On the uncapacitated plant location problem II : facets and lifting theorems

D. Chinhyung Cho, M.W. Padberg, M.R. Rao

IRIA

CENTRE DE ROCQUENCOURT

Institut National
de Recherche
en Informatique
et en Automatique

Domaine de Voluceau
Rocquencourt
B.P. 105
78153 Le Chesnay Cedex
France
Tél. 954 90 20

Rapports de Recherche

N° 87

ON THE UNCAPACITATED PLANT LOCATION PROBLEM II : FACETS AND LIFTING THEOREMS

D. Chinhyung CHO
Manfred W. PADBERG
M. R. RAO

Juillet 1981

ON THE UNCAPACITATED PLANT LOCATION PROBLEM II :

FACETS AND LIFTING THEOREMS

by

D. CHINHYUNG CHO*)

Manfred W. PADBERG**)

M.R. RAO***)

*) GBA, New York University, New York, N.Y.

**) GBA, New York University, New York, N.Y. and INRIA, Rocquencourt, France.

***) Indian Institute of Management, Bangalore, India.

1. INTRODUCTION

In the previous paper [2], several facets of (PLP) were discussed. We showed that non-negative constraints and clique constraints are trivial facets of (PLP). We also showed that, for the case of 3 plants and 3 or more destinations, all of the non-trivial facets of (PLP) correspond to chordless odd-cycles of length 9 in the associated intersection graph.

We start this paper by discussing several lifting theorems for set-packing problems in section 2. The so-called claw-inequality [6] and its variations are discussed in this section.

In section 3, we consider the following questions for (PLP):

1) Can a facet for a smaller dimensional problem be also a facet for a larger dimensional problem?

2) Is it possible to generate a different facet from a known facet and, if so, how?

In section 3, we give an affirmative answer to question 1. Theorem 3.4 in section 3, answers question 2 partially. Is is proven that, if the known facet-defining inequality has 0-1 coefficients, we can generate a new facet by adding a plant that supplies the same destinations as one of the plants with positive coefficient in the known facet-defining inequality. Finally, theorem 3.5 in the same section gives necessary and sufficient conditions for non-trivial facets of (PLP) with 0-1 coefficients.

The results of [2] and those obtained in section 2 and section 3 are applied to derive a family of facts in section 4. Then, in section 5, we describe all facets for the case of 3 or more plants and 3 destinations. Contrary to the case of 3 plants and 3 or more destinations treated in [2], a non-trivial facet for the case of 3 or more plants and 3 destinations contains an odd-cycle of length 9 but not necessarily without chords.

For notations and definitions, see section 2 of the previous paper [2].

2. Lifting theorems for the set-packing problem.

In this section, we will discuss some of the lifting theorems for the set-packing problem and, in the following sections, we will discuss some more special lifting theorems for (PLP). Several lifting theorems for the set-packing problem can be found in [6].

In this section, we use the terminology and notation as defined in [6]. More precisely, we will use π and t instead of (π,μ) and (x,y), respectively. In other words, for $i = 1,2,\ldots,p$ and $j = 1,2,\ldots,d$, we let $\pi_{ij} = \pi_{(i-1)d+j}$, $x_{ij} = t_{(i-1)d+j}$, and for $i = 1,2,\ldots,p$ we let $\mu_i = \pi_{pd+i}$, $y_i = t_{pd+i}$. Let $P = \{t \in R^n \mid At \leq e,\ 0 \leq t \leq e_n\}$ with A being a 0-1 matrix where e, e_n are vectors of ones and $P_I^S = P_I \cap \{t \in R^n \mid t_j = 0 \text{ for all } j \notin S\}$ where $P_I = \text{conv}\{t \in P \mid t \text{ integer}\}$. Also, let S be the support of π, i.e. $S = \{j \in N \mid \pi_j > 0\}$ and $G^S = (S, E^S)$ be the induced subgraph of G with a node set S and an edge set $E^S \subseteq E$.

Definition. [6]

A vertex-induced subgraph $G^S = (S, E^S)$ of $G = (N, E)$ with node set $S \subseteq N$ is facet-producing if there exists an inequality $\pi t \leq \pi_0$ with non-negative integer components π_j such that (i) $\pi t \leq \pi_0$ is a facet of P_I^S and (ii) $\pi t \leq \pi_0$ is not a facet for $P_I^T = P_I \cap \{t \in R^n \mid t_j = 0 \text{ for all } j \notin T\}$ where T is any subset of S such that $|T| = |S| - 1$. A subgraph G^S of G is called strongly facet-producing if there exists an inequality $\pi t \leq \pi_0$ such that (i) holds and (ii) holds for all $T \subset S$ satisfying $|T| \leq |S| - 1$. A subgraph G^S of G is facet-defining if there exists an inequality $\pi t \leq \pi_0$ such that (i) holds and (iii) such that $\pi_j > 0$ for $j \in S$.

