proof of this theorem that

iii) $\lim_{n\to+\infty} \|\dot{u}_n-\dot{u}_{n+1}\|_{V/Z} = 0$.

Let us check that properties i)-iii) yield the strong convergence of $\{\dot{u}_n\}$ in V/Z. We denote by $\mathcal{U}_j$, j=1,2,...,p the cluster values of $\{\dot{u}_n\}$. Let N be a balanced neighbourhood of zero such that, for each j, $\mathcal{U}_j$ is the only $\mathcal{U}_k$ contained in $\mathcal{U}_j$+3N.

According to iii), for n great enough, say $n \geq n_o$, we have $(\dot{u}_{n+1}-\dot{u}_n) \in N$. Because of i) and ii) there is n_1 such that for $n \geq n_1$, $\dot{u}_n$ belongs to one of the sets $(\mathcal{U}_j+N)$, $j=1,\ldots,p$. Suppose that for some $n \geq n_o, n_1$ we have $\dot{u}_n \in (\mathcal{U}_{j_o}+N)$. Since $(\dot{u}_{n+1}-\dot{u}_n) \in N$, we have $\dot{u}_{n+1} \in (\mathcal{U}_{j_o}+2N)$ but the choice of N yields $(\mathcal{U}_{j_o}+2N) \cap (\mathcal{U}_k+N) = \emptyset$ for $k \neq j_o$, so that $\dot{u}_{n+1}$ also belongs to $(\mathcal{U}_{j_o}+N)$. Thus $\mathcal{U}_{j_o}$ is the only cluster value of $\{\dot{u}_n\}$, and by compactness it is also its limit. Consequently, the cluster values of $\{u_n\}$ all belong to the same class modulo Z. But if $z \in Z$ and if u and u+z are solutions of problem (1.5), for every $\lambda \in (0,1)$ $u+\lambda z$ is also a solution. Problem (1.5) then admits infinitely many solutions unless z=0. This proves that the sequence $\{u_n\}$ has also an only cluster value which is therefore its limit. Let us now check that $u+\lambda z$ is actually a solution. Since $z \in Z$, for $L \in \partial F(u) \cap \partial G(u)$ we have

$$F(u+\lambda z+v) \geq F(u)+L(v),$$

$$G(u+\lambda z+v) \geq G(u)+L(v).$$

Since u and (u+z) belong to K (cf. (1.6)), it is clear that $F(u) = F(u+z) = \operatorname*{Inf}_{z' \in Z} F(u+z')$. The convexity of F then yields $F(u) = F(u+\lambda z)$ for $\lambda \in (0,1)$. But we also have $G(u+\lambda z) = G(u)$ for all λ so that for $\lambda \in (0,1)$ we obviously have $L \in \partial F(u+\lambda z) \cap \partial G(u+\lambda z)$, whence $(u+\lambda z)$ is a solution of (1.5).

Concerning the case when Z is finite dimensional, the argument is the same as in Remark 1.5.

2. - DISCRETIZATION OF THE ABSTRACT SUBDIFFERENTIAL PROBLEM.

2.1. The discrete problem.

In the same framework as in Sec. 1, our purpose is now to give a discretization of problem (1.7) or more generally of problem (1.5). Let us recall that solutions of (1.7) are said variational solutions of (1.5).

We suppose given an internal convergent approximation of the space V in a classical manner (cf. P.G. CIARLET [7], R. TEMAM [22]) that is to say a family $\{V_h\}_{h \in \mathcal{H}}$ (1) of finite dimensional subspaces of V satisfying the convergence condition

(1) The set $\mathcal{H}$ of indices is a basis of a filter that converges to 0.

$$(2.1) \qquad \lim_{h\to 0} \{ \inf_{v_h\in V_h} \|v-v_h\| \} = 0, \text{ for every } v\in V.$$

We suppose moreover that the space Z (appearing in (1.3)-(1.4)) is contained in every V_h.

We denote by F_h, G_h, J_h the restrictions of the functionals F,G,J to the space V_h and we call K_h the set

$$(2.2) \qquad K_h = \{v_h\in V_h : F_h(v_h) \le F_h(v_h+z) \text{ for all } z\in Z\} ,$$

which is nothing else than $K\cap V_h$.

We associate with problem (1.5) the following family of discrete problems : for fixed $h\in \mathcal{H}$, find $u_h\in V_h$ such that

$$(2.3)_h \qquad \partial F_h(u_h) \cap \partial G_h(u_h) \neq \emptyset.$$

We also associate with problem (1.7) the following problems : for fixed $h\in\mathcal{H}$, find u_h such that

$$(2.4)_h \qquad u_h\in K_h \text{ and } J_h(u_h) \le J_h(v_h) \text{ for all } v_h\in K_h.$$

Since conditions (1.1)-(1.4) and (1.8)-(1.10) are satisfied by F and G, they are clearly true for their restrictions F_h and G_h ; thus according to Theorem 1.1, every solution of Ineq. $(2.4)_h$ is also a (variational) solution of $(2.3)_h$ and there is at least one such solution.

Remark 2.1 : In the quadratic case $G(v) = \frac{1}{2}\, a(v,v)$, the discrete problem $(2.3)_h$ can also be written : find $u_h\in V_h$ such that

$$(2.3)'_h \qquad a_h(u_h,\cdot) \in \partial F_h(u_h).$$

2.2. A convergence result.

Concerning the convergence of a sequence $\{u_h\}_{h\in\mathcal{H}}$ of solutions of $(2.4)_h$, we cannot expect a global convergence because of the non-uniqueness of solutions. However, we have

<u>Theorem 2.1</u> : *Assume that hypotheses* (1.1)-(1.4), (1.8)-(1.10) *are satisfied and that the approximation* $\{V_h\}_{h\in\mathcal{H}}$ *of the space* V *is convergent in the sense of* (2.1) *; then, if* $\{u_h\}_{h\in\mathcal{H}}$ *is a family of variational solutions of* $(2.3)_h$, *there exists a subsequence* $\{u_{h'}\}_{h'}$ *which converges strongly in* V *to a variational solution of* (1.5). *Moreover, the limit* u *of any weakly convergent subsequence of solutions (resp. variational solutions)* $\{u_{h'}\}_{h'}$ *of* $(2.3)_h$ *is a solution (resp. variational solution) of* (1.5).

<u>Proof of theorem 2.1</u> :

i) <u>First step</u> : the variational solutions u_h of $(2.3)_h$ are bounded independently of h.

The minimization of F on the space Z provides some $z_o \in K \cap Z$. Because of the inclusion $Z \subset V_h$, z_o belongs to each K_h. Any solution u_h of $(2.4)_h$ then satisfies $J(u_h) \leq J(z_o)$. This estimate and the coercivity assumption (1.10) yield there exists $C > 0$ independent of h such that $\|u_h\| \leq C$ for all h.

Given solutions u_h of $(2.4)_h$ for infinitely many values of h, there is therefore at least one weakly converging subsequence $\{u_{h'}\}$.

ii) <u>Second step</u> : the weak limit u of a sequence $\{u_{h'}\}$ of variational solutions of $(2.3)_h$ is a variational solution of (1.5).

At first, we have $u \in K$ since K is weakly closed and each $u_{h'}$ belongs to K. Now, we prove that $J(u) \leq J(v)$ for all $v \in V$. Given $v \in V$, the convergence condition (2.1) allows us to introduce a family $\{v_h\}_{h\in\mathcal{H}}$ of elements of the spaces V_h such that $v_h \to v$. Then, by (1.15) we know that there is $z_h \in Z$ such that $w_h = v_h + z_h$ belongs to K_h. And since $u_{h'}$ satisfies $(2.4)_h$, we have

$$(2.5) \qquad J(u_{h'}) \leq J(w_{h'}).$$

But we noticed in Sec. 1.3 that J is weakly l.s.c. on V, whence

$$J(u) \leq \liminf_{h' \to 0} J(u_{h'}) \leq \liminf_{h' \to 0} J(w_{h'}).$$

To prove that u is a solution of (1.7) as announced, it is now enough to show

$$(2.6) \qquad \lim_{h\to 0} J(w_h) = J(v).$$

On the one hand, we have $G(w_h) = G(v_h)$ because of (1.3), and $\lim_{h\to 0} G(v_h) = G(v)$.

