An end to end approach to the resequencing problem

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Abstract :

The resequencing or reserialisation problem is of basic interest in distributed systems and computer communication systems. This is because a flow of packets, messages or updates entering a communication system in chronological order from the same port or from different ports may be disordered. The receiving port must then insure that these objects are resequenced in the appropriate order before they are fed to the output of the system. In this paper we analyze the end-to-end delay incurred by objects traversing such a system, including both the disordering delay, the delay introduced by the resequencing algorithm and the delay due to the output server at the receiving port. The analysis is carried out via spectral factorization methods.

Résumé :

Les phénomènes de reséquencement sont de première importance dans les systèmes à contrôle réparti (bases de données réparties, réseaux de processeurs, réseaux de télécommunication). En effet, la répartition du contrôle ne permet pas, en règle générale, de préserver l'ordre chronologique des émissions de messages (demandes de mise à jour, messages de synchronisation, paquets de données) pouvant avoir lieu en chacun des noeuds du système à destination d'un noeud terminal donné. Les algorithmes de reséquencement implémentés en ce dernier noeud sont destinés à restaurer l'ordre chronologique des messages arrivant au processeur terminal. Dans cette étude, nous analysons le délai de bout en bout subi par les messages traversant un tel système : délai de transmission, délai de reséquencement, délai de traitement. Des équations récursives sont établies qui permettent de réduire l'analyse de ce délai à des problèmes de factorisation spectrale étudiés en détail sur des exemples.

AN END TO END APPROACH TO THE RESEQUENCING PROBLEM

1. INTRODUCTION

The resequencing problem is a fundamental issue in networks and in distributed systems. Let us first give an abstract statement and then provide examples of some practical occurrences of the problem.

Consider a sequence of objects $\{\phi_n\}_{n \in \mathbb{N}}$, where $\mathbb{N}$ denotes the set of all non-negative integers. They enter a communication system at instants $\{a_n\}_{n \in \mathbb{N}}$ where a_n corresponds to ϕ_n.

Each ϕ_n is then delayed by some time D_n, $n \in \mathbb{N}$. Thus at the output point of the system the objects appear at instants $\{a_n + D_n\}_{n \in \mathbb{N}}$: these instants are not necessarily in chronological order any more (i.e. it is possible that $a_n + D_n > a_\ell + D_\ell$ for $\ell > n$).

These objects are then processed by the Resequencing Algorithm (RA) where ϕ_n will receive some service of duration S_n and will then depart at time d_n. However this service can only be given in the same order as that of the external arrival instants ; i.e. , ϕ_n must be served after ϕ_{n-1} and before ϕ_{n+1}, which does not necessarily coincide with the order of arrival to the RA. We would like to know what all this would cost, and in particular we would like to evaluate the total effective delay (d_n-a_n).

This problem has multiple practical incarnations : let us now outline a few areas of application.

I.1. Packet switching networks

Packet switching networks are well known practical computer communication systems [1] used to interconnect computer systems or subsystems by transmitting streams of data in the form of standardized packets. Thus packets belonging to the same message, or to the same logical stream (e.g. in the case of a file transfer) have to be reassembled at the receiving station in the proper order, identical to the order at the transmitting station. Yet the packet switching network may not necessarily guarantee this particular order if it allows adaptive routing, or alternate routing, in order to handle congestion problems or temporary unavailability of certain routes. Therefore, the output nodes of the network may themselves insure proper packet reassembly before they provide an output ; in this case, the resequencing delay must be included in the end-to-end delay evaluation for the network.

I.2. Distributed data bases

Several consistency preserving mechanisms or algorithms [2,3,4] for distributed data bases use a logical or physical time-stamping mechanism in order to determine the order in which updates must be carried out on the data. For obvious reasons, updates do not necessarily arrive at the system sites in that order : these updates may originate at distinct nodes, thus they often arrive out of sequence even though the network may be able (and this too is not always the case) to transmit updates in sequence between a given pair of nodes. Thus the desequencing introduced by the system, and the corresponding resequencing problem, are basic issues in time-stamp oriented algorithms. It is this particular application which has motivated the first analyses of the problem [4, 5, 7].

I.3. The Mathematical problem

Consider the system shown on Figure 1. The objects $\{\phi_n\}$ which enter into the system at instants $\{a_n\}$ and which depart (in order) at $d_1 \leq d_2 \leq \ldots \leq d_n \leq \ldots$ must be processed by the resequencing algorithm before being transferred to the output server queue. Notice that S_n is a service time inside the RA.

