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RESUME

On introduit dans ce rapport une méthode d'éléments finis simpliciaux pour la résolution de problèmes d'élasticité incompressible en dimension n, n=2 ou 3. Une structure asymétrique des fonctions-test par rapport au barycentre du simplex la rend particulièrement efficace pour le cas de grandes déformations dans lequel la condition d'incompressibilité est non linéaire.

On prouve en particulier que sous certaines conditions d'assemblage des éléments très peu restrictive on a toujours l'existence d'une solution aux problèmes approchés linéaire et non linéaire. Dans ce cas on établit aussi des résultats de convergence qui s'appliquent au cas de matériaux incompressibles linéaires.

ABSTRACT

This paper deals with a class of simplicial finite elements for solving incompressible elasticity problems in n dimensional space, n=2 or 3. An asymmetric structure of the shape functions with respect to the centroid of the simplex renders them particularly suitable for the large strain case, in which the incompressibility condition is nonlinear.

We prove that under certain assembling conditions of the elements, there exists a solution to both nonlinear and linear discrete problems, and in this case we also give convergence results for linear incompressible materials.

1. Introduction

In this work we introduce and discuss a new class of finite element methods for solving incompressible elasticity problems. We recall that in elasticity, incompressibility means that the measure of every part of an elastic body in any deformed state induced by loading conditions is invariant.

More precisely our problem, can be described as follows :

Ω being a bounded set of $\mathbb{R}^n$, for every open subset D of Ω, we shall denote by $\|\cdot\|_{m,r,D}$ and by $|\cdot|_{m,r,D}$ the usual norm and semi-norm respectively, of the Sobolev space $W^{m,r}(D)$ (see e.g.[1]), $m,r \in \mathbb{R}$, $m \geq 0$ and $1 \leq r \leq \infty$, with $W^{0,r}(D) = L^r(D)$. Similarly in the case $r = 2$ we denote by $(.,.)_{m,D}$ the usual inner product of $W_0^{m,2}(D) \equiv H_0^m(D)$ and by $|\cdot|_{m,D} = |\cdot|_{m,2,D}$ the corresponding norm, while we will represent the norm of $W^{m,2}(\Omega) = H^m(\Omega)$ by $\|\cdot\|_{m,D}$ instead of $\|\cdot\|_{m,2,D}$. In all cases we shall drop the subscript D whenever D is Ω itself.

For every space of functions V defined on D , $\underset{\sim}{V}$ will represent the space of vector fields whose n components belong to V. In the case where V is $W^{m,r}(D)$ or $W_0^{m,r}(D)$, we define the norm, semi-norm and inner product (if r=2) for $\underset{\sim}{V}$, by introducing obvious modifications in the scalar case, and keeping the same notations.

We shall denote by $x.y$ the euclidian inner product of two vectors x and y of $\mathbb{R}^\ell$ and by $|.|$ the corresponding norm. ℓ will be either equal to n in the case of vectors of $\mathbb{R}^n$, or equal to n^2 in the case of tensors of $\mathbb{R}^{n\times n}$.

Finally for every function or vector field y defined over a certain set D, we shall denote by $y/_S$ its restriction to a subset S , $S \subset D$.

Now we are given an elastic body represented by a bounded domain $\Omega \subset \mathbb{R}^n$, $n = 2,3$, with a smooth boundary Γ. Keeping fixed a part Γ_0 of Γ with $meas\ (\Gamma_0) \neq 0$, we consider a loading of Ω consisting of body forces acting on set $\Gamma^* \subset \Gamma$, such that $meas(\bar{\Gamma}_0 \cap \bar{\Gamma}^*) = 0$ and $\bar{\Gamma}^* \cup \bar{\Gamma}_0 = \Gamma$, having a density $\underset{\sim}{g}$ per unit of measure of Γ^* .

The effect of $\underset{\sim}{f}$ and $\underset{\sim}{g}$ is to deform Ω into an equilibrium configuration defined by a displacement vector field that we will denote by $\underset{\sim}{u}$. In this way, the new position of every point $\underset{\sim}{x}$ of Ω is given by $\underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x})$.

Now the fact that every element of Ω is measure invariant in its deformed state can be expressed mathematically by :

(1.1) $J[\underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x})] = 1$ for almost every $\underset{\sim}{x} \in \Omega$,
where $J[\underset{\sim}{v}(\underset{\sim}{x})]$ denotes the Jacobian of a vector field $\underset{\sim}{v}$ at point $\underset{\sim}{x}$.

(1.1) is called the incompressibility condition in finite elasticity and we shall often rewrite it as :

(1.1)' $det(\underset{\approx}{I} + \underset{\sim}{\nabla}\underset{\sim}{v}) = 1$ a.e. in Ω ,

where $\underset{\approx}{I}$ is the identity tensor $n \times n$ and $\underset{\sim}{\nabla}$ represents the gradient operator.

The incompressibility condition (1.1) is obviously nonlinear but in the case of small strains, that is to say, when

$$\max_{\underset{\sim}{x} \in \Omega} |\underset{\sim}{\nabla}\, \underset{\sim}{u}(\underset{\sim}{x})| << 1$$

one can reglect products of derivatives of $\underset{\sim}{u}$ of order higher than one.(1.1) becomes then the well-known linear incompressibility condition arising in infinitesimal elasticity or in fluid mechanics, namely :

(1.2) $div\, \underset{\sim}{u}(\underset{\sim}{x}) = 0$ for a.e. $\underset{\sim}{x} \in \Omega$.

Although there is a rather large range of incompressible materials, in this work we would like to focus our study to the case of Mooney-Rivlin materials, because they are particularly representative of the class of materials for which (1.1) holds. We note by the way that among Mooney-Rivlin materials rubber is a typical case.

For a Mooney-Rivlin material the elastic energy for a certain admissible displacement vector field $\underset{\sim}{v}$ is given by [20] :

$$(1.3)_2 \quad \tilde{W}(\underset{\sim}{v})=\frac{C_1}{2}\left[\int_\Omega |\underset{\approx}{I} + \underset{\sim}{\nabla}\, \underset{\sim}{v}|^2 d\underset{\sim}{x} - 2\right] - \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, d\underset{\sim}{x} - \int_{\Gamma^*} \underset{\sim}{g} \cdot \underset{\sim}{v}\, ds \quad \text{for } n = 2$$

$$(1.3)_3 \quad \tilde{W}(\underset{\sim}{v})=\frac{C_1}{2}\left[\int_\Omega |\underset{\approx}{I} + \underset{\sim}{\nabla}\, \underset{\sim}{v}|^2 d\underset{\sim}{x} - 2\right] + \frac{C_2}{2}\left[\int_\Omega |adj(\underset{\approx}{I}+\underset{\sim}{\nabla}\underset{\sim}{v})|^2 d\underset{\sim}{x}-3\right] - \int_\Omega \underset{\sim}{f}\cdot \underset{\sim}{v}\, d\underset{\sim}{x} - \int_{\Gamma^*} \underset{\sim}{g}\cdot \underset{\sim}{v}\, ds \quad \text{for } n = 3$$

where $adj\ A$ denotes the transpose of the matrix of cofactors of an $n\times n$ matrix A and C_1 and C_2 are positive physical constants.

Taking into account (1.1) and the fact that $\tilde{W}$ must be finite, it is natural to choose the following set of admissible displacement vector fields :

$$\tilde{X} = \{\underset{\sim}{v}/\underset{\sim}{v} \in \underset{\sim}{W}^{1,r}(\Omega) \ ,\ \underset{\sim}{v}_{/\Gamma_0} = \underset{\sim}{0} \ ,\ det[\underset{\approx}{I} + \underset{\sim}{\nabla}\,\underset{\sim}{v}(\underset{\sim}{x})] = 1 \quad \text{a.e. in } \Omega\}$$

with $r \geq 2(n-1)$,whereas we shall assume that $\underset{\sim}{f} \in L^2(\Omega)$ and $\underset{\sim}{g} \in H^{1/2}(\Gamma^*)$.

The problem we want to solve can now be stated as follows :

$$(\tilde{P}) \quad \begin{cases} \text{Find } \underset{\sim}{u} \in \tilde{X} \text{ such that} \\ \tilde{W}(\underset{\sim}{u}) \leq \tilde{W}(\underset{\sim}{v}) \qquad \forall\ \underset{\sim}{v} \in \tilde{X} \end{cases}$$

It is interesting to note that $\tilde{X}$ is a non convex set and that it is a subset of the vector space $\underset{\sim}{V}^r$ defined by :

$$\underset{\sim}{V}^r = \{\underset{\sim}{v}/\underset{\sim}{v} \in \underset{\sim}{W}^{1,r}(\Omega) \ ,\ \underset{\sim}{v}_{/\Gamma_0} = \underset{\sim}{0}\}$$

which can be normed by the semi-norm $|\cdot|_{1,r}$ (Ω being connected [17]).

It will be useful in the sequel to consider a "linearized" finite incompressible model that corresponds to a small strain deformed state. In this case, taking into account that every displacement vector field must satisfy (1.2),the usual expression of the energy for linear elastic behavior in terms of the tensor of strains $\underset{\approx}{\varepsilon}$, is given by :

$$(1.4) \qquad \hat{W}(\underset{\sim}{v}) = \frac{C}{2}\int_{\Omega} \left|\underset{\approx}{\varepsilon}(\underset{\sim}{v})\right|^2 d\underset{\sim}{x} - \int_{\Omega} \underset{\sim}{f}\cdot\underset{\sim}{v}\, d\underset{\sim}{x} - \int_{\Gamma^*} \underset{\sim}{g}\cdot\underset{\sim}{v}\, ds$$

where $\varepsilon_{ij}(\underset{\sim}{v}) = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)$ and C depends on C_1 and C_2 .

The set of admissible displacement vector fields is then the linear vector space :

$$\hat{X} = \{\underset{\sim}{v} \;/\; \underset{\sim}{v} \in \underset{\sim}{V} \;,\; div\, \underset{\sim}{v} = 0 \quad \text{a.e. in } \Omega\}$$

where $\underset{\sim}{V} = \underset{\sim}{V}^2$.

The problem to solve for this linearized model would then be :

$$(\hat{P}) \quad \begin{cases} \text{Find } \underset{\sim}{u} \in \hat{X} \quad \text{such that} \\ \hat{W}(\underset{\sim}{u}) \leq \hat{W}(\underset{\sim}{v}) \quad \forall\, \underset{\sim}{v} \in \hat{X} \end{cases}$$

Although the above linearization appears to be rather unrealistic in practice as far as Mooney-Rivlin materials are concerned, it is not only fundamental for the convergence and existence analysis that will be given later on, but it will allow us to treat the case of many incompressible materials for which (1.2) does hold. Furthermore it will enable us to treat implicitly the case of viscous fluids, by simply considering null surface forces, together with $\Gamma^* = \phi$.

Now, the coercivity of $\hat{W}$ over $\hat{X}$ equipped with the $|\cdot|_1$ norm [6] implies the existence and uniqueness of a solution $\underset{\sim}{u}$ to $(\hat{P})$. However, due to the well-known difficulties in defining a well-posed discrete finite element analogue of $(\hat{P})$, in this work we shall be concerned with a mixed formulation of this problem obtained in the classical way, by dualization of (1.2). This means that we introduce a Lagrange multiplier, that is nothing else than a hydrostatic pressure p.

The saddle-point $(\underset{\sim}{u},p)$ arising from this dualization is characterized as the solution to the following problem :

$$(\acute{P}') \begin{cases} \text{Find } (\underset{\sim}{u},p) \in \underset{\sim}{V} \times Q \quad \text{such that} \\ \acute{a}(\underset{\sim}{u},\underset{\sim}{v}) + \acute{b}(\underset{\sim}{v},p) = \acute{L}(\underset{\sim}{v}) & \forall\ \underset{\sim}{v} \in \underset{\sim}{V} \\ \acute{b}(\underset{\sim}{u},q) = 0 & \forall\ q \in Q \end{cases}$$

where $\acute{a}$, $\acute{b}$ and $\acute{L}$ are given by :

$$(1.5) \qquad \acute{a}\,(\underset{\sim}{u},\underset{\sim}{v}) = C \sum_{i,j=1}^{n} \int_\Omega \varepsilon_{ij}(\underset{\sim}{u})\ \varepsilon_{ij}(\underset{\sim}{v})\ d\underset{\sim}{x}$$

$$(1.6) \qquad \acute{b}(\underset{\sim}{v},q) = \int_\Omega q\ div\ \underset{\sim}{v}\ d\underset{\sim}{x}$$

$$(1.7) \qquad \acute{L}(\underset{\sim}{v}) = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\ d\underset{\sim}{x} - \int_{\Gamma^*} \underset{\sim}{g} \cdot \underset{\sim}{v}\ ds$$

and $Q = L^2(\Omega)$ in case $meas\ (\Gamma^*) \neq 0$, and $Q = L^2_0(\Omega)$ otherwise, the latter being the subspace of $L^2(\Omega)$ of functions q such that $(q,1)_0 = 0$.

It is trivial to prove that the bilinear form $\acute{a}$, obtained by differentiation of the quadratic part of $\acute{W}$ along $\underset{\sim}{v}$ over $\underset{\sim}{V}$, satisfies the following inequalities :

$$(1.8) \qquad \acute{a}(\underset{\sim}{u},\underset{\sim}{v}) \leq \|\acute{a}\|\ |\underset{\sim}{u}|_1\ |\underset{\sim}{v}|_1 \quad \text{with} \quad \|\acute{a}\| = 1$$

$$(1.9) \qquad \acute{a}\ (\underset{\sim}{v},\underset{\sim}{v}) \geq \alpha |\underset{\sim}{v}|_1^2$$

where $\alpha > 0$ is the constant of coercivity of $\acute{W}$ (see e.g. [6])

Therefore, according to [7] , we have existence and uniqueness of (u,p), $\underset{\sim}{u}$ being the solution of $(\acute{P})$.

Now, if we assume that $\underset{\sim}{f}$, $\underset{\sim}{g}$ and $\underset{\sim}{\Omega}$ are regular enough, we can say that $(\underset{\sim}{u},p)$ satisfies the following equations :

$$(1.10) \qquad \begin{cases} - C\ \Delta\underset{\sim}{u} - \underset{\sim}{\nabla} p = \underset{\sim}{f} & \text{in } \Omega \\ div\ \underset{\sim}{u} = 0 & \text{in } \Omega \\ C\ \underset{\approx}{\varepsilon}(\underset{\sim}{u}) \cdot \underset{\sim}{n} + p\ \underset{\sim}{n} = \underset{\sim}{g} & \text{on } \Gamma^* \\ \underset{\sim}{u} = \underset{\sim}{0} & \text{on } \Gamma_0 \end{cases}$$

where $\underset{\sim}{n}$ denotes the outer unit normal vector with respect to Γ^* , in the sense of suitable function spaces about which we are going to be more specific later on .

As for problem $(\tilde{P})$, similarly to the case of (P), we will consider the following weak formulation obtained by dualization of (1.1)' with the help of a multiplier p , and by differentiation of $\tilde{W}(\underset{\sim}{u})$ along $\underset{\sim}{v}$ over $\underset{\sim}{V}^r$.

$$(\tilde{P}')\quad \begin{cases} \text{Find } (\underset{\sim}{u},p)\in \underset{\sim}{V}^r\times Q^t \text{ such that} \\ \tilde{a}(\underset{\sim}{u},\underset{\sim}{v}) + \tilde{b}(\underset{\sim}{u},\underset{\sim}{v},p) = \tilde{L}(\underset{\sim}{v}) & \forall\ \underset{\sim}{v}\in \underset{\sim}{V}^r \\ \tilde{b}(\underset{\sim}{u},q) = 0 & \forall\ q\in Q^t \end{cases}$$

where $Q^t = L^t(\Omega)$, with t such that $n/r + 1/t \le 1$, and

$$(1.11)\quad \tilde{a}(\underset{\sim}{u},\underset{\sim}{v}) = C_1\int_\Omega \nabla\underset{\sim}{u}\cdot\nabla\underset{\sim}{v}\,d\underset{\sim}{x} + C'\int_\Omega adj(\underset{\approx}{I} + \underset{\sim}{\nabla}\underset{\sim}{w})\cdot[adj(\underset{\approx}{I} + \underset{\sim}{\nabla}\underset{\sim}{u} + \underset{\sim}{\nabla}\underset{\sim}{v}) - adj\,\underset{\sim}{\nabla}\,\underset{\sim}{v}]\,d\underset{\sim}{x} \quad \text{with } C'=0 \text{ if } n=2 \text{ and } C'=C_2 \text{ if } n=3,$$

$$(1.12)\quad \tilde{b}'(\underset{\sim}{u},\underset{\sim}{v},q) = \int_\Omega q[adj(\underset{\approx}{I}+\underset{\sim}{\nabla}\underset{\sim}{u})^T\cdot\ \underset{\sim}{\nabla}\,\underset{\sim}{v}]d\underset{\sim}{x}$$

$$(1.13)\quad \tilde{b}(\underset{\sim}{v},q) = \int_\Omega q[det(\underset{\approx}{I} + \underset{\sim}{\nabla}\,\underset{\sim}{v}) - 1]\,d\underset{\sim}{x}$$

$$(1.14)\quad \tilde{L}(\underset{\sim}{v}) = \int_\Omega \underset{\sim}{f}\cdot\underset{\sim}{v}\,d\underset{\sim}{x} + \int_{\Gamma^*}\underset{\sim}{g}\cdot\underset{\sim}{v}\,ds - C_1\int_\Omega div\,\underset{\sim}{v}\,d\underset{\sim}{x}$$

REMARK : For the above problem we will not consider the case $meas(\Gamma^*) = 0$. □

According to results by Le Tallec [16], under reasonable assumptions, there exists a hydrostatic pressure p, with $p\in L^t(\Omega)$, associated with every solution $\underset{\sim}{u}$ to problem $(\tilde{P})$, and in this case $(\underset{\sim}{u},p)$ is a solution to $(\tilde{P}')$.

At this stage we would like to point out that in practice, it seems unwise to use formulation $(\tilde{P}')$ for numerical computations with mixed finite elements, such as those we are going to treat here. Indeed, there are other mixed formulations of $(\tilde{P})$ much more suitable for such a purpose and in this respect we refer to [9] , for instance. However, for the sake of clearness, we prefer to consider $(\tilde{P}')$ in this work, as it appears to be the most natural formulation of all.

Bearing in mind that our mixed finite element methods apply to other mixed formulations of $(\acute{P})$ and $(\tilde{P})$as well, we shall from now on, consider that we are actually going to approximate problems $(\acute{P}')$ and $(\tilde{P}')$. For this purpose we will define two finite dimensional spaces $\underset{\sim}{V}_h$ and Q_h aimed at approximating $\underset{\sim}{V}$(resp. $\underset{\sim}{V}^{t}$) and Q (resp. Q^{t}), associated with two n-simplicial finite elements for $n = 2$ and $n = 3$, respectively. We note that the three-dimensional element can be viewed as a certain generalization of the two-dimensional one. It should be mentioned that the latter was first introduced in [21] and discussed in more details in [22]. Nevertheless, we intend to reconsider in this work some aspects of the two-dimensional case, in order to be able to give a systematic analysis applying to both elements in a fairly analogous way.

An outline of the paper is as follows :

In section 2 we introduce the two displacement-pressure finite elements of asymmetric type. In the same section we recall some abstract approximatio results derived in [23], for linear problems discretized with mixed finite elements that are nonconforming in the first variable, as the three-dimensional element is nonconforming in displacements.

In section 3 we examine in detail some basic properties of both elements that justify a priori their adequacy for the numerical solution of problem $(\tilde{P})$.

In section 4 we give a convergence analysis for problem $(\acute{P}')$ and finally in section 5 we give existence results related to the approximation of $(\tilde{P}')$.

2. Definition of the elements

In this section, together with the two subsequent ones, we consider Ω to be a domain of $\mathbb{R}^n$, $n = 2,3$, having a polyhedral boundary Γ . For the case $n=3$ we also assume that $\bar{\Gamma}^* \cap \bar{\Gamma}_0$ is a set of spacial polygonal lines.

We are given a family $(\tau_h)_h$ of partitions of Ω into n-simplices, satisfying, besides the classical assembling rules for the finite element method, some additional compatibility conditions for our asymmetric elements to be specified hereafter. We also assume that Γ^* and Γ_0 can be viewed as the union of faces of elements of τ_h and that $(\tau_h)_h$ is regular in the following sense :

Denoting by h_K the diameter of the circumscribed sphere and by ρ_K the diameter of the inscribed sphere of element K , $K \in \tau_h$ and setting

$$h = \max_{K \in \tau_h} h_K \qquad \text{and} \qquad \rho = \min_{K \in \tau_h} \rho_K \ ,$$

there exists a strictly positive constant c such that $\rho h^{-1} > c \quad \forall h$.

With each partition τ_h we associate the finite dimensional spaces Q_h and $\underset{\sim}{V}_h$, approximations of Q and $\underset{\sim}{V}$(resp. Q^t and $\underset{\sim}{V}^t$) respectively. We define Q_h to be the space of functions q_h that are constant over each element of τ_h , such that $\int_\Omega q_h \, d\underset{\sim}{x} = 0$ if $\Gamma^* = \emptyset$, and we clearly have $Q_h \subset Q$(resp. $Q_h \subset Q^t$). For convenience we consider the degrees of freedom of Q_h to be the functional values at the centroid G of the elements. V_h in turn consists of functions whose restriction to each simplex $K \in \tau_h$ belongs to a space P_a defined as follows :

Let S_i denote the vertices of a simplex $K \subset \tau_h$, $i = 1,2,\ldots,n+1$. We first assign to K a priviledged face, say the face opposite to vertex S_{n+1} , that will be called the basis B^K of K , and let F_i^K be the face opposite to vertex S_i , $i = 1,2,\ldots,n$. The F_i^K's will be called the lateral faces of K :

Let λ_i denote the area coordinate of K associated with vertex S_i , $i = 1,2,\ldots,n+1$ and S_{n+2} denote the centroid of B^K .

Now we define P_a to be the $(n+2)$-dimensional space spanned by the functions $\lambda_i = i = 1,2,\ldots,n+1$ and φ , where :

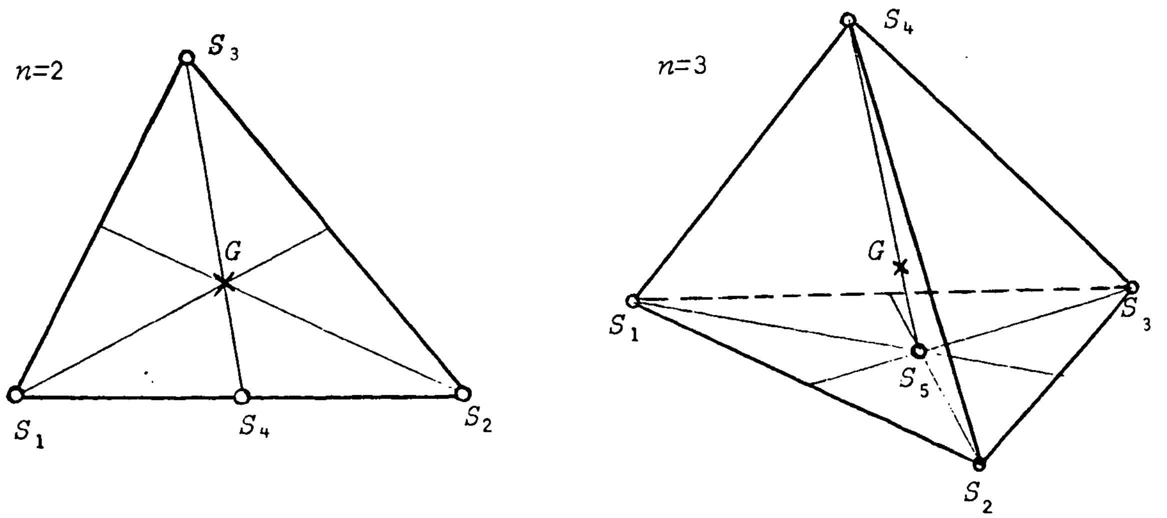
(2.1) $$\varphi = \sum_{\substack{j,k=1\\ j<k}}^{n} \lambda_j\lambda_k$$

One can easily verify that the set of degrees of freedom $\{a_i\}_{i=1}^{n+2}$, where a_i is the value of the function at point S_i , is P_a-unisolvent and that the associated basis functions are given by :

(2.2) $$\begin{cases} p_i = \lambda_i - \frac{2}{n-1}\varphi & i = 1,2,\ldots,n \\ p_{n+1} = \lambda_{n+1} & \\ p_{n+2} = \frac{2n}{n-1}\varphi & \end{cases}$$

In figure 2.1. we illustrate the so-defined asymmetric finite elements where o represents degrees of freedom for $\underset{\sim}{V}_h$ and × represents those for Q_h.