On the other hand, we have $F(w_h) \le F(v_h)$ because $w_h \in K_h$, so that $\limsup_{h\to 0} F(w_h) \le F(v)$. Besides that, since $v \in K$ there exists $L \in \partial F(v) \cap Z^o$ and we have

$$F(w_h) \ge F(v)+L(v_h-v) \quad ,$$

which yields $\liminf_{h\to 0} F(w_h) \ge F(v)$. We finally obtain $\lim_{h\to 0} F(w_h) = F(v)$ which completes the proof of (2.6).

iii) Third step : the weak limit u of a sequence of solutions $\{u_{h'}\}$ of $(2.3)_h$ is a solution of (1.5).

Let $\{L_{h'}\}$ be a corresponding sequence of continuous linear forms on $V_{h'}$ such that $L_{h'} \in \partial F_{h'}(u_{h'}) \cap \partial G_h(u_{h'})$. We introduce below extensions $\tilde{L}_{h'} \in V^*$ of the forms $L_{h'}$ and prove that they have a cluster value $L \in \partial F(u) \cap \partial G(u)$, which shows that u is a solution of (1.5).

Using Hahn Banach theorem as in the proof of lemma 1.2, we see that $L_{h'}$ can be extended into $\tilde{L}_{h'} \in V^*$ satisfying

$$\tilde{L}_{h'} \in \partial F(u_{h'}).$$

The weakly convergent sequence $\{u_{h'}\}$ being bounded, the arguments used to show i) ii) and iii) in the proof of Theorem 1.2 can be repeated to prove that a subsequence of $\{L_{h'}\}$ converges strongly in V^* to some $L \in \partial F(u)$.

On the other hand, since $\tilde{L}_{h'}|_{V_{h'}} = L_{h'} \in \partial G_{h'}(u_{h'})$ we have

$$G(v_{h'}) \ge G(u_{h'})+\tilde{L}_{h'}(v_{h'}-u_{h'}) \text{ for all } v_{h'} \in V_{h'}.$$

Let $\{v_{h'}\}$ be a sequence converging strongly to v. Since G is weakly l.s.c. on V, we get at the limit

$$G(v) \ge G(u)+L(v-u) \text{ for all } v \in V,$$

that is to say $L \in \partial G(u)$. Hence $L \in \partial G(u) \cap \partial F(u)$.

iv) Fourth step : any weakly converging sequence $\{u_{h'}\}$ of solutions of $(2.3)_h$ converges in fact strongly to its limit u. According to Remark 1.5, we only need to prove the strong convergence in V/Z. We argue by contradiction and assume that $\{u_{h''}\}$ is some subsequence such that $\|\dot{u}_{h''}-\dot{u}\|_{V/Z} \geq \varepsilon > 0$ for all h". Repeating the arguments of step iii) we may also suppose that corresponding forms $L_{h''} \in \partial F_{h''}(u_{h''}) \cap \partial G_{h''}(u_{h''})$ have been extended into $\tilde{L}_{h''} \in V^*$ and that the sequence $\{\tilde{L}_{h''}\}$ converges strongly to $L \in \partial G(u) \cap \partial F(u)$. Let $\bar{u}_{h''} \in V_{h''}$ be such that $\bar{u}_{h''} \to u$ strongly (such a sequence exists thanks to (2.1)). We then have

$$G(\bar{u}_{h''}) \geq G(u_{h''})+L_{h''}(\bar{u}_{h''}-u_{h''})$$

$$\geq \text{(by (1.4))}$$

$$\geq G(u)+L(u_{h''}-u)+\tilde{L}_{h''}(\bar{u}_{h''}-u_{h''})+\alpha\|\dot{u}-\dot{u}_{h''}\|^2_{V/Z} \,.$$

Taking the lim sup of the two members of this inequality we get

$$0 \geq \alpha \limsup_{h''\to 0} \|\dot{u}-\dot{u}_{h''}\|^2_{V/Z} \geq \alpha\varepsilon^2$$

in contradiction with $\varepsilon > 0$. This proves that the convergence of $\{u_{h'}\}$ to u is strong and completes the proof of Theorem 2.1. ∎

2.3. The discrete iteration scheme (for fixed $h \in \mathcal{H}$).

The starting point is some $u_o \in K_h$; once $u_1 \in K_h, \ldots, u_n \in K_h$ have been obtained, we get $u_{n+1} \in K_h$ in the following way.

Step 1 : let L_n belong to the (non-empty convex) set $\partial F_h(u_n) \cap Z^o$; we define $\mathcal{U}_{n+1} \in V_h/Z$ as the solution of the minimization problem

$$\operatorname*{Inf}_{\mathcal{U}\in V_h/Z} \{G_h(\mathcal{U})-L_n(\mathcal{U})\}.$$

Step 2 : We then take for u_{n+1} a solution of the minimization problem

$$\operatorname*{Inf}_{u\in\mathcal{U}_{n+1}} F_h(u).$$

For fixed $h \in \mathcal{H}$, since $Z \subset V_h$, conditions (1.1)-(1.4) and (1.8)-(1.10) that are fulfilled by F and G on the space V, are also satisfied by their restrictions F_h and G_h on the space V_h. Consequently, as well as in the continuous case, a sequence $\{u_n\}$ can be obtained by this algorithm and its convergence properties are described by Theorem 1.2 and Corollary 1.1 which are still valid on V_h.

3. - A SET-VALUED ELLIPTIC DIFFERENTIAL EQUATION.

3.1. The problem. Equivalence with an abstract subdifferential problem.

Given a bounded domain $\Omega \subset \mathbb{R}^N$ with a smooth boundary Γ, let

$$\mathcal{L}u = - \sum_{i,j=1}^{N} \frac{\partial}{\partial x_j} (a_{ij}(x) \frac{\partial u}{\partial x_i})$$

be a second order uniformly-strongly elliptic differential operator (with constant of ellipticity $\eta > 0$). We suppose that $a_{ij} \in C^1(\bar{\Omega})$ and that $a_{ij} = a_{ji}$. Let $\frac{\partial}{\partial \nu_{\mathcal{L}}}$ be the outward conormal derivative associated with the operator $\mathcal{L}$

$$\frac{\partial u}{\partial \nu_{\mathcal{L}}} = \sum_{i,j=1}^{N} a_{ij}(x) \cos(\nu, e_i) \frac{\partial u}{\partial x_j}$$

where ν denotes the unit outward normal to Γ.

Let $\Phi : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function which satisfies

(3.1)
- i) for all $s \in \mathbb{R}$, $\Phi(\cdot, s)$ is Lebesgue-measurable,
- ii) for almost all $x \in \Omega$, $\Phi(x, \cdot)$ is convex,
- iii) $\Phi(\cdot, 0) \in L^\infty(\Omega)$ and for almost all $x \in \Omega, \Phi(x,s) = \Phi(x,0)$ for $s \leq 0$,
- iv) $\lim_{s \to +\infty} \frac{\|\Phi(\cdot, s)\|_\infty}{s^{\alpha+1}} = 0$ where α is $\frac{N}{N-2}$ if $N > 2$ and some number >1 if $N \leq 2$.

<u>Remark 3.1</u> : Assumption ii) yields that for almost all $x \in \Omega$ the function $\Phi(x, \cdot)$ is continuous, whence Φ satisfies the Coratheodory condition.

Let us now define the left and right derivatives of $\Phi(x, \cdot)$

$$\underline{\phi}(x,s) = \lim_{t \to s-0} \frac{\Phi(x,t) - \Phi(x,s)}{t-s} \leq \bar{\phi}(x,s) = \lim_{t \to s+0} \frac{\Phi(x,t) - \Phi(x,s)}{t-s} .$$

It is easy to check that the functions $\underline{\phi}$ and $\bar{\phi}$ are nonnegative a.e., nondecreasing with respect to s, and that we have the asymptotic condition

$$(3.2) \qquad \lim_{s \to +\infty} \frac{\|\bar{\phi}(\cdot, s)\|_\infty}{s^\alpha} = 0 \text{ , with } \alpha \text{ given in (3.1) iv).}$$

Moreover, if $u : \Omega \to \mathbb{R}$ is any Lebesgue-measurable function, the functions

$\underline{\phi}(\cdot,u(\cdot))$ and $\bar{\phi}(\cdot,u(\cdot))$ are also Lebesgue-measurable.