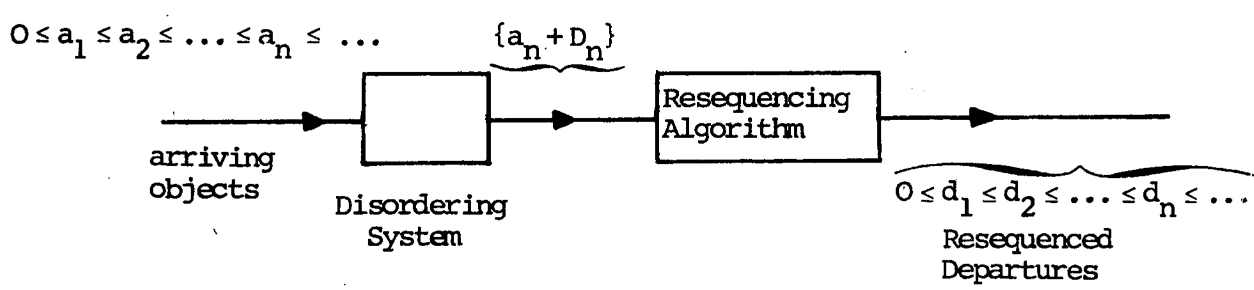

Figure 1 : Structure of the resequencing problem

Kamoun, Kleinrock and Muntz [5] have analyzed the distribution of the number of objects in the "Resequencing Algorithm" box of Figure 1 under the assumption that the arrival instants constitute a Poisson process, that the delays $\{D_n\}_{n \geq 1}$ are independent identically distributed (iid) random variables of exponential distribution (this corresponds to the case when the disorder is provoked by an M/M/∞

type system) and when the processing time is zero ($S_n=0$). This analysis allows them also to obtain the average delay due to the resequencing algorithm. Harrus and Plateau [6] generalize this approach to the case where the delays $\{D_n\}$ are iid random variables of general distribution. Baccelli and Znati [7] have considered an "approximate" model of the output queue in which the resequencing is represented by interruptions in the service mechanism. Finally Feuvre [8] has also developed an approximate approach by modelling it as a queue-length dependent bulk service mechanism.

In this paper we take a formal "end-to-end" approach to the problem, which allows to include the effect of the output service mechanism. We obtain certain "existence" type results with very weak assumption on the various variables involved, and then examine cases where the $\{D_n\}$ are, or are not, iid. Our approach provides a complete solution to cases previously considered in [5, 6], and also allows analysis of a case where the delays $\{D_n\}$ have a form of dependency : this is an important extension for the following reason.

In the $\{D_n\}$ represent delays of successive packets or updates travelling through a network or a distributed system, delays experienced by successive packets will be independent only if the paths traversed by successive packets or updates are disjoint, or if the message traffic rate is very low. In other cases, some form of dependence will naturally exist.

□

2. A WAITING TIME OR TOTAL DELAY EQUATION FOR THE RESEQUENCING ALGORITHM

The resequencing algorithm (RA) operates as follows. It delays an object ϕ_n until all of the objects ϕ_j, $j < n$, have been released by the RA. The RA will release objects sequentially, taking S_n units of time to release ϕ_n, only after ϕ_{n-1} has been released.

Thus the RA may be viewed as a "real-time sorting algorithm". We are interested in the end-to-end properties of this system, from the arrival instant of an object to its instant of release by the RA.

<u>Remark 1</u>. In the other models considered in the literature, and in particular the one treated by Kamoun, Kleinrock, Muntz [5] and Harrus, Plateau [6] the processing time S_n of the resequencing algorithm is not represented. Our basic model reduces to their's simply by setting $S_n = 0$, $n \geq 1$.

Let us call $T_n = d_n - a_n$ the total delay separating the instant a_n at which the object ϕ_n arrives to the system from the instant d_n at which it is released by the RA.

We may write

$$T_n = D_n + S_n + W_n$$

where W_n is the time during which ϕ_n has to wait in the RA before it is serviced and released.

<u>Lemma 1.</u> We have $T_1 = D_1 + S_1$, and for $n \geq 2$:

$$T_n = \begin{cases} D_n + S_n & \text{if } D_n > T_{n-1} - (a_n - a_{n-1}) \\ D_n + S_n + [T_{n-1} - (a_n - a_{n-1}) - D_n] & \text{, otherwise} \end{cases}$$