Theorem 2.1. If the inequality

$$\sum_{j \in S} \pi_j t_j \leq 1 \qquad (2.1)$$

is a facet of P_I^S, then there exist non-negative numbers α_j, $0 \leq \alpha_j \leq 1$, such that

$$\sum_{j \in S} \pi_j t_j + \sum_{j \in N-S} \alpha_j t_j \leq 1 \qquad (2.2)$$

is a facet of P_I.

Proof: See [5 , theorem 3.3].

A facet (2.2) is called a lifting of (2.1). There are several ways that one can lift a facet. One way is to consider one potential node $i \in N-S$ at a time in a sequence, called a sequential lifting, and the other way is to consider several nodes at a time. Some versions of the latter procedure are illustrated below.

Remark 2.1. In a sequential lifting, π_j for $j \in S$ remains the same and different orders of the nodes in N-S may yield different facets. The calculation of the coefficients, α_j, $j \in N-S$, is carried out iteratively, see [5]. The first step of the calculation is as follow:

For $k \in N-S$ let

$$z_k = \max \{ \sum_{j \in S} \pi_j t_j \mid t \in P_I^{S-k},\ t_k = 1 \}$$

Then in an ordering where variable t_k appears in the first position $\alpha_k = 1 - z_k$. For later reference we call this number the first-order lifting coefficient of variable t_k or, simply, the lifting coefficient of variable t_k.

Construction 1. Let G be any facet-defining graph with a node set $N = \{1,2,\ldots,n\}$ where $n \geq 2$ and define $G^* = (N^*,E^*)$ as follows.

$$N^* = N \cup \{n+1, n+2,\ldots,2n,2n+1\}$$

$$E^* = E \cup \{(1,n+1),(2,n+2),\ldots,(n,2n)\} \cup \{(n+1,2n+1),(n+2,2n+1),\ldots,(2n,2n+1)\}$$

See figure 2.1 where the construction is carried out for an odd cycle of length 5. Let $\pi t \leq 1$ be a facet defined by G. Define $\pi^* \in R^{2n+1}$ as

$$\pi_j^* = \pi_j \quad \text{for } j = 1,2,\ldots,n$$

$$\pi_{n+i}^* = \pi_i \quad \text{for } i = 1,2,\ldots,n$$

$$\pi_{2n+1}^* = \sum_{i=1}^{n} \pi_{n+i}^* - 1$$

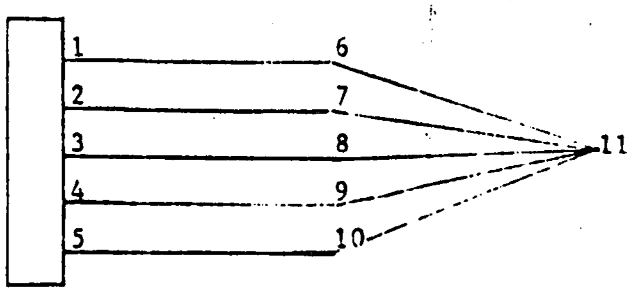


Figure 2.1.

Theorem 2.2. Let $G = (N, E)$ be a facet-defining graph on $n \geq 2$ nodes and let $\pi t \leq 1$ be a facet defined by G. Denote by $G^* = (N^*, E^*)$ the graph obtained from G as indicated above. Then, G^* strongly produces the facet

$$\pi t + \sum_{i=1}^{n+1} \pi_{n+i}^* t_{n+i} \leq \sum_{i=1}^{n} \pi_{n+i}^* .$$

Proof: See [6, theorem 6].

We extend this theorem as follows.