We consider the following set-valued equation : given some positive constant I, we look for solutions $u \in H^2(\Omega)$ of

$$(3.3) \qquad \begin{cases} \mathcal{L}u \in [\underline{\phi}(x,u(x)),\ \bar{\phi}(x,u(x))] \text{ a.e. in } \Omega, \\ u = \text{const. (unknown) a.e. on } \Gamma, \\ -\displaystyle\int_\Gamma \frac{\partial u}{\partial \nu_{\mathcal{L}}}\, d\sigma = I\ . \end{cases}$$

The following part of this section studies the equivalence of problem (3.3) with an abstract subdifferential problem entering the framework of Sec. 1.

Let V and H be the spaces

$$V = \{v \in H^1(\Omega) \ :\ v = \text{const. a.e. on } \Gamma\},$$
$$H = L^2(\Omega)$$

which are Hilbert spaces when equipped respectively with the scalar products

$$((u,v)) = \int_\Omega \nabla u \cdot \nabla v \, dx + u(\Gamma)\cdot v(\Gamma),$$
$$(u,v) = \int_\Omega uv \, dx,$$

where $v(\Gamma)$ denotes the constant value of v on Γ.

We associate with the operator $\mathcal{L}$ the nonnegative symmetric bilinear form on V

$$a(u,v) = \sum_{i,j=1}^{N} \int_\Omega a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}\, dx.$$

Its kernel Z consists of all the functions that are constant on Ω.

It is easy to check that conditions i)-ii) of the quadratic case in Sec.1.1 are fulfilled by V,H,a.

Now, let us study the two convex functionals

$$F_o(v) = \int_\Omega \Phi(x,v(x))dx,$$
$$F(v) = F_o(v) - I\cdot v(\Gamma).$$

It follows from (3.1) that for some constant $C>0$ we have $|\Phi(x,s)| \leq C+|s|^{\alpha+1}$ for all s and almost all x. Then, if $v \in L^{\alpha+1}(\Omega)$, we have $\phi(\cdot,v(\cdot)) \in L^1(\Omega)$. Moreover, according to a theorem of KRASNOSEL'SKII (Th. I.2.1 of [11]) $v \to \phi(\cdot,v(\cdot))$ maps continuously $L^{\alpha+1}(\Omega)$ into $L^1(\Omega)$ so that the functional F_o is continuous on $L^{\alpha+1}(\Omega)$. Sobolev's theorem asserts that the embedding $H^1(\Omega) \subset L^q(\Omega)$ is continuous for $1 \leq q \leq 2\alpha$ and completely continuous for $1 \leq q < 2\alpha$, especially for $q=\alpha+1$. Hence, the functional F_o is weakly continuous on V, and so is F. The subdifferential $\partial F_o(v)$ (and consequently $\partial F(v)$) can be determined thanks to arguments contained in I. EKELAND - R. TEMAM [9]. By definition, $v^* \in L^{\alpha+1/\alpha}(\Omega) = (L^{\alpha+1}(\Omega))'$ belongs to $\partial F_o(v)$ if and only if

$$(3.4) \qquad F_o(v)+F_o^*(v^*) = \int_\Omega v(x)v^*(x)\,dx,$$

where the conjugate function of F_o is given by

$$F_o^*(v^*) = \sup_{v \in L^{\alpha+1}(\Omega)} \{\int_\Omega vv^* dx - F_o(v)\}.$$

By Prop. IX 2.1 of [9], $F_o^*(v^*)$ can be expressed in the form

$$F_o^*(v^*) = \int_\Omega \Phi^*(x,v^*(x))dx$$

where $\Phi^*(x,s) = \sup_{t \in \mathbb{R}} \{st-\Phi(x,t)\}$.

Equality (3.4) then means

$$\int_\Omega [\Phi(x,v(x))+\Phi^*(x,v^*(x))-v(x)v^*(x)]\,dx = 0,$$

but the bracket being nonnegative, this is also equivalent to

$$(3.5) \qquad \Phi(x,v(x))+\Phi^*(x,v^*(x)) = v(x)v^*(x) \text{ a.e. in } \Omega.$$

By definition of subdifferentials, we see that $v^*(x)$ belongs to the subdifferential of $\Phi(x,\cdot)$ at v(x) for almost all x, i.e. $v^*(x) \in [\underline{\phi}(x,v(x)),\bar{\phi}(x,v(x))]$ a.e. Conversely, any mesurable function v^* which satisfies this latter relation belongs to $L^{\alpha+1/\alpha}(\Omega)$ because of (3.2) and to $\partial F_o(v)$ because of (3.4)-(3.5). Then, by Prop. I.5.7 of [9] the subdifferential $\partial F(v)$ of F at $v \in V$ consists of all the linear forms L such that

$$L(v) = \int_\Omega v^*(x)v(x)dx - I\cdot v(\Gamma)$$

where $v^* \in L^{\alpha+1/\alpha}(\Omega)$ satisfies $v^*(x) \in [\underline{\phi}(x,v(x)), \overline{\phi}(x,v(x))]$ a.e. in Ω.

As we check it below, problem (3.3) can be put in the following abstract subdifferential form : find $u \in V$ wuch that $a(u,\cdot) \in \partial F(u)$, that is to say

$$(3.6) \qquad \begin{cases} \text{find } u \in V \text{ such that} \\ a(u,v) = \int_\Omega \rho(x)v(x)dx - I\cdot v(\Gamma) \text{ for all } v \in V \\ \text{with } \underline{\phi}(x,u(x)) \le \rho(x) \le \overline{\phi}(x,u(x)) \text{ a.e. in } \Omega. \end{cases}$$

If $u \in H^2(\Omega)$ satisfies (3.3), then $u \in L^{\alpha+1}(\Omega)$ and $\rho = \mathcal{L}u \in L^{\alpha+1/\alpha}(\Omega)$ since $\underline{\phi}(x,u(x)) \le \rho(x) \le \overline{\phi}(x,u(x))$ a.e. in Ω. If we multiply the equality $\rho = \mathcal{L}u$ by $v \in V$ and integrate over Ω, we get by Green's formula

$$a(u,v) = \int_\Omega \rho(x)v(x)dx + v(\Gamma)\int_\Gamma \frac{\partial u}{\partial \nu_{\mathcal{L}}} d\sigma ,$$

so that (3.6) is satisfied since $I = -\int_\Gamma \frac{\partial u}{\partial \nu_{\mathcal{L}}} d\sigma$.

Conversely, if $u \in V$ satisfies (3.6) we have $\mathcal{L}u = \rho$ in the sense of distributions, and u is constant on Γ. By Sobolev's embedding $H^1(\Omega) \subset L^{2\alpha}(\Omega)$, we have $u \in L^{2\alpha}(\Omega)$ so that $\rho \in L^2(\Omega)$. Then, classical regularity results yield $u \in H^2(\Omega)$, since the boundary Γ and the coefficients $a_{ij}(x)$ are smooth enough. Green's formula is therefore valid and leads to $-\int_\Gamma \frac{\partial u}{\partial \nu_{\mathcal{L}}} d\sigma = I$, whence u satisfies (3.3).

We now conclude this section with an explicit characterization of the set K, defined as in Sec. 1.1 by

$$K = \{v \in V \ : \ F(v) \le F(v+z) \text{ for all } z \in \mathbf{R}\} ,$$

which contains every solution of (3.3).