Note that the following inclusions hold : $P_1 \subset P_a \subset P_2$, where P_k denotes the space of polynomials of degree less or equal to k defined over K(*).



The asymmetric quasilinear elements

Figure 2.1.

(*) Like in [22] , a can be viewed as a rational subscript. In the present case we have $a = (C_{n+1}^2 + 1)/C_{n+1}^2$.

It should be clear that, due to the asymmetric structure of P_a , one cannot expect to associate those elements with arbitrary partitions of Ω. Indeed τ_h must respect the following compatibility condition :

Every face of τ_h belonging to two neighboring simplices must be either a basis or a lateral face for both simplices.

As a matter of fact to obtain such a partition, which will be called a *compatible partition*, it suffices to assign a priori, faces that are going to be bases, in a systematic way. Notice that in practice this does not represent a real restriction in programming with these elements. In order to illustrate our assertion we propose below two simple constructions of compatible partitions of an arbitrary domain :

Partition τ_h^1 : In the two-dimensional case we first construct a partition of Ω into artibrary convex quadrilaterals (like in the case of the bilinear Q_1 element). Next, every quadrilateral is subdivided into two triangles by an arbitrarily chosen diagonal. Those diagonals will be the only basis of the elements of the so-generated triangulation.

In the three-dimensional case we first construct a partition of Ω into arbitrary convex hexahedrons having quadrilateral faces . Now we refer to figure 2.2b where we show a classical subdivision of a hexahedron into 6 tetrahedrons that are precisely elements of τ_h^1 . The bases of this partition are faces *BDA'* , *BDC'* and *DBD'* of each hexahedron . Note that, like in the two-dimensional case, the diagonal *B'D* of the hexahedron, which determines its partition into 6 tetrahedrons, can be chosen in an entirely arbitrary way.

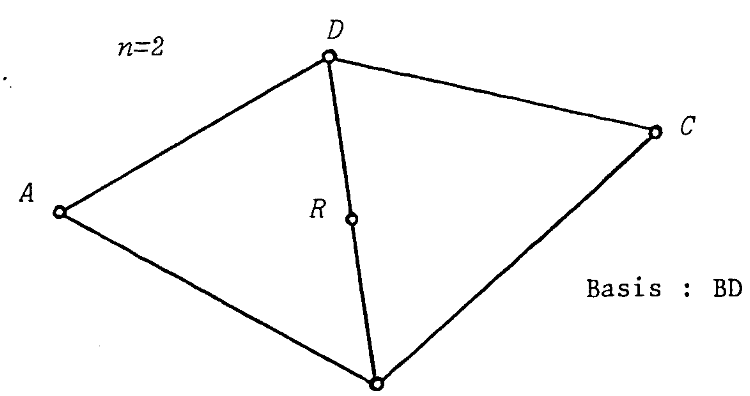


Figure 2.2a

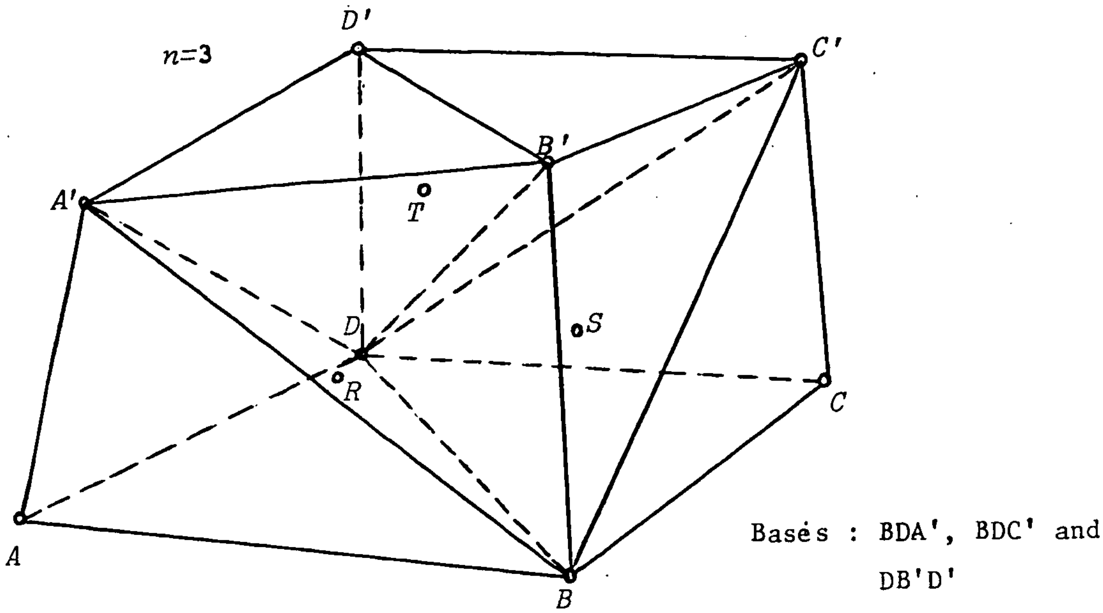


Fig.2.2b

An illustration of the compatible partition τ_h^1
Figure 2.2.

Partition τ_h^2 : We first construct an arbitrary partition τ_h of Ω into n-simplices K. Then we subdivide each $K \in \tau_h$ into $n+1$ simplices having a common vertex situated in $\mathring{K}$.

This subpartition of τ_h becomes the compatible partition τ_h^2 if we define its bases to be the faces of τ_h . Note that the interior point of the simplex $K \in \tau_h$ can be arbitrary, although in this work we will choose it to be the centroid G(see figure 4.1).

With the above considerations, we define the degrees of freedom of V_h to be the functional values at the vertices and at the centroid of the bases of a compatible partition τ_h of Ω , except the values at those nodes lying on $\bar{\Gamma}_0$, where a function of V_h vanishes necessarily.

With the above definition of V_h we can say that $\underset{\sim}{V}_h \subset \underset{\sim}{V}$ if $n=2$. However, if $n=3$ this inclusion does not hold and therefore we have a nonconforming element. Indeed, in this case a function of V_h is necessarily continuous only along the bases of the partition, as it can be easily verified.

Now as far as problem $(\acute{P}')$ is concerned, since we cannot define an approximate problem in the case $n=3$, by simply replacing $\underset{\sim}{V}$, Q , $\underset{\sim}{u}$ and p by $\underset{\sim}{V}_h$, Q_h , $\underset{\sim}{u}_h$ and p_h as usual, we need approximations of $\hat{a}$ and $\hat{b}$ as follows :

$$(2.3) \qquad \hat{a}_h(\underset{\sim}{u}_h , \underset{\sim}{v}_h) = \sum_{K\in \tau_h} C \int_K \underset{\approx}{\varepsilon}(\underset{\sim}{u}_h) \cdot \underset{\approx}{\varepsilon}(\underset{\sim}{v}_h)\, d\underset{\sim}{x} , \qquad \underset{\sim}{u}_h , \underset{\sim}{v}_h \in \underset{\sim}{V}_h$$

$$(2.4) \qquad \hat{b}_h(\underset{\sim}{v}_h , q_h) = \sum_{K\in \tau_h} \int_K q_h \, div\, \underset{\sim}{v}_h \, d\underset{\sim}{x} \qquad \underset{\sim}{v}_h \in \underset{\sim}{V}_h \text{ and } q_h \in Q_h$$

Note that $\hat{a}_h$ and $\hat{b}_h$ are also defined over $\underset{\sim}{V} \times \underset{\sim}{V}$ and $\underset{\sim}{V} \times Q$, and that whenever $\underset{\sim}{u}, \underset{\sim}{v} \in \underset{\sim}{V}$ we have $\hat{a}_h(\underset{\sim}{u},\underset{\sim}{v}) = \hat{a}(\underset{\sim}{u},\underset{\sim}{v})$ and $\hat{b}_h(\underset{\sim}{v},q) = \hat{b}(\underset{\sim}{v},q)$, $q \in Q$.

We will need discrete H^1_α -inner product and norm defined as follows :

$$(2.5) \qquad (\underset{\sim}{u},\underset{\sim}{v})_h = \sum_{K\in \tau_h} (\underset{\sim}{u},\underset{\sim}{v})_{1,K} \quad \text{with} \quad \|\underset{\sim}{v}\|_h = (\underset{\sim}{v},\underset{\sim}{v})_h^{1/2} .$$

The fact that the degrees of freedom attached to Γ_0 vanish, together with the continuity of the functions at the vertices of τ_h , imply that $\|.\|_h$ is actually a norm for $\underset{\sim}{V}_h$. Note also that $(\underset{\sim}{u},\underset{\sim}{v})_h = (\underset{\sim}{u},\underset{\sim}{v})_1 \; \forall \underset{\sim}{u},\underset{\sim}{v} \in \underset{\sim}{V}$.

It is easy to verify that $\hat{a}_h$ satisfies inequalities analogous to (1.8) and (1.9) with the norm $|\cdot|_1$ replaced by $\|.\|_h$, and that $\hat{b}_h$ satisfies :

$$\hat{b}_h(\underset{\sim}{v}_h,q_h) \le \|\hat{b}\| \; \|\underset{\sim}{v}_h\|_h |q_h|_0 \qquad \text{with} \quad \|\hat{b}\| = 2 .$$

Thefore, if we are able to prove that $\hat{b}_h$ also satisfies the so-called discrete Brezzi condition [4] , namely, the existence of a strictly positive constant β_h such that :

$$(2.6) \qquad \sup_{\underset{\sim}{v}_h \in \underset{\sim}{V}_h} \frac{b_h(\underset{\sim}{v}_h,q_h)}{\|\underset{\sim}{v}_h\|_h} \ge \beta_h |q_h|_0 \qquad \forall q_h \in Q_h ,$$

we have existence of a solution to the approximate problem :

$$(\hat{P}_h) \quad \begin{cases} \text{Find } (\underset{\sim}{u}_h , p_h) \in \underset{\sim}{V}_h \times Q_h \quad \text{such that} \\ \hat{a}_h(\underset{\sim}{u}_h , \underset{\sim}{v}_h) + \hat{b}_h(\underset{\sim}{v}_h , p_h) = \hat{L}(\underset{\sim}{v}_h) \qquad \forall \underset{\sim}{v}_h \in \underset{\sim}{V}_h \\ \hat{b}_h(\underset{\sim}{u}_h , q_h) = 0 \qquad \forall q_h \in Q_h . \end{cases}$$

In this case, we can also apply the following estimates given in [23] .

$$(2.7)\quad \|\underset{\sim}{u} - \underset{\sim}{u}_h\|_h \leq \left(1 + \frac{\|\hat{a}\|}{\alpha}\right)\left(1 + \frac{\|\hat{b}\|}{\beta_h}\right) \inf_{\underset{\sim}{v}_h \in \underset{\sim}{V}_h} \|\underset{\sim}{u} - \underset{\sim}{v}_h\|_h + \frac{\|\hat{b}\|}{\alpha} \inf_{q_h \in Q_h} |p - q_h|_0 + \frac{1}{\alpha} \sup_{\underset{\sim}{w}_h \in \underset{\sim}{V}_h} \frac{|E_h(\underset{\sim}{u}, p, \underset{\sim}{w}_h)|}{\|\underset{\sim}{w}_h\|_h}$$

and

$$(2.8)\quad |p - p_h|_0 \leq \frac{\|\hat{a}\|}{\beta_h} \|\underset{\sim}{u} - \underset{\sim}{u}_h\|_h + \left(1 + \frac{\|\hat{b}\|}{\beta_h}\right) \inf_{q_h \in Q_h} |p - q_h|_0 + \frac{1}{\beta_h} \sup_{\underset{\sim}{w}_h \in \underset{\sim}{V}_h} \frac{|E_h(\underset{\sim}{u}, p, \underset{\sim}{w}_h)|}{\|\underset{\sim}{w}_h\|}$$

where $E_h : \underset{\sim}{V} \times Q \times \underset{\sim}{V}_h \rightarrow R$ is given by

$$(2.9)\quad E_h(\underset{\sim}{u}, p, \underset{\sim}{w}_h) = \hat{L}(\underset{\sim}{w}_h) - \hat{a}_h(\underset{\sim}{u}, \underset{\sim}{w}_h) - \hat{b}_h(\underset{\sim}{w}_h, p) \quad .$$

REMARK : $E_h(\underset{\sim}{u}, p, \underset{\sim}{w}_h)$ vanishes identically whenever $\underset{\sim}{V}_h \subset \underset{\sim}{V}$. ■

As for problem $(\tilde{P})$ we will consider an approximate problem defined in a similar way in section 5 .

3. Some properties of the asymmetric elements related to the nonlinear case.

In this section we intend to justify our proposal of the elements of asymmetric type for the numerical solution of problem $(\tilde{P}')$.

First of all let us briefly recall some a priori arguments already considered in [21] and [22] .

If a vector field of an approximation space $\underset{\sim}{V}_h$ of $\underset{\sim}{V}^h$ is such that each component restricted to an element K of τ_h is a polynomial of P_k , its Jacobian is a polynomial of $P_{n(k-1)}$ over K. This implies that one must satisfy constraint (1.1) in a relatively large number of points of K in order to represent the incompressibility condition within a satisfactory degree of accuracy. Note that this question becomes particularly critical in the three-dimensional case. However, the number of constraints to be satisfied in the discrete problem associated to $(\tilde{P}')$ - which is precisely $dim\ Q_h$ - should not exceed the total number of displacement degrees of freedom, i.e. $dim\ \underset{\sim}{V}_h$, otherwise we would be dealing with an ill-posed problem. This condition is usually expressed numerically by requiring that the following asymptotic ratio :

$$\theta = \lim_{h\to 0} \frac{dim\ Q_h}{dim\ \underset{\sim}{V}_h}$$

be strictly less than one (actually in practice θ should not be too close to one).

On the other hand, from a mathematical point of view, it is not appropriate to choose a space Q_h satisfying continuity requirements at points situated on the interface of the elements. This fact prevents one from reducing the dimension of Q_h significantly like in the case of linear problems solved with the so-called Taylor-Hood elements [11] .

Let us also add that $\underset{\sim}{V}_h$ should be preferably conforming. Indeed, even if condition (1.1) is properly satisfied elementwise, the nonconformity may lead to a meaningless representation of the incompressibility phenomenon at the global level, unless one can prove that the resulting interpenetrations of neighboring deformed elements cancel each other or are negligible.

Summing up all the above considerations, we can say that, except for a very few cases, one cannot expect to approximate succesfully problem $(\tilde{P}')$ by using standard spaces $\underset{\sim}{V}_h$ and Q_h, such as those that work well for fluid problems or for linear incompressible elasticity. Therefore, a solution that seems reasonable, is to construct $\underset{\sim}{V}_h$ by means of spaces of special polynomials of degree k, for which the Jacobian is of maximal degree significantly less than $n(k-1)$. As we show hereafter this is precisely the case of $\underset{\sim}{P}_a$.

Theorem 3.1. If $\underset{\sim}{v} = (v_1,\ldots,v_n)$ defined over K is such that $v_i \in P_a$ $\forall i$, then $J[\underset{\sim}{x} + \underset{\sim}{v}(\underset{\sim}{x})]$ is a polynomial of P_1.

Proof : According to (2.2), each component v_i can be written as :

$$v_i = \sum_{j=1}^{n+1} \alpha_j^i \lambda_j + \beta^i \varphi$$

where the α_j^i's and the β^i's are scalars and φ is the quadratic function given by (2.1). We have :

$$(3.1)\quad J[\underset{\sim}{x} + \underset{\sim}{v}(\underset{\sim}{x})] = \begin{vmatrix} c_{11} + \beta^1 \frac{\partial\varphi}{\partial x_1} & c_{12} + \beta^1 \frac{\partial\varphi}{\partial x_2} & \cdots & c_{1n} + \beta^1 \frac{\partial\varphi}{\partial x_n} \\ c_{21} + \beta^2 \frac{\partial\varphi}{\partial x_1} & c_{22} + \beta^2 \frac{\partial\varphi}{\partial x_2} & \cdots & c_{2n} + \beta^2 \frac{\partial\varphi}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ c_{n1} + \beta^n \frac{\partial\varphi}{\partial x_1} & c_{n2} + \beta^n \frac{\partial\varphi}{\partial x_2} & \cdots & c_{nn} + \beta^n \frac{\partial\varphi}{\partial x_n} \end{vmatrix}$$

where constant c_{ij} is the x_j - derivative of the linear part of $x_i + v_i(\underset{\sim}{x})$.

Now we expand the above determinant into a sum of 2^n determinants whose j-th column is either $(c_{1j}, c_{2j}, \ldots, c_{nj})^T$ or $\frac{\partial\varphi}{\partial x_j}(\beta^1, \beta^2 \ldots, \beta^n)^T$. As one can easily see, the only determinants of this expansion that do not vanish identically are those having at most one column with linear entries $\frac{\partial\varphi}{\partial x_j} \underset{\sim}{\beta}^T$, and the results follows. q.e.d.

An immediate consequence of Theorem 3.1, is the fact that it suffices to satisfy (1.1) at the centroid G of K to have incompressible elements in the following weak sense :

The measure of K in deformed state induced by $\underset{\sim}{u} \in \underset{\sim}{P}_a$ is invariant.

Indeed, if we denote by $\tilde{A}$ the deformed state induced by $\underset{\sim}{u}$ of any subset A of K , $K \in \tau_h$, according to a well-known numerical quadrature formula, we have :

$$meas(\tilde{K}) = \int_K J[x + \underset{\sim}{u}(\underset{\sim}{x})] \, d\underset{\sim}{x} = J[G + \underset{\sim}{u}(G)] \; meas(K) = meas(K) \; .$$

This shows that the space Q_h defined in section 2 is a proper choice for these asymmetric elements.

Let us add that using the some arguments as in [22], we can conclude that in both cases $n=2$ and $n=3$, we have $\theta = 1/2$, which is a reasonable asymptotic ratio.

REMARKS :

i) In the two-dimensional case, the standard $Q_1 \times P_0$ element has the same properties as the quasi-linear asymmetric element, as far as the degree of the Jacobian and θ are concerned. It can actually give satisfactory numerical results as shown by many examples in [15]. However, in the three-dimensional case, the property of Theorem 3.1 no longer holds for the Q_1 element.

ii) Another generalization to the case $n=3$ of the two-dimensional asymmetric element satisfying the property of theorem 3.1, was presented in [21]. This element has the advantage of being conforming, but the value of the asymptotic ratio is rather high, namely $\theta = 4/5$ or $\theta = 8/11$ in the most favorable case of partitions. This explains the introduction of the present nonconforming generalization.

Now, having proved that the incompressibility can be properly treated for each element, we would like to assert that the same is true for Ω .

More precisely, letting A denote any subset of Ω , setting $A_K = A \cap K$, $K \in \tau_h$ and defining

$$\tilde{A} = \bigcup_{K \in \tau_h} \tilde{A}_K \quad \text{with} \quad \tilde{A}_K = \underset{\sim}{u}(A_K) \; ,$$

where $\underset{\sim}{u}_{/K} \in \underset{\sim}{P}_a$, we would like to verify that

$$meas(\widetilde{K}) = meas(K) \quad \forall K \in \tau_h \quad \Rightarrow \quad meas(\widetilde{\Omega}) = meas(\Omega) \quad ,$$

or yet that

$$meas(\widetilde{\Omega}) = \sum_{K \in \tau_h} meas(\widetilde{K})$$

Actually letting $\widehat{\Omega}$ be the deformed state of Ω induced by $\underset{\sim}{u}$ to be defined hereafter, we will prove that :

$$(3.2) \qquad meas(\widehat{\Omega}) = \sum_{K \in \tau_h} meas(\widetilde{K})$$

In the two-dimensional case it will be convenient to set $\widehat{\Omega} = \widetilde{\Omega}$. Indeed if $J[\underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x})] \geq 0 \quad \forall \ \underset{\sim}{x} \in \Omega$,(3.2) is trivially satisfied since $\underset{\sim}{V}_h$ is conforming and therefore the elements in deformed state do not interpenetrate However even under the above assumption, this is not necessarily the case of a nonconforming $\underset{\sim}{V}_h$. That is why for n=3 we will set $\Omega = \bigcup_{K \in \tau_h} \hat{K}$, where $\hat{K}$ denotes the deformed state of K induced by the vector field $\pi \underset{\sim}{u}$ at the vertices of the elements of τ_h . In this way $\widehat{\Omega}$ can be viewed as a certain interpolation of $\widetilde{\Omega}$ at the points $\widetilde{S}$, S being a vertex of an element of τ_h . In so doing we can prove that (3.2) is exactly satisfied for some kind of partitions, whereas in the general case it is satisfied up to an $O(h^2)$ term.

Before giving the proofs, let us say that, wherever the above Jacobian is negative for some $\underset{\sim}{x} \in \Omega$, we must define $\widetilde{A}$ for $A \subset \Omega$, not as the union of the $\widetilde{A}_K'$s , but with modifications taking into account the interpenetrations of the elements in deformed state that occur in the general case. This can be achieved by assigning a subtractive meaning to the sets $\widetilde{A}$ such that $J[\underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x})] < 0 \quad \forall \ \underset{\sim}{x} \in A$, $A \subset K$. In so doing, all the above assertions for the so-defined Ω would be true, and in particular (3.2) with or without a perturbation term. Bearing this in mind, in order to simplify the notations we shall assume in this section that $\underset{\sim}{u} \in \underset{\sim}{V}_h$ is such that :

(3.3) $\quad J[\underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x})] \geq 0 \qquad$ for a.e. $\underset{\sim}{x} \in \Omega$

(3.4) $\quad J[G_K + \underset{\sim}{u}(G_K)] = 1 \qquad \forall K \in \tau_h$, where G_K is the centroid of K .

Now we note that $(\underset{\sim}{x} + \pi\underset{\sim}{u})_{/K}$ is nothing else than the linear part of $(\underset{\sim}{x}+\underset{\sim}{u})_{/K}$. Therefore, since $\dfrac{\partial\varphi}{\partial x_j}$ vanishes at vertex S_{n+1} , $j = 1,2,\ldots,n$, and recalling (3.1) we have :

$$J[S_{n+1} + \underset{\sim}{u}(S_{n+1})] = J[\underset{\sim}{x} + \pi\, \underset{\sim}{u}(\underset{\sim}{x})] \quad \forall\, \underset{\sim}{x} \in K \ .$$

Since $meas(\acute{K}) = \int_K J[\underset{\sim}{x} + \pi\underset{\sim}{u}(\underset{\sim}{x})]d\underset{\sim}{x}$, assumption (3.3) implies that $meas(\acute{K}) \geq 0$, which in this case means that the K's are oriented in the same way as the K's , or yet that the K's do not interpenetrate.

Let us now consider the particular case n=3. We further define $\acute{A}$ to be the deformed state induced by $\pi\underset{\sim}{u}$ of every subset A of Ω. Notice that we are actually defining $\widehat{\Omega} = \acute{\Omega}$.

We first need the following :

<u>*Lemma 3.1*</u> *:* Let K be a tetrahedron and $\underset{\sim}{n}_K$ denote the outer unit normal vector with respect to ∂K, the boundary of K. Let $\underset{\sim}{\psi}$ be a vector field defined over K such that $\underset{\sim}{\psi} = \underset{\sim}{\beta}\,\varphi$, with $\underset{\sim}{\beta} \in \mathbb{R}^3$ and φ be given by (2.1). We then have :

$$\int_K div\ \underset{\sim}{\psi}\ d\underset{\sim}{x} = \frac{2}{3}\int_{B^K} \underset{\sim}{\psi} \cdot \underset{\sim}{n}_K\ ds$$

<u>*Proof*</u> *:* From Stokes' formula we have :

$$\int_K div\ \underset{\sim}{\psi}\ d\underset{\sim}{x} = \int_{\partial K} \underset{\sim}{\psi} \cdot \underset{\sim}{n}_K\ ds = \sum_{i=1}^{3} \int_{F_i^K} \underset{\sim}{\psi} \cdot \underset{\sim}{n}_K\ ds + \int_{B^K} \underset{\sim}{\psi} \cdot \underset{\sim}{n}_K\ ds\ .$$

Since $\varphi_{/F_i^K} = \lambda_j\lambda_k$, with i,j and k distinct, we have,

$$\sum_{i=1}^{3} \int_{F_i^K} \underset{\sim}{\psi} \cdot \underset{\sim}{n}_K\ ds = \frac{1}{12} \sum_{i=1}^{3} \underset{\sim}{\beta} \cdot \underset{\sim}{n}_{K/F_i} \quad area(F_i^K)$$

where $\underset{\sim}{n}_{K/F}$ denotes the outer unit normal vector with respect to face F^K of ∂K .