<u>Proof</u> : ϕ_n will only be released after ϕ_{n-1}'s release instant which is

$$a_{n-1} + T_{n-1}$$

but obviously not before

$$a_n + D_n + S_n$$

Thus ϕ_n cannot be processed by the RA before instant

$$a_n + D_n$$

Hence if

$$a_n + D_n > a_{n-1} + T_{n-1}$$

then ϕ_n will be released at instant

$$a_n + D_n + S_n$$

yielding $T_n = D_n + S_n$. Otherwise, if

$$a_n + D_n < a_{n-1} + T_{n-1}$$

it follows that ϕ_{n-1} will wait in the RA for a duration of

$$a_{n-1} + T_{n-1} - (a_n+D_n)$$

and will only be released at

$$D_n + S_n + [a_{n-1} + T_{n-1} - (a_n+D_n)]$$

hence the result. □

We shall write the formula of Lemma 1 in the more convenient form

$$(1) \quad \begin{cases} W_1 = 0 \\ W_n = [W_{n-1} - A_n + S_{n-1} - (D_n-D_{n-1})]^+ , \quad n \geq 2 \end{cases}$$

where $W_n + S_n$ is the total time spent by ϕ_n in the RA, $[X]^+$ denotes $\max[0,X]$ as usual, and $A_n = a_n-a_{n-1}$.

Lemma 2 . Consider the sequence $\{W_n\}_{n \geq 1}$ defined by equation (1). Define the random variable ξ_n by

$$\xi_n = S_{n-1} - A_n - (D_n - D_{n-1}), \quad n \geq 2$$

so that we can write

$$(2) \quad \begin{cases} W_1 = 0 \\ W_n = [W_{n-1} + \xi_n]^+ , \quad n \geq 2 \end{cases}$$

If $\{\xi_n\}_{n \geq 2}$ is a sequence of identically distributed (but not necessarily independent) random variables, then when $E[\xi_n] < 0$ it follows that

$$W_n \underset{P}{\rightarrow} W \text{ a proper random variable}$$

Proof : Notice that (2) is identical in form to the well known waiting time formula for the G/G/1 queue. The result then follows directly (see for instance Borovkov [9]), since it is identical to the corresponding result for the G/G/1 system. □

Remark 2. If we view the RA as a "real-time sorting algorithm", then we see that the condition $E[\xi_n] < 0$, or equivalently

$$E[A_n] > E[S_{n-1}] + E[D_{n-1} - D_n], \text{ or } E[a_n + D_n] > E[a_{n-1} + D_{n-1}] + E[S_{n-1}]$$

is a sufficient (and in fact necessary) condition for the sorting algorithm to keep up with its workload.

□

3. THE CASE WHEN THE DELAY IS AN INDEPENDENT INCREMENT PROCESS

3.1. The waiting time

In this section we make two assumptions which are quite common : the $(A_n)_{n \geq 1}$ and $(S_n)_{n \geq 1}$ are iid and mutually independent.

Furthermore we assume that the $(D_n)_{n \geq 1}$ are <u>not</u> independent. We shall assume however that the $(D_n)_{n \geq 1}$ have independent increments. That is, that

$$\theta_n = D_{n-1} - D_n, \; n \geq 1$$

$$D_0 \equiv 0$$

where the $\{\theta_n\}_{n \geq 1}$ are iid and independent of the $(A_n)_{n \geq 1}$, $(S_n)_{n \geq 1}$.

Physically this is an appealing assumption since it states that D_n is "nearly" equal to D_{n-1}, but differs from it by an "error term" given by $-\theta_n$.

<u>Lemma 3.</u> Under these assumptions, $\{W_n, n \geq 0\}$ is a Markov chain with state space $\mathbb{R}^+$ satisfying the following transition equation

$$(1) \begin{cases} W_n = [W_{n-1} + \xi_n]^+ \\ n \geq 1 \end{cases}$$

<u>Proof</u> : this is simply because , ξ_n is independent of the $\{W_i, i \leq n-1\}$. □

<u>Lemma 4.</u> Under these assumptions, the Laplace Stieltjes transform of the stationary distribution of the W_n's, $F^*(s)$ satisfies the following functional equation if $E[\xi_n] < 0$:

$$(2) \begin{cases} \text{For } Re(s) = 0 \\ [1 - A^*(-s)\Theta^*(s).S^*(s)]\, F^*(s) = U^*(s) \end{cases}$$

where $U^*(s)$ is some unknown function which is analytic and bounded in the domain $Re(s) \leq 0$.

<u>Proof</u> : Let $F(x)$ be $P[W_n \leq x]$ in steady state and $C(x)$ be $P[\xi_n \leq x]$. We derive from (1) :

$$F(x) = \begin{cases} 0 & x < 0 \\ \int_{-\infty}^{x} F(x-y)\,dC(y) & x \geq 0 \end{cases}$$

Define now

$$G(x) = \begin{cases} 0 & x \geq 0 \\ \int_{-\infty}^{x} F(x-y)\,dC(y) & x < 0 \end{cases}$$