According to lemma 1.2, $v \in K$ if (and only if) $\partial F(v) \cap Z^0 \neq \emptyset$, that is to say if $I = \int_\Omega v^*(x)dx$ for some function v^* such that $\underline{\phi}(x,v(x)) \le v^*(x) \le \overline{\phi}(x,v(x))$ a.e. in Ω. Considering for example the function $\theta\underline{\phi} + (1-\theta)\overline{\phi}$ for an appropriate

$\theta \in (0,1)$ it is easy to see that

$$\text{(3.7)} \qquad K = \{v \in V : \int_\Omega \underline{\phi}(x,v(x))\,dx \le I \le \int_\Omega \bar{\phi}(x,v(x))\,dx\} .$$

3.2. An iteration scheme.

We note that $H_o^1(\Omega)$ is a topological complement of Z in V ; it is therefore isomorphic to V/Z. For this reason the iteration scheme described in Sec. 1.4 becomes : the starting point is some $u_o \in K$; once $u_1, \ldots, u_n \in K$ have been obtained, we get $u_{n+1} \in K$ as follows

Step 1 : Let $\rho_n \in L^2(\Omega)$ be defined by

$$\rho_n(x) = \theta\underline{\phi}(x,u_n(x)) + (1-\theta)\,\bar{\phi}(x,u_n(x)),$$

with $\theta \in (0,1)$ chosen such that $\int_\Omega \rho_n(x)\,dx = I$; we then determine $\tilde{u}_{n+1}$ as the unique solution in $H_o^1(\Omega)$ of

$$\mathcal{L}\tilde{u}_{n+1} = \rho_n \text{ a.e. in } \Omega.$$

Step 2 : We set $u_{n+1} = \tilde{u}_{n+1} + z_{n+1}$ where $z_{n+1} \in \mathbb{R}$ is a solution of

$$F(\tilde{u}_{n+1} + z_{n+1}) \le F(\tilde{u}_{n+1} + z) \text{ for all } z \in \mathbb{R}.$$

Remark 3.2 :

i) wherever $\underline{\phi}(x,u_n(x)) = \bar{\phi}(x,u_n(x))$ we have $\rho_n(x) = \bar{\phi}(x,u_n(x))$.

ii) we could also take for ρ_n any function such that

$$\int_\Omega \rho_n(x)\,dx = I \text{ and } \underline{\phi}(x,u_n(x)) \le \rho_n(x) \le \bar{\phi}(x,u_n(x)) \text{ a.e. in } \Omega.$$

iii) we can equivalently obtain z_{n+1} by solving

$$\int_\Omega \underline{\phi}(x,\tilde{u}_{n+1}(x)+z_{n+1})\,dx \le I \le \int_\Omega \bar{\phi}(x,\tilde{u}_{n+1}(x)+z_{n+1})\,dx.$$

iv) If $\tilde{V}_h$ is some finite dimensional subspace of $H_o^1(\Omega)$ (for example a finite element approximation of $H_o^1(\Omega)$) and $V_h = \tilde{V}_h \oplus \mathbb{R}$, the discrete iteration scheme of Sec. 2.3 consists in computing $\rho_n \in L^2(\Omega)$ as above, then $\tilde{u}_{n+1} \in \tilde{V}_h$ in solving

$$a(\tilde{u}_{n+1},\tilde{v}_h) = (\rho_n,\tilde{v}_h) \text{ for all } \tilde{v}_h \in \tilde{V}_h,$$

and finally in setting $u_{n+1} = \tilde{u}_{n+1}+z_{n+1}$ with z_{n+1} determined as above.

3.3. Existence of solutions and convergence of the iteration scheme.

Theorem 3.1 : *The domain* Ω *and the operator* $\mathcal{L}$ *being those defined in Sec.* 3.1, *we assume that the function* Φ *satisfies condition* (3.1) *and that the constant* I *satisfies* $0 < I < I_\infty$ (with $I_\infty = \int_\Omega \lim_{s\to+\infty} \underline{\phi}(x,s)dx$). *Then*

i) *Eq.* (3.3) *admits at least one solution* $u \in H^2(\Omega)$;

ii) *for every* $u_o \in K$, *the sequence* $\{u_n\}$ *obtained with the iteration scheme described in Sec.* 3.2 *admits a subsequence which converges in* $H^1(\Omega)$ *to a solution of* (3.3), *and the limit of any other converging subsequence is also a solution of* (3.3) ;

iii) *furthermore, if the solutions of Eq.* (3.3) *form a finite set, the whole sequence* $\{u_n\}$ *converges to one of them.*

Proof : We have to check conditions (1.8),(1.9),(1.10) of Sec. 1.1.

i) Verification of (1.8)

Let us recall that in this section, the space Z consists of all real constants. We study separetely $z \to +\infty$ and $z \to -\infty$.

If $z \to -\infty$, we have $\phi(x,u(x)+z) \to \Phi(x,0)$ a.e. in Ω so that by Lebésgue's theorem of dominated convergence

$$\int_\Omega \phi(x,u(x)+z)dx \to \int_\Omega \Phi(x,0)dx$$

whence $F(u+z) \to +\infty$ for $z \to -\infty$.

By assumption, there is $\beta > 0$ such that $\int_\Omega \lim_{s\to+\infty} \underline{\phi}(x,s)dx > I+2\beta$, and we obtain by the theorem of monotonous convergence that there exists $C > 0$ such that

$$\int_\Omega \underline{\phi}(x,u(x)+c)dx \geq I+\beta \ . \tag{3.8}$$

For almost all x, it is easy to see that

$$\phi(x,s) \geq (s-s_o)\underline{\phi}(x,s_o) \text{ for all } s,s_o.$$

Then, if we set $s = u(x)+z$, $s_o = u(x)+C$ and integrate over Ω, we get

$$F(u+z)+(u(\Gamma)+z)I \geq (z-C)\int_\Omega \underline{\phi}(x,u(x)+C)\,dx.$$

According to (3.8) we have for some constant C'

$$F(u+z) \geq \beta z + C',$$

whence $F(u+z) \to +\infty$ for $z \to +\infty$.

ii) <u>Verification of</u> (1.9).

A broad use of techniques contained in the proofs of lemma 1.1 [21] of R. TEMAM and lemmas 3.2, 3.3 [3] of H. BERESTYCKI - H. BREZIS - in which $\Phi(x,\cdot)$ is a function of class C^1 - leads to

<u>Lemma 3.2</u> : *Under the hypotheses of Th.* 3.1, *there exists constants* $\eta' > 0$ *and* $C' > 0$ *such that*

$$\frac{1}{2}a(u,u)-F(u) \geq \eta'|\nabla u|^2_{L^2} - C' \text{ for all } u \in K. \tag{3.9}$$

<u>Proof</u> : The first step is to see that for every $\varepsilon > 0$ there are positive constants $C_{i,\varepsilon}$ (i=1,3) such that for all $v \in V$

$$0 \leq \underline{\phi}(x,s) \leq \bar{\phi}(x,s) \leq C_{1,\varepsilon} + \varepsilon(s_+)^\alpha \text{ for all } s \text{ and almost all } x^{(1)} \tag{3.10}$$

$$|\underline{\phi}(\cdot,v(\cdot))|_{L^2} \leq \{C_{2,\varepsilon} + \varepsilon|v_+|^{2\alpha}_{L^{2\alpha}}\}^{1/2} \tag{3.11}$$

$$|\Phi(\cdot,v(\cdot))|_{L^1} \leq \{C_{3,\varepsilon}+\varepsilon|v_+|^2_{L^{2\alpha}}\}\cdot|\underline{\phi}(\cdot,v(\cdot))|^{1-\frac{1}{\alpha}}_{L^1} + |\Phi(\cdot,0)|_{L^1} \tag{3.12}$$

Relation (3.10) is an easy consequence of (3.2).

We then set $s=v(x)$ and integrate the square of $\underline{\phi}(x,v(x))$ on Ω ; in this way, we get (3.11) since the domain Ω is bounded.

It follows from the relation $\Phi(x,s) = \int_0^s \underline{\phi}(x,t)dt + \Phi(x,0)$ that

$$|\Phi(x,s)| \leq s_+\cdot\underline{\phi}(x,s) + |\Phi(x,0)|$$

---

(1) Hereinafter s_+ denotes max(s,0) for numbers and functions as well.

whence by Hölder inequality

$$(3.13)\qquad |\Phi(\cdot,v(\cdot)|_{L^1} \le |v_+|_{L^{2\alpha}}\cdot|\phi(\cdot,v(\cdot))|_{L^q} + |\Phi(\cdot,0)|_{L^1}$$

where q is the conjugate of 2α (given by $\frac{1}{2\alpha}+\frac{1}{q}=1$). We have $1<q<2$ and $\frac{1}{q}=\frac{\theta}{2}+\frac{1-\theta}{1}$ with $\theta=\frac{1}{\alpha}$ so that by Hölder inequality again

$$(3.14)\qquad |\phi(\cdot,v(\cdot))|_{L^q} \le |\phi(\cdot,v(\cdot))|_{L^2}^{1/\alpha}|\phi(\cdot,v(\cdot))|_{L^1}^{1-1/\alpha}.$$

Then, we use (3.11),(3.13) and (3.14) to obtain (3.12).