On the other hand, for a constant valued vector field $\underset{\sim}{\beta}$ we have :

$$0 = \int_K div\, \underset{\sim}{\beta}\, d\underset{\sim}{x} = \sum_{i=1}^{3} \underset{\sim}{\beta} \cdot \underset{\sim}{n}_{K/F_i}\, area(F_i^K) + \underset{\sim}{\beta} \cdot \underset{\sim}{n}_{K/B}\, area\,(B^K)\ .$$

Thus we have :

$$\int_K div\ \underset{\sim}{\psi}\, d\underset{\sim}{x} = \int_{B^K} \underset{\sim}{\psi} \cdot \underset{\sim}{n}_K\, ds - \frac{1}{12}\underset{\sim}{\beta} \cdot \underset{\sim}{n}_{K/B}\ area(B^K)$$

The results follows from the fact that

$$\int_{B^K} \underset{\sim}{\psi} \cdot \underset{\sim}{n}_K\, ds = \frac{1}{4}\underset{\sim}{\beta} \cdot \underset{\sim}{n}_{K/B}\ \ area\,(B^K)$$

q.e.d.

Now we note that since $\pi\underset{\sim}{u}$ is conforming we clearly have :

$$vol(\hat{\Omega}) = \sum_{K \in \tau_h} vol(\hat{K})$$

Actually we can prove that, under a reasonable assumption the above equality also holds if the $\hat{K}$'s are replaced by the $\tilde{K}$'s .

Theorem 3.2 : If τ_h is a compatible partition of Ω that has no basis on Γ^* we have :

$$vol(\overline{\Omega}) = \sum_{K \in \tau_h} vol(\tilde{K})$$

REMARK : It is interesting to note that partition τ_h^1 defined in section 2 satisfies the assumptions of this theorem.

Proof : A partition satisfying the assumptions of the theorem can be viewed as a subpartition of a first partition χ_h of Ω , consisting of hexahedrons having triangular faces. Each hexahedron H of χ_h generates two tetrahedrons of τ_h , say K_1 and K_2 , having a common basis lying in the interior of H, and lateral faces coinciding with the faces of the hexahedron (see figure 3.1).

Since $\underset{\sim}{u}$ is continuous over B, the common basis of K_1 and K_2 , we have :

$$vol(\tilde{H}) = vol(\tilde{K}_1) + vol\,(\tilde{K}_2)$$

Now we want to prove that we actually have

$$vol(\tilde{H}) = vol(\hat{H}) \qquad \forall H \in \chi_h$$

which will yield the result we are looking for, since

$$vol(\hat{\Omega}) = \sum_{H \in \chi_h} vol(\hat{H}) .$$

For this purpose we introduce a new variable $\hat{\underset{\sim}{x}}$ with the help of the following affine transformation over each K :

$$\underset{\sim}{x} \rightarrow \hat{\underset{\sim}{x}} = \underset{\sim}{x} + \pi\underset{\sim}{u}(\underset{\sim}{x})$$

In this way $\tilde{K}$ can be regarded as a deformed state of $\hat{K}$ obtained by the application of the displacement vector field $\hat{\underset{\sim}{\psi}}$ defined by :

$$\hat{\underset{\sim}{\psi}}(\hat{\underset{\sim}{x}}) = \underset{\sim}{\psi}(\underset{\sim}{x})$$

where $\underset{\sim}{\psi} = \underset{\sim}{\beta}\varphi$, with $\underset{\sim}{\beta} = (\underset{\sim}{u})_5 - [\sum_{i=1}^{3} (\underset{\sim}{u})_i] /3$, $(\underset{\sim}{u})_i$ being the value of $\underset{\sim}{u}$ at S_i , $i = 1,2,\ldots,5$.

If we denote by $\hat{\lambda}_i(\hat{\underset{\sim}{x}})$ the area coordinates of $\hat{K}$, we have necessarily $\hat{\lambda}_i(\hat{\underset{\sim}{x}}) = \lambda_i(\underset{\sim}{x})$, which means that $\hat{\underset{\sim}{\psi}} = \underset{\sim}{\beta}\hat{\varphi}$ where

$$\hat{\varphi} = \sum_{\substack{j,k=1 \\ j < k}}^{n} \hat{\lambda}_j\hat{\lambda}_k .$$

Now we have :

$$vol(\tilde{K}) = \int_{\hat{K}} \hat{J}[\hat{\underset{\sim}{x}} + \hat{\underset{\sim}{\psi}}(\hat{\underset{\sim}{x}})] \, d\hat{\underset{\sim}{x}}$$

where $\hat{J}$ represents the Jacobian with respect to the new variable $\hat{\underset{\sim}{x}}$.

Expanding the integrand above, we have :

$$vol(\tilde{K}) = vol(\hat{K}) + \int_{\hat{K}} \hat{div}\, \hat{\underset{\sim}{\psi}} \, d\hat{\underset{\sim}{x}} + \int_{\hat{K}} [\sum_{\ell=1}^{3} [\hat{J}(\hat{\underset{\sim}{\psi}}_\ell) + \hat{J}(\hat{\underset{\sim}{\psi}})] \, d\hat{\underset{\sim}{x}}$$

where $\hat{\underset{\sim}{\psi}}_\ell$ is the vector field obtained by replacing the ℓ-th component of $\hat{\varphi}$ by x_ℓ and $\hat{div}$ represents the divergence operator associated with $\hat{x}$

Since each Jacobian of the second integrand above has at least two columns of form $\beta^T\hat{\varphi}$, they vanish identically.

On the other hand, according to Lemma 3.1 we have :

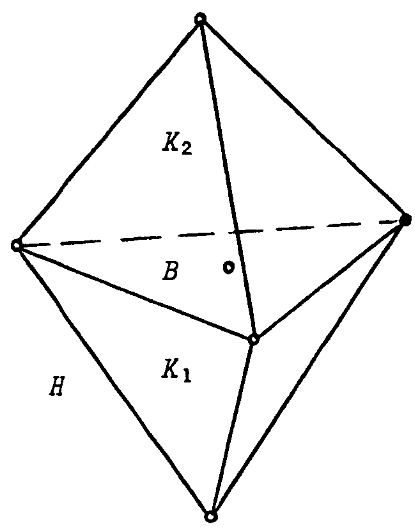
$$\int_{\hat{K}} \hat{div}\ \hat{\psi}\ (\hat{x})\ d\hat{x} = \frac{2}{3} \int_{\hat{B}} \hat{\psi} \cdot n_{\hat{K}}\ d\hat{s} \quad .$$

However, since πu is conforming, $\hat{B}$ coincides for both K_1 and K_2 together with $\hat{\psi}_{/\hat{B}}$, whereas $n_{/\hat{K}_1/\hat{B}} = -n_{/\hat{K}_2/\hat{B}}$.

Therefore we have :

$$vol(\tilde{H}) = vol(\tilde{K}_1) + vol(\tilde{K}_2) = vol(\hat{K}_1) + vol(\hat{K}_2) = vol\ (\hat{H}) \ .$$

q.e.d.



A hexahedron of partition χ_h

Figure 3.1

Now for the general case we have :

<u>*Theorem 3.3*</u> : For any compatible family $(\tau_h)_h$ of partitions of Ω we have :

$$\left| vol(\bar{\Omega}) - \sum_{K \in \tau_h} vol(\tilde{K}) \right| \leq C\, h^2 |u|_{2,\infty}$$

where C is a constant independent of h.(*)

(*) From now on the letter C, with or without subscripts, will represent various constants independent of the discretization parameter h.

Proof : According to Theorem 3.2, all we have to do is proving that

$$\left| \sum_{K \in \tau_h^*} [vol(\acute{K}) - vol(\tilde{K})] \right| \leq C\, h^2 |\underset{\sim}{u}|_{2,\infty}$$

where $\tau_h^* = \{K/K \in \tau_h \;\;, meas\ (B^K \cap \Gamma^*) \neq 0\}$

By a direct computation of the increments of volume of $\acute{K}$ over its faces, due to the quadratic component $\underset{\sim}{\beta}\,\varphi$ of $\underset{\sim}{u}$, we get :

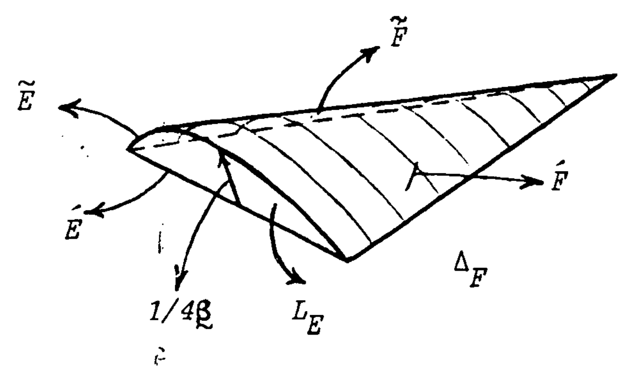
$$vol(\tilde{K}) - vol(\acute{K}) = \int_{\partial \acute{K}} \underset{\sim}{\psi}(\acute{\underset{\sim}{x}}) \cdot \underset{\sim}{n}_{\acute{K}}\, d\acute{x}$$

According to Lemma 3.1 we get :

$$vol(\tilde{K}) - vol(\acute{K}) = -2 \sum_{i=1}^{3} \int_{\acute{F}_i} \acute{\underset{\sim}{\psi}}(\acute{\underset{\sim}{x}}) \cdot \underset{\sim}{n}_{\acute{K}}\, d\acute{\underset{\sim}{x}}$$

Now, F being a lateral face of element K , we define the set Δ_F as follows :

Let E be the edge of F belonging to the basis of tetrahedron K and let L_E be the plane surface delimited by $\acute{E}$ and $\tilde{E}$. Δ_F is defined to be the solid delimited by $\tilde{F}$, $\acute{F}$ and L_E as illustrated in Figure 3.2 below.



A perturbation of $\acute{F}$ due to the quadratic components of $\underset{\sim}{u}$.

Figure 3.2

Using classical arguments, if $(\tau_h)_h$ is regular we can estimate :

$$vol(\Delta_F) \leq C\, h^4 |\underset{\sim}{u}|_{2,\infty} \qquad \forall F$$

Now noting that

$$\left| vol(\Delta_{F_i}) \right| = \left| \int_{\hat{F}_i} \hat{\underset{\sim}{\psi}}\,(\hat{\underset{\sim}{x}}) \cdot \underset{\sim}{n}_{\hat{K}}\, d\hat{\underset{\sim}{x}} \right|$$

we have :

$$vol(\tilde{K}) - vol(\hat{K}) \leq 6\, C\, h^4 |\underset{\sim}{u}|_{2,\infty}$$

Since card $\tau_h^* \leq C\, h^{-2}$ the result follows. q.e.d.

4. Convergence results for the linear case

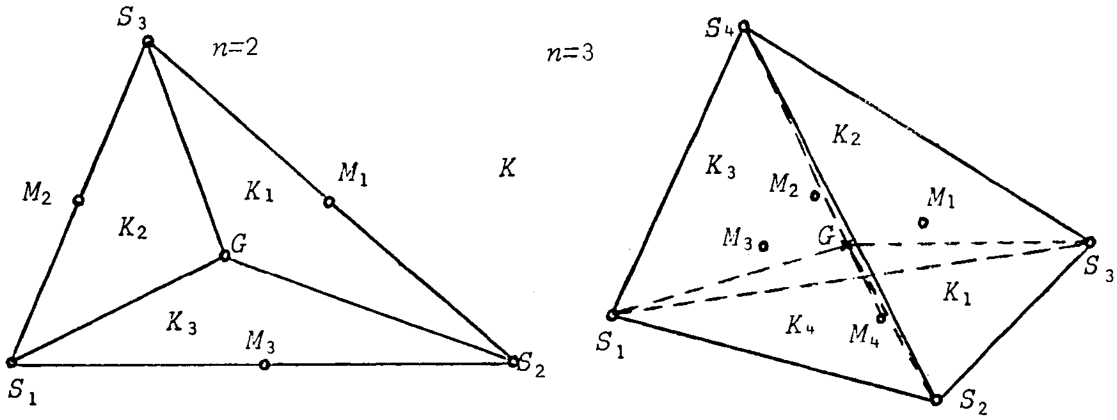
In this section we give a convergence analysis for the two asymmetric finite elements introduced in Section 2, related to problem (P'). We shall confine ourselves to the case where $(\tau_h)_h$ is a family of compatible partitions of type τ_h^2 defined in Section 2 (see Figure 4.1), due to the following reason :

We recall that the essential difficulty for proving, not only convergence but existence results for problem $(\acute{P}_h)$ lies in deriving the Brezzi inequality (2.6). For this purpose, if we consider other partitions we have to face at least the same difficulties as in the case of the classical $Q_1 \times P_0$ element. For example, in the case of partition τ_h^1 the nodes R, S and T lying in the interior of the quadrilateral or hexahedral superelements of the first partition of Ω illustrated in Figure 2.2, become useless as degrees of freedom if the pressure is taken to be constant over the whole superelement. In this case the situation becomes the same as the one of the $Q_1 \times P_0$ element, for which the convergence properties that one can expect to obtain are rather weak.

For instance, assuming that Ω is a rectangle and Γ^* is empty (two-dimensional Stokes problem), Johnson and Pitkäranda [13] proved that by using a uniform rectangular $M \times N$ grid, with M and N even, the following estimate holds :

$$|\underset{\sim}{u}-\underset{\sim}{u}_h|_1 \leq C\, h[|\underset{\sim}{u}|_2 + |\underset{\sim}{u}|_{3,r} + |p|_1] \text{ for some } real\ r, r > 1 \quad .$$

Their result, that seems to reflect reality, requires a strict regularity of the solution to problem $(\hat{P})$ and gives no information about the convergence of the pressure. The latter actually does not always converge as it has been verified in practice.



Construction of partition τ_h^2 for the asymmetric elements

Figure 4.1

With simple modifications, for a partition of type τ_h^1 associated with the rectangular grid considered by Johnson & Pitkäranta, their result also applies to the case of the two-dimensional quasi-linear element. For the three-dimensional case however, no similar results seem to be available yet.

Now, for the case of partition τ_h^2 we can verify that (2.6) holds with $\beta_h = O(h^{2-s})$, where the value of s is to be taken in the interval $[3/2-\varepsilon,2]$ (*) , and depends on the regularity of the solution to the following mixed Dirichlet-Neumann problem :

(*) The symbol ε, as usual, denotes a strictly positive real number arbitrarily small.

$$(4.1) \quad \begin{cases} -\Delta z = f & \text{in } \Omega \quad \text{for } f \in L^2(\Omega) \\ \dfrac{\partial z}{\partial n} = 0 & \text{on } \Gamma^* \\ z = 0 & \text{on } \Gamma_0 \end{cases}$$

According to the results of Grisvard [12] and Raugel[19], we have the following for the two-dimensional case :

Let Ω be a convex polygon and $\{A_i\}_{i=1}^{I}$ be the set of points of Γ at which $\bar{\Gamma}_0$ and $\bar{\Gamma}^*$ intersect. If Γ^* is empty, we consider I to be zero.

Let now π/α_i be the angle associated with A_i in such a way that it is equal to the angle of the corner of Γ if A_i is a vertex of Ω and $\alpha_i=1$ otherwise. Then we have $z \in H^s(\Omega)$ with $s = 1 + \alpha/2 - \delta$, where :

$$\alpha = 2 \quad \text{and} \quad \delta = 0 \quad \text{if} \quad I = 0$$

$$\alpha = \min[2, \max(1, \min_{1 \le i \le I} \alpha_i)] \quad \text{and} \quad \delta = \varepsilon \quad \text{if} \quad I > 0 \quad .$$

Moreover, there exists a constant C independent of f such that

$$(4.2) \qquad \| z \|_s \le C \| f \|_0$$

In the three-dimensional case, similar regularity results are given in [12]. However we shall confine our study to the cases in which one can assert that $z \in H^2(\Omega)$. This is because the non-conformity of the three-dimensional element makes the analysis very complicated in general, but if $z \in H^2(\Omega)$ for every $f \in L^2(\Omega)$, the problem can be considerably simplified. Notice that we actually have the above regularity for z in many important cases, such as $\Gamma^* = \emptyset$ with Ω convex [14] .

Nevertheless, as we will point out later on, one can give proofs of the existence and uniqueness of the solution to problem (P_h) that do not depend on the regularity of z .

Summing up all the above remarks, we will simply assume in this section that domain Ω has a polyhedral boundary consisting of two parts Γ_0 and Γ^*, in such a way that the solution to problem (4.1) lies in $H^s(\Omega)$ for some s, $3/2 - \varepsilon \le s \le 2$, and that (4.2) holds.

Now we give some preparatory materials :

First we recall the so-called Ciarlet's Lemma for bilinear forms :

Lemma 4.1 [6] : Let $\sigma : H^{k+1}(D) \times P \to R$ be a bilinear form, $k \in \mathbb{N}$ where P is such that $P_\ell \subset P \subset H^{\ell+1}(D)$, $\ell \in \mathbb{N}$, P being normed by $\|\cdot\|_{\ell+1,D}$. If σ satisfies :

$$\|\sigma\| = \sup_{(y,w)\in H^{k+1}(D)\times P} \frac{\sigma(y,w)}{\|y\|_{k+1,D}\ \|w\|_{\ell+1,D}} < \infty$$

$$\sigma(y,w) = 0 \qquad \forall\ y \in H^{k+1}(D) \text{ and } \forall\ w \in P_\ell.$$

$$\sigma(y,w) = 0 \qquad \forall\ y \in P_k \qquad \text{and} \quad \forall\ w \in P$$

there exists a constant C that only depends on D, such that :

$$|\sigma(y,w)| \le C\ \|\sigma\|\ |y|_{k+1,D} |w|_{\ell+1,D} \qquad \forall\ (y,w) \in H^{k+1}(D) \times P \qquad \square$$

Let us consider for a moment the three-dimensional case alone :

Lemma 4.2 : Let $n=3$, and $(\tau_h)_h$ be a regular family of partitions of Ω and $\underset{\sim}{z}$ be a vector field such that z_i is the solution to problem (4.1) for $f = f_i$, $\underset{\sim}{f} = (f_1, f_2, \ldots, f_n) \in \underset{\sim}{L}^2(\Omega)$. Assuming that $\underset{\sim}{z} \in \underset{\sim}{H}^2(\Omega)$ we have :

$$(4.3) \quad \sum_{K\in\tau_h} \int_{\partial K} \frac{\partial \underset{\sim}{z}}{\partial n} \cdot \underset{\sim}{w}_h\, ds \le C\, h\, |\underset{\sim}{z}|_2\ \|\underset{\sim}{w}_h\|_h \quad ,$$

where $\dfrac{\partial g}{\partial n_K}$ denotes the outer normal derivative of a function g, with respect to the boundary ∂K of K.

Proof : Since $\underset{\sim}{z} \in \underset{\sim}{H}^2(\Omega)$, $\frac{\partial z}{\partial n_K} \in \underset{\sim}{H}^{1/2}(\partial K)$, which means that the left hand side of (4.3) has a sense. Recalling operator Π of Section 3 we have

$$\sum_{K \in \tau_h} \int_{\partial K} \frac{\partial \underset{\sim}{z}}{\partial n_K} \cdot \underset{\sim}{w}_h \, ds = \sum_{K \in \tau_h} \int_{\partial K} \frac{\partial \underset{\sim}{z}}{\partial n} \cdot (\underset{\sim}{w}_h - \Pi \underset{\sim}{w}_h) \, ds$$

for $\Pi \underset{\sim}{w}_h$ vanishes on Γ_0 and $\frac{\partial z}{\partial n}$ on Γ^* . Moreover we have :

$$\sum_{K \in \tau_h} \int_{\partial K} \frac{\partial \underset{\sim}{z}}{\partial n_K} \cdot (\underset{\sim}{w}_h - \Pi \underset{\sim}{w}_h) \, ds = \sum_{K \in \tau_h} \sum_{i=1}^{3} \int_{F_i^K} \frac{\partial \underset{\sim}{z}}{\partial n_K} \cdot (\underset{\sim}{w}_h - \Pi \underset{\sim}{w}_h) ds \; +$$

$$+ \sum_{K \in \tau_h} \int_{B^K} \frac{\partial \underset{\sim}{z}}{\partial n_K} \cdot (\underset{\sim}{w}_h - \Pi \underset{\sim}{w}_h) \, ds$$

where the F_i^K's are the lateral faces, and B^K is the base of K .

Since $\underset{\sim}{w}_h$ is continuous over the bases of τ_h, the second summation on the right hand side above vanishes, and we can write :

$$\sum_{K \in \tau_h} \int_{\partial K} \frac{\partial \underset{\sim}{z}}{\partial n_K} \cdot \underset{\sim}{w}_h ds = \sum_{K \in \tau_h} \left| \sum_{i=1}^{3} \int_{F_i^K} \frac{\partial \underset{\sim}{z}}{\partial n_K} \cdot (\underset{\sim}{w}_h - \Pi \underset{\sim}{w}_h) ds + \frac{1}{3} \int_{B_K} \frac{\partial \underset{\sim}{z}}{\partial n_K} \cdot (\underset{\sim}{w}_h - \Pi \underset{\sim}{w}_h) ds \right|$$

Now we define the bilinear form : $\sigma_K : H^2(K) \times P_a \rightarrow \mathbb{R}$ by

$$(4.4) \quad \sigma_K(u,w) = \frac{1}{3} \int_{B^K} \underset{\sim}{\nabla} u . \underset{\sim}{n}_K (w - \Pi w) \, ds + \sum_{i=1}^{3} \int_{F_i^K} \underset{\sim}{\nabla} u . \underset{\sim}{n}_K (w - \Pi w) ds$$

First we note that by a linear affine transformation A of K into the reference element $\hat{K}$, we can define a bilinear form $\hat{\sigma}_K : H^2(\hat{K}) \times \hat{P}_a \rightarrow \mathbb{R}$ by

$$\hat{\sigma}_K(\hat{u},\hat{w}) = h^{-1} \sigma_K(u,w)$$

where $\hat{v}[A(\underset{\sim}{x})] = v(\underset{\sim}{x})$, $\underset{\sim}{x} \in K$ and $\hat{P}_a = \{\hat{v} / \hat{v}[A(x)] = v(\underset{\sim}{x}) \; \forall v \in P_a\}$.

Setting $\hat{B} = A(B^K)$ and $\hat{F}_i = A(F_i^K)$ we can write :

$$\hat{\sigma}_K(\hat{u},\hat{w}) = \frac{C_B}{3}\int_{\hat{B}} \underset{\sim}{\hat{\nabla}}\hat{u} \cdot \underset{\sim}{\nu}_B(\hat{w} - \Pi\hat{w})\, d\hat{s} + \sum_{i=1}^{3} C_i \int_{\hat{F}_i} \underset{\sim}{\hat{\nabla}}\hat{u} \cdot \underset{\sim}{\nu}_i(\hat{w} - \Pi\hat{w})\, d\hat{s}$$

where $|\underset{\sim}{\nu}_B| = |\underset{\sim}{\nu}_i| = 1 \quad , \quad i = 1,2,3$.

Since $(\tau_h)_h$ is regular we can say that there exists a constant C such that :

$$max\ (C_B, C_1, C_2, C_3) \leq C \qquad K \in \tau_h \ .$$

which implies in turn the existence of $\hat{C}$ such that :

$$\| \hat{\sigma}_K \| = \sup_{\hat{u} \in H^2(\hat{K}), \hat{w} \in \hat{P}_a} \frac{\hat{\sigma}_K(\hat{u},\hat{w})}{\|\hat{u}\|_{2,\hat{K}} \|\hat{w}\|_{1,\hat{K}}} \leq \hat{C} \qquad \forall K \in \tau_h$$

for $\hat{P}_a$ is a finite dimensional space.

Clearly we have :

$$\hat{\sigma}_K(\hat{u},\hat{w}) = 0 \qquad \forall\ \hat{u} \in H^2(\hat{K}) \quad \text{and} \quad \forall \hat{w} \in P_0$$

On the other hand, if $\hat{u} \in P_1$, we can say that $\underset{\sim}{\beta} = \mu\, \underset{\sim}{\nabla} u$ is a constant vector over K, μ being such that $w - \Pi w = \mu\varphi$, φ given by (2.1).