We have, $\forall\ x \in \mathbb{R}$:

$$F(x) + G(x) = \int_{-\infty}^{x} F(x-y)\,dC(y)$$

Taking the Fourier Stieltjes transforms of both sides of this equation yields :

$$\begin{cases} F^*(s) = F^*(s)A^*(-s)S^*(s)\Theta^*(s) + U^*(s) \\ \text{for } \operatorname{Re}(s) = 0 \end{cases}$$

where

$$\begin{cases} F^*(s) \triangleq \int_{0^-}^{\infty} e^{-sx}\,dF(x) \\ \text{for any } s \text{ with } \operatorname{Re}(s) \geq 0 \end{cases}$$

and

$$\begin{cases} U^*(s) \triangleq -\int_{-\infty}^{0^+} e^{-sx} \int_{-\infty}^{x} F(x-y)\,dC(y) \\ \text{for any } s \text{ with } \operatorname{Re}(s) \leq 0 \end{cases}$$

Notice that $F^*(s)$ (resp $U^*(s)$) is analytic and bounded in the domain $\operatorname{Re}(s) \geq 0$ (resp $\operatorname{Re}(s) \leq 0$). □

Hence the analytical solution will be obtained by the spectrum factorization of the function

$[1 - A^*(-s)\Theta^*(s)S^*(s)]$

In the special case where $\Theta^*(s)$ may be factorized as

$$\begin{cases} \Theta^*(s) = D^+(s).D^-(-s) , \quad Re(s) = 0 \\ \text{where both } D^+(s) \text{ and } D^-(s) \text{ are analytic in } Re(s) \geq 0 \end{cases}$$

then $A^*(-s)\Theta^*(s)S^*(s)$ may be factorized as $(A^*(-s)D^-(-s))(S^*(s)D^+(s))$ so that the problem reduces to the analysis of the GI/GI/1 queue with $A^*(s).D^-(s)$ (resp $S^*(s).D^+(s)$) interarrival (resp service time) distribution.

As an example, we now analyze the case where the delay increments have a bilateral exponential distribution function and the arrival process is poisson of parameter λ. (However, the techniques developed in the proofs may be generalized to any bilateral coxian distribution for the increments and coxian distribution for the interarrival times).

Theorem 5. In the case where $\Theta^*(s) = \frac{\Theta^2}{\Theta^2-s^2}$, $Re(s) = 0$,
If $\lambda E[S] < 1$, then
$\exists\, F(x) = \lim_{n\to\infty} P[W_n \leq x]$, with Laplace Stieltjes transform

$$F^*(s) = \frac{\Theta}{\zeta}.(1-\lambda E[S]).\frac{s(s+\Theta)(s-\zeta)}{(\lambda-s)(\Theta^2-s^2)-\lambda\Theta^2 S^*(s)} \tag{3}$$

where ζ is the unique strictly positive real solution of the equation

$$(\lambda-s)(\Theta^2-s^2) = \lambda\Theta^2 S^*(s)$$

Proof : Define

$$R(s) = (\lambda-s)(\Theta-s)F^*(s)-\lambda F^*(s)S^*(s).\frac{\Theta^2}{\Theta+s}$$

for $Re(s) \geq 0$

$$L(s) = U^*(s)\ (\Theta-s).(\lambda-s)$$

for $Re(s) \leq 0$

We derive from (5) that R(s) and L(s) are equal for Re(s) = 0. Thus, R(s) is the analytic continuation of L(s). We denote as G(s) this unique function which is analytic in the whole complex plane.

$$G(s) = \begin{cases} R(s) & \text{for } Re(s) \geq 0 \\ L(s) & \text{for } Re(s) \leq 0 \end{cases}$$

Let H(s) be

$$H(s) \triangleq \frac{G(s)-G(\Theta)}{s-\Theta}$$

and I(s) be

$$I(s) \triangleq \frac{H(s)-H(\lambda)}{s-\lambda}$$

I(s) is analytic in the whole complex plane. Furthermore, I(s) is bounded for Re(s) ≤ 0 since in this domain :

$$I(s) = -\frac{H(\lambda)}{s-\lambda} - \frac{G(\Theta)}{(s-\lambda)(s-\Theta)} + U^*(s)$$

I(s) is also bounded for Re(s) ≥ 0 since in this domain :

$$\begin{cases} I(s) = F^*(s) + \dfrac{\dfrac{E[s]-E[\Theta]}{s-\Theta} - \dfrac{E[\lambda]-E[\Theta]}{\lambda-\Theta}}{s-\lambda} \\ \text{where } E(s) \triangleq -\lambda F^*(s)\ S^*(s) . \dfrac{\Theta^2}{\Theta+s} \end{cases}$$

Therefore, by Liouville's theorem I(s) is constant. Hence, $U^*(s)$, initially defined for Re(s) ≤ 0 may be continued for Re(s) ≥ 0 by the function

$$U^*(s) = C + \frac{H(\lambda)}{s-\lambda} + \frac{G(\Theta)}{(s-\lambda)(s-\Theta)}$$