From now on, we only consider the case $u\in K$. It follows from (3.12) that

$$(3.15)\qquad |\Phi(\cdot,u(\cdot))|_{L^1} \le I^{1-1/\alpha}\{C_{3,\varepsilon}+\varepsilon|u_+|^2_{L^{2\alpha}}\} + |\Phi(\cdot,0)|_{L^1}.$$

Now, we consider separately the cases $u(\Gamma)\le 0$ and $u(\Gamma)>0$.

Case $u(\Gamma)\le 0$.

First, we have $u_+\in H^1_o(\Omega)$ and Sobolev embedding's theorem yields

$$(3.16)\qquad |\Phi(\cdot,u(\cdot))| \le C_{4,\varepsilon}+\varepsilon|\nabla u|^2_{L^2},$$

for some constant $C_{4,\varepsilon}$ depending on $\varepsilon>0$.

From (3.10) it follows that for some $C_{5,\varepsilon}$

$$\gamma^\alpha\bar{\phi}(x,s) \le C_{5,\varepsilon} + \varepsilon(\gamma+s_+)^{2\alpha} \text{ for all } \gamma>0.$$

Taking $s=u(x)$ and $\gamma=-u(\Gamma)$, we obtain by integration over $\Omega_+=\{u\ge 0\}$

$$|u(\Gamma)|\cdot|\bar{\phi}(\cdot,u(\cdot))|^{1/\alpha}_{L^1} \le C_{6,\varepsilon} + \varepsilon|u-u(\Gamma)|^2_{L^{2\alpha}}.$$

The left-hand side member of this inequality is greater than $|u(\Gamma)|\cdot I^{1/\alpha}$ since $u\in K$, and applying again Sobolev's theorem to $u-u(\Gamma)\in H^1_o(\Omega)$ we get

$$(3.17)\qquad |u(\Gamma)| \le C_{7,\varepsilon} + \varepsilon|\nabla u|^2_{L^2}.$$

From (3.16),(3.17) and the coercivity of $a(\cdot,\cdot)$ on $H^1_o(\Omega)$, choosing ε small enough, we get (3.9) in the case $u(\Gamma)\le 0$.

Case $u(\Gamma) > 0$.

We have $u-u(\Gamma) \in H_0^1(\Omega)$ and by definition of K

$$F(u) \leq F(u-u(\Gamma)).$$

Since $u \in K$ and ϕ est nondecreasing, we have

$$\int_\Omega \phi(x,u(x)-u(\Gamma))dx \leq \int_\Omega \phi(x,u(x))dx \leq I.$$

Then, applying (3.12) to $u-u(\Gamma)$ we get

$$|\Phi(\cdot,u-u(\Gamma))|_{L^1} \leq C_{8,\varepsilon} + \varepsilon|\nabla u|^2_{L^2}$$

thanks to Sovolev's theorem.

Finally, we have

$$-F(u) \geq -F(u-u(\Gamma)) \geq -\varepsilon|\nabla u|^2_{L^2} - C_{8,\varepsilon}$$

whence (3.9) follows.

iii) Verification of (1.10).

We argue by contradiction and assume that a sequence $\{u_n\}$ satisfies

$$u_n \in K, \ \|u_n\| \to +\infty \text{ and } \frac{1}{2} a(u_n,u_n)-F(u_n) \leq C$$

for some constant C.

It follows from (3.9) and (3.17) respectively that

$$|\nabla u_n|_{L^2} \leq C \ , \ -C \leq u_n(\Gamma)$$

for another constant C. Hence, the sequence $u_n-u_n(\Gamma)$ which is bounded in $H_0^1(\Omega)$ admits a converging subsequence in $L^2(\Omega)$ and we may even suppose that

$$u_{n'}-u_{n'}(\Gamma) \to w \text{ a.e. in } \Omega.$$

Since on the other hand we must have $u_n(\Gamma) \to +\infty$,

$$u_{n'} \to +\infty \quad \text{a.e. in } \Omega$$

and Fatou's lemma yields

$$\int_\Omega \lim_{s \to +\infty} \phi(x,s)\,dx \le \liminf_{n' \to +\infty} \int_\Omega \phi(x,u_{n'}(x))\,dx$$

in contradiction with the assumption $I < I_\infty$.

This completes the proof of Theorem 3.1.■

4. - APPLICATION I : FREE BOUNDARY EQUILIBRIA FOR AXIALLY SYMMETRIC PLASMAS.

4.1. Stokes-Grad-Shafranov equation :

The last two sections of this paper are devoted to physical applications ; the main one is the determination of free boundary equilibria for axially symmetric plasmas, the formulation of which we briefly recall now.

We use a cylindrical coordinate system (x_1,θ,x_2) whose axis Ox_2 coincides with the center line of the toroid. In the meridional plane (Ox_1,Ox_2) we denote by Ω (with $x_1 \ge \alpha > 0$ for every $(x_1,x_2) \in \Omega$) the domain bounded by the shell Γ, by Ω_p the plasma, by Γ_p the interface plasma-vacuum and by Ω_v the vacuum.

According to the Magnetohydrodynamic theory (see G. BATEMAN [1]), the physical quantities B (magnetic field) and p (pressure of every axisymmetric equilibrium can be expressed in terms of the poliodal flux function $u(x_1,x_2)$ by

$$B = \frac{\nabla u}{x_1}\, e_\theta + \frac{f(u)}{x_1}\, e_\theta ,$$

$$p = p(u),$$

the functions p,f,u being related in the plasma by Stokes-Grad-Shafranov equation

$$\mathcal{L}u = x_1 p'(u) + \frac{1}{x_1} f(u)f'(u),$$

where $\mathcal{L}$ denotes the operator $-\frac{\partial}{\partial x_1}\left(\frac{1}{x_1}\frac{\partial}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{1}{x_1}\frac{\partial}{\partial x_2}\right)$, whilst in the vacuum $p=0$, $f = \text{const.}$ and the function u satisfies $\mathcal{L}u = 0$.

Following a method of R. TEMAM (cf. [20],[21]), we look for axisymmetric equilibria by specifying the whole domain Ω, the functions $p = p(u)$, $f = f(u)$ and the total current crossing the plasma $I = \int_\Omega j_\theta \, dx_1 dx_2$ [1]. When the functions

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[1] $j_\theta = \mathcal{L}u$ is the toroidal component of the plasma.

p and f are smooth (with p(0) = 0), the unknown poloidal flux has to satisfy

$$(4.1)\qquad \begin{cases} \mathcal{L}u = \Phi'_s(\cdot,u(\cdot)) \text{ in } \Omega, \\ u = \text{const. (unknown) on } \Gamma, \\ -\displaystyle\int_\Gamma \frac{1}{x_1}\frac{\partial u}{\partial \nu}\, d\sigma = I, \end{cases}$$

where we have set for $x = (x_1,x_2)$

$$(4.2)\qquad \Phi(x,s) = \begin{cases} x_1\, p(x) + \dfrac{1}{2x_1}\, f^2(s) & \text{if } s > 0, \\ \dfrac{1}{2x_1}\, f^2(0) & \text{if } s \le 0, \end{cases}$$

and ν denotes the unit outward normal to Γ. The region Ω_p accupied by the plasma, not known beforehand, is given by

$$\Omega_p = \{x \in \Omega : u(x) > 0\} .$$

But in many cases of physical interest, the functions p and f are not regular, so that, in the case where $s \to \Phi(x,s)$ is convex[1], it seems natural to consider the generalized problem

$$(4.3)\qquad \begin{cases} \mathcal{L}u \in [\underline{\phi}(\cdot,u(\cdot)),\overline{\phi}(\cdot,u(\cdot))] \text{ a.e. in } \Omega, \\ u = \text{const. (unknown) a.e. on } \Gamma, \\ -\displaystyle\int_\Gamma \frac{1}{x_1}\frac{\partial u}{\partial \nu}\, d\sigma = I, \end{cases}$$

where $\underline{\phi}(x,\cdot)$ and $\overline{\phi}(x,\cdot)$ are the left and right derivatives of $\Phi(x,\cdot)$.