Construction 2. Let G be partitioned into $q \geq 2$ node-disjoint complete subgraphs C_i satisfying $|C_i| \geq 1$ for $i = 1,2,\ldots,q$ and define $G^* = (N^*, E^*)$ as follows.

$$N^* = N \cup \{n+1, n+2, \ldots, n+q+1\} \tag{2.3}$$

$$E^* = E \cup \bigcup_{i=1}^{q} \{(j,n+i) \mid j \in C_i\} \cup \{(n+1,n+q+1),(n+2,n+q+1),\ldots,(n+q,n+q+1)\} \tag{2.4}$$

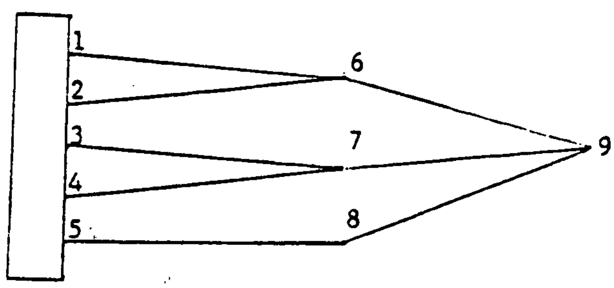
See figure 2.2 where the construction is carried out for an odd cycle of length 5. Let $\pi t \leq 1$ be a facet of G. Finally, let $z = (t_1, t_2, \ldots, t_n, t_{n+1}, \ldots, t_{n+q+1})$ and define $\pi^* \in R^{n+q+1}$ as follows.

$$\pi^*_j = \pi_j \qquad \text{for } j = 1,2,\ldots,n \tag{2.5}$$

$$\pi^*_{n+i} = \max_{j \in C_i} \pi_j \qquad \text{for } i = 1,2,\ldots,q \tag{2.6}$$

$$\pi^*_{n+q+1} = \sum_{i=1}^{q} \pi^*_{n+i} - 1 \tag{2.7}$$

$$\pi^* z = \pi^*_{n+q+1} + 1 = \sum_{i=1}^{q} \pi^*_{n+i} \tag{2.8}$$



$C_1 = \{1,2\}$

$C_2 = \{3.4\}$

$C_3 = \{5\}$

Figure 2.2.

Remark 2.2. Since $\pi t \leq 1$ is a facet of $P_I(G)$ and $q \geq 2$, there exists a 0-1 vector $t \in P_I(G)$ satisfying $\pi t = 1$ and $\sum_{j \in C_1} t_j = 0$, say, i.e. $t_j = 0$ for all $j \in C_1$. (Otherwise, $\pi t = \sum_{j \in C_1} t_j = 1$ which implies that the facet will be $\sum_{j \in C_1} t_j \leq 1$ and this is a contradiction to the fact that G is a facet defining graph with $q \geq 2$.) Furthermore, C_i's are complete subgraphs and thus $\sum_{j \in C_i} \pi_j t_j \leq \pi^*_{n+i}$, which implies $\sum_{i=2}^{q} \sum_{j \in C_i} \pi_j t_j \leq \sum_{i=2}^{q} \pi^*_{n+i}$. Now, since C_i's partition G, we have $\sum_{i=2}^{q} \sum_{j \in C_i} \pi_j t_j = \sum_{k \notin C_1} \pi_k t_k = 1$. Therefore, $\sum_{i=2}^{q} \pi^*_{n+i} \geq 1$ and since $\pi^*_{n+1} > 0$, we have $\sum_{i=1}^{q} \pi^*_{n+i} > 1$, i.e. $\pi^*_{n+q+1} > 0$.

Theorem 2.3. Let $\pi t \leq 1$ be a facet defined by G and $\pi_j > 0$ for $j = 1,2,\ldots,n$. Let $G^* = (N^*, E^*)$ be defined as in (2.3)-(2.4) with respect to some partitioning of G into $q \geq 2$ disjoint complete subgraphs C_i, $i = 1,2,\ldots,q$. Let z, π^* and π^*_0 be defined as in (2.5)-(2.8). Then, $\pi^* z \leq \pi^*_0$ defines a facet of $P_I(G^*)$, i.e. $\pi t + \sum_{i=1}^{q} \pi^*_{n+i} t_{n+i} + (\sum_{i=1}^{q} \pi^*_{n+i} - 1) t_{n+q+1} \leq \sum_{i=1}^{q} \pi^*_{n+i}$ defines a facet of $P_I(G^*)$.

Proof:

Let $z \in P_I(G^*)$ be the incidence vector of any independent node set in G^*. If $t_{n+q+1} = 1$, then $t_{n+i} = 0$ for $i = 1,2,\ldots,q$ and thus z satisfies the inequality $\pi^* z \leq \pi^*_0$. If $t_{n+q+1} = 0$, then by construction we have $\pi^* z \leq \pi^*_0$. Thus, the inequality $\pi^* z \leq \pi^*_0$ is a valid inequality for $P_I(G^*)$.