Hence, we derive from equation (2)

$$F^*(s) = \frac{C(\Theta^2-s^2)(\lambda-s)-H(\lambda)(\Theta^2-s^2)+G(\Theta)(\Theta+s)}{(\Theta^2-s^2)(\lambda-s) - \lambda\Theta^2 S^*(s)}$$

The necessary condition

$$F^*(s) \underset{s\to\infty}{\to} F(O)$$

yields the relationship

$$F(O) = C$$

The necessary condition

$$F^*(O) = 1$$

yields the following :

$$\begin{cases} \Theta[F(O)\lambda - H(\lambda)] = -G(\Theta) \\ \text{and (using l'Hospital's rule)} \\ F(O) = \frac{\Theta}{\Theta+\lambda}(1 - \lambda E[S]) - H(\lambda) \end{cases}$$

Thus, $F^*(s)$ may be writen as follows :

$$F^*(s) = \frac{s(F(O).s-a)}{(\lambda-s)(\Theta-s)-\lambda\Theta S^*(s).\frac{\Theta}{\Theta+s}}$$

where

$$a \triangleq \Theta[1-\lambda E(S)]$$

Let N(s) (resp. M(s)) be the numerator (resp the denominator) of the expression up there. N(s) has two real roots in $Re(s) \geq O$: O and $\frac{a}{F(O)}$ (a is positive since $\lambda E[S] < 1$). M(s) has at least two real roots in $Re(s) \geq O$: the first one is O and the existence of at least another real positive root is a consequence of :

$$\begin{cases} M(O) = O \\ M(+\infty) = +\infty \\ M'(O) = \Theta(E[S]\lambda-1) < O \end{cases}$$

The analyticity of $F^*(O)$ for $Re(s) \geq O$ implies that M(s) has at most one supplementary root in this domain, say ζ and that this root coincides with $\frac{a}{F(O)}$ completing the proof. □

Remark 3. A special case of interest is when $\Theta \to \infty$ (ie $D_1=D_n$, $n \geq 1$). We derive then from theorem 5 :

$$(5) \qquad F^*(s) = \frac{s(\Theta+s)[F(\Theta)s-\Theta(1-\lambda E[S])]}{(\lambda-s)(\Theta^2-s^2)-\lambda\Theta^2 S^*(s)} \underset{\Theta\to\infty}{\to} \frac{(1-\lambda E[S])s}{s+\lambda(S^*(s)-1)}$$

which coincides, as expected, with the waiting time transform in the M/G/1 system.

Remark 4. If $S_n = 0$, $n \geq 1$ then

$$(6) \qquad \zeta = \frac{\lambda+\sqrt{\lambda^2+4\Theta^2}}{2}$$

Remark 5. Another quantity of interest is the average value E[W] of the waiting time at steady state. After some algebra, we obtain as a consequence of theorem 5 :

$$(7) \qquad E[W] = \frac{\Theta.(1-\lambda E[S])(\Theta-\zeta)+\lambda.\zeta.(1+\Theta^2\frac{E[S^2]}{2})}{(1-\lambda E[S]).\Theta^2.\zeta}$$

when $S_n = 0$, this becomes :

$$(8) \qquad E[W] = \frac{\Theta^2+\zeta(\lambda-\Theta)}{\Theta^2\zeta}$$

3.2. The number in the system

So far, we have been exclusively interested in properties of the delay incurred by an object in traversing the system. We shall now consider properties of the number of objects in the system.

Let N_n denote the number of objects in the system just after the departure of ϕ_n. Since the order of departure with respect to the order of arrival is first-in-first-out, this is simply the number of arrivals in the interval $]a_n,a_n+T_n]$. Since $T_n = D_n + W_n + S_n$, and because the arrival instants after a_n do not depend on these quantities (W_n depends on the arrivals before a_n, see (1)), we can write

$$P[N_n = k] = \int_0^\infty \frac{(\lambda x)^k}{k!} e^{-\lambda x} \, dP[T_n \le x]$$

using the Poisson arrival assumption. We therefore have

(9) $$E[N_n] = \lambda E[T_n] = \lambda E[D_n] + \lambda E[W_n] + \lambda E[S_n]$$

Theorem 6. Under the assumptions of theorem 5, the average number of objects E[Q] waiting in the RA in steady state is given by

(10) $$E[Q] = \lambda E[S] + \lambda \frac{\Theta(1-\lambda E[S])(\Theta-\xi)+\lambda\zeta(1+\Theta\frac{E[S^2]}{2})}{\zeta.\Theta^2.(1-\lambda E[S])}$$ □

Notice that, since D_n and W_n are not independent, it is not easy to obtain the distribution of Q.