Moreover, we shall see that the solutions of (4.3) are in some sense solutions of (4.1). More precisely, the convexity of $\Phi(x,\cdot)$ yields that ϕ'_s is well defined on $\Omega \times (\mathbb{R}-D)$ (where D denotes an at most countable set $D \subset \mathbb{R}_+$), and choosing the following extension for Φ'_s on $\Omega \times \mathbb{R}$

$$(4.4)\qquad \text{for all } x \in \Omega,\ \phi(x,s) = \begin{cases} \Phi'_s(x,s) \text{ for } s \notin D, \\ 0 \text{ for } s=0 \text{ if } 0 \in D, \\ \text{any value}^{(2)} \text{ for } s \neq 0 \text{ such that } s \in D, \end{cases}$$

(1) which is essentially a mathematical assumption.

(2) Let us stress that this value does not depend on x. This ensures that $\phi(\cdot,u(\cdot))$ is measurable for all measurable functions u.

we shall see that every solution u of (4.3) is also solution of

(4.5)
$$\begin{cases} \mathcal{L}u = \phi(\cdot,u(\cdot)) \text{ a.e. in } \Omega, \\ u = \text{const. (unknown) a.e. on } \Gamma, \\ -\int_\Gamma \frac{1}{x_1}\frac{\partial u}{\partial \nu}\, d\sigma = I. \end{cases}$$

Remark 4.1 : If we look for cylindrical plasma equilibria instead of axisymmetric equilibria, the magnetic field B and the pressure p can be expressed in terms of a flux function $u(x_1,x_2)$ by

$$B = \nabla u \times e_3 + f(u)e_3,$$
$$p = p(u),$$

and the problem (4.1) becomes

(4.6)
$$\begin{cases} -\Delta u = \phi'(u) \text{ in } \Omega, \\ u = \text{const. (unknown) on } \Gamma, \\ -\int_\Omega \frac{\partial u}{\partial \nu}\, d\sigma = I , \end{cases}$$

where we have set

(4.7)
$$\Phi(s) = \begin{cases} p(s) + \frac{1}{2} f^2(s) & \text{if } s > 0, \\ \frac{1}{2} f^2(0) & \text{if } s \le 0. \end{cases}$$

It is immediate to rewrite the generalized problem (4.3) and the single-valued problem (4.5) in that case.

4.2. Existence and computation of solutions.

Assume that Ω is a bounded domain $\Omega \subset \mathbb{R}^2$, with a smooth boundary Γ, such that

$$0 < \alpha \le x_1 \le \beta < +\infty \text{ for all } (x_1,x_2) \in \Omega,$$

and that $p : \mathbb{R}_+ \to \mathbb{R}_+$, $f : \mathbb{R}_+ \to \mathbb{R}$ are two functions satisfying

$$(4.8)\quad \begin{cases} \text{i)}\ \ p + \dfrac{1}{2\alpha^2} f^2 \text{ and } p + \dfrac{1}{2\beta^2} f^2 \text{ are convex functions,} \\ \text{ii)}\ \ p(0) = 0 \text{ and } 2\alpha^2 p(s) + f^2(s) > f^2(0) \text{ for all } s > 0, \\ \text{iii)}\ \ \lim\limits_{s \to +\infty} \dfrac{p(s) + \frac{1}{2\alpha^2} f^2(s)}{s^\gamma} = 0 \text{ for some } \gamma > 2. \end{cases}$$

Corollary 4.1 : *Under these hypotheses, and the assumption that the constant* I *satisfies* $0 < I < I_\infty$ (with $I_\infty = \int_\Omega \lim\limits_{s \to +\infty} (x_1 p'(s) + \frac{1}{x_1} f(s)f'(s))\, dx$ [1]), *then*

i) *Eq.* (4.5), *with* ϕ *defined by* (4.4), *admits at least one solution* $u \in H^2(\Omega)$;

ii) *for every* $u_o \in K$ *(defined by* (3.7)), *the sequence* $\{u_n\}$ *obtained with the iteration scheme described in Sec.* 3.2 *admits a subsequence which converges in* $H^1(\Omega)$ *to a solution of* (4.5), *and the limit of any other converging subsequence is also a solution of* (4.5) ;

iii) *furthermore, if the solutions of Eq.* (4.5) *form a finite set, the whole sequence* $\{u_n\}$ *converges to one of them.*

Proof : This result is due to theorem 3.1 and the equivalence between the set-valued equation (4.3) and the single-valued equation (4.5). In order to apply theorem 3.1 we only need to check condition (3.1) ii), since the others are obviously satisfied.

i) Verification of assumption (3.1) ii). Given $x \in \Omega$, there is $\theta \in [0,1]$ such that $\frac{1}{x_1^2} = \frac{\theta}{\alpha^2} + \frac{1-\theta}{\beta^2}$, whence by $(4.8)_{i)}$

$$(4.9)\qquad \Phi(x,s) = x_1[\theta(p(s) + \frac{1}{2\alpha^2} f(s)^2) + (1-\theta)(p(s) + \frac{1}{2\beta^2} f(s)^2)]$$

is convex with respect to s.

ii) Equivalence between Eq. (4.3) and Eq. (4.5). The two convex functions $p + \frac{1}{2\alpha^2} f^2$ and $p + \frac{1}{2\beta^2} f^2$ are continuous on $\mathbb{R}_+$ and differentiable except perhaps on an atmost countable set $D \subset \mathbb{R}_+$.

Then it follows from (4.9) that $\Phi(x,\cdot)$ is also differentiable on $\mathbb{R}_+ - D$ for all $x \in \Omega$ so that we may introduce the function ϕ by (4.4).

([1]) As we shall see in Remark 4.2, p and f are differentiable on $R_+ - D$.

Now, let $u \in H^2(\Omega)$ be a solution of Eq. (4.3). We associate with u a partition of Ω in measurable sets

$$\Omega_1 = \{x\in\Omega : u(x) \notin D\},$$

$$\Omega_2 = \{x\in\Omega : u(x) = 0\} \text{ when } 0 \in D \text{ and } \emptyset \text{ otherwise},$$

$$\Omega_3 = \{x\in\Omega : u(x) \neq 0 \text{ and } u(x) \in D\}.$$

If $x \in \Omega_1$, $\Phi(x,\cdot)$ is differentiable at $s = u(x)$, whence $\underline{\phi}(x,u(x)) = \bar{\phi}(x,u(x)) = \Phi'_s(x,u(x))$ and it turns out that

$$\mathcal{L}u(x) = \phi(x,u(x)) \text{ a.e. in } \Omega_1.$$

Now, given some $s \in \mathbf{R}$ we introduce the measurable set $E_s = \{x \in \Omega : u(x) = s\}$. Since $u \in H^2(\Omega)$, a well-known result of G. STAMPACCHIA [19] shows that $\mathcal{L}u = 0$ a.e. in E_s. We have therefore $\underline{\phi}(x,s) \leq 0$ a.e. in E_s. But, for $s > 0$, it follows from (4.8) ii) that $\underline{\phi}(x,s) > 0$ a.e. in Ω, whence E_s is a set of measure 0, and so is Ω_3. On $\Omega_2 \equiv E_o$ we have $\mathcal{L}u = 0 = \phi(\cdot,0)$ a.e. . This shows that u is a solution of (4.5). Conversely, if $u \in H^2(\Omega)$ is a solution of (4.5), we introduce the same partition of Ω as above and observe again that Ω_3 is a set of measure 0. Thus, u satisfies (4.3). ∎

<u>Remark 4.2</u> : Under the assumption (4.8)i), it can be shown that the functions p and f are continuous on $\mathbf{R}_+$ and differentiable except perhaps on an atmost countable set D' which contains D. Hence, if we substitute D' to D in (4.4), we may differentiate $\Phi(x,\cdot)$ term by term.