Using Lemma 3.1 we have :

$$\hat{\sigma}_K(\hat{u},\hat{w}) = 0 \qquad \forall\ \hat{u} \in P_1 \quad \text{and} \quad \forall\ \hat{w} \in \hat{P}_a$$

Therefore, σ_K satisfies all the assumptions of Lemma 4.1 with $k = 1$ and $\ell = 0$, and we have :

$$\hat{\sigma}_K(\hat{u},\hat{w}) \leq C|\hat{u}|_{2,\hat{K}}|\hat{w}|_{1,\hat{K}}$$

which, by standard transformations gives :

$$|\ \sigma_K(u,w)| \leq C\ h|u|_{2,K}|w|_{1,K} \qquad K \in \tau_h \ .$$

Now since the right hand side of (4.3) is nothing else than

$\sum_{K\in\tau_h}\sum_{i=1}^{3}\sigma_K(z_i,w_{h_i})$ the result follows . q.e.d.

Let now $r_h : \underset{\sim}{V} \to \underset{\sim}{V}_h$ be the orthogonal projection operator for the inner product $(.,.)_h$.

Lemma 4.3 : Let $\underset{\sim}{e}_h = \underset{\sim}{v} - r_h\,\underset{\sim}{v}$ be the error of the projection of $\underset{\sim}{v} \in \underset{\sim}{V}$. We assume that the solution to problem (4.1) lies in $H^s(\Omega)$, where $s = 2$ if $n = 3$ and $3/2 - \varepsilon \le s \le 2$ for $n = 2$. We then have :

$$|\underset{\sim}{e}_h| \le C\,h^{s-1}|\underset{\sim}{v}|_1$$

Proof : First we note that

$$|e_h|_0 = \sup_{\underset{\sim}{f}\in \underset{\sim}{L}^2(\Omega)} \frac{(\underset{\sim}{f},\underset{\sim}{e}_h)_0}{|\underset{\sim}{f}|_0}$$

Let M be the space $H^s(\Omega)\cap V$. According to the existence results for problem (4.1) and to (4.2) we have by classical duality arguments :

$$|e_h|_0 = C \sup_{\underset{\sim}{z}\in \underset{\sim}{M}} \frac{-(\Delta\underset{\sim}{z},\underset{\sim}{e}_h)_0}{\|\underset{\sim}{z}\|_s}$$

Using Green's formula we have

$$(4.5)\quad |(\Delta\underset{\sim}{z},\underset{\sim}{e}_h)_0| \le |(\underset{\sim}{z}\,,\,\underset{\sim}{e}_h)_h| + \Big|\sum_{K\in\tau_h}\int_{\partial K}\frac{\partial\underset{\sim}{z}}{\partial n_K}\cdot\underset{\sim}{e}_h\,ds\Big|$$

We note that the second term on the right hand side above vanishes identically in the case $n = 2$. Since $\underset{\sim}{v} \in \underset{\sim}{V}$, for $n = 3$ we have :

$$\Big|\sum_{K\in\tau_h}\int_{\partial K}\frac{\partial\underset{\sim}{z}}{\partial n_K}\cdot\underset{\sim}{e}_h\,ds\Big| = \Big|\sum_{K\in\tau_h}\int_{\partial K}\frac{\partial\underset{\sim}{z}}{\partial n_K}\cdot r_h\,\underset{\sim}{v}\,ds\Big|$$

Now using Lemma 4.2 we get

(4.6) $$\left| \sum_{K\in\tau_h} \int_{\partial K} \frac{\partial \underset{\sim}{z}}{\partial n_K} \cdot \underset{\sim}{e}_h \, ds \right| \le C\, h^{s-1} |\underset{\sim}{z}| \, \| r_h \underset{\sim}{v} \|_h \quad \text{with } s = 2 .$$

Moreover according to the Sobolev imbedding Theorems [1], taking into account our assumptions for s, we have $\underset{\sim}{z} \in \underset{\sim}{C}^{o}(\overline{\Omega})$. Therefore it has a sense to define the Π-interpolate of $\underset{\sim}{z}$ at the vertices of τ_h and since $\Pi \underset{\sim}{z} \in \underset{\sim}{V}_h$ we can write , for both cases $n = 2$ and $n = 3$:

$$|(\underset{\sim}{z}, \underset{\sim}{e}_h)_h| = |(\underset{\sim}{z} - \Pi \underset{\sim}{z} , \underset{\sim}{e}_h)_h| \le |\underset{\sim}{z} - \Pi \underset{\sim}{z}|_1 \, \|\underset{\sim}{e}_h\|_h$$

According to standard approximation results (see e.g.[3] and [10, Vol. 2, page 11] , we have :

$$|\underset{\sim}{z} - \Pi \underset{\sim}{z}|_1 \le C\, h^{s-1} \, \| \underset{\sim}{z} \|_s$$

which together with (4.5) and (4.6) gives :

$$|\underset{\sim}{e}_h| \le C\, h^{s-1} \, [\|\underset{\sim}{e}_h\|_h + \| r_h \underset{\sim}{v}\|_h] \quad .$$

Since we have $\| r_h \underset{\sim}{v}\|_h \le |\underset{\sim}{v}|_1$ and therefore $\|\underset{\sim}{e}_h\|_h \le 2|\underset{\sim}{v}|_1$, the result follows. q.e.d.

Lemma 4.4 : With every vector field $\underset{\sim}{v} \in \underset{\sim}{V}$ we can associate $\underset{\sim}{v}_h \in \underset{\sim}{V}_h$ such that :

(4.7) $$\sum_{k=1}^{n+1} \int_{K_k} div\, \underset{\sim}{v}_h \, d\underset{\sim}{x} = \int_K div\, \underset{\sim}{v} \, d\underset{\sim}{x} \qquad \forall K \in \tau_h$$

(4.8) $$\|\underset{\sim}{v}_h\|_h \le C_1 \, h^{s-2} |\underset{\sim}{v}|_1$$

where $(\tau_h)_h$ is the family of partitions from which $(\tau_h^2)_h$ is constructed and the K_k' s are the elements of τ_h^2 contained in $K \in \tau_h$.

Proof : We first refer to Figure 4.1 for the notations.

Let B_k be the face of K opposite to vertex S_k, $1 \le k \le n+1$, i.e., the basis of K_k. By a straight forward calculation we get, for every g belonging to space P_a associated with element K_k :

$$(4.9) \quad \int_{B_k} g \, ds = \frac{1}{6(n-1)} \left| \sum_{\substack{j=1 \\ j \neq k}}^{n+1} g(S_j) + (5n-6)\, g(M_k) \right| \, meas(B_k) \, .$$

Thefore we can uniquely define a field $\underset{\sim}{v}_h \in \underset{\sim}{V}_h$ such that $\forall K \in \tau_h$ we have :

$$v_{h_i}(S_j) = r_h \, v_i(s_j) \qquad j = 1,2,\ldots,n+1$$

$$(4.10) \quad \frac{5-n}{3} \int_{B_k} v_{h_i} \, ds + \frac{n-2}{3} \int_{B_k} \Pi \, v_{h_i} \, ds = \int_{B_k} v_i \, ds \qquad k = 1,2,\ldots,n+1$$

Setting $\underset{\sim}{\mu}_k = \underset{\sim}{n}_{/K_k/B_k}$, we multiply each side of (4.10) by μ_{k_i} and we sum up with respect to i , $1 \le i \le n$.

For $n = 2$ we get :

$$\int_{B_k} \underset{\sim}{v}_h \cdot \underset{\sim}{\mu}_k \, ds = \int_{B_k} \underset{\sim}{v} \cdot \underset{\sim}{\mu}_k \, ds \, .$$

For $n = 3$ we have :

$$\frac{2}{3} \int_{B_k} (\underset{\sim}{v}_h - \Pi \underset{\sim}{v}_h) \cdot \underset{\sim}{\mu}_k \, ds + \int_{B_k} \Pi \underset{\sim}{v}_h \cdot \underset{\sim}{\mu}_k \, ds = \int_{B_k} \underset{\sim}{v} \cdot \underset{\sim}{\mu}_k \, ds$$

But according to Lemma 3.1 we have :

$$\frac{2}{3} \int_{B_k} (\underset{\sim}{v}_h - \Pi \underset{\sim}{v}_h) \cdot \underset{\sim}{\mu}_k \, ds = \int_{K_k} div(\underset{\sim}{v}_h - \Pi \underset{\sim}{v}_h) \, d\underset{\sim}{x}$$

Therefore, summing up with respect to k and using Stokes' formula, we get (4.7), for both cases $n = 2$ and $n = 3$.

Let us now prove (4.8), which we do only for $n = 3$. As a matter of fact, using Lemma 4.3 the proof in the case $n = 2$ becomes a trivial variant of Lemma 2.5 of [8] .

First we note that $\|\underset{\sim}{v}_h\|_h \le \|r_h \underset{\sim}{v} - \underset{\sim}{v}_h\|_h + |\underset{\sim}{v}|_1$. We have :

$$\|r_h\underset{\sim}{v} - \underset{\sim}{v}_h\|_h = \left| \sum_{K\in\tau_h} \sum_{k=1}^{4} |r_h \underset{\sim}{v}(M_k) - \underset{\sim}{v}_h(M_k)|^2 \, |p_{k+4}|^2_{1,K_k} \right|^{1/2}$$

where p_{k+4} is the basis function of P_a associated with the centroid M_k of B_k .

According to standard estimates we have :

$$|p_{k+4}|_{1,K_k} \le C\, h^{1/2}$$

On the other hand, using (4.9) and (4.10) we get :

$$|r_h \underset{\sim}{v}(M_k) - \underset{\sim}{v}_h(M_k)| = \frac{4}{3} \frac{\left| \int_{B_k} r_h \underset{\sim}{v}\, ds - \int_{B_k} \underset{\sim}{v}_h\, ds \right|}{\int_{B_k} ds} \le$$

$$\le \frac{4}{3} \frac{\left| \frac{3}{2} \int_{B_k} \underset{\sim}{e}_h\, ds \right| + \frac{1}{2} \left| \int_{B_k} (r_h \underset{\sim}{v} - \Pi \underset{\sim}{v}_h)\, ds \right|}{\int_{B_k} ds}$$

Now use the affine transformation $A : K_k \to \hat{K}$ and we get :

$$|r_h\underset{\sim}{v}(M_k) - \underset{\sim}{v}_h(M_k)| \le \frac{8}{3} \left[\frac{3}{2} \left| \int_{\hat{B}} \hat{\underset{\sim}{e}}_h\, ds \right| + \frac{1}{2} \left| \int_{\hat{B}} (\widehat{r_h\underset{\sim}{v}} - \Pi\hat{\underset{\sim}{v}}_h)\, ds \right| \right]$$

Using the Trace Theorem we get

$$|r_h \underset{\sim}{v}(M_k) - \underset{\sim}{v}_h(M_k)| \le C \left[\|\hat{\underset{\sim}{e}}_h\|_{1,\hat{K}} + \|\widehat{r_h\underset{\sim}{v}} - \Pi\hat{\underset{\sim}{v}}_h\|_{1,\hat{K}} \right|$$

Now noting that $\Pi\hat{\underset{\sim}{v}}_h$ interpolates $\hat{r_h \underset{\sim}{v}}$ at the vertices of $\hat{K}$, by standard approximation results we get :

$$|r_h \underset{\sim}{v}(M_k) - \underset{\sim}{v}_h(M_k)| \leq C \left[\|\hat{\underset{\sim}{e}}_h\|_{1,\hat{K}} + |\hat{r_h \underset{\sim}{v}}|_{1,\hat{K}} \right]$$

Going back to element K_k we get :

$$\left| r_h \underset{\sim}{v}(M_k) - \underset{\sim}{v}(M_k) \right| \leq C\, h^{-1/2} (h^{-2}|\underset{\sim}{e}_h|^2_{0,K_k} + |\underset{\sim}{e}_h|^2_{1,K_k} + |r_h \underset{\sim}{v}|^2_{1,K_k})^{1/2}$$

which yields :

$$|r_h\underset{\sim}{v}-\underset{\sim}{v}_h|^2_{1,K_k} \leq C\, (h^{-2}|\underset{\sim}{e}_h|^2_{0,K_k} + |\underset{\sim}{e}_h|^2_{1,K_k} + |r_h \underset{\sim}{v}|^2_{1,K_k})$$

Summing up over k and τ_h and using Lemma 4.3 we get :

$$\|r_h\underset{\sim}{v} - \underset{\sim}{v}_h\|_h \leq \tilde{C}(1 + h^{s-2})(\|\underset{\sim}{e}_h\|_h + \|r_h \underset{\sim}{v}\|_h)$$

and the result follows, with $C_1 = 1 + 6\tilde{C}$ q.e.d.

As a consequence we have :

<u>*Lemma:4.5*</u> : If $\underset{\sim}{V}_h$ is associated with partition τ_h^2 and Q_h^o is the space of functions that are constant over each element of τ_h, for each $q_h \in Q_h^o$, $\underset{\sim}{v}_h \in \underset{\sim}{V}_h$ such that :

(4.11) $$b_h(\underset{\sim}{v}_h, q_h) \geq C_0 |q_h|_0^2$$

(4.12) $$\|\underset{\sim}{v}_h\|_h \leq C_2\, h^{s-2} |q_h|_0$$

<u>*Proof*</u> : Let $q_h \in Q_h^0$. According to [7], Lemma $C2$ there exist constants C' and C'' independent of q_h and $\underset{\sim}{v} \in \underset{\sim}{V}$ such that :

$$b(\underset{\sim}{v}, q_h) = |\underset{\sim}{v}|_1^2$$

$$C'|q_h|_0 \leq |\underset{\sim}{v}|_1 \leq C''|q_h|_0$$

Since q_h is constant over $\bigcup_{k=1}^{4} K_k = K$, $K \in \tau_h$, if we associate to $\underset{\sim}{v}$ a vector field $\underset{\sim}{v}_h$ of $\underset{\sim}{V}_h$ in the way prescribed in Lemma 4.4 , we have:

$$b_h(\underset{\sim}{v}_h, q_h) = \sum_{K \in \tau_h} \sum_{k=1}^{4} \int_{K_k} q_h \, div \, \underset{\sim}{v}_h \, d\underset{\sim}{x} = b(\underset{\sim}{v}, q_h)$$

which yields (4.11) with $C_a = (C')^2$

Finally, recalling (4.8), we have (4.12) with $C_2 = C_1 C''$, which proves the theorem. q.e.d.

Now let us introduce an orthogonal basis $\gamma = \bigcup_{K \in \tau_h} \{\gamma_1^K, \gamma_2^K, \ldots, \gamma_{n+1}^K\}$ of Q_h , defined by $supp(\gamma_i^K) \subset K$, $1 \le i \le n+1$, and by :

For $n = 2$

$\gamma_1^K(\underset{\sim}{x}) = 1$ for $\underset{\sim}{x} \in K$

$$\gamma_2^K(\underset{\sim}{x}) = \begin{cases} 1/2 & \text{for } \underset{\sim}{x} \in K_1 \cup K_3 \\ -1 & \text{for } \underset{\sim}{x} \in K_2 \end{cases}$$

$$\gamma_3^K(\underset{\sim}{x}) = \begin{cases} -1 & \text{for } \underset{\sim}{x} \in K_1 \\ 0 & \text{for } \underset{\sim}{x} \in K_2 \\ +1 & \text{for } \underset{\sim}{x} \in K_3 \end{cases}$$

For $n = 3$

$\gamma_1^K(\underset{\sim}{x}) = 1$ for $\underset{\sim}{x} \in K$

$$\gamma_2^K(\underset{\sim}{x}) = \begin{cases} +1 & \text{for } \underset{\sim}{x} \in K_1 \cup K_2 \\ -1 & \text{for } \underset{\sim}{x} \in K_3 \cup K_4 \end{cases}$$

$$\gamma_3^K(\underset{\sim}{x}) = \begin{cases} +1 & \text{for } \underset{\sim}{x} \in K_1 \cup K_3 \\ -1 & \text{for } \underset{\sim}{x} \in K_2 \cup K_4 \end{cases}$$

$$\gamma_4^K(\underset{\sim}{x}) = \begin{cases} +1 & \text{for } \underset{\sim}{x} \in K_1 \cup K_4 \\ -1 & \text{for } \underset{\sim}{x} \in K_2 \cup K_3 \end{cases}$$

Let ξ_i^K be the components of $q_h \in Q_h$ with respect to γ. We then have :

$$q_h = \sum_{i=1}^{n+1} q_i \quad \text{where}$$

$$q_i = \sum_{K\epsilon\tau_h} \xi_i^K \gamma_i^K \quad , \qquad 1 \leq i \leq n+1 \qquad .$$

Notice that Q_h^o is the space spanned by $\gamma_1 = \bigcup_{K\epsilon\tau_h} \gamma_1^K$.

Now we are ready to prove :

Theorem 4.1 : If we use partition τ_h^2 for the construction of the spaces $\underset{\sim}{V}_h$ and Q_h defined in Section 2, the discrete Brezzi condition (2.6) is satisfied with $\beta_h = \beta_0 h^{2-s}$, where β_0 is a constant independent of h.

Proof : If the space of pressures was Q_h^o , the problem would be already solved, for in this case (2.6) becomes a trivial consequence of (4.11) and (4.12). Therefore we will try to construct a suitable $\underset{\sim}{z} \in \underset{\sim}{V}_h$ associated with the pressures spanned by γ_i^K , $2 \leq i \leq n+1$ in order to prove (2.6) for the whole Q_h .

For this purpose it will be useful to set $\underset{\sim}{m}_i = \overrightarrow{S_i M_i}$, *i.e*, the oriented median of K with respect to S_i , with $m_i = |\underset{\sim}{m}_i|$, $i = 1,2,\ldots, n+1$.

Let also

$$q = \sum_{i=2}^{n+1} q_i$$

Now over each simplex K , $\underset{\sim}{z}$ will be such that $\underset{\sim}{z}_{/K_i} \in \underset{\sim}{P}_a$ and $\underset{\sim}{z}(S_i) = \underset{\sim}{z}(M_i) = 0$, $1 \leq i \leq n+1$.

Let us first consider the case $n = 2$.

Dropping the superscript K for simplicity and letting G be the centroid of K we set :

$$\underset{\sim}{z}(G) = \frac{2}{3} (\xi_2 \underset{\sim}{m}_2 - \xi_3 \underset{\sim}{m}_3) \quad .$$

It is easy to see that $\underset{\sim}{z} \in \underset{\sim}{P}_1$ over each K_i , $\underset{\sim}{z} \in \underset{\sim}{C}^o(\overline{\Omega})$ and by a straightforward calculation we get (see also Figure 4.2)

$$|q|^2_{0,K} = \left(\frac{3\xi_2^2}{2} + 2\xi_3^2 \right) \frac{meas(K)}{3}$$

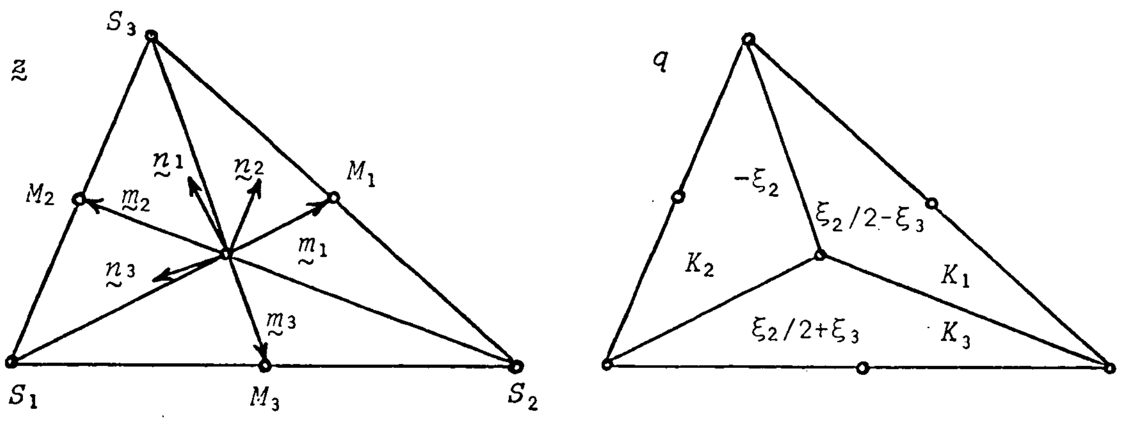
and

(4.13) $|\underset{\sim}{z}|_{1,K} \le C|q|_{0,K}$

Let now $\underset{\sim}{n}_i$ be the unit vector orthogonal to $\underset{\sim}{m}_i$ and oriented as indicated in Figure 4.2 .

We have :

$$\int_K q \; div \; \underset{\sim}{z} \; d\underset{\sim}{x} = \sum_{i=1}^{3} \int_{\partial K_i} q \; \underset{\sim}{z} \cdot \underset{\sim}{n}_{K_i} \; ds$$



Field $\underset{\sim}{z}$ associated with $q \in Q_h$

Figure 4.2

Using the elementary identity

$$\frac{2}{3} m_i |\underset{\sim}{m}_j \cdot \underset{\sim}{n}_i| \equiv meas(K) \qquad \text{whenever} \quad i \neq j$$

and performing simple calculations we get

$$\int_K q \; div \; \underset{\sim}{z} \; d\underset{\sim}{x} = \frac{meas(K)}{3} \left(3\xi_2^2 + \frac{3\xi_2\xi_3}{2} + 3\xi_3^2 \right) \ge |q|^2_{0,K} \; ,$$

which yields :

$$(4.14)\qquad \acute{b}_h\,(\underset{\sim}{z},q) \geq |q|_0^2 \quad .$$

For the case $n = 3$ we choose

$$z(G) = \frac{3}{2}\,(-\xi_2\, \underset{\sim}{m}_2 + \xi_3\, \underset{\sim}{m}_3 + \xi_4\, \underset{\sim}{m}_4)$$

Notice that here also we have (4.13) and that

$$|q|^2_{0,K} = (\xi_2^2 + \xi_3^2 + \xi_4^2\,)\ meas(K)$$

Now thanks to the identity :

$$|\underset{\sim}{m}_k \cdot \underset{\sim}{n}_{ij}|\ meas(F_{ij}) \equiv meas(K) \qquad 1 \leq k \leq 4\ ,\ 1 \leq i < j \leq 4\ ,\ i,j \neq k\ ,$$

where $\underset{\sim}{n}_{ij}$ is the unit normal vector to F_{ij} oriented in a suitable way, F_{ij} being the face whose vertices are S_i , S_j and G , we get (4.14) again, using the same techniques as in the two-dimensional case.

Let us now note that

$$\acute{b}_h(\underset{\sim}{z}\ ,\ q_1) = 0 \qquad\qquad \forall\, q_1 \in Q_h^o$$

For a given $q_1 \in Q_h^o$, let then $\underset{\sim}{y} \in \underset{\sim}{V}_h$ be a vector field satisfyin (4.11) and (4.12)

Now we set $v_h = \theta \underset{\sim}{h}^{2(s-2)}\ \underset{\sim}{z} + \underset{\sim}{y}$ where θ is a parameter independent of h , to be determined in such a way that there exists $C_4 > 0$ for which we have :

$$(4.15)\qquad \acute{b}_h(\underset{\sim}{v}_h\ ,\ q_h) \geq C_4(h^{2(s-2)}\ |q|_0^2 + |q_1|_0^2\ )$$

Since

$$\acute{b}_h(\underset{\sim}{v}_h\ , q_h) = \theta h^{2(s-2)}\ \acute{b}_h(\underset{\sim}{z}\ , q) + \acute{b}_h(\underset{\sim}{y}\ ,\ q_1) + \acute{b}_h(\underset{\sim}{y}\ ,\ q)$$

and $\acute{b}_h(\underset{\sim}{y}\ ,\ q) \leq 2\, \|\underset{\sim}{y}\|_h\ |q|_0$ we have, according to (4.11),(4.12) and (4.14) :

$$\hat{b}_h(\underset{\sim}{v}_h, q_h) \geq \theta\, h^{2(s-2)} |q|_0^2 + C_0 |q_1|_0^2 - 2\, C_2\, h^{s-2} |q_1|_0 |q|_0$$

Therefore if we choose $\theta = \dfrac{4\, C_2^2}{C_0}$ we have (4.15) with

$C_4 = min\ \left(\dfrac{2C_2^2}{C_0}\ ,\ \dfrac{C_0}{2} \right)$.