Let A be a n x n non-singular matrix of incident vectors of independent node sets in G satisfying $\pi t = 1$ and consider

$$A^* = \begin{pmatrix} A & 0\ldots0 & 1 \\ F & E-I & 0 \\ 0 & e & 0 \end{pmatrix}$$

where E-I is the q x q matrix having zeroes on the diagonal and ones every where else, F is the q x n matrix having exactly one 1 per row and zeroes else where, 1 is the column vector with n entries equal to one, and e is a row vector with q entries equal to one. The one entry in row i of F is chosen for that node of C_i (i.e. those columns of A) where the $\max_{j \in C_i} \pi_j$ is assumed. It follows that A* is the (n+q+1) x (n+q+1) incidence matrix of n+q+1 independent node sets of G* all of which satisfy $\pi^* z \leq \pi_0^*$ at equality. But

$$\begin{aligned} |A^*| &= |A| \left| \begin{pmatrix} E-I & 0 \\ e & 0 \end{pmatrix} - \begin{pmatrix} F \\ 0 \end{pmatrix} A^{-1} (0\ldots0\ 1) \right| \\ &= |A| \left| \begin{pmatrix} E-I & 0 \\ e & 0 \end{pmatrix} - \begin{pmatrix} F \\ 0 \end{pmatrix} (0\ldots0\ \pi) \right| \\ &= |A| \begin{vmatrix} E-I & -\pi^*_{n+i} \\ e & 0 \end{vmatrix} \\ &= |A| \begin{vmatrix} -I & -\pi^*_{n+i} \\ e & 0 \end{vmatrix} \\ &= |A| \begin{vmatrix} -I & -\pi^*_{n+i} \\ 0 & -\sum_{i=1}^{q} \pi^*_{n+i} \end{vmatrix} \\ &= |A| \, (-1)^{q+1} \sum_{i=1}^{q} \pi^*_{n+i} \\ &\neq 0 \end{aligned}$$

implies that $\pi^* z \leq \pi_0^*$ is a facet of $P_I(G^*)$. Δ

Construction 3. Let $C_1, C_2, \ldots, C_q$, $q \geq 2$ be any node-disjoint complete subgraphs of G satisfying $|C_i| \geq 2$ for $i = 1,2,\ldots,q$, i.e. they do not necessarily partition G and thus it is possible to have $N - \bigcup_{i=1}^{q} C_i \neq \emptyset$. Define $G^* = (N^*, E^*)$ as follows.

$$N^* = N \cup \{n+1, n+2, \ldots, n+q+1\} \quad (2.9)$$

$$E^* = E \cup \bigcup_{i=1}^{q} \{(j,n+i) \mid j \in C_i\} \cup \{(n+1,n+q+1),(n+2,n+q+1),\ldots,(n+q\ n+q+1)\} \quad (2.10)$$

See figure 2.3 where the construction is carried out for an odd cycle of length 5. Let $z = (t_1, t_2, \ldots, t_n, t_{n+1}, \ldots, t_{n+q+1})$ and define $\pi^* \in R^{n+q+1}$ as follows.

$$\pi_j^* = (q-1)\pi_j \qquad \text{for } j = 1,2,\ldots,n \quad (2.11)$$

$$\pi_{n+i}^* = 1 - c \qquad \text{for } i = 1,2,\ldots,q+1 \quad (2.12)$$

$$\pi_0^* = q - c \quad (2.13)$$

where $c = \max \{\pi t \mid t \in P_I(G),\ \sum_{j \in C_i} t_j = 0 \text{ for all } i = 1,2,\ldots,q\}$.

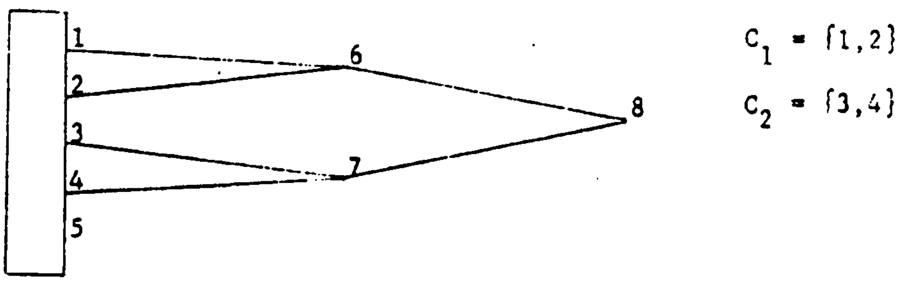


Figure 2.3.