4. ANALYTICAL SOLUTION FOR THE CASE OF INDEPENDENT DELAYS OF EXPONENTIAL DISTRIBUTION

In [5], and later in [6], the case where the $\{D_n\}_{n \geq 1}$ are iid has been analyzed. The case when delays are exponential was treated in [5], while general independent delays were considered in [6]. In both cases, the number of customers in the R.A as well as analytical properties of the output process of the RA were computed (this last being considered as a bulk departure process). However, this approach leads to some difficulties for analyzing the end queue, since there is a correlation between the interarrival times and the bulk sizes. In the present section, we show that the end-to-end approach yields a tractable approach for analyzing the whole system including the end queue.

We first write the expression derived in lemma 1 in a form better suited to the present assumptions. Notice that when the $\{D_n\}_{n \geq 1}$ are iid, $\{W_n\}_{n \geq 1}$ given by (1) is not a Markov chain anymore.
Therefore, define :

$$Y_n \equiv T_n - S_n , \quad n \geq 1$$

where $T_n = D_n + S_n + W_n$; then we can write

$$Y_1 = D_1$$

and for $n \geq 2$:

$$Y_n = \begin{cases} D_n \text{ if } & D_n > Y_{n-1} + S_{n-1} - A_n \\ Y_{n-1} + S_{n-1} - A_n , & \text{otherwise} \end{cases}$$

or

$$(11) \qquad Y_n = \max[D_n, (Y_{n-1} + S_{n-1} - A_n)] , \quad n \geq 2$$

as a direct consequence of Lemma 1.

We shall assume that for all $n \geq 2$

$$E[\xi_n] = E[S_{n-1} - A_n] < 0$$

(notice now that $E[D_n] = E[D_{n-1}]$) so that

$$W_n \underset{P}{\Rightarrow} W \quad (\text{ a proper random variable})$$

Since $Y_n = D_n + W_n$, we have also

$$(12) \qquad Y_n \underset{P}{\Rightarrow} Y \quad (\text{ a proper random variable})$$

Denote

$$D(x) = P[D_n \leq x], \quad Y(x) = P[Y \leq x]$$

$$C(x) = P[\xi_n \leq x]$$

From (11) and (12) we can write

$$(13) \qquad Y(x) = \begin{cases} D(x) \int_{-\infty}^{x} Y(x-z)dC(z), & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

Now define

$$U(x) = \begin{cases} Y(x)/D(x) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

Since $W_n \geq 0$, it follows that for all $x \geq 0$

$$P[Y \leq x] \leq P[D \leq x]$$

Thus $Y(x) \leq 1$ for all $x \geq 0$; furthermore, since Y and D are proper random variables

$$\lim_{x \to \infty} U(x) = 1$$

Furthermore by (13), $U(x)$ satisfies

$$U(x) = \begin{cases} \int_{-\infty}^{x} U(x-z)D(x-z)dC(z) & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

so that it is non-decreasing ; hence $U(x)$ is a probability distribution function (In fact $U(x)$ is $P[Y_n \le x | D_n \le x]$). Define now

$$U_+(x) = \begin{cases} U(x) & \text{if } x \ge 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad U_-(x) = \begin{cases} 0 & \text{if } x \ge 0 \\ \int_{-\infty}^{x} U(x-z)D(x-z)dC(z) & \text{if } x < 0 \end{cases}$$

and their respective Laplace-Stieltjes transforms

$$U_+^*(s) = \int_{-\infty}^{\infty} e^{-sx} \, d\, U_+(x), \quad Re(s) \ge 0$$

$$U_-^*(s) = \int_{-\infty}^{+\infty} e^{-sx} \, d\, U_-(x), \quad Re(s) \le 0$$

Let $S^*(s)$, $A^*(s)$, be the Laplace-Stieltjes transforms of the densities of S and of A, respectively. We can write using (14) for <u>all</u> x :

$$U_+(x) + U_-(x) = \int_{-\infty}^{x} U(x-z)D(x-z)dC(z)$$

so that when $Re(s) = 0$:

(15) $$U_+^*(s) + U_-^*(s) = S^*(s)\, A^*(-s)\, K^*(s)$$

where

$$K^*(s) = \int_{0^-}^{\infty} d[U_+(x)D(x)]\, e^{-sx}, \quad Re(s) \ge 0 \quad .$$