Now, according to (4.12) and (4.13) we have :

$$|\underset{\sim}{v}_h|_1 \leq C_5 h^{s-2} \left[h^{s-2} |q|_0 + |q_1| \right] .$$

Thus from (4.15) we get

$$\frac{\hat{b}_h(\underset{\sim}{v}_h, q_h)}{|\underset{\sim}{v}_h|_1} \geq C_6\, h^{2-s}\ \frac{h^{2(s-2)} |q|_0^2 + |q_1|_0^2}{h^{s-2}|q|_0 + |q_1|} \geq \frac{C_6}{2}\, h^{2-s} (h^{s-2} |q|_0 + |q|_1)$$

Recalling that $s \leq 2$, we set $C_7 = min[1, \underset{(\tau_h)_h}{min}\ h^{s-2}\ ]$.

Finally noting that

$$|q|_0 + |q_1| \geq (|q|_0^2 + |q_1|_0^2)^{1/2} = |q_h|_0$$

we have (2.6) with $\beta_h = \dfrac{C_6 C_7}{2}\ h^{2-s}$ q.e.d.

Theorem 4.1 provides all the essential tools for obtaining error estimates in the two-dimensional case. For the three-dimensional element however we still need to derive suitable bounds for the term of non-conformity $E_h(\underset{\sim}{u}\ , p\ , \underset{\sim}{w}_h)$ given by (2.9), which we do in the following :

Lemma 4.6 : Assume that the solution $(\underset{\sim}{u}, p)$ of problem $(\tilde{P}')$ belongs to $(\underset{\sim}{H}^2(\Omega) \cap \underset{\sim}{V}) \times H^1(\Omega)$. Then we have :

$$(4.16)\quad E_h(\underset{\sim}{u}, p, \underset{\sim}{w}_h) \leq C\ [h(|\underset{\sim}{u}|_2 + |p|_1) + h^{1/2} |\underset{\sim}{g}|_{0,\Gamma^*}\ ]\ \|\underset{\sim}{w}_h\|_h$$

Proof : Recalling (2.3),(2.4) and (1.7), and using Green's formula we have :

$$E_h(\underset{\sim}{u},p,\underset{\sim}{w}_h) = \int_\Omega (C\,\Delta\underset{\sim}{u} + \nabla p + \underset{\sim}{f})\cdot\underset{\sim}{w}_h\,d\underset{\sim}{x} + \sum_{K\in\tau_h^2}\int_{\partial K}\left|C\,\underset{\approx}{\varepsilon}(\underset{\sim}{u})\cdot\underset{\sim}{n}_K + p\underset{\sim}{n}_K\right|\cdot\underset{\sim}{w}_h\,ds +$$

$$+ \int_{\Gamma^*} \underset{\sim}{g}\cdot\underset{\sim}{w}_h\,ds$$

According to (1.10) and recalling that $\underset{\sim}{w}_h$ is continuous over the bases of τ_h^2 we have :

$$E_h(\underset{\sim}{u},p,\underset{\sim}{w}_h) = \sum_{K\in\tau_h^2}\sum_{i=1}^{3}\int_{F_i^K}(C\,\underset{\approx}{\varepsilon}(\underset{\sim}{u})\cdot\underset{\sim}{n}_K + p\,\underset{\sim}{n}_K)\cdot\underset{\sim}{w}_h\,ds\,.$$

Using again the continuity of $\underset{\sim}{w}_h$ over the basis B^K and (1.10), together with the continuity of $\Pi\underset{\sim}{w}_h$ over all the faces of τ_h^2, like in Lemma 4.2 we can rewrite $E_h(\underset{\sim}{u},p,\underset{\sim}{w}_h)$ as :

$$E_h(\underset{\sim}{u},p,\underset{\sim}{w}_h) = \sum_{K\in\tau_h^2}\left\{\sum_{i=1}^{3} C\,\tau_K[\underset{\sim}{\varepsilon}_i(\underset{\sim}{u}),\, w_{h_i}] + \zeta_K(p,\underset{\sim}{w}_h)\right\} - \frac{1}{3}\int_{\Gamma^*}\underset{\sim}{g}\cdot(\underset{\sim}{w}_h - \Pi\underset{\sim}{w}_h)\,ds$$

where $\underset{\sim}{\varepsilon}_i(u) = [\varepsilon_{i1}(\underset{\sim}{u}),\varepsilon_{i2}(\underset{\sim}{u}),\varepsilon_{i3}(\underset{\sim}{u})]$, and $\tau_K : \underset{\sim}{H}^1(K)\times P_a \to \mathbb{R}$

and $\zeta_K : H^1(K)\times\underset{\sim}{P}_a \to \mathbb{R}$ are given respectively by :

$$\tau_K(\underset{\sim}{z},w) = \sum_{i=1}^{3}\int_{F_i^K}\underset{\sim}{z}\cdot\underset{\sim}{n}_K(w-\Pi w)\,ds + \frac{1}{3}\int_{B^K}\underset{\sim}{z}\cdot\underset{\sim}{n}_K(w-\Pi w)\,ds$$

$$\zeta_K(p,\underset{\sim}{w}) = \sum_{i=1}^{3}\int_{F_i^K} p\,\underset{\sim}{n}_K\cdot(\underset{\sim}{w}-\Pi\underset{\sim}{w})\,ds + \frac{1}{3}\int_{B^K} p\underset{\sim}{n}_K\cdot(\underset{\sim}{w}-\Pi\underset{\sim}{w})\,ds$$

Now we note that τ_K and ζ_K can be treated essentially in the same way as we did for σ_K in Lemma 4.2, which allows us to conclude that :

$$\tau_K(\underset{\sim}{z},w) \le C\,h|\underset{\sim}{z}|_{1,K}|w|_{1,K} \qquad \forall\,(\underset{\sim}{z},w)\in\underset{\sim}{H}^1(K)\times P_a$$

and $\zeta_K(p,\underset{\sim}{w}) \le C\,h\,|p|_{1,K}\,|\underset{\sim}{w}|_{1,K} \qquad \forall\,(p,\underset{\sim}{w})\in H^1(K)\times\underset{\sim}{P}_a$.

Since $\sum_{i=1}^{3}|\underset{\sim}{\varepsilon}_i(\underset{\sim}{u})|_{1,K}\,|w_{h_i}|_{1,K} \le C\,|\underset{\sim}{u}|_{2,K}\,|\underset{\sim}{w}_h|_{1,K}$

we obtain :

$$(4.17)\quad E_h(\underset{\sim}{u},p,\underset{\sim}{w}_h) \le C\, h\left|\,|\underset{\sim}{u}|_2 + |p|_1\right| + \left|\frac{1}{3}\int_{\Gamma^*} \underset{\sim}{g}\cdot(\underset{\sim}{w}_h - \Pi\underset{\sim}{w}_h)\, ds\right|$$

Now all that remains to be done is to estimate the above integral over Γ^*. We have :

$$\int_{\Gamma^*} \underset{\sim}{g}\cdot(\underset{\sim}{w}_h - \Pi\underset{\sim}{w}_h)\, ds = \sum_{B\subset\Gamma^*}\int_B \underset{\sim}{g}\cdot(\underset{\sim}{w}_h - \Pi\underset{\sim}{w}_h)\, ds$$

B being the bases of τ_h^2 .

Let $\eta_K : P_a \longrightarrow \mathbb{R}$ be given by

$$\eta_K(w) = \int_B g(w - \Pi w)\, ds \quad , \text{ where } g \in L^2(B)$$

Using the customary affine transformation $A : K \to \hat{K}$ we define $\hat{\eta} : \hat{P}_a \to \mathbb{R}$ by $\hat{\eta} = \int_{\hat{B}} \hat{g}(\hat{w} - \Pi\hat{w})\, d\hat{s}$.

Clearly we have :

$$|\eta_K(w)| \le C\, h^2\, |\hat{\eta}(\hat{w})| \qquad K \in \tau_h^2 \;,$$

and

$$||\hat{\eta}|| = \sup_{\hat{w}\in\hat{P}_a} \frac{\hat{\eta}(\hat{w})}{\|\hat{w}\|_{1,\hat{K}}} \le C\, |\hat{g}|_{0,\hat{B}} \;.$$

Now we note that $\hat{\eta}(\hat{w}) = 0 \quad \forall\ \hat{w} \in P_\ell$, $\ell = 0$ or 1 . Hence by the Bramble-Hilbert Lemma [6] we get :

$$|\hat{\eta}(\hat{w})| \le \hat{C}\, |\hat{g}|_{0,\hat{B}}\, |\hat{w}|_{1,\hat{K}}$$

which by standard transformations gives :

$$|\eta_K(w)| \le C\, h^{1/2} |g|_{0,B} |w|_{1,K}$$

Now setting successively $g = g_i$ and $w = w_{h_i}$, $i = 1,2,3$ we get

$$\int_{\Gamma^*} \underset{\sim}{g}\cdot(\underset{\sim}{w}_h - \Pi\underset{\sim}{w}_h)\, ds \le C\, h^{1/2} |\underset{\sim}{g}|_{0,\Gamma^*} \|\underset{\sim}{w}_h\|_h$$

which, together with (4.17), implies (4.16) . q.e.d.

REMARK : Using the same arguments as in the proof of Lemma 4.6 and the results of [24] for nonconforming finite element approximation of elliptic boundary value problems, one can prove that for the problem

$$\begin{cases} \text{Find } z \in V \text{ such that} \\ (\nabla z \, , \, \nabla v)_0 = (f, v)_0 + (g, v)_{0,\Gamma^*} \quad \forall v \in V \end{cases}$$

approximated by

$$\begin{cases} \text{Find } z_h \in V_h \text{ such that} \\ (z_h \, , \, v_h)_h = (f, v_h)_0 + (g, v_h)_{0,\Gamma^*} \quad \forall v_h \in V_h \end{cases}$$

the following estimate holds :

$$\| z - z_h \|_h \le C\, h \, | z |_2 + \overline{C}\, h^{1/2} |g|_{0,\Gamma^*} \quad .$$

The optimality of this error estimate has been confirmed by numerical experiments . □

Now, in order to obtain final convergence results we notice that if solution (u,p) to problem $(\hat{P}')$ is so that $u \in H^d(\Omega)$, we have $p \in H^{d-1}(\Omega)$.

Since $P_1 \subset P_a \subset P_2$, assuming that $1 \le d \le 2$, we can use standard approximation results (see e.g. [2] and [10, Vol. 2, page 16]) , and we get :

$$\inf_{v_h \in V_h} \| u - v_h \|_h \le C\, h^{d-1} \| u \|_d$$

$$\inf_{q_h \in Q_h} | p - q_h |_0 \le C\, h^{d-1} \| p \|_{d-1}$$

Now recalling (2.7) and (2.8) we obtain the following error estimates for the case where $(\hat{P}')$ is approximated by $(\hat{P}_h)_h$ with $(V_h)_h$ and $(Q_h)_h$ associated with a regular family of partitions $(\tau_h^2)_h$ defined in Section 2 :

<u>*Theorem 4.2.*</u> : Let $n = 2$ and Γ_0 and Γ^* be such that the solution to

problem (4.1) belongs to $H^{s}(\Omega)$, $3/2 - \varepsilon \leq s \leq 2$, and the solution $(\underset{\sim}{u},p)$ to problem $(\acute{P}')$ belongs to $\underset{\sim}{H}^{d}(\Omega) \times H^{d-1}(\Omega)$, $1 \leq d \leq 2$. Then the couple $(\underset{\sim}{u}_h,p_h)$ solution to $(\acute{P}_h)$ satisfies :

$$(4.18) \quad |\underset{\sim}{u}-\underset{\sim}{u}_h|_1 \leq C \,[\, h^{d+s-3}\|\underset{\sim}{u}\|_d + h^{d-1}\|p\|_{d-1} \,]$$

$$(4.19) \quad |p-p_h|_0 \leq C \,[\, h^{d+2s-5}\, \|\underset{\sim}{u}\|_d + h^{d+s-3}\|p\|_{d-1} \,] \qquad \square$$

Theorem 4.3 : Let $n = 3$ and Γ_0 and Γ^* be such that the solution to problem (4.1) belongs to $H^2(\Omega)$ and the solution to $(\acute{P}')$ belongs to $\underset{\sim}{H}^2(\Omega) \times H^1(\Omega)$. Then the sequence $((\underset{\sim}{u}_h , p_h))_h$ converges to $(\underset{\sim}{u},p)$ as $h \to 0$, in the following sense :

$$(4.20) \quad \| \underset{\sim}{u}-\underset{\sim}{u}_h \|_h + |p-p_h|_0 \leq C\, h\, [|\underset{\sim}{u}|_2 + |p|_1] + \bar{C}\, h^{1/2}|\underset{\sim}{g}|_{0,\Gamma^*} \qquad \square$$

It is clear that (4.18) ~ (4.20) imply convergence if $s = d = 2$, but the assumptions we have to make on Γ for such a regularity are rather restrictive. Nevertheless we would like to mention two important cases where this optimal $O(h)$ estimate for the $\underset{\sim}{H}^1 \times L^2$ error can be attained.

First, if for $n = 3$ no surface forces are acting on Γ^* , the $O(h^{1/2})$ term vanishes. Secondly the same happens if we simply have $\Gamma^* = \emptyset$. Notice that the latter situation includes the Stokes problem as a special case. It should also be noted that if $\Gamma^* = \emptyset$, both assumptions of Theorem 4.3 and the values $s = d = 2$ in theorem 4.2 are realistic, specially if Ω is convex [14] .

A further remark for the case $n = 3$ is the following. As we mentioned before, by introducing simple modifications in the analysis that was carried out in this section, we could prove the existence of $\beta_h > 0$ such that the discrete Brezzi condition (2.6) holds for $n = 3$, irrespective of the regularity of the solution of (4.1). The key to the problem is to define the vector field $\underset{\sim}{v}_h$ of Lemma 4.4 independently of the orthogonal projection $r_h\, \underset{\sim}{v}$, in a way proposed by J.M. Thomas [25]. Since the same technique will be used in Section 5, we prefer to omit the proof here.

REMARKS 1. If $\Gamma^* \neq \phi$ only in few cases one can expect the solution of (4.1) to belong to $H^2(\Omega)$. Among those we mention the case where both $\overline{\Gamma}^*$ and $\overline{\Gamma}_a$ are the disjoint boundaries of two convex domains such that one lies in the interior of the other. Notice that such a situation appears to be unlikely to occur in practice for $n = 3$, even if in the two-dimensional case this would correspond to a simplified model of a tire.

2. A further word of caution related to regularity results is as follows : It seems reasonable to consider that $(\underset{\sim}{u},p) \in \underset{\sim}{H}^2(\Omega)\times H^1(\Omega)$ if Γ_a and Γ^* are not only as above but also if $\Gamma^* = \phi$. However, only for the latter case (Stokes problem) a rigorous justification of this regularity result seems to be available [18]. □

Let us now discuss in detail the two-dimensional case. Although Theorem 4.2 has a scope much wider then Theorem 4.3, we must be careful in asserting the convergence of $(\underset{\sim}{u}_h)_h$ and $(p_h)_h$. Just to have a clear look at this question, let us assume that Ω is convex and that $d = s$ (for such Ω this is actually to be expected in many cases including the Stokes problem).

Now if $s \leq 3/2$ we cannot guarantee that the displacements converge. This immediately rules out the case where $\overline{\Gamma}_0$ and $\overline{\Gamma}^*$ intersect at a point that is not a vertex of Γ. However, except for this case, according to the regularity results for the solution of (4.1) [12], we have convergence of the displacements whichever the angle of the corner of Γ where $\overline{\Gamma}_0$ intersects $\overline{\Gamma}^*$. In particular, if none of those angles is greater than $\Pi/2$ we have :

$$|\underset{\sim}{u} - \underset{\sim}{u}_h|_1 \leq C\, h^{1-\varepsilon}\, [\, \|\underset{\sim}{u}\|_{2-\varepsilon} + \|p\|_{1-\varepsilon}\,]$$

For example, if Ω is a rectangle having one or more edges fixed, we have practically optimal convergence rates.

Now, as for the pressure, the condition of convergence is more stringent, for we need $s < 5/3$. This value corresponds to an upper bound of $3\Pi/4$ for the angle of Γ at a vertex where $\overline{\Gamma}_0$ intersects $\overline{\Gamma}^*$.

Summing up, it is important to emphasize that for a convex Ω , in both cases $n = 2$ and $n = 3$, optimal $C\, h[|\underset{\sim}{u}|_2 + |p|_1]$ error bounds for displacements and pressure are obtained if $\Gamma^* = \emptyset$. This confirms the superiority of τ_h^2 over triangulation τ_h^1 (see e.g.[13] and [15]) .

5. The discrete problem in the nonlinear case.

The purpose of this section is the study of a discrete analogue of $(\widetilde{P}')$, with an emphasis on the asymmetric elements defined in Section 3. For the definition of the discrete problem below we will follow the main lines of the work of Le Tallec [16] .

We first need a finite dimensional space of displacement vector fields $\underset{\sim}{V}_h$ to approximate $\underset{\sim}{V}^{t}$. For the sake of simplicity we shall confine ourselves to the case where $\underset{\sim}{V}_h \subset \underset{\sim}{V}^{t}$, although the nonconforming case can be treated in a similar way to the one which is given here. Notice that in so doing, we are ruling out the study of our three-dimensional asymmetric element.

We are next given a finite dimensional space of pressures Q_h to approximate Q^t , with $Q_h \subset Q^t$. Now, like in the linear case, we weaken the requirement that the approximation $\underset{\sim}{u}_h \in \underset{\sim}{V}_h$ of the solution $\underset{\sim}{u}$ to problem $(\widetilde{P})$ satisfy exactly (1.1), in the following way :

The incompressibility condition is to be satisfied only at those points of Ω to which we attach the degrees of freedom of Q_h . This is equivalent to require that $\underset{\sim}{u}_h$ belong to an approximation $\widetilde{X}_h$ of $\widetilde{X}$ defined by :

$$\widetilde{X}_h = \{ \underset{\sim}{v}_h \;/\; \underset{\sim}{v}_h \in \underset{\sim}{V}_h \,,\; \widetilde{b}_h(\underset{\sim}{v}_h, q_h) = 0 \qquad \forall\, q_h \in Q_h\}$$

where $\widetilde{b}_h$ is a suitable approximation of $\widetilde{b}$ given by (1.13) .

A natural way of defining $\tilde{b}_h$ is to set

$$(5.1)\qquad \tilde{b}_h(\underset{\sim}{v}_h, q_h) = \sum_{K\in\tau_h} b_K(\underset{\sim}{v}_h, q_h)$$

where b_K corresponds to an approximation of the integral of (1.13), restricted to element K, whose quadrature points are those associated with the degrees of freedom of Q_h. We consider two possibilities of performing this numerical quadrature, according to the way of defining the elements of τ_h.

To be more specific, if the domain Ω is a polygon like we have considered so far, the elements of the partition τ_h are as prescribed in Section 2. Notice that in this case we have :

<u>*Case i)*</u> Every $K \in \tau_h$ is the reciprocal image of the usual reference element $\hat{K}$ (see Figure 5.1) by an affine transformation $A_K : K \to \hat{K}$.

In this case we define the approximation of $\int_K q_h\,[det(\underset{\approx}{I} + \underset{\sim}{\nabla}\,\underset{\sim}{v}_h) - 1]\,d\underset{\sim}{x}$ to be :

$$(5.2)\qquad \tilde{b}_K(\underset{\sim}{v}_h, q_h) = \sum_{j=1}^{m} \omega_j\, q_h(x_j^K)\,[det\,(\underset{\approx}{I} + \underset{\sim}{\nabla}\,\underset{\sim}{v}_h) - 1]_{/\underset{\sim}{x}_j^K}\quad meas(K)$$

where $\{x_j^K\}_{j=1}^m$ is the set of points used to define $q_{h/K}$, and the $\omega_j^{\prime}s$ are the weights of the numerical quadrature formula.

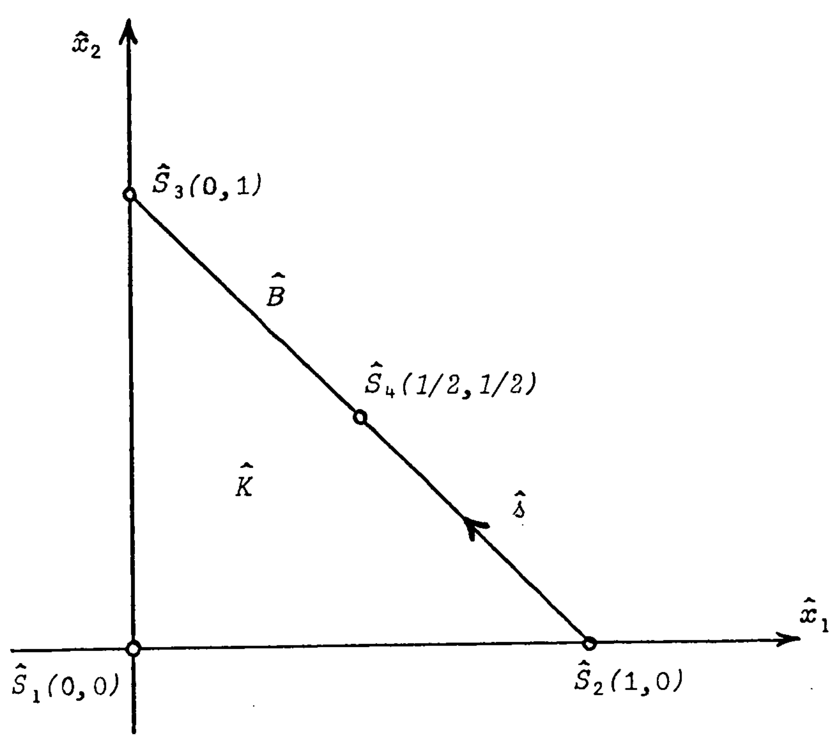
On the other hand, if Ω has a curved boundary, it may be interesting to partition it into curved elements defined in the classical way, namely :

<u>*Case ii)*</u> Every $K \in \tau_h$ is the reciprocal image of $\hat{K}$ by a bijective isoparametric transformation $\mathcal{A}_K : K \to \hat{K}$. This means that $\mathcal{A}_K^{-1}(\hat{\underset{\sim}{x}}) = [a_1^K(\hat{\underset{\sim}{x}}), \ldots, a_n^K(\hat{\underset{\sim}{x}})]$, where $a_i^K \in \hat{P}$, $1 \leq i \leq n$, $\hat{P}$ being a space of shape functions defined over $\hat{K}$, such that $\hat{v}_h = v_{h/K} \circ \mathcal{A}_K \in \hat{P}$ $\forall\, v_h \in V_h$ and $\forall\, K \in \tau_h$.

In this case the approximation of $\int_K q_h[det(\underset{\approx}{I} + \underset{\sim}{\nabla}\,\underset{\sim}{v}_h) - 1]\,d\underset{\sim}{x}$ is given by :

(5.3) $\mathcal{b}_K(\underset{\sim}{v}_h, q_h) = \sum_{j=1}^{m} \omega_j \, \hat{q}_h(\hat{\underset{\sim}{x}}_j)[\det \hat{\underset{\sim}{\nabla}}(\hat{\underset{\sim}{v}}_h + \mathcal{A}_K^{-1}) - \det \hat{\underset{\sim}{\nabla}} \mathcal{A}_K^{-1}]/\hat{\underset{\sim}{x}}_j \; meas(\hat{K})$

where $\{\hat{\underset{\sim}{x}}_j\}_{j=1}^m$ is the set of points of $\hat{K}$ whose reciprocal images through $\mathcal{A}_K$ are the points of K to which we attach the degrees of freedom of Q_h.



The reference element $\hat{K}$ for $n=2$

Figure 5.1

Now, taking into consideration (5.1), we can verify that in both *cases* *i)* and *ii)* we have for all $\underset{\sim}{v}_h \in \tilde{X}_h$:

(5.4) $\det(\underset{\approx}{I} + \underset{\sim}{\nabla}\, \underset{\sim}{v}_h)/\underset{\sim}{x}_j^K = 1 \qquad \forall j \,,\ 1 \le j \le m \quad \text{and} \quad \forall\, K \in \tau_h .$

Indeed, in *case i)* this is trivial provided *meas (K)* in nonzero for all $K \in \tau_h$. On the other hand, from the well-known formula of Calculus [5] we have :

$$J(\underset{\sim}{v}) = \hat{J}(\hat{\underset{\sim}{v}})\, J(A) \quad \text{where } \hat{\underset{\approx}{x}} = A(\underset{\approx}{x}) \text{ and } \hat{\underset{\sim}{v}} \circ A(\underset{\sim}{x}) = \underset{\sim}{v}(\underset{\sim}{x})$$

Thus we see that (5.4) also holds for *case ii)* by setting $\underset{\sim}{v}(\underset{\sim}{x}) = \underset{\sim}{v}_h(\underset{\sim}{x}) + \underset{\sim}{x}$ and $A = \mathcal{A}_K$, and taking into account the identity $J^{-1}(A) = \hat{J}(A^{-1})$.