Theorem 2.4. Let $\pi t \leq 1$ be a facet defined by G and $\pi_j > 0$ for $j = 1,2,\ldots,n$. Let $G^* = (N^*, E^*)$ be defined as in (2.9)-(2.10) with respect to a collection of $q \geq 2$ node-disjoint complete subgraphs C_i, $i = 1,2,\ldots,q$. Let c, π^* and π_0^* be defined as in (2.11)-(2.13), and, for a subset K of $\{1,2,\ldots,q\}$ such that $1 \le |K| \le q$, let $c_K = \max\{\pi t \mid t \in P_I(G),\ t_j = 0 \text{ for } j \in \bigcup_{i \in K} C_i\}$ and $c^k = \max\{c_K \mid |K| = k\}$. (note: $c^q = c$.) Suppose $c < 1$ and $c^k \le 1 - \frac{(k-1)(1-c)}{q-1}$ then $\pi^* z \le \pi_0^*$ or equivalently $(q-1)\pi t + (1-c)\sum_{i=1}^{q+1} t_{n+i} \le q-c$ defines a facet of $P_I(G^*)$.

Proof:

Let $z \in P_I(G^*)$ be the incidence vector of any independent node-set in G*. If $t_{n+q+1} = 1$ then $t_{n+i} = 0$ for $i = 1,2,\ldots,q$ and z satisfies the inequality $\pi^* z \le \pi_0^*$ because $\pi t \le 1$. If $t_{n+q+1} = 0$, then $\pi^* z \le \pi_0^*$ follows from the construction. For if z is such that $\sum_{j \in C_i} t_j = 0$ for exactly one index i, then $t_{n+k} = 0$ for $k \ne i$ and $(q-1)\pi t + (1-c)\sum_{i=1}^{q+1} t_{n+i} \le q-1+1-c = q-c = \pi_0^*$, while if $\sum_{j \in C_i} t_j = 0$ for $k \ge 2$ indices of i then we have $(q-1)\pi t + (1-c)\sum_{i=1}^{q+1} t_{n+i} \le (q-1)c^k + k(1-c) \le (q-1)+(1-c) = \pi_0^*$. Thus, $\pi^* t \le \pi_0^*$ is a valid inequality for $P_I(G^*)$.

Let A be a n x n non-singular matrix whose rows are incidence vectors of independent node-sets in G satisfying $\pi t = 1$ and consider

$$A^* = \begin{pmatrix} A & 0\ldots0 & 1 \\ F & I & 0 \\ a & e & 0 \end{pmatrix}$$

where I is the q x q identity matrix, F is a q x n matrix and i-th row of F corresponds to an incidence vector of an independent node-sets in G satisfying $\pi t = 1$ and $\sum_{j \in C_i} t_j = 0$, e is a row vector with q entries equal to one, 1 is a column vector with n entries equal to one, and a is the incidence vector of some independent node-set in G satisfying $\pi t = c$ and $\sum_{j \in C_i} t_j = 0$ for all $i = 1,2,\ldots,q$

It follows that every row of A* defines an incidence vector z of some independent node set in G* satisfying $\pi^* z \le \pi_0^*$ with equality. Furthermore,

$$\begin{aligned} |A^*| &= |A| \left| \begin{pmatrix} I & 0 \\ e & 0 \end{pmatrix} - \begin{pmatrix} F \\ a \end{pmatrix} A^{-1}(0\ldots0\ 1) \right| \\ &= |A| \left| \begin{pmatrix} I & 0 \\ e & 0 \end{pmatrix} - \begin{pmatrix} F \\ a \end{pmatrix} (0\ldots0\ \pi) \right| \\ &= |A| \begin{vmatrix} I & -1 \\ e & -c \end{vmatrix} \\ &= |A|\ (q-c) \\ &\neq 0. \end{aligned}$$

Consequently, $\pi^* z \le \pi_0^*$ defines a facet of $P_I(G^*)$. Δ

The question whether or not Constructions 2 and 3 yield facet-producing or even strongly facet-producing graphs is left for future research.

3. Facets of (PLP) with 0-1 coefficients.

In this section, we will discuss necessary and sufficient conditions for facets of (PLP) with 0-1 coefficients.