4.1. The case of Poisson arrivals and exponential distribution

Equation (15) was obtained assuming that the $\{A_n\}_{n \geq 1}$, $\{D_n\}_{n \geq 1}$, $\{S_n\}_{n \geq 1}$ are mutually independent sequences. To proceed further with the analytical solution we must take specific instances of the distribution. Assuming Poisson arrivals of parameter λ and exponentially distributed delays of parameter μ, we have :

$$\begin{cases} A^*(s) = \frac{\lambda}{\lambda+s} \\ D^*(s) = \frac{\mu}{\mu+1} \end{cases}$$

so that

$$K^*(s) = U_+^*(s) - U_+^*(s+\mu) + U_+^*(s+\mu).\frac{\mu}{s+\mu}$$

$$= U_+^*(s) - \frac{s}{s+\mu} U_+^*(s+\mu) \;, \quad \mathrm{Re}(s) = 0$$

yielding, using (15), for Re(s) = 0 :

$$U_+^*(s) + U_-^*(s) = S^*(s) \frac{\lambda}{\lambda-s} [U_+^*(s) - \frac{s}{s+\mu} U_+^*(s+\mu)]$$

which is

$$(16) \quad U_-^*(s)(\lambda-s) = U_+^*(s)(s-\lambda) + \lambda S^*(s) [U_+^*(s) - \frac{s}{s+\mu} U_+^*(s+\mu)]$$

for Re(s) = 0.

Notice now that the Left-hand-side (LHS) of (16) is analytic for $\mathrm{Re}(s) \leq 0$, while its RHS is analytic for $\mathrm{Re}(s) \geq 0$, and by (16) they are equal for Re(s) = 0. Thus the RHS of (16) is the analytic continuation of the LHS

$$U_-^*(s)(\lambda-s)$$

for Re(s) > 0. Clearly then $U_-^*(s)$ can have at most one pole for Re(s) > 0 at $s=\lambda$, so that we may write

$$U_-^*(s) = \frac{\alpha}{\lambda-s} + g(s)$$

where $g(s)$ is some analytic function both for $Re(s) \le 0$ and for $Re(s) \ge 0$, and α is a constant. Furthermore, since $U_-^*(s)$ and $U_+^*(s)$ are bounded on their respective domains, so is $g(s)$. Therefore by Liouville's theorem $g(s)$ must be a constant. To obtain it, set $s=0$ in (15) ; since

$$U_+^*(0) = A^*(0) = S^*(0) = K^*(0) = 1$$

it follows that $U_-^*(0) = 0$, yielding

$$g(s) = g(0) = -\alpha/\lambda$$

Therefore (16) becomes

$$U_+^*(s)\ [s-\lambda+\lambda S^*(s)] = \frac{\lambda s S^*(s)}{s+\mu} U_+^*(s+\mu) + s\alpha/\lambda$$

or

(17) $$U_+^*(s) = \frac{\alpha}{\lambda} \cdot m(s) + \ell(s) \cdot U_+^*(s+\mu), \quad Re(s) \ge 0$$

where

(18) $$\begin{cases} m(s) = \dfrac{s}{s+\lambda(S^*(s)-1)} \\ \\ \ell(s) = \dfrac{\lambda s S^*(s)}{(s+\mu)[s+\lambda(S^*(s)-1)]} \end{cases}$$

Let us first determine α, using the fact that $U_+^*(0) = 1$. After some algebra, necessitated by the indeterminate forms which arise, we have

(19) $$\alpha = \lambda(1-\lambda E[S]) - \frac{\lambda^2}{\mu} U_+^*(\mu)$$

where $U_+^*(\mu)$ is still unknown. Notice also that $\frac{\alpha}{\lambda}$ is exactly $U(0)$.

We are now ready to prove the main result of this section.

<u>Theorem 7.</u> If $\lambda E[S] < 1$, then $U_+^*(s)$ the Laplace-Stieltjes transform of $U(x) \triangleq Y(x)/D(x)$ is given by :

$$(20) \qquad U_+^*(s) = U(0).\xi(s)$$

where $\xi(s) = m(s) + \sum_{n=1}^{\infty} m(s+n\mu) \prod_{i=0}^{n-1} \ell(s+i\mu)$

and

$$U(0) = (1-\lambda E[S]).\frac{1}{1+\frac{\lambda}{\mu}.\xi(\mu)} \qquad \square$$

<u>Proof</u> : By applying (17) recursively to itself we obtain for any $N > 0$

$$(21) \qquad U_+^*(s) = U(0)[m(s) + \sum_{n=1}^{N} m(s+n\mu) \prod_{i=0}^{n-1} \ell(s+i\mu)] + \prod_{i=0}^{N} \ell(s+i\mu).U_+^*(s+(N+1)\mu)$$