<u>Remark 4.3</u> : In the cylindrical case, assumptions (4.8)i)-iii) become

i) $p+\frac{1}{2}f^2$ is a convex function, ii) p=0 and $p(s)+\frac{1}{2}f^2(s) > \frac{1}{2}f^2(0)$ for $s > 0$,

iii) $\lim\limits_{s\to+\infty} \dfrac{p(s)+\frac{1}{2}f^2(s)}{s^\gamma} = 0$ for some $\gamma > 2$,

whereas I_∞ is now given by $I_\infty = |\Omega|\cdot \lim\limits_{s\to+\infty} (p'(s)+f(s)f'(s))$, with $|\Omega|$ denoting the measure of the domain Ω.

4.3. Numerical results.

The example studied numerically corresponds to the following problem (cylindrical case)

$$(4.10)\quad \begin{cases} -\Delta u = \lambda H(u) \text{ a.e. in } \Omega, \\ u = \text{const. (unknown) on } \Gamma, \\ -\int_\Gamma \frac{\partial u}{\partial \nu}\, d\sigma = I, \end{cases}$$

where H denotes the Heaviside function ($H(z) = 0$ for $z \leq 0$, $H(z) = 1$ for $z > 0$), and λ, I are two prescribed constants such that $0 < I < \lambda|\Omega|$ (1). This problem is discretized by piecewise linear finite elements.

Fig. 1 shows the domain Ω and the triangulation used for the computations, and Fig. 2 shows the computed interface vacuum-plasma and flux-surfaces for a prescribed value of λ.

Remark 4.4 : The domain Ω is symmetrical with respect to the line $x_2 = 0$ but the solution is asymmetrical. As mentioned in [18] it is possible to obtain a symmetrical equilibrium by solving (with the same iteration scheme) problem (4.10) on the half domain with a Neumann condition on the boundary $x_2 = 0$.

5. - APPLICATION II : SHAPE OF A HEAVY ROTATING STRING.

An inelastic string of uniform cross-section, suspended with one end-point fixed, lies in a vertical plane rotating with angular velocity ω. The only forces are gravity forces and tension. Without loss of generality, we may suppose the length of the string to be 1. I.I. KOLODNER [10] showed that the horizontal displacement as a function of the arclength (2) is given by $\lambda^{-1}u'(s)$ where λ is the constant $\omega^2 g^{-1}$ (g denotes the acceleration of gravity) and $u \in H^1(0,1)$ is a solution of

$$(5.1)\quad \begin{cases} -u''(s) = \lambda \dfrac{u(s)}{\sqrt{s^2+u^2(s)}}, \quad s \in (0,1), \\ u(0) = 0 \;,\; u'(1)=0. \end{cases}$$

---

(1) In (4.10) $-\Delta u = \lambda H(u)$ a.e. is equivalent to $-\Delta u \in \lambda\beta(u)$ where β is the set-valued mapping $\beta(s) = \{0\}$ for $s < 0$, $\beta(0) = [0,1]$ and $\beta(s) = \{1\}$ for $s > 0$.

(2) The fixed end-point of the string corresponds to s=1.

We observe that $u \equiv 0$ is always a solution (vertical string) and that if u is a solution, so is $-u$. I.I. KOLODNER gives the exact number of non-trivial solutions. If $\mu_n < \lambda \leq \mu_{n+1}$, where $\{\mu_n\}$ denotes the sequence of eigenvalues of

$$(5.2) \qquad \begin{cases} -\phi''(s) = \mu \dfrac{\phi(s)}{s} , \\ \phi(0) = 0, \ \phi'(1) = 0, \end{cases}$$

then Eq. (5.1) has exactly n couples $(u,-u)$ of non-trivial solutions.

This problem was one of the motivations of the already quoted paper [24] by J.F. TOLAND. It we set, following J.F. TOLAND,

$$G(u) = \int_0^1 u'(s)^2 \, ds,$$

$$F(u) = \int_0^1 \sqrt{s^2+u(s)^2} ds \ ;$$

the solutions of (5.1) are the critical points of $J = G-F$ on the space $V = \{v \in H^1(0,1), \ v(0) = 0\}$.

One checks immediately that conditions (1.1)-(1.4) and (1.8)-(1.10) of Sec. 1 are satisfied, with $Z = \{0\}$.

Iteration scheme.

The starting point is some $u_o \in V$; once $u_1, \ldots, u_n$ have been obtained, we set $u_{n+1} \in V$ as the unique solution of

$$(5.3) \qquad \begin{cases} -u''_{n+1}(s) = \lambda \dfrac{u_n(s)}{\sqrt{s^2+u_n(s)^2}} , \ s \in (0,1), \\ u_{n+1}(0) = 0, \ u'_{n+1}(1) = 0. \end{cases}$$

Since the set of solutions of (5.1) is finite, we may apply Corollary 1.1 and we obtain

Theorem 5.1 : *For every fixed* $\lambda > 0$, *for every* $u_o \in V$, *the sequence* $\{u_n\}$ *obtained through the iteration scheme* (5.3) *converges strongly in* $H^1(0,1)$ *to some solution* u *of* (5.1).

Numerical results.

The problem is discretized by piecewise linear finite elements (40 elements). Fig. 3 shows the shape of the string for various values of λ.

Remark 5.1 :

i) For $0 < \lambda \leq \mu_1$ ($\mu_1 \simeq 1.446$), $u \equiv 0$ is the only solution of (5.1). For $\lambda > \mu_1$, we obtain a non-trivial solution using a non-trivial initialization.

ii) For any fixed $\lambda > \mu_1$, iteration scheme (5.3) provides very rapidly a solution of Eq. (5.3) which has the expected shape of a stable equilibrium (i.e. the string does not meet a second time the vertical axis of origin the fixed end-point). Other solutions (crossing the vertical axis) could be obtained by a continuation method.

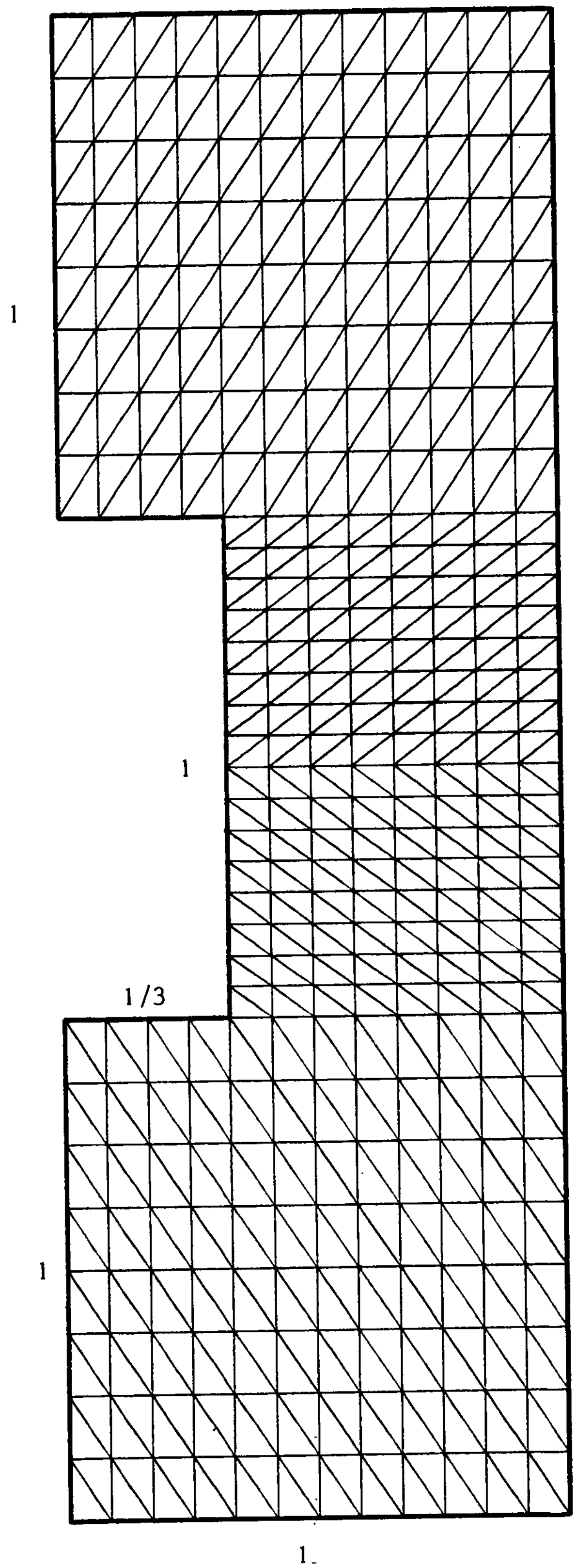

Fig. 1 - Triangulation : 640 triangles and 369 vertices.