We note that if a non-trivial facet $\pi x + \mu y \leq \pi_0$ has 0-1 coefficients, then there exists a pd-subgraph G^S of G whose node set corresponds to the support of this facet, i.e. $N^S = \{x_{ij} | \pi_{ij} > 0\} \cup \{y_i | \mu_i > 0\}$, and S is an $|I^S| \times |J^S|$ adjacency matrix. In fact, from 2) and 3) of theorem 5.1 in [2], each row must contain at least two non-zero elements and each column must contain at least two non-zero elements. Also, from theorem 3.1 in [2], it follows that $\pi_0 = \alpha(G^S)$.

For our discussion below, we need the following theorem by Chvatal [1].

<u>Theorem 3.1. If all of the edges in $G^S = (N^S, E^S)$ are critical and G^S is connected, then the inequality</u>

$$\sum_{i \in I^S} \sum_{j \in J^S} s_{ij} x_{ij} + \sum_{i \in I^S} y_i \leq \alpha(G^S) \qquad (3.1)$$

<u>is a facet of L_I^S, where $s_{ij} = 0$ or 1.</u>

<u>Proof</u>: See [1, theorem 4.2.].

Cornuejols and Thizy (C-T) [3] proved that the sequential lifting coefficients of an odd-cycle facet are either 0 or 1. We prove the following more general result.

<u>Theorem 3.2. Let S be an $|I^S| \times |J^S|$ adjacency matrix where $I^S \subseteq P$ and $J^S \subseteq D$ so that the inequality</u>

$$\sum_{i \in I^S} \sum_{j \in J^S} s_{ij} x_{ij} + \sum_{i \in I^S} y_i \le \alpha(G^S) \tag{3.2}$$

is a facet of L_I^S. Then, there exists a $p \times d$ adjacency matrix U whose elements are either 0 or 1 with $u_{ij} \ge s_{ij}$ for $i \in I^S$, $j \in J^S$ and $u_{ij} = 0$ elsewhere such that the inequality

$$\sum_{i \in I^S} \sum_{j \in J^S} u_{ij} x_{ij} + \sum_{i \in I^S} y_i \le \alpha(G^S) \tag{3.3}$$

is a facet of L_I^{pd}.

Proof:

We distinguish four cases.

Case 1. $v = x_{ij}$ where $i \in P - I^S$, $j \in D$. Since (3.2) is not a trivial facet there exists a solution vector $(x,y) \in L_I^S$ such that $\sum_{i \in I^S} x_{ij} = 0$ which satisfies (3.2) with equality. Hence we can set $x_{ij} = 1$ and the lifting coefficient of x_{ij} is equal to 0.

Case 2. $v = x_{ij}$ where $i \in I^S$, $j \in D-J^S$. Since (3.2) is not a trivial facet there exists a solution vector $(x,y) \in L_I^S$ such that $y_i = 0$ which satisfies (3.2) with equality. Hence we can set $x_{ij} = 1$ and the lifting coefficient of x_{ij} is equal to 0.

Case 3. $v = y_i$ where $i \in P-I^S$.

We can simply let $y_i = 1$ without altering any solution vector $(x,y) \in L_I^S$ which satisfies (3.2) with equality. Hence, the lifting coefficient of y_i is equal to 0.

Case 4. $v = x_{ij}$ where $i \in I^S$, $j \in J^S$ such that $s_{ij} = 0$.

Since (3.2) is not a trivial facet there exists a solution vector $(x,y) \in L_I^S$ such that $\sum_{i \in I^S} x_{ij} = 0$ which satisfies (3.2) with equality. Hence setting $x_{ij} = 1$ reduces $\sum_{i \in I^S} y_i$ by at most one and thus the lifting coefficient of x_{ij} is either 0 or 1. Δ

From theorem 3.2 we obtain the following corollary.

Corollary 3.1. For $I^S \subseteq P$ and $J^S \subseteq D$, let the inequality

$$\sum_{i \in I^S} \sum_{j \in J^S} a_{ij} x_{ij} + \sum_{i \in I^S} y_i \leq \alpha(G^S) \tag{3.4}$$

be a facet of L_I^{pd} where $a_{ij} = 0$ or 1. If $P \subset P^*$ and $D \subseteq D^*$ with $|P^*| = p^*$ and $|D^*| = d^*$, then (3.4) is also a facet of $L_I^{p^*d^*}$.