REMARK : If the $\hat{\underset{\sim}{x}}_j$'s are the points of a quadrature formula that integrates exactly functions of form $\hat{J}(\hat{\underset{\sim}{v}}_h)$ over $\hat{K}$ $\forall \hat{\underset{\sim}{v}}_h \in \hat{P}$, then like in [22] we can draw the following conclusion :

If (5.4) holds and $\sum_{j=1}^{m} \omega_j = 1$, we have $meas(\tilde{K}) = meas(K)$ $\forall K \in \tau_h$, $\tilde{K}$ being the deformed state of K induced by $\underset{\sim}{v}_h$. □

Now we further set

$$(5.5) \quad \tilde{b}_h'(\underset{\sim}{u}_h,\underset{\sim}{v}_h,q_h) = \frac{\partial \tilde{b}_h}{\partial \underset{\sim}{\nabla} \underset{\sim}{u}_h} \cdot \underset{\sim}{\nabla}\, \underset{\sim}{v}_h$$

and we define the discrete mixed formulation of problem $(\tilde{P})$ to be :

$$(\tilde{P}_h) \quad \begin{cases} Find (\underset{\sim}{u}_h , p_h) \in \underset{\sim}{V}_h \times Q_h \text{ such that} \\ \tilde{a}(\underset{\sim}{u}_h , \underset{\sim}{v}_h) + \tilde{b}_h'(\underset{\sim}{u}_h , \underset{\sim}{v}_h , q_h) = \tilde{L}(\underset{\sim}{v}_h) & \forall \underset{\sim}{v}_h \in \underset{\sim}{V}_h \\ \tilde{b}_h(\underset{\sim}{u}_h , q_h) = 0 & \forall q_h \in Q_h \end{cases}$$

According to [16], the existence of a solution to problem $(\tilde{P}_h)$ is directly dependent on the validity of a nonlinear discrete Brezzi-type compatibility condition between the spaces $\underset{\sim}{V}_h$ and Q_h . However now this condition must be expressed in terms of the vector field $\underset{\sim}{u}_h$ itself. Since $\underset{\sim}{u}_h$ is supposed to minimize the energy $\tilde{W}$ in some sense, the following result proved in [16], Theorem 4.1 is of crucial importance :

The problem :

(5.6) *Find* $\underset{\sim}{u}_h \in \tilde{X}_h$ to minimize $\tilde{W}(\underset{\sim}{u}_h)$ over $\tilde{X}_h$ has a solution .

Now, let $\underset{\sim}{u}_h$ be a local minimum of $\tilde{W}$. Let also $\|.\|$ be the norm of $\underset{\sim}{V}_h$ and $|.|$ be the norm of Q_h induced respectively by $\underset{\sim}{V}$ and $L^2(\Omega)$. The nonlinear compatibility condition can be stated as follows :

There exists $\beta_h > 0$ such that

$$(5.7)\quad \sup_{\underset{\sim}{v}_h \in \underset{\sim}{V}_h} \frac{\tilde{b}'_h(\underset{\sim}{u}_h, \underset{\sim}{v}_h, q_h)}{\|\underset{\sim}{v}_h\|} \geq \beta_h |q_h| \qquad \forall q_h \in Q_h$$

According to [16], Theorem 4.3, if condition (5.7) is fulfilled, there exists a unique pressure $p_h \in Q_h$ such that $(\underset{\sim}{u}_h , p_h)$ is a solution to $(\tilde{P}_h)$.

Let us now examine the particular case of the spaces $\underset{\sim}{V}_h$ and Q_h defined in Section 2 (for $n = 2$). In this case we have $m = 1$, $\omega_1 = 1$, and the quadrature point $\underset{\sim}{x}_1^K$ is the centroid of K in *case i)* and the image of the centroid of $\hat{K}$ through transformation $\mathcal{A}_K^{-1}$ in *case ii)* .

It is then possible to verify, using arguments already developped in Section 3, that in both cases we have :

$$\tilde{b}_h(\underset{\sim}{u}_h , q_h) = \int_{\Omega_h} q_h[det(\underset{\approx}{I} + \underset{\sim}{\nabla}\, \underset{\sim}{u}_h) -1]\, d\underset{\approx}{x}$$

and

$$\tilde{b}'_h(\underset{\sim}{u}_h , \underset{\sim}{v}_h , q_h) = \int_{\Omega_h} q_h\, [adj(\underset{\approx}{I} + \underset{\sim}{\nabla}\, \underset{\sim}{u}_h)]^{T} \cdot \underset{\sim}{\nabla}\, \underset{\sim}{v}_h\, d\underset{\approx}{x}$$

where $\overline{\Omega}_h = \bigcup_{K\in\tau_h} K$ and $\Omega_h = \overset{o}{\overline{\Omega}}_h$. This means that, at least when $\Omega_h = \Omega$ we have $\tilde{b}_h \equiv \tilde{b}$ and $\tilde{b}'_h \equiv \tilde{b}'$.

REMARKS : 1) Definition (5.1) of the approximate functionals $\tilde{b}_h$ allows us to take into account implicitly the case of a nonconforming space $\underset{\sim}{V}_h$. Notice that in the case $n = 3$ the above identities would not hold only because of the nonconformity of $\underset{\sim}{V}_h$.

2) Strictly speaking, if $\Omega \neq \Omega_h$ we should refefine problem $(\tilde{P}_h)$ by replacing $\tilde{a}$ and $\tilde{L}$ by approximate functionals $\tilde{a}_h$ and $\tilde{L}_h$ that take into account integration over Ω_h rather than over Ω . □

Let us now prove that, under suitable assumptions on $\underset{\sim}{u}_h$, the compatibility condition (5.7) is satisfied. We treat separately *cases i)* and *ii)* for the two-dimensional asymmetric element. Like in Section 4 we shall confine ourselves to the case of a partition of type τ_h^2. Nevertheless for partition τ_h^1, in some particular cases, the existence of a solution to problem $(\tilde{P}_h)$ can be proved by introducing simple modifications in the analysis of [15] given in Chapter 4 for the $Q_1 \times P_0$ element. Actually, at the end of this section we give a short account of the arguments used in [15] together with these modifications.

For the sake of simplicity we will work with the linear manifold $\underset{\sim}{V}_h^x$ of $\underset{\sim}{V}_h$, defined to be $\underset{\sim}{x} + \underset{\sim}{V}_h$. We also define the following subset of $\underset{\sim}{V}_h^x$:

$$\tilde{X}_h^x = \{\underset{\sim}{u}_h^x \;/\; \underset{\sim}{u}_h^x - \underset{\sim}{x} \in \tilde{X}_h\}$$

In both cases *i)* and *ii)* we shall prove the validity of (5.7) under the following basic assumption on $\underset{\sim}{u}_h$.

ASSUMPTION A) Let $\Pi\underset{\sim}{u}_h^x$ denote the piecewise linear interpolate of $\underset{\sim}{u}_h^x$ defined in Section 3. The triangulation $\hat{\tau}_h^2$ of $\hat{\Omega}_h = \Pi\, \underset{\sim}{u}_h^x(\Omega_h)$ defined to be :

$$\hat{\tau}_h^2 = \{\hat{K} \;/\; \hat{K} = \Pi\, \underset{\sim}{u}_h^x\,(K) \quad , K \in \tau_h^2\} \quad ,$$

is such that there exists a constant $\alpha > 0$ for which we have :

$$\frac{1}{\alpha}\; area(K) \geq area(\hat{K}) \geq \alpha\; area(K) \qquad \forall\; K \in \tau_h^2 \;. \qquad \square$$

Notice that Assumption A) implies that $J(\Pi\underset{\sim}{u}_h^x) > 0$ a.e. in Ω_h. It also implies that $\hat{\tau}_h^2$ belongs to a regular family of partitions $\{\hat{\tau}_h^2\}_h$, whenever $\underset{\sim}{u}_h$ belongs to a bounded subset of $W^{1,\infty}(\Omega_h)$ $\forall\; h$.

Indeed, in this case if we set :

$$\hat{h} = \max_{K \in \tau_h^2} \{\, \hat{h}_K = diameter\ of\ \hat{K}\}$$

and

$$\hat{\rho} = \min_{K \in \tau_h^2} \{\hat{\rho}_K = \text{diameter of the inscribed circle in } \hat{K}\} \quad ,$$

we have $\hat{\rho}\,\hat{h}^{-1} \geq \hat{c} \quad \forall h$, where $\hat{c}$ is given by $\frac{2\alpha}{3}\,\frac{c^2}{U^2}$ with $U = \max_h |\underset{\sim}{u}_h|_{1,\infty}$, c being the constant such that $\rho h^{-1} \geq c > 0$ $\forall \tau_h^2 \in \{\tau_h^2\}_h$, as one can easily verify.

We now consider *case i)* :

In this case both $\Omega = \Omega_h$ and $\hat{\Omega} = \hat{\Omega}_h$ are polygons. Thus, since $\Pi \underset{\sim}{u}_h^x$ defines an affine transformation over each triangle K onto $\hat{K}$, we can define a space $\hat{\underset{\sim}{V}}_h$ over $\hat{\Omega}_h$ associated with $\hat{\tau}_h^2$ in the same way as $\underset{\sim}{V}_h$ is associated with τ_h^2, and $\hat{\underset{\sim}{V}}_h$ will have the same structure as $\underset{\sim}{V}_h$.

Also we define $\hat{Q}_h$ to be the space of pressures analogous to Q_h for triangulation $\hat{\tau}_h^2$.

Let us first consider the subspace $\hat{Q}_h^0$ of those pressures that are constant over $\hat{K}$, K being a triangle of τ_h. According to [7] Lemma $C2$, if $\hat{\underset{\sim}{V}} = \{ \hat{\underset{\sim}{v}} \,/\, \hat{\underset{\sim}{v}} \in H^1(\hat{\Omega})$, $\underset{\sim}{v} = \underset{\sim}{0}$ on $\hat{\Gamma}_0 \equiv \Gamma_0 \}$, $\forall\, \hat{q}_h^o \in Q_h^o$, $\exists\, \hat{\underset{\sim}{v}} \in \hat{\underset{\sim}{V}}$ such that

(5.8) $\quad (\hat{div}\, \hat{\underset{\sim}{v}} \,,\, \hat{q}_h^o)_{0,\hat{\Omega}} \geq \hat{\beta}_0 \,|\, \hat{q}_h^o \,|^2_{0,\hat{\Omega}}$

(5.9) $\quad |\hat{\underset{\sim}{v}}|_{1,\hat{\Omega}} \leq C_0 |\hat{q}^o|_{0,\hat{\Omega}}$

where $\hat{\beta}_0 > 0$ and C_0 are independent of $\hat{q}_h^o$.

<u>*Lemma 5.1*</u>. There exist constants $\hat{\beta}_0 > 0$ and $\hat{C}_0$ such that with every $\hat{q}_h^o \in \hat{Q}_h^o$ we can associate a $\hat{\underset{\sim}{w}}_h \in \hat{\underset{\sim}{V}}_h$ that satisfies :

(5.10) $\quad \hat{\underset{\sim}{w}}_h(\hat{S}) = 0$ for all vertices $\hat{S}$ of a supertriangle $\bigcup_{i=1}^{3} \hat{K}_i$, the K_i's being the triangles of a supertriangle $K \subset \tau_h$, where τ_h is the first partition of Ω upon which τ_h^2 is constructed.

(5.11) $\quad (\hat{div}\, \hat{\underset{\sim}{w}}_h \,,\, \hat{q}_h^o\,)_{0,\hat{\Omega}} \geq \hat{\beta}_0 |\hat{q}_h^o|_{0,\hat{\Omega}}$

(5.12) $\quad |\hat{\underset{\sim}{w}}_h|_{1,\hat{\Omega}} \le \hat{C}_0 \, |\hat{q}_h^o|_{0,\hat{\Omega}}$

Proof : Let $\hat{\underset{\sim}{v}} \in \hat{\underset{\sim}{V}}$ satisfy (5.8) and (5.9). We associate with $\hat{\underset{\sim}{v}}$ a vector field $\hat{\underset{\sim}{w}}_h \in \hat{\underset{\sim}{V}}_h$ such that $\hat{w}_{h/\hat{K}}$ satisfies $\forall K \in \tau_h$:

$$\hat{\underset{\sim}{w}}_h(\hat{S}) = 0 \text{ if } \hat{S} \text{ is a vertex of } \hat{K}_i \quad i = 1,2,3,$$

$$\hat{\underset{\sim}{w}}_h(\hat{M}_i) = \frac{3}{2} \frac{\int_{\hat{B}_i} \hat{\underset{\sim}{v}} \, ds}{meas(\hat{B}_i)}$$

where $\hat{B}_i$ is the basis of $\hat{K}_i$ and $\hat{M}_i$ is its mid-point. (5.10) is thus fulfilled.

Like in Lemma 4.4, letting $\hat{\tau}_h$ be the partition of $\hat{\Omega}$ into supertriangles $\hat{K}$, $K \in \tau_h$, we have :

$$\int_{\hat{K}} \hat{div} \, \hat{\underset{\sim}{w}}_h \, d\underset{\sim}{x} = \int_{\hat{K}} \hat{div} \, \hat{\underset{\sim}{v}} \, d\underset{\sim}{x} \qquad \forall \hat{K} \in \hat{\tau}_h$$

This yields :

$$(\hat{div} \, \hat{\underset{\sim}{w}}_h \, , \, \hat{q}_h^o)_{0,\hat{\Omega}} = (\hat{div} \, \hat{\underset{\sim}{v}} \, , \, \hat{q}_h^o)_{0,\hat{\Omega}}$$

which in turn gives (5.11), taking into account (5.8) .

In order to prove (5.12) we first use the Trace Theorem and we get :

$$|\hat{\underset{\sim}{w}}_h|_{1,\hat{K}_i} = \hat{\underset{\sim}{w}}_h(\hat{M}_i) \, C(\hat{K}_i) \le C'(\hat{K}_i) \, \|\hat{\underset{\sim}{v}}\|_{1,\hat{K}_i} \quad ,$$

which according to Assumption A) yields :

$$|\hat{\underset{\sim}{w}}_h|_{1,\hat{\Omega}} \le C \, (\hat{\Omega} \, , \, \hat{\underset{\sim}{u}}_h) \, |\hat{\underset{\sim}{v}}|_{1,\hat{\Omega}} \qquad \text{with} \quad C < \infty$$

Thus using (5.9) we get (5.12) with $\hat{C}_0 = C_0 C$. q.e.d.

Let now $s_i = area \, (\hat{K}_i)$ $\quad i = 1,2,3$. Without loss of generality we can assume that $s_1 \ge s_2 \ge s_3$.

Let $\hat{Q}_h^1$ be the subspace of $\hat{Q}_h$ generated by the set of orthogonal functions $\{\eta_2^K , \eta_3^K\}_{K\in\tau_h}$ such that $supp(\eta_i^K) \subset \hat{K}$ $i = 2,3$, with :

$$\begin{cases} \eta_2^K = -1 & \text{if } \hat{x} \in \hat{K}_1 \\ \eta_2^K = \dfrac{s_1}{s_2 + s_3} & \text{if } \hat{x} \in \hat{K}_2 \cup \hat{K}_3 \end{cases}$$

$$\begin{cases} \eta_3^K = 0 & \text{if } \hat{x} \in \hat{K}_1 \\ \eta_3^K = -1 & \text{if } \hat{x} \in \hat{K}_2 \\ \eta_3^K = \dfrac{s_2}{s_3} & \text{if } \hat{x} \in \hat{K}_3 \end{cases}$$

As one can easily verify we have $\eta_i^K \perp q_h^o$ $\forall q_h^o \in \hat{Q}_h^o$, $i = 2,3$, and $\hat{Q}_h = \hat{Q}_h^o \oplus \hat{Q}_h^1$.

Let now q_h^1 be any function of $\hat{Q}_h^1$. We can write :

(5.13) $$q_h^1 = \sum_{K\in\tau_h} (q_2^K \eta_2^K + q_3^K \eta_3^K)$$

where the q_i^K's are given scalars.

Lemma 5.2 : If Assumption A) holds, for every $\hat{q}_h \in \hat{Q}_h$ there exists $\hat{v}_h \in \hat{V}_h$ satisfying (5.10) together with

$$\frac{(\hat{div}\, \hat{v}_h , \hat{q}_h)_{0,\hat{\Omega}}}{|\hat{v}_h|_{1,\hat{\Omega}}} \geq \hat{\beta}_h |\hat{q}_h|_{0,\hat{\Omega}}$$

for some $\hat{\beta}_h > 0$ independent of $\hat{q}_h$.

Proof : Let $\hat{q}_h^0 = q_h^0 + q_h^1$ where $q_h^0 \in \hat{Q}_h^0$ and $q_h^1 \in \hat{Q}_h^1$.

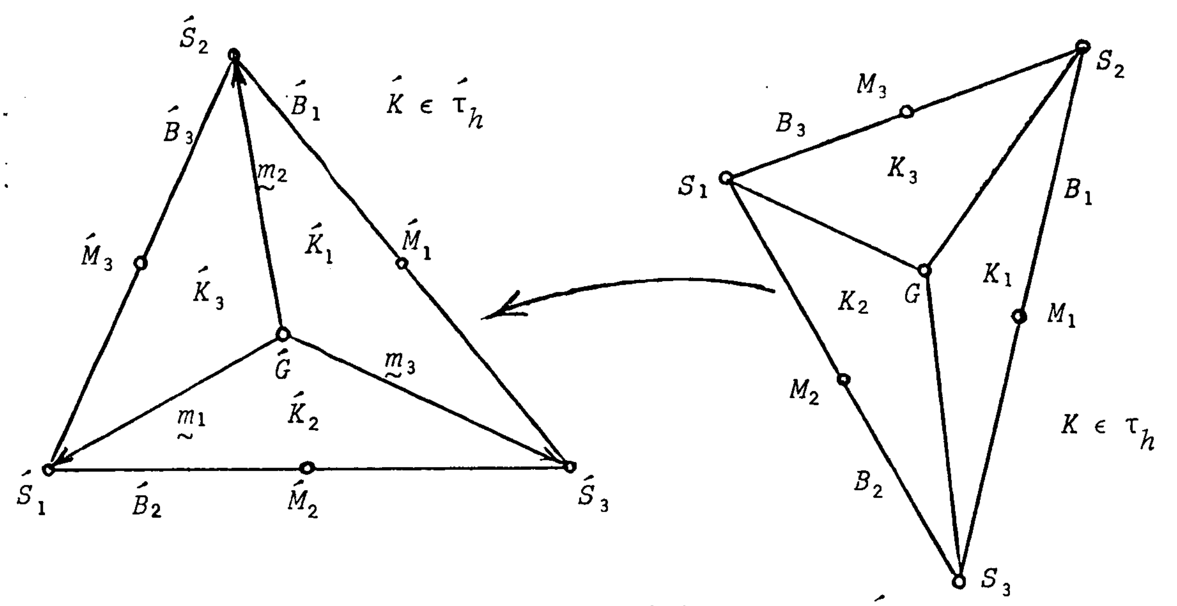
We first construct a vector field $\hat{z}_h \in \hat{V}_h$ satisfying (5.10) in the following way :

$\underset{\sim}{\hat{z}}_h(\hat{M}) = \underset{\sim}{0}$ for every mid-point $\hat{M}$ of the basis of $\hat{K}$, $\hat{K} \in \hat{\tau}_h^2$.

If $\hat{G}$ is the common vertex of $\hat{K}_1$, $\hat{K}_2$ and $\hat{K}_3$ we set :

$$\underset{\sim}{\hat{z}}_h(\hat{G}) = -q_2^K \underset{\sim}{m}_2 + q_3^K \underset{\sim}{m}_3$$

where the $\underset{\sim}{m}_i$'s are the oriented edges $\hat{G}\hat{S}_i$ of the $\hat{K}_i'$ s ,as indicated in Figure 5.2 below, and we refer to (5.13) for the meaning of q_i^K.



Superelements of partitions τ_h and $\hat{\tau}_h$

Figure 5.2

First of all, using Assumption A) one can easily estimate :

(5.15) $$|\underset{\sim}{\hat{z}}_h|_{1,\hat{\Omega}} \leq C(\underset{\sim}{u}_h)\ |q_h^1|_{0,\hat{\Omega}}$$

Now dropping the superscript K we get after simple calculations :

$$|q_h^1|^2_{0,\hat{K}} = q_2^2 |n_2|^2_{0,\hat{K}} + q_3^2 |n_3|^2_{0,\hat{K}} \leq 2 \frac{\delta_1^3}{\delta_3^2} (q_2^2 + q_3^2)$$

Since Assumption A) implies that $\delta_3 \geq \alpha^2 \delta_1$ we have

$$|q_h^1|^2_{0,\hat{K}} \leq \frac{2\delta_1}{\alpha^4} (q_2^2 + q_3^2)$$

Now we prove that

(5.16) $\quad (\hat{div}\ \hat{z}_h\ ,\ q_h^1)_{0,\hat{\Omega}} \geq C'|q_h^1|^2_{0,\hat{\Omega}} \quad , \quad c' > 0 \quad .$

A straightforward calculation gives :

$$\int_{\hat{K}} \hat{div}\ \hat{z}_h\, q_h^1\, d\hat{x} = (s_1 + s_3 + \frac{s_1^2 + s_1 s_3}{s_2 + s_3})\, q_2^2 + (s_1 + \frac{s_1 s_2 + s_2^2}{s_3})\, q_3^2 +$$

$$+ (\frac{s_1 s_2}{s_2 + s_3} + 2s_2 - s_1) q_2\, q_3 \quad .$$

Thus we have :

$$\int_{\hat{K}} \hat{div}\ \hat{z}_h\, q_h^1\, d\hat{x} \geq (2s_2 + s_3)\, q_2^2 + (2s_1 + s_2) q_3^2 + (2s_2 - s_1 + \frac{s_1 s_2}{s_2 + s_3}) q_2 q_3$$

Now as one can easily check, for any $\alpha > 0$ we have :

$$\left| \frac{s_1 s_2}{s_2 + s_3} + 2s_2 - s_1 \right|^2 < 4(s_1 + s_2)(s_1 + s_3)$$

which yields

$$\int_{\hat{K}} \hat{div}\ \hat{z}_h\, q_h^1\, d\hat{x} \geq s_1 (q_2^2 + q_3^2)$$

This in turn implies (5.16) with $c' = \alpha^4/2$.

Finally we proceed like in Theorem 4.1, namely we set $\hat{v}_h = \theta \hat{w}_h + \hat{z}_h$, where $\hat{w}_h$ is defined in Lemma 5.1 and $\theta > 0$. From (5.11),(5.12), (5.15) and (5.16) it is clear that for θ sufficiently small there exists $\hat{\beta}_h > 0$ such that (5.14) holds together with (5.10). q,e.d.

Now we further prove :

<u>*Lemma 5.3*</u> : With every $q_h \in Q_h$ we can associate $v_h \in V_h$ that satisfies

(5.17) $\quad v_h(S) = 0 \quad$ for every vertex S of a supertriangle K , $K \in \tau_h$.

$$(5.18) \qquad \frac{\tilde{b}_h'(\Pi \underset{\sim}{u}_h\, ,\, \underset{\sim}{v}_h\, ,\, q_h)}{\|\underset{\sim}{v}_h\|} \geq \beta_h\, |q_h|$$

where β_h is a strictly positive parameter independent of q_h .

Proof : Using an identity encountered in [15, page 108] we obtain :

$$(5.19) \qquad \tilde{b}_h'(\Pi \underset{\sim}{u}_h\, ,\, \underset{\sim}{v}_h\, ,\, q_h) = \int_{\hat{\Omega}} q_h\, \hat{div}\, \hat{\underset{\sim}{v}}_h\, d\hat{\underset{\sim}{x}}$$

where $\hat{\underset{\sim}{v}}_h(\hat{\underset{\sim}{x}}) = \underset{\sim}{v}_h(\underset{\sim}{x})$.