Notice first that $m(s)$ and $\ell(s)$ are analytic for $Re(s) \geq 0$. To show this it suffices to examine their poles for $Re \geq 0$, which are simply the zeros (see (18)) of

$$s + \lambda(S^*(s) - 1)$$

Now clearly for $Re(s) \geq 0$

$$\left|\frac{d}{ds} S^*(s)\right| \leq E[S]$$

so that, if $\lambda E[S] < 1$

$$\left|\lambda \int_{0^-}^{s} d\, S^*(s)\right| = \lambda|S^*(s) - 1| \leq \lambda E[S]|s| < |s|$$

Therefore by Rouché's theorem $s+\lambda(S^*(s)-1)$ has exactly the same number of zeros for $Re(s) \geq 0$ as s, namely one at $s=0$, which cancels with the zero of $m(s)$ or of $\ell(s)$. Hence $m(s)$ and $\ell(s)$ have no poles for $Re(s) \geq 0$ and are thus analytic. Clearly

$$\lim_{n\to\infty} |m(s+n\mu)| = 1$$

while

$$\lim_{n\to\infty} |U_+^*(s+n\mu)| = U(0) < 1$$

and for some positive constant K,

$$\prod_{i=0}^{n} \ell(s+n\mu) \le \frac{\lambda^n}{\mu^n} \frac{K}{n!}$$

for large enough n. Thus as we take $N \to \infty$ in (21) the last term vanishes and we remain with (20). The unknown constant $U_+^*(\mu)$ is determined by solving (20) at $s=\mu$.

□

A few remarks concerning the form of (17) are worth mentioning.

Remark 6. We see that m(s) is very similar to the waiting time distribution transform of an M/G/1 queue (it suffices to replace α/λ by $(1-\lambda E[S])$ in the numerator). Therefore U can be viewed as the sum of two random variables, the first of which is similar in form to the waiting time of an M/G/1 system.

Remark 7. As $\mu \to \infty$ (the delay D is zero), (19) yields

$$\alpha = \lambda(1-\lambda E[S])$$

since $\ell(s) \to 0$; hence (17) becomes

$$U_+^*(s) = \frac{s(1-\lambda E[S])}{s+\lambda(S^*(s)-1)}$$

which is precisely, as expected, the transform of the waiting time in the M/G/1 system.

More generally since

$$U(x) = Y(x)/D(x), \quad x > 0$$

it follows that when D is exponentially distributed

$$Y(x) = U(x) - e^{-\mu x} U(x)$$

so that

$$Y^*(s) = U_+^*(s) - U_+^*(s+\mu)$$

From (17) we then have

$$U_+^*(s) = U(0)\ m(s) + \ell(s)[U_+^*(s) - Y^*(s)]$$

or

$$(22) \qquad Y^*(s) = U(0) \frac{m(s)}{\ell(s)} + U_+^*(s) \left[\frac{\ell(s)-1}{\ell(s)}\right]$$

A special case of interest corresponds to the model analyzed in [5] when we set $S_n = 0,\ n \geq 1$. We then have :

<u>Corollary 8</u> Under the conditions of Theorem 7, if $S_n=0,\ n \geq 1$, it follows that

$$(23) \qquad U_+^*(s) = e^{-\lambda/\mu} \left(1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \left(\frac{\lambda}{s+\mu i}\right)\right)$$

<u>Remark 8.</u> For $\mu >> \lambda$ (the average value of D is very small) we have

$$(24) \qquad U_+^*(s) \cong [1 + \lambda/(s+\mu)] / [1 + \lambda/\mu]$$

or

$$U(x) \cong 1 - \frac{\lambda/\mu}{1+\lambda/\mu} e^{-\mu x}, \quad x \geq 0$$

so that

$$Y(x) \cong 1 - e^{-\mu x}\left(1 + \frac{\lambda/\mu}{1+\lambda/\mu}\right) + e^{-2\mu x}\,\frac{\lambda/\mu}{1+\lambda/\mu}$$

yielding

$$E[Y] \cong \frac{1}{\mu}\left(1 + \frac{\lambda/\mu}{1+\lambda/\mu}\right) - \frac{1}{2\mu}\cdot\frac{\lambda/\mu}{1+\lambda/\mu}$$

or

$$E[Y] \cong \frac{1}{\mu} + \frac{1}{2\mu}\cdot\frac{\lambda/\mu}{1+\lambda/\mu}$$

Remark 9. The average number of customers in the system just after a departure may be obtained in the case of Poisson arrival using equation (8) which remains satisfied here.

CONCLUSIONS

Resequencing algorithms have been analyzed in this paper for two types of disordering systems of interest for modelling communication networks and distributed systems. The case where the disordering delay is an independent increment process, as well as that of independent identically distributed delays, are considered. The approach taken is that of examining the end to end delay encountered by each object in the system. These leads to some new analytical results concerning both the "pure" resequencing problem and concerning the effect of an "output service time" on resequencing delays.

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