Proof:

We make the same case definition as in the proof of Theorem 3.2. The corresponding first three cases are identical and lead to lifting coefficients of zero. In the fourth case it follows that the lifting coefficient is zero because (3.4) is assumed to be a facet of L_I^{pd}. Δ

Corollary 3.1 can be further generalized to the case where the coefficients are not 0-1 integers.

Corollary 3.2 For $I^S \subseteq P$ and $J^S \subseteq D$, let the inequality

$$\sum_{i \in I^S} \sum_{j \in J^S} \pi_{ij} x_{ij} + \sum_{i \in I^S} \mu_i y_i \leq \pi_0 \tag{3.5}$$

be a facet of L_I^{pd} where π_{ij} and μ_i are non-negative integers. If $P \subseteq P^*$ and $D \subseteq D^*$ with $|P^*| = p^*$ and $|D^*| = d^*$, then (3.5) is a facet of $L_I^{p^*d^*}$.

Proof:

The proof is similar to the proof of corollary 3.1. Δ

The two corollaries state that a facet of the polyhedron L_I^{pd} stays a facet if either p of d or both are increased. The following theorem gives one way of constructing a class of facets from a given facet with 0-1 coefficients.

Theorem 3.3. (Replicating theorem) Let S be an $|I^S| \times |J^S|$ adjacency matrix corresponding to the type (3.4), which is a facet of L_I^{pd}. Suppose we append one more row to S so that the new row k replicates some other row q already in S, and call this new matrix S'. Then, the inequality

$$\sum_{i \in I^{S'}} \sum_{j \in J^{S'}} s'_{ij} x_{ij} + \sum_{i \in I^{S'}} y_i \le \alpha(G^{S'})$$

or equivalently,

$$\sum_{i \in I^S} \sum_{j \in J^S} s_{ij} x_{ij} + \sum_{i \in I^S} y_i + \sum_{j \in J_q} x_{kj} + y_k \le \alpha(G^S) + 1 \qquad (3.6)$$

is a facet of $L_I^{|I^{S'}||J^{S'}|}$ where $J_q = \{j \mid s_{qj} = 1 \text{ for some } q \in I^S\}$ and $k \in I^{S'} - I^S$.

Proof:

First, we note that $\alpha(G^{S'}) = \alpha(G^S) + 1$ because, due to the construction, the covering number remains the same while there is one additional plant and no change in the number of destinations. Since $|J_q| \ge 2$, we choose u and v in J_q so that $u \ne v$, and consider the following inequality.

$$\sum_{i \in I^S} \sum_{j \in J^S} s_{ij} x_{ij} + \sum_{i \in I^S} y_i + x_{ku} + x_{kv} + y_k \le \alpha(G^S) + 1 \qquad (3.7)$$

Using the construction 3 and theorem 2.4 in section 2, we prove that (3.7) is a facet of L_I^{G*}, where $G*$ is a subgraph of $G^{S'}$ induced by $N^S \cup \{x_{ku}, x_{kv}, y_k\}$.

Let $C_1 = \{x_{iu} | i \in I^S \text{ and } s_{iu} = 1\}$ and $C_2 = \{x_{iv} | i \in I^S \text{ and } s_{iv} = 1\}$. Then, the maximal node packing of G with $\sum_{x_{ij} \in C_1 \cup C_2} x_{ij} = 0$ is $\alpha(G^S)-1$. This is so because there must exist a solution vector $(x,y) \in L_I^{pd}$ satisfying (3.4) with equality such that $\sum_{i \in I_j} x_{ij} = 0$ for $j = u$ or $j = v$. Hence, the maximal node packing of G with $\sum_{x_{ij} \in C_1 \cup C_2} x_{ij} = 0$ is at least $\alpha(G^S)-1$. On the other hand, since $u,v \in J_q$ and $u \neq v$, we have $I_u \cap I_v \neq \emptyset$ and so there exists no solution vector (x,y) satisfying (3.4) with equality such that $\sum_{i \in I_u} x_{iu} = 0$ and $\sum_{i \in I_v} x_{iv} = 0$. Otherwise, for $q \in I_u \cap I_v$, we can let $y_q = 0$, $x_{qu} = 1$, $x_{qv} = 1$ and others same so that it is still feasible but the left hand side of (3.4) becomes $\alpha(G^S)+1$. This is a contradiction. Therefore, the maximal node packing of G with $\sum_{x_{ij} \in C_1 \cup C_2} x_{ij} = 0$ is $\alpha(G^S)-1$ and, from theorem 2.4 noting the fact that $q = 2$ and $c = \frac{\alpha(G^S)-1}{\alpha(G^S)}$, (3.7) is a facet of $L_I^{G^*}$.