On the other hand, from Assumption A) it is straightforward to establish the existence of a constant $C(\Omega, \underset{\sim}{u}_h)$ such that :

$$\|\underset{\sim}{v}_h\| \leq C\, (\Omega, \underset{\sim}{u}_h)\, |\hat{\underset{\sim}{v}}_h|_{1,\hat{\Omega}}$$

whereas

$$|q_h| \leq \alpha^{-1/2} |\hat{q}_h|_{0,\hat{\Omega}}$$

Now, if $\hat{\underset{\sim}{v}}_h$ is the field defined in Lemma 5.2 we have (5.17) and (5.18) with $\beta_h = \alpha^{1/2}\, C^{-1}\, \hat{\beta}_h > 0$. q.e.d.

As a final preparatory result we have :

Lemma 5.4 : Under Assumption A), for any $\underset{\sim}{v}_h \in \underset{\sim}{V}_h$ satisfying (5.17) we have :

$$\tilde{b}_h'\, (\underset{\sim}{u}_h\, ,\, \underset{\sim}{v}_h\, ,\, q_h) = \tilde{b}_h'(\Pi\, \underset{\sim}{u}_h\, ,\, \underset{\sim}{v}_h\, ,\, q_h) \qquad \forall\, q_h \in Q_h\ .$$

Proof : Taking into account the definitions of $\tilde{b}_h'$ and Q_h if we prove :

$$\int_K adj^T\, \underset{\sim}{\nabla}\, \underset{\sim}{u}_h^x \cdot \underset{\sim}{\nabla}\, \underset{\sim}{v}_h\, d\underset{\sim}{x} = \int_K adj^T\, \underset{\sim}{\nabla} \Pi \underset{\sim}{u}_h^x \cdot \underset{\sim}{\nabla}\, \underset{\sim}{v}_h\, d\underset{\sim}{x} \quad \forall\, K \in \tau_h^2\ .$$

we have the Lemma. In order to prove the above equality we rewrite :

$$\int_K adj^T\, \underset{\sim}{\nabla} \underset{\sim}{u}_h^x \cdot \underset{\sim}{\nabla}\, \underset{\sim}{v}_h\, d\underset{\sim}{x} = \lim_{\theta \to 0} \frac{1}{\theta} \int_K \Big[J(\underset{\sim}{x} + \Pi \underset{\sim}{u}_h + \underset{\sim}{\beta}\, \varphi + \theta\, \underset{\sim}{v}_h) - $$

$$- J(\underset{\sim}{x} + \Pi \underset{\sim}{u}_h + \underset{\sim}{\beta}\, \varphi) \Big]\, d\underset{\sim}{x}$$

where φ is given by (2.1) and $\underset{\sim}{\beta} = 4\,\underset{\sim}{u}_4 - 2(\underset{\sim}{u}_1 + \underset{\sim}{u}_2)$, $\underset{\sim}{u}_i$ being the value of $\underset{\sim}{u}_h$ at node S_i of $K \in \tau_h^2$ (See Figure 2.1)

Passing to element $\hat{K}$ using the affine transformation and notations already encountered in Section 3 we get :

$$\int_K adj^{\mathsf{T}}\, \underset{\sim}{\nabla}\, \underset{\sim}{u}_h^x \cdot \underset{\sim}{\nabla}\, \underset{\sim}{v}_h\, d\underset{\sim}{x} = \lim_{\theta \to 0} \frac{1}{\theta} \int_{\hat{K}} [\hat{J}(\hat{x} + \underset{\sim}{\beta}\,\hat{\varphi} + \theta\,\hat{\underset{\sim}{v}}_h) - \hat{J}(\hat{\underset{\sim}{x}} + \underset{\sim}{\beta}\,\hat{\varphi})] d\hat{\underset{\sim}{x}}$$

Expanding the right hand side above and taking the limit we have :

$$(5.20) \quad \int_K adj^{\mathsf{T}}\, \underset{\sim}{\nabla}\, \underset{\sim}{u}_h^x \cdot \underset{\sim}{\nabla} \underset{\sim}{v}_h\, d\,\underset{\sim}{x} = \int_{\hat{K}} \Big[d\hat{i}v\, \hat{\underset{\sim}{v}}_h + \beta_1 \Big(\frac{\partial \hat{v}_2}{\partial \hat{x}_2} \frac{\partial \hat{\varphi}}{\partial \hat{x}_1} - \frac{\partial \hat{v}_2}{\partial \hat{x}_1} \frac{\partial \hat{\varphi}}{\partial \hat{x}_2} \Big) +$$

$$\cdot \quad \beta_2 \Big(\frac{\partial \hat{v}_1}{\partial \hat{x}_1} \frac{\partial \hat{\varphi}}{\partial \hat{x}_2} - \frac{\partial \hat{v}_1}{\partial \hat{x}_2} \frac{\partial \hat{\varphi}}{\partial \hat{x}_1} \Big) \Big] d\hat{\underset{\sim}{x}}, \qquad \hat{\underset{\sim}{v}}_h = (\hat{v}_1, \hat{v}_2) .$$

Now we note that for $i = 1,2$ we have :

$$\int_{\hat{K}} \Big(\frac{\partial \hat{v}_i}{\partial \hat{x}_2} \frac{\partial \hat{\varphi}}{\partial \hat{x}_1} - \frac{\partial \hat{v}_i}{\partial \hat{x}_1} \frac{\partial \hat{\varphi}}{\partial \hat{x}_2} \Big) d\hat{\underset{\sim}{x}} = \int_{\partial \hat{K}} \frac{\partial \hat{v}_i}{\partial \hat{s}} \hat{\varphi}\, d\hat{s} = \int_{\hat{S}_k}^{\hat{S}_j} \frac{\partial \hat{v}_i}{\partial \hat{s}} \hat{\varphi}\, d\hat{s}$$

where $\hat{S}_k$ and $\hat{S}_j$ are the vertices of the basis $\hat{B}$ of $\hat{K}$.

Since $\underset{\sim}{v}_h$ satisfies (5.17), $\dfrac{\partial \hat{v}_h}{\partial \hat{s}}$ over this basis is of form $\chi \dfrac{\partial \hat{\varphi}}{\partial \hat{s}}$ for a constant vector χ. This gives :

$$\int_{\hat{K}} \Big(\frac{\partial v_i}{\partial x_2} \frac{\partial \hat{\varphi}}{\partial \hat{x}_1} - \frac{\partial \hat{v}_i}{\partial \hat{x}_1} \frac{\partial \varphi}{\partial x_2} \Big) d\hat{\underset{\sim}{x}} = \gamma_i \int_{\hat{S}_k}^{\hat{S}_j} \hat{\varphi} \frac{\partial \hat{\varphi}}{\partial \hat{s}} = \frac{\gamma_i}{2}\, \hat{\varphi}^2 \Big/_{\hat{S}_k}^{\hat{S}_j} = 0$$

and the result follows taking into account (5.19) and (5.20). q.e.d.

Now, as an immediate consequence of Lemmas 5.2,5.3 and 5.4 we have :

Theorem 5.1: If $\underset{\sim}{u}_h$ satisfies Assumption A) for any $\alpha > 0$, (5.7) holds in *case i)*. □

Let us now turn to *case ii)*.

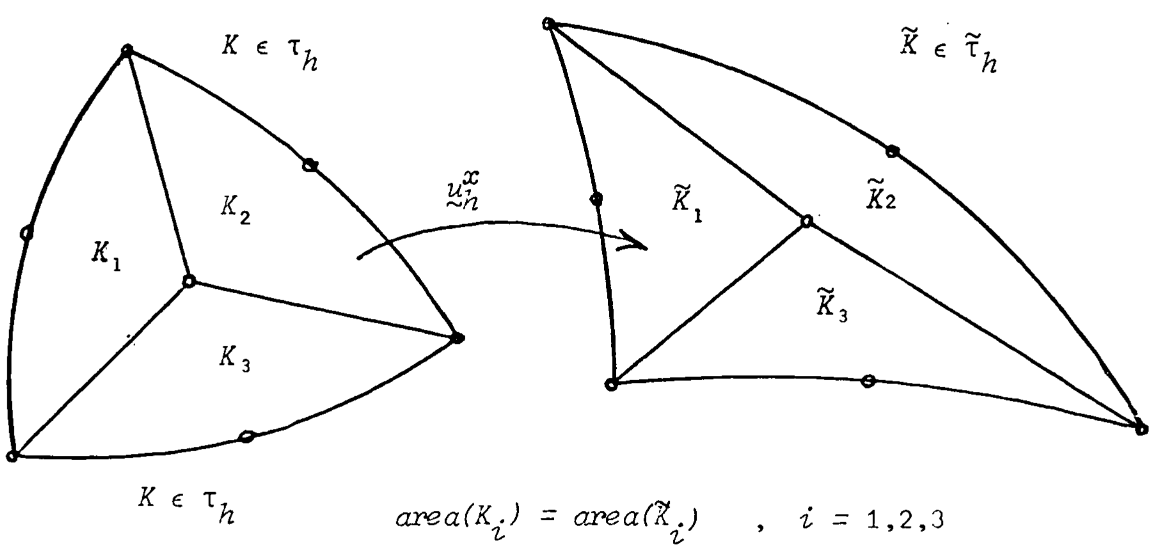
In this case Ω_h will be the union of triangles with one parabolic edge, such that its boundary Γ_h coincides with Γ at least at the nodes of those triangles that have a parabolic edge (basis) on Γ_h . Let also Γ_{0_h} be the portion of Γ_h consisting of such parabolic edges that have its three nodes on $\overline{\Gamma}_0$.

Now instead of Assumption A) we make a stronger one, namely :

ASSUMPTION B) $J(\underset{\sim}{u}_h^x) > 0$ almost everywhere in Ω_h . □

Taking into account (3.5), the above assumption implies Assumption A). Moreover, it allows us to say that $\underset{\sim}{u}_h^x$ is a bijection between Ω_h and $\tilde{\Omega}_h = \underset{\sim}{u}_h^x(\Omega_h)$. In this case $\tilde{\Omega}_h$ is a domain that has the same structure as Ω_h , in the sense that it can also be viewed as the union of isoparametric elements $\tilde{K}$, where $\tilde{K} = \underset{\sim}{u}_h^x(K)$, $K \in \tau_h^2$.

Let then $\tilde{\tau}_h^2$ be the triangulation of $\tilde{\Omega}_h$ consisting of the $\tilde{K}$'s . Similarly, let τ_h be the set of curved superelements $K = \bigcup_{i=1}^{3} K_i$ upon which τ_h^2 is constructed, and let $\tilde{\tau}_h$ be the partition of $\tilde{\Omega}_h$ into curved superelements $\tilde{K}$ where $\tilde{K} = \underset{\sim}{u}_h^x(K)$, $K \in \tau_h$ (see Figure 5.3)



$area(K_i) = area(\tilde{K}_i)$, $i = 1,2,3$

Supertriangles of partitions τ_h and $\tilde{\tau}_h$

Figure 5.3

For simplicity we consider the case where $\forall K \in \tau_h$ $area(K_1) = area(K_2) = area(K_3)$, although the more general case can be treated without major difficulties.

Now if $\hat{K}_i = \Pi \underset{\sim}{u}_h^x(K_i)$, Assumption B), hence A), implies

$$\frac{1}{\alpha}\, area(\tilde{K}) \geq 3\; area\; (\hat{K}_i) \geq \alpha\; area\; (\tilde{K})\ , \qquad 1 \leq i \leq 3\ ,$$

Since $area\ (\tilde{K}_i) = area\ (K_i) = \frac{1}{3}\, area\ (K)$ $\qquad \forall K \in \tau_h$.

Let us now define the following spaces of functions defined over $\tilde{\Omega}_h$:

$$\tilde{Q}_h = \{\tilde{q}_h \ / \ \tilde{q}_h \circ \underset{\sim}{u}_h^x = q_h \ , \ q_h \in Q_h\}$$

$$\tilde{\underset{\sim}{V}}_h = \{\tilde{\underset{\sim}{v}}_h \ / \ \tilde{\underset{\sim}{v}}_h \circ \underset{\sim}{u}_h^x = \underset{\sim}{v}_h \ , \ \underset{\sim}{v}_h \in \underset{\sim}{V}_h\}$$

We equip $\tilde{\underset{\sim}{V}}_h$ and $\tilde{Q}_h$ with the norms $\|\cdot\|$ and $|\cdot|$ given respectively by $\|\tilde{\underset{\sim}{v}}_h\| = |\tilde{\underset{\sim}{v}}_h|_{1,\tilde{\Omega}_h}$, $\tilde{\underset{\sim}{v}}_h \in \tilde{\underset{\sim}{V}}_h$ and $|\tilde{q}_h| = |\tilde{q}_h|_{0,\tilde{\Omega}_h}$, $\tilde{q}_h \in \tilde{Q}_h$

(Since $\underset{\sim}{v}_h = \underset{\sim}{0}$ on $\Gamma_{0_h} \equiv \tilde{\Gamma}_{0_h}$, $\|.\|$ is actually a norm).

Let us also denote by $\tilde{\underset{\sim}{x}}$ the new variable $\underset{\sim}{u}_h^x\,(\underset{\sim}{x})$.

More generally, for every function f defined over Ω_h we denote by $\tilde{f}$ the function defined over $\tilde{\Omega}_h$ such that $\tilde{f}\ [\underset{\sim}{u}_h^x(\underset{\sim}{x})] = f(\underset{\sim}{x})$ $\forall \underset{\sim}{x} \in \Omega_h$.

In order to prove that (5.7) holds, we use the following theorem given by Le Tallec :

<u>*Theorem 5.2*</u> *:* [16, Theor. 4.5] : Under Assumption B) (5.7) is equivalent to :

$$\exists\ \tilde{\beta}_h > 0 \quad \text{such that}$$

$$(5.21) \qquad \sup_{\tilde{\underset{\sim}{v}}_h \in \tilde{\underset{\sim}{V}}_h} \frac{\int_{\tilde{\Omega}_h} \tilde{q}_h\ \tilde{div}\ \tilde{\underset{\sim}{v}}_h\ d\,\tilde{\underset{\sim}{x}}}{\|\tilde{\underset{\sim}{v}}_h\|} \geq \tilde{\beta}_h\ |\tilde{q}_h|$$

where $\tilde{div}$ represents the divergence operator with respect to the $\tilde{\underset{\sim}{x}}$ variable.

□

The above result states that if suffices to prove the linear discrete compatibility condition between spaces $\underset{\sim}{\tilde{V}}_h$ and $\tilde{Q}_h$ to have existence of a solution to $(\tilde{P}_h)$ in the isoparametric case.

Now, in order to prove (5.21) for the asymmetric triangle we give the following lemmas :

Lemma 5.5. : Let $\tilde{Q}_h^0$ be the subspace of $\tilde{Q}_h$ of those functions that are constant over $\tilde{K}$ $\forall \tilde{K} \in \tilde{\tau}_h$. Then for every $\tilde{q}_h \in \tilde{Q}_h^0$ there exists a vector field $\underset{\sim}{\tilde{w}}_h \in \underset{\sim}{\tilde{V}}_h$ such that :

$$(5.22) \quad \int_{\tilde{\Omega}_h} q_h \, \tilde{div} \, \tilde{w}_h \, d\,\underset{\sim}{\tilde{x}} \geq \tilde{\beta}_0 \, |\tilde{q}_h|^2$$

$$(5.23) \quad \|\tilde{w}_h\| \leq \tilde{C}_h \, | \, \tilde{q}_h \, |$$

where $\tilde{\beta}_0$ and $\tilde{C}_h$ are strictly positive constants independent of $\tilde{q}_h$.

Proof : According to [7] , Lemma C2 , for a given $\tilde{q}_h \in \tilde{Q}_h^0$, there exists $\underset{\sim}{v} \in \underset{\sim}{H}^1(\tilde{\Omega}_h)$ with $\underset{\sim}{v} = \underset{\sim}{0}$ on Γ_{0h} such that

$$\int_{\tilde{\Omega}_h} \tilde{q}_h \quad \tilde{div} \, \underset{\sim}{v} \quad d\underset{\sim}{\tilde{x}} \geq \tilde{\beta}_0 |\tilde{q}_h|^2$$

and $$|\underset{\sim}{v}|_{1,\tilde{\Omega}_h} \leq \tilde{C} \, |\tilde{q}_h| \quad .$$

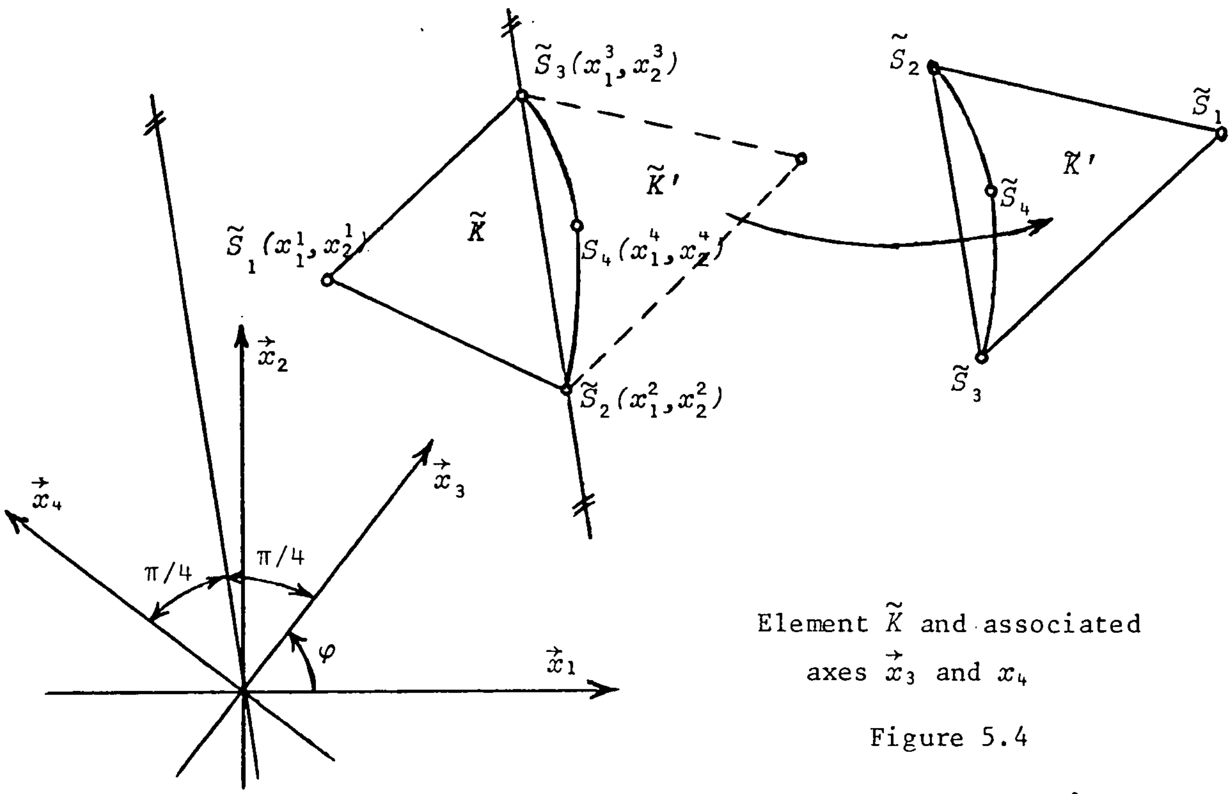
Now we construct a vector field $\underset{\sim}{\tilde{w}}_h \in \underset{\sim}{\tilde{V}}_h$ associated with $\underset{\sim}{v}$ in the following way :

For each triangle $\tilde{K} \in \tilde{\tau}_h^2$, we define two perpendicular axes $\vec{x}_3^K$ and $\vec{x}_4^K$ oriented in such a way that they correspond to rotations of the reference cartesian axes $\vec{x}_1$ and $\vec{x}_2$ of an angle φ^K .

Dropping the supercript K for simplicity, we determine φ in such a way that the straight line passing through nodes $\tilde{S}_2$ and $\tilde{S}_3$ of $\tilde{K}$ forms an angle of $\Pi/4$ with both $\vec{x}_3$ and $\vec{x}_4$.

Let x_j be the variable with respect to axis $\vec{x}_j$, $1 \leq j \leq 4$.

Clearly $\vec{x}_3$ and $\vec{x}_4$ will coincide for any pair of elements of $\tilde{\tau}_h^2$ that have a basis $\tilde{B}$ as a common edge. Let the local numbering of the vertices of each element respect the usual permutation convention (in this way, $\tilde{S}_2$ and $\tilde{S}_3$ interchange within each element of such a pair, as shown in Figure 5.4). Now for each $\tilde{K} \in \tilde{\tau}_h^2$, let s be the curved abcissa along $\tilde{B}$ with origin in $\tilde{S}_2$ and $\underset{\sim}{n}(s)$ denote the outer unit normal vector along $\tilde{B}$ with respect to $\tilde{K}$. We also denote by $n_j(s)$ the component of $\underset{\sim}{n}$ with respect to $\vec{x}_j$.



Element $\tilde{K}$ and associated axes $\vec{x}_3$ and x_4

Figure 5.4

Let $\underset{\sim}{w} = (w_1,w_2)$, $\underset{\sim}{w} = \underset{\sim}{w}_{h/\tilde{K}}$ and w_3 and w_4 be given by

$$w_3 = w_1 \cos\varphi + w_2 \sin\varphi$$

$$w_4 = -w_1 \sin\varphi + w_2 \cos\varphi$$

Now we check that we can uniquely define w_3 and w_4 (and consequently $\underset{\sim}{w}$) in the following way :

The values of w_3 and w_4 at the vertices of $\tilde{K}$ are given by

$$w_3(\tilde{S}_i) = w_4(\tilde{S}_i) = 0 \qquad i = 1,2,3$$

The value of w_3 and w_4 at node $\tilde{S}_4$ are such that

$$\int_{\tilde{B}} w_j \, n_j(s) ds = \int_{\tilde{B}} v_j \, n_j(s) ds \qquad j = 3,4$$

where v_j is the component of $\underset{\sim}{v}$ with respect to $\vec{x}_j$, $j = 3 , 4$.

Since $\underset{\sim}{u}_h \in \underset{\sim}{V}_h$ we can compute the coordinates x_3 and x_4 in terms of the reference coordinates $\hat{x}_1$ and $\hat{x}_2$ (see Figure 5.1) used for defining $\hat{P}_a$ over $\hat{K}$, in the following way :

$$x_3 = [4\,\xi_1^4 - 2(\xi_1^2 + \xi_1^3)]\,\hat{x}_1\,\hat{x}_2 + \xi_1^2\,\hat{x}_1 + \xi_1^3\,\hat{x}_2 + x_3^1$$

$$x_4 = [4\,\xi_2^4 - 2(\xi_2^2 + \xi_2^3)]\,\hat{x}_1\,\hat{x}_2 + \xi_2^2\,\hat{x}_1 + \xi_2^3\,\hat{x}_2 + x_4^1$$

where $\xi_1^i = x_3^i - x_3^1$ and $\xi_2^i = x_4^i - x_4^1 \qquad i = 2,3,4$ and

$$x_3^i = x_1^i \cos\varphi + x_2^i \sin\varphi$$

$$x_4^i = x_2^i \sin\varphi + x_2^i \cos\varphi \qquad i = 1,2,3,4 .$$

Using the above relations we make a change of variables in the integral $\int_{\tilde{B}} w_j \, n_j(s)\, ds$, $j = 3,4$, namely from s to $\hat{s}$, where $\hat{s}$ is the abcissa along the edge $\hat{B}$ of $\hat{K}$ with origin in $\hat{S}_2$ (see Figure 5.1) .

Since we have $n_3(s) = \dfrac{d\,x_4}{ds}$ and $n_4(s) = -\dfrac{d\,x_3}{ds}$, for a vector field $\underset{\sim}{f}$ defined over $\tilde{B}$, whose components with respect to $\vec{x}_j$ are f_j , $j = 3,4$, we have for the x_3-component :

$$\int_{\tilde{B}} f_3 n_3(s)\, ds = \int_{\tilde{S}_2}^{\tilde{S}_3} f_3(s) \frac{dx_4}{ds}\, ds = \int_{\hat{S}_2}^{\hat{S}_3} \hat{f}_3(\hat{s}) \left[\frac{\partial x_4}{\partial \hat{x}_1} \frac{d\hat{x}_1}{d\hat{s}} + \frac{\partial x_4}{\partial \hat{x}_2} \frac{d\hat{x}_2}{d\hat{s}}\right]_{\hat{x}_1+\hat{x}_2=1} d\hat{s}$$

where $\hat{f}_j(\hat{s}) = f_j(s)$. Since $\dfrac{d\hat{x}_i}{d\hat{s}} = (-1)^i \dfrac{\sqrt{2}}{2}$ we have :

$$\int_{\tilde{B}} f_3 n_3(s) ds = \int_0^{\sqrt{2}} \frac{1}{\sqrt{2}}\, \hat{f}_3(\hat{s}) \{(\xi_2^3 - \xi_2^3) + [4\xi_2^4 - 2(\xi_2^3 + \xi_2^2)](1 - \hat{s}\sqrt{2})\}\, d\hat{s} .$$

whereas an entirely analogous relation holds for the x_4- component.