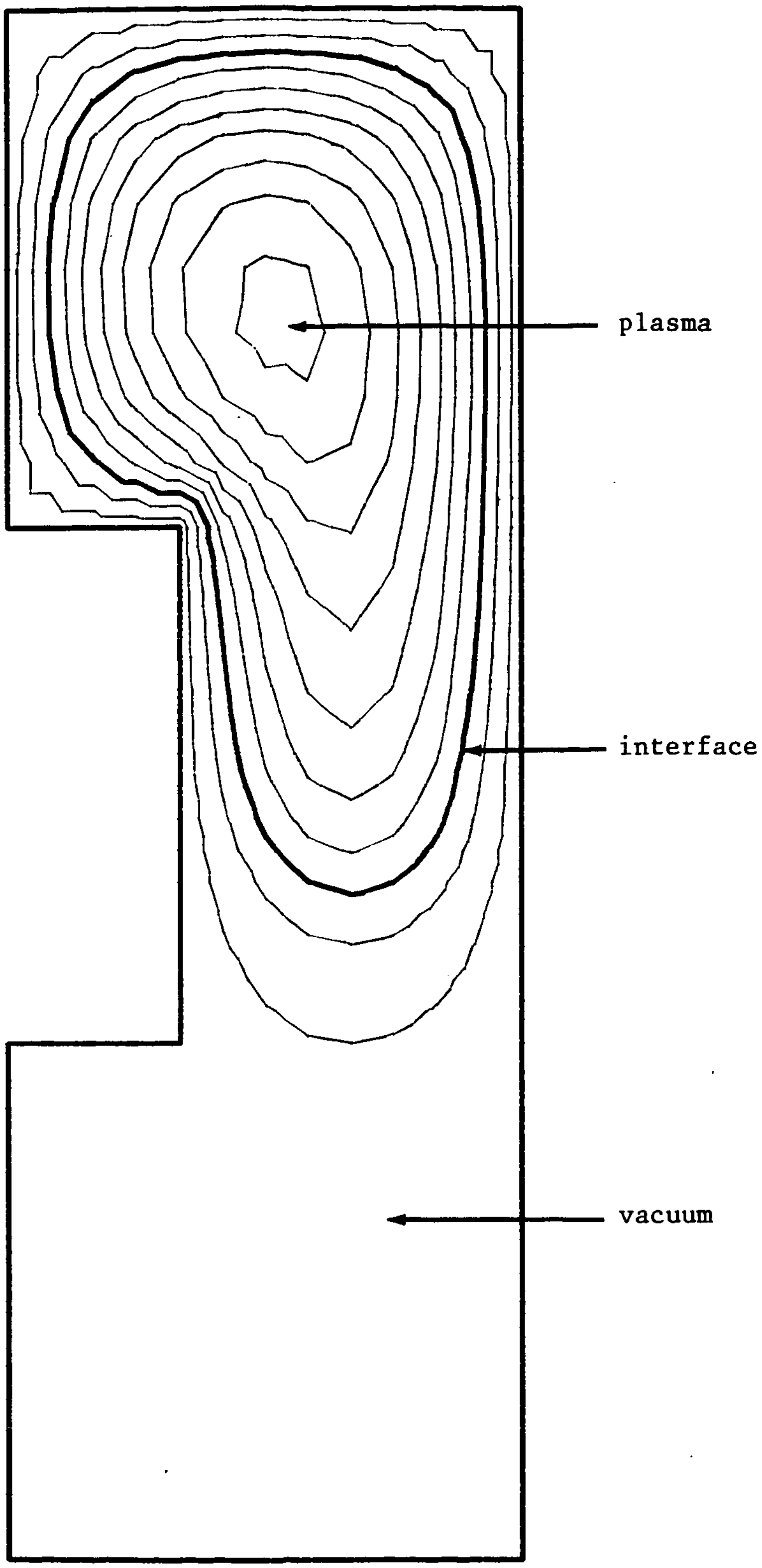

Fig. 2 - One of the solutions for $\lambda = I = 1$ ($|\Omega| = 2.666$)

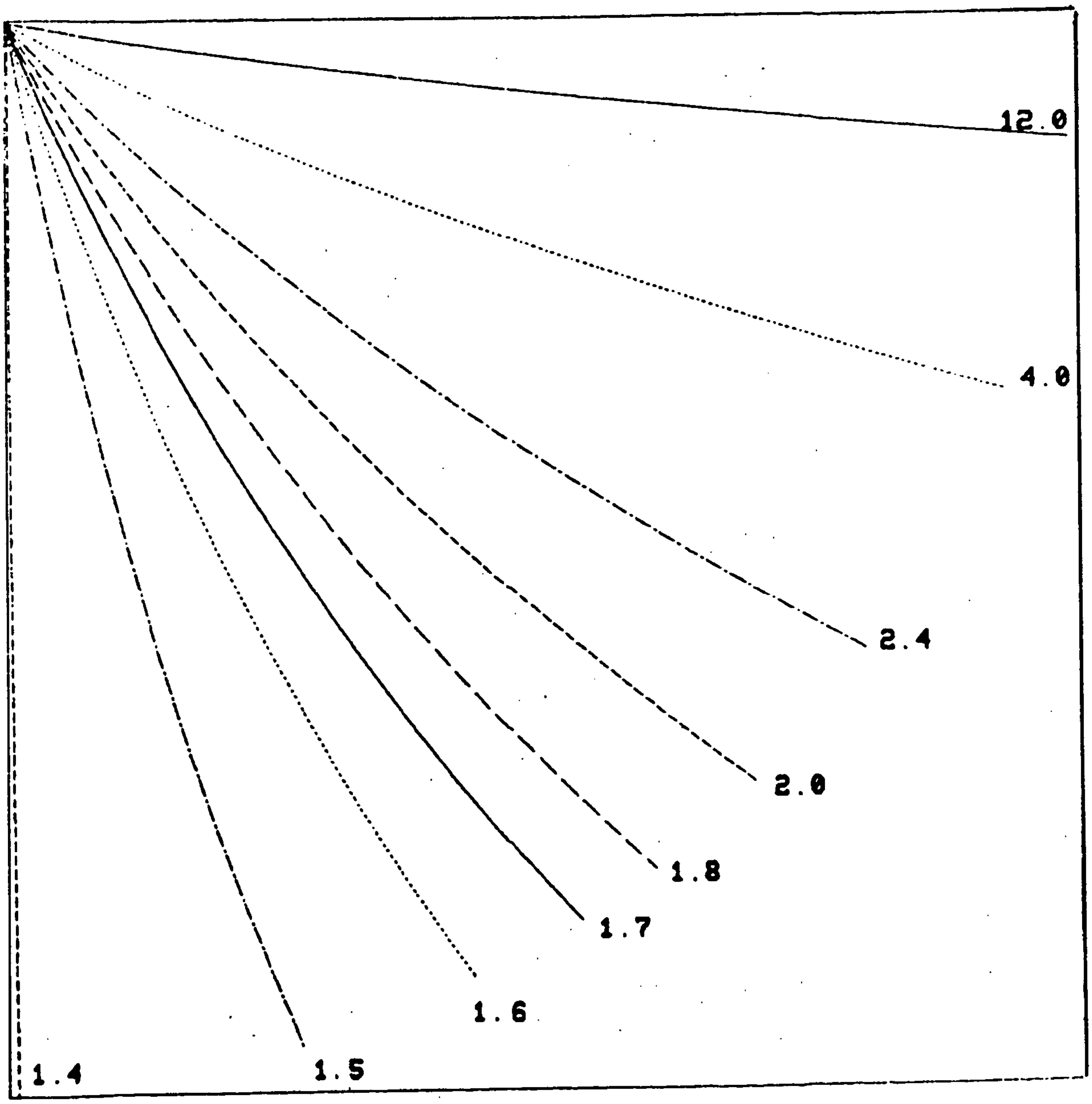

Fig. 3 - Shape of the string for various values of λ

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