Now from the fact that the row corresponding to the plant k is a replicate of the row corresponding to the plant q, it follows by sequential lifting that s'_{kj} equals 1 for $j \in J_q$. Since by assumption (3.4) defines a facet of L_I^{pd}, S is a maximal pd-adjacency matrix. On the other hand, if $s'_{kj} = 1$ for some $j \notin J_q$, then the covering number drops by one. Thus, $s'_{kj} = 0$ for $j \notin J_q$. Δ

Remark 3.1. If a new plant is added to the (PLP) with p plants and d destinations theorem 3.4 enables us to generate a facet of $L_I^{(p+1)d}$ from a facet of L_I^{pd} by making a new plant k to play the same role as another plant q so that $J_k = J_q$. This procedure can be repeated as many times as there are additional plants to be added. In each replication, the right hand side of the inequality increases by one. However, this result does not carry over for the case of adding a new destination because, from 4) of theorem 5.1 in [2], a facet must have distinct I_j's.

Next we state several useful properties of maximal adjacency matrices.

Lemma 3.1. A pd-adjacency matrix S is maximal if and only if S satisfies the following properties:

1) $\beta \geq 2$ where β is the covering number of the associated pd-subgraph G^S.

2) For each row and column, there exist at least β-1 zero elements. Furthermore, if $s_{i^*j^*} = 0$, then there exist β-1 plants among $\overline{I_{j^*}}$ ($\equiv I^S - I_{j^*}$) including i* such that all of the destinations except j* are covered.

3) Suppose that the rows i and k are distinct. Then, there exist β-2 plants different from i and k so that together with i and k cover all destinations.

4) For any row i in I^S, there exist at least two rows different from the row i.

5) The support of any row can not be a proper subset of the support of another row.

6) No two columns are same and the support of any column can not be a proper subset of the support of another column.

7) $|J_i| \geq 2$ for all $i \in I^S$.

8) $|I_j| \geq 2$ for all $j \in J^S$.

<u>Proof:</u> Let S satisfy properties 1) - 8). If S is not maximal then there exists an entry s_{ij} which can be changed from 0 to 1 without changing β. This contradicts property 2) and thus S is maximal. We prove now that any maximal matrix S satisfies all of the eight properties.

1) If $\beta = 1$, then there exist at least one row whose elements are all one. This implies that all of the elements in S must be equal to 1 because S is maximal. But, this is a contradiction to the fact that each column of S must have at least one zero element.

2) Suppose $|\overline{J_i}| < \beta - 1$ where $\overline{J_i} = J^S - J_i$. Then, since there must exist at least one non-zero element in each column, the covering number will be less than β because we need at most $|\overline{J_i}|$ plants different from i to cover all of the destinations j such that $s_{ij} = 0$. This is a contradiction to the fact that the covering number is β. Thus, there exist at least β-1 zero elements for each row.

Since S is a pd-adjacency matrix, $|\overline{I_j}| \geq 1$ for each $j \in J^S$ where $\overline{I_j} = I^S - I_j$. Without loss of generality, let $s_{i*j*}=0$. Since S is maximal, if we let $s_{i*j*} = 1$, then the covering number will reduce to β-1. Furthermore, these β-1 plants must include the plant i* because, if not, then the covering number of S (with $s_{i*j*} = 0$) will be β-1 leading to the contradiction. Thus, for each column, there must exist at least β-1 zero elements.

Also, if any one of these β-1 plants covers the destination j*, then the covering number of S(with $s_{i*j*} = 0$) will be β-1 leading to the contradiction. Thus, if $s_{i*j*} = 0$, then there exist β-1 plants among $\overline{I_{j*}}$ including i* such that all of the destinations except j* are covered.

3) Let $j \in J_i$ but $j \notin J_k$. Then, from 2), there exist β-1 plants among $\overline{I_j}$ including k so that they cover all of the destinations except j. Since the plant i can cover the destination j, the proof is complete.

4) First, we note that, since S is a pd-adjacency matrix, the pd-subgraph G^S is connected and $|I^S| \geq 3$ and $|J^S| \geq 3$.

Suppose all of the rows are same with the row i. Then, there will be a column whose elements are all zeroes. This is a contradiction.

Suppose there is only one row that is different from the row i. Without loss of generality, let k be the row that is different from the row i. Then, there exists no column