Now since $\hat{w}_{j/\hat{B}} = 2\, w_j(\hat{S})\ \hat{s}\ (\sqrt{2} - \hat{s})$ we have :

$$(5.24)\quad \int_{\hat{B}} w_3 n_3\ (s)\ ds = \frac{2}{3}\ (\xi_2^3 - \xi_2^2)\ w_3(\tilde{S}_4)$$

and analogously

$$(5.25)\quad \int_{\hat{B}} w_4 n_4 (s) ds = \frac{2}{3}(\ \xi_1^2 - \xi_1^3)\ w_4(\tilde{S}_4) \quad .$$

Since by construction $|\xi_1^3 - \xi_1^2\ | = |\xi_2^3 - \xi_2^2| = \frac{\sqrt{2}}{2}$ length$(\hat{B}) \neq 0$, $\underset{\sim}{w}$ can be defined uniquely.

Furthermore, proceeding in the same way for every element we can define a vector field $\underset{\sim}{\tilde{w}}_h \in \underset{\sim}{\tilde{V}}_h$ such that :

$$\int_{\tilde{B}} \tilde{w}_h \cdot \underset{\sim}{n}(s)\ ds = \int_{\tilde{B}} \underset{\sim}{v} \cdot \underset{\sim}{n}(s) ds \quad \text{for every basis } \tilde{B} \text{ of } \tilde{K} \in \tilde{\tau}_h \ .$$

This yields :

$$\int_{\tilde{\Omega}_h} \tilde{q}_h\ d\tilde{i}v\ \underset{\sim}{\tilde{w}}_h\ d\underset{\sim}{\tilde{x}} = \int_{\tilde{\Omega}_h} \tilde{q}_h\ d\tilde{i}v\ \underset{\sim}{\tilde{v}}\ d\underset{\sim}{\tilde{x}} \qquad \forall\ \tilde{q}_h \in \tilde{Q}_h^0 \ ,$$

and consequently (5.22) holds .

On the other hand we have

$$\|\underset{\sim}{\tilde{w}}_h\|^2 = \sum_{\tilde{K}\in\tilde{\tau}_h^2} \int_{\tilde{K}} (|\underset{\sim}{\nabla}\ w_1|^2 + |\underset{\sim}{\nabla}\ w_2|^2) d\underset{\sim}{\tilde{x}} = \sum_{\tilde{K}\in\tilde{\tau}_h^2} \int_{\tilde{K}} (|\underset{\sim}{\nabla} w_3\ |^2 + |\underset{\sim}{\nabla}\ w_4|^2) d\underset{\sim}{\tilde{x}} \ .$$

But

$$\int_{\tilde{K}} |\underset{\sim}{\nabla} wj|^2\ d\underset{\sim}{\tilde{x}} = w_j^2\ (\tilde{S}_4) \int_{\tilde{K}} |\underset{\sim}{\nabla}\tilde{p}_4|^2\ d\underset{\sim}{\tilde{x}} \ , \quad j = 3,4 \ ,$$

where $\tilde{p}_4(\underset{\sim}{\tilde{x}}) = \hat{p}_4(\underset{\sim}{\hat{x}})$, $\hat{p}_4(\underset{\sim}{\hat{x}}) = 4\ \hat{x}_1\hat{x}_2$, $\underset{\sim}{\tilde{x}} = \underset{\sim}{u}^x_{h/K}\ [\mathcal{A}_K^{-1}\ (\underset{\sim}{\hat{x}})]$

Now, according to Assumption B)and standard estimates we have :

$$\int_{\tilde{K}} |\underset{\sim}{\nabla}\, \tilde{p}_4|^2 \, d\underset{\sim}{x} \leq C \; \frac{h^2 |\underset{\sim}{u}_h|^2_{1,\infty}}{\tilde{\rho}^2_K}$$

where $\tilde{\rho}_K$ denotes the diameter of the inscribed circle in $\tilde{K}$.

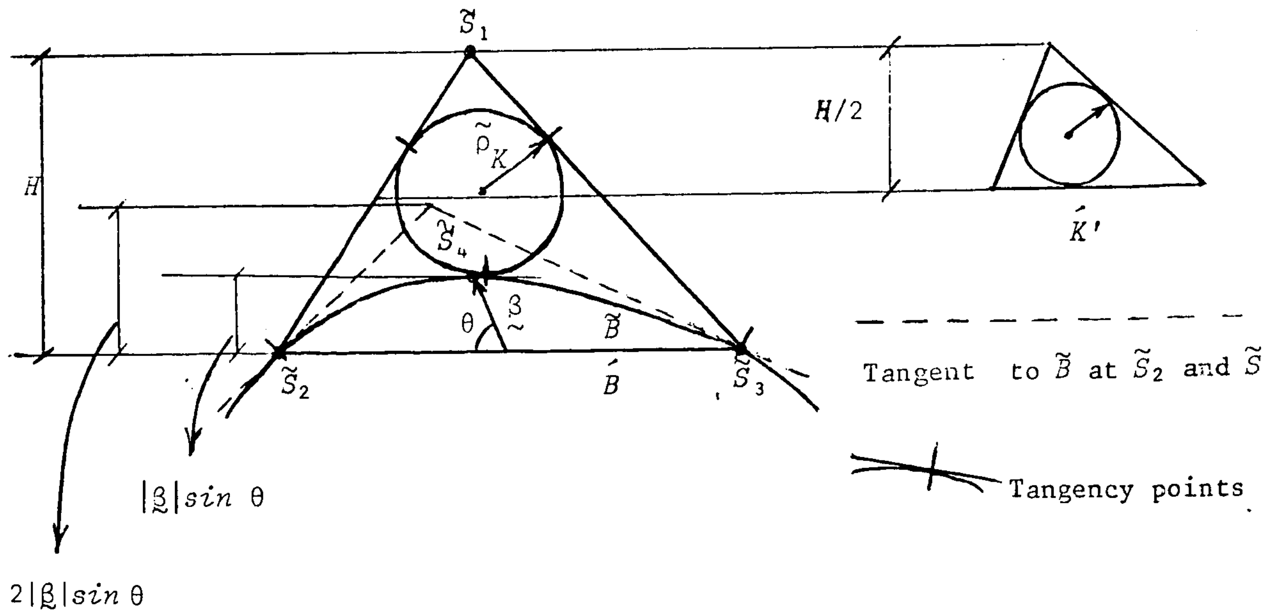
Now if $area(\tilde{K}) \geq area(\acute{K})$ we clearly have :

$$\tilde{\rho}_K \geq \acute{\rho}_K \geq \frac{2area(\acute{K})}{3\acute{h}_K} \geq \frac{2\alpha \; area(K)}{3h|\underset{\sim}{u}_h|_{1,\infty}}$$

If $area(\tilde{K}) \leq area(\acute{K})$ we use Assumption B) together with geometrical arguments sketched in self-explanatory Figure 5.5 (we omit details for the sake of conciseness). It is then possible to prove that $\tilde{\rho}_K$ is greater than the diameter of the inscribed circle in a triangle $\acute{K}'$, defined to be the homotetical reduction of $\acute{K}$ with ratio 1/2 .

Hence we have in this case :

$$\tilde{\rho}_K \geq \frac{\frac{1}{4}\, area(\acute{K})}{\frac{3}{4}\, \acute{h}_K} \geq \frac{1}{3} \; \frac{area(K)}{h|\underset{\sim}{u}_h|_{1,\infty}}$$



Triangles $\tilde{K}$ and $\acute{K}$ when $area(\tilde{K}) \leq area(\acute{K})$

Figure 5.5.

This gives :

$$\int_{\tilde{K}} |\underset{\sim}{\nabla} \tilde{p}_4|^2 \, d\tilde{\underset{\sim}{x}} \leq \frac{C}{\frac{2}{3}\, min(\alpha, 1/2)} \; \frac{h^4 |\underset{\sim}{u}_h|^4_{1,\infty}}{[area(K)]^2} \leq \frac{C'}{c^4} |\underset{\sim}{u}_h|^4_{1,\infty}$$

where c is the constant of regularity of $\{\tau_h^2\}_h$ (see Section 2) .

On the other hand, by construction , (5.24), (5.25) and the Trace Theorem we have :

$$|w_j(\tilde{S}_4)| \leq \frac{\int_{\tilde{B}} |v_j| ds}{\frac{\sqrt{2}}{3} \acute{\rho}_K} \leq C \; \frac{\|\underset{\sim}{v}\|_{1,\tilde{K}} \; |u_h|_{1,\infty}}{h}$$

Therefore

$$\|\underset{\sim}{w}_h\| \leq C\, h^{-1} |\underset{\sim}{u}_h|^3_{1,\infty} \|\underset{\sim}{v}\|_{1,\tilde{\Omega}_h} \leq C(\tilde{\Omega}_h)\; h^{-1} |\underset{\sim}{u}_h|^3_{1,\infty} \; |\tilde{\underset{\sim}{v}}|_{1,\tilde{\Omega}_h}$$

which proves (5.23) with $\tilde{C}_h = C\,(\tilde{\Omega}_h)\; |\underset{\sim}{u}_h|^3_{1,\infty}/h$. q.e.d.

Let us now construct a vector field $\tilde{\underset{\sim}{z}}_h \in \tilde{\underset{\sim}{V}}_h$ associated with the subspace $\tilde{Q}_h^1$ of $\tilde{Q}_h$, such that $\tilde{Q}_h = \tilde{Q}_h^o \oplus \tilde{Q}_h^1$. Like space Q_h^1 of Section 4, $\tilde{Q}_h^1$ is spanned by a set of orthogonal basis functions $\{\gamma_2^K, \gamma_3^K\}_{\tilde{K} \in \tilde{\tau}_h}$ defined in an entirely analogous way. Now we prove :

<u>*Lemma 5.6*</u> : Let $\tilde{q}_h^1$ be a function of $\tilde{Q}_h^1$ whose components with respect to γ_2^K and γ_3^K are respectively ξ_2^K and ξ_3^K , $\tilde{K} \in \tilde{\tau}_h$. Under Assumption B), the vector field $\tilde{\underset{\sim}{z}}_h \in \tilde{\underset{\sim}{V}}_h$ that vanishes at all the vertices of $\tilde{\tau}_h$ and whose value at the common vertex $\tilde{G}$ of $\tilde{K}_i$, 1,2,3, $\tilde{K}_i \subset \tilde{K}$ is given by (refer to Figure 5.6)

$$\tilde{z}_h(\tilde{G}) = -\xi_2^K \underset{\sim}{m}_2 + \xi_3^K \underset{\sim}{m}_3$$

satisfies :

$$(5.26) \qquad \|\tilde{\underset{\sim}{z}}_h\| \leq C(\underset{\sim}{u}_h) |\tilde{q}_h^1| \;, \qquad C\,(\underset{\sim}{u}_h) < \infty$$

(5.27) $$\int_{\tilde{\Omega}_h} \tilde{q}_h^1 \, \widetilde{div} \, \tilde{\underset{\sim}{z}}_h \, d\tilde{\underset{\sim}{x}} \geq \tilde{\beta}_1 |\tilde{q}_h^1|^2 \qquad \text{with } \tilde{\beta}_1 > 0$$

Proof : (5.26) is a trivial consequence of the definition of $\tilde{\underset{\sim}{z}}_h$.

On the other hand a straight forward computation gives :

$$\int_{\tilde{K}} \tilde{q}_h^1 \, \widetilde{div} \, \tilde{\underset{\sim}{z}}_h \, d\tilde{\underset{\sim}{x}} = (s_3 + s_1) \frac{3\xi_2^2}{2} + (2s_1+s_2) \xi_3^2 + (3s_2+s_3-s_1)\xi_2 \xi_3$$

where $s_i = area(K_i)$ $\quad i = 1,2,3$.

Assuming again that the local numbering of the nodes of $\tilde{K}$ is such that $s_1 \geq s_2 \geq s_3$ we have :

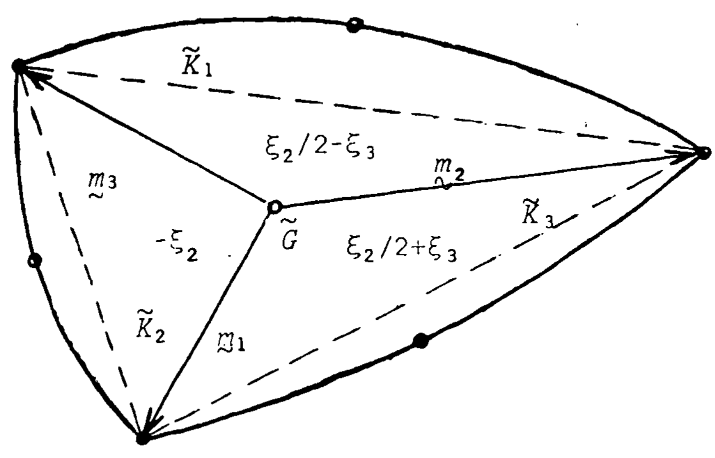
$$\frac{1}{4} \left(\frac{3}{2} s_2 + s_3 + s_1 \right)^2 - \frac{1}{2} (s_1 + 3s_3)(s_1 + s_2) < 0$$

if we just have $s_1 \geq \alpha \, area(K)/3 > 0$.

Thus we can write :

$$\int_{\tilde{K}} \tilde{q}_h^1 \, \widetilde{div} \, \tilde{\underset{\sim}{z}}_h \, d\tilde{\underset{\sim}{x}} \geq \frac{\alpha}{3} \left(\xi_2^2 + \xi_3^2 \right) area(K)$$

which yields (5.27) with $\tilde{\beta}_1 = \frac{2}{9}\alpha$ q.e.d.



Superelement $\tilde{K}$

Figure 5.6

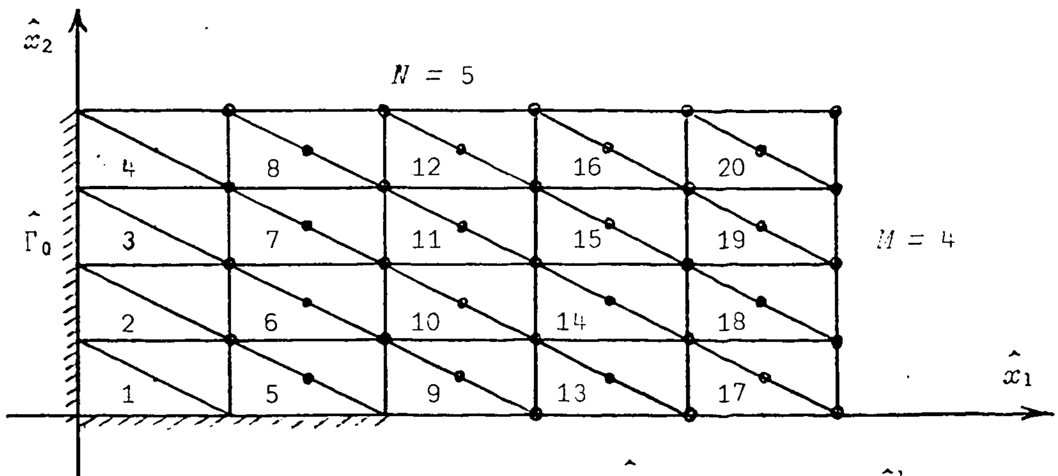
Now defining $\tilde{v}_h = \theta\, \tilde{w}_h + \tilde{z}_h$, from (5.22),(5.23),(5.26) and (5.27) we have (5.18) just like in Lemma 5.2, for a sufficiently small θ. Hence, as an immediate consequence of Theorem 5.2 and Lemmas 5.5 and 5.6 we have:

Theorem 5.3 : Under Assumption B) the compatibility condition (5.7) holds for case ii). □

REMARK : Assumptions B) and A) with $\alpha > 0$ express in particular the fact that the area delimited by the basis of the triangles in deformed states $\tilde{B}$ and $\hat{B}$ do not account for the whole of $area(\tilde{K}) = area(K)$. This fact was crucial for the assertion of the existence results in both *cases i)* and *ii)* . □

Let us finally consider the existence of a solution to problem $(\tilde{P}_h)$ when one uses a partition of type τ_h^1 for a special case described below :

Let Ω be a domain that can be viewed as the image of a rectangle $\hat{\Omega}$ with boundary $\hat{\Gamma}$, through a mapping $A : \hat{x} + \hat{w}(\hat{x})$. Here $\hat{w}$ is an element of a reference vector space $\hat{V}_h$ such that $det\,(I + \hat{\nabla}\, \hat{w}(x)) > 0$ a.e in $\hat{\Omega}$. $\hat{V}_h$ is defined in the same way as V_h in Section 2, for a compatible partition $\hat{\tau}_h^1$ of $\hat{\Omega}$ into equal triangles illustrated in Fig.5.7. $\hat{\tau}_h^1$ is constructed upon a first partition χ_h of $\hat{\Omega}$ into rectangles by means of a uniform $M \times N$ grid, in such a way that the edges of $\hat{\tau}_h^1$ over which $\hat{v}_h \in \hat{V}_h$ is necessarily linear, are the edges parallel to the reference axes $\vec{\hat{x}}_1$ and $\vec{\hat{x}}_2$.



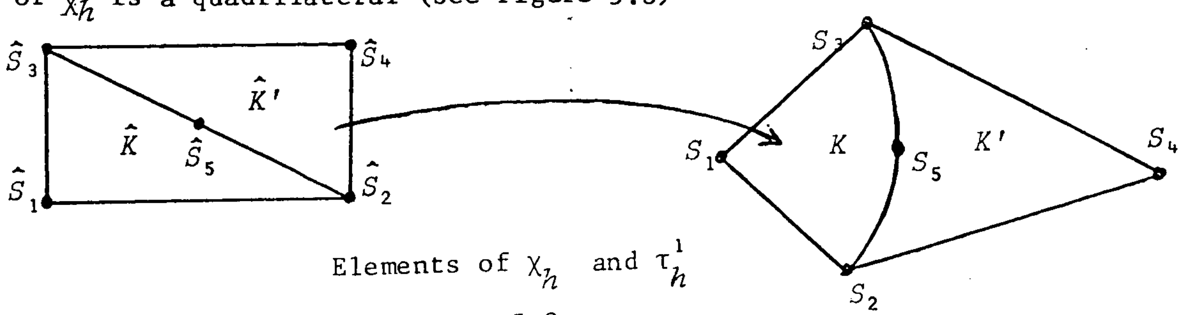
Reference rectangle $\hat{\Omega}$ and partition $\hat{\tau}_h^1$

Figure 5.7

We assume that the fixed portion $\hat{\Gamma}_0$ of $\hat{\Gamma}$ over which $\hat{\underset{\sim}{v}}_h \in \hat{V}_h$ vanishes, is the union of edges of rectangles of $\hat{\chi}_h$. If we define Γ_0 to be the image through A of $\hat{\Gamma}_0$ it is clear that Γ_0 consists of polygonal lines (eventually disjoint), just like $\Gamma = A(\hat{\Gamma})$.

Now we define τ_h^1 to be the partition of Ω into isoparametric elements K that are the images of $\hat{K}$ through A , $\forall \hat{K} \in \hat{\tau}_h^1$. Similarly we define χ_h to be the partition of Ω into elements that are the images through A of rectangles of $\hat{\chi}_h$.

Notice that the union of a pair of elements K and K' of τ_h^1 that are the images of two triangles of $\hat{\tau}_h^1$ contained in a given rectangle of $\hat{\chi}_h$, is a quadrilateral (with four straight edges). Therefore every element of χ_h is a quadrilateral (see Figure 5.8)



Elements of χ_h and τ_h^1

Figure 5.8

Now, according to [16], Theorem 4.5, it suffices to prove (5.21) to assert the existence of a solution to $(\tilde{P}_h)$, assuming of course that $J(\underset{\sim}{u}_h^x) > 0$ a.e. in Ω .

Let us denote the quadrilaterals of χ_h by R_i , $i = 1,2,\ldots, M \times N$, where $R_i = A(\hat{R}_i)$. $\hat{R}_i$ are the rectangles of $\hat{\chi}_h$ that we mumber in a systematic way along the columns, row by row, as indicated in Fig. 5.7.

Let $\eta = \{\eta_i, \eta_{i+M\times N}\}_{i=1}^{M\times N}$ be the basis of the space of pressures Q_h associated with τ_h^1 , in such a way that $supp(\eta_i) \subset R_i$ and $supp(\eta_{i+M\times N}) \subset R_i$, $1 \le i \le M \times N$, with :

$$\begin{cases} \eta_i(\underset{\sim}{x}) = 1 & \forall \underset{\sim}{x} \in K_i \\ \eta_i(\underset{\sim}{x}) = -1 & \forall \underset{\sim}{x} \in K_i' \\ \eta_{i+M\times N}(\underset{\sim}{x}) = 1 & \forall \underset{\sim}{x} \in R_i \end{cases},$$

where K_i and K_i' are the curved triangles into which R_i is subdivided.

Let also $\nu = \{\underset{\sim}{\nu}_i\}_{i=1}^{2NN}$ be the usual basis of $\underset{\sim}{V}_h$, where NN is the number of free nodes of τ_h^1 . Each $\underset{\sim}{\nu}_i$ is associated with a degree of freedom of $\underset{\sim}{V}_h$ which are assigned to two different blocks. The first one corresponds to the $2M\times N$ components of a field of $\underset{\sim}{V}_h$ that are associated with the nodes lying in the interior of $R_i \in \chi_h$, while the remaining degrees of freedom are assigned to the second block. Now we number the degrees of freedom of $\underset{\sim}{V}_h$ in such a way that those in the first block carry the numbers from one to $2M\times N$ and those in the second block the numbers from $2M\times N+1$ to $2NN$.

Finally, let B_h be the $(2NN) \times (2M\times N)$ matrix whose entry at the i-th row and j-th column is given by

$$\int_\Omega \eta_i \, div \, \underset{\sim}{\nu}_j \, d\underset{\sim}{x} \quad .$$

According to [15], Lemma 5.1, the existence of $\tilde{\beta}_h > 0$ such that (5.21) holds is equivalent to the rank of B_h being equal to $dim \; Q_h = 2M \times N$.

In order to examine this rank condition, it is convenient to split B_h into four rectangular matrices, according to the pattern below :

$i=$ \ $j=$	$1 \ldots\ldots 2M\times N$	$2M\times N+1 \ldots\ldots 2NN$
$1 \ldots M\times N$	B_h^1	B_h^4
$M\times N+1 \ldots 2M\times N$	B_h^3	B_h^2

First we notice that all the terms of B_h^3 vanish, since the basis functions of $\underset{\sim}{V}_h$ associated with nodes lying in the interior of the quadrilaterals have zero flux along its boundary.

Secondly, recalling (5.24) and (5.25), we can say that the entries of B_h^1 in the positions $j = 2i-1$ or $j = 2i$, $1 \leq i \leq M \times N$, are given by expressions of the form $\pm\ 2(x_k^3 - x_k^2)/3$, $k = 1,2$, where (x_1^ℓ, x_2^ℓ) , $\ell = 2,3$ are the coordinates of the vertices of the curved diagonal of R_i . Since those vertices are necessarily distinct, at least one of the above terms of B_h^1 is nonzero.

Finally we notice that matrix B_h^2 has exactly the same entries as the matrix studied by Le Tallec for the $Q_1 \times P_0$ element associated with a partition of Ω into quadrilaterals, like χ_h .

With the above considerations it is easy to conclude that the rank of B_h is $2M \times N$, provided the rank of B_h^2 is $M \times N$. Therefore the condition of existence and uniqueness of p_h such that $(\underset{\sim}{u}_h , p_h)$ is a solution to $(\tilde{P}_h)$ becomes the same as in the case of the $Q_1 \times P_0$, at least for domains defined as above. That is why we refer to the work of Le Tallec [15] for the proper answer to this question in various situations depending on the shape of Γ_0 .

Nevertheless, with the purpose of giving a brief illustration of his results we mention here the following case :

If Γ_0 is contained in a set that is the image through A of two non disjoint edges of $\hat{\Omega}$, then the above existence and uniqueness result is guaranteed. If on the other hand Γ_0 does not fall in this category this can only be asserted under some restrictive condition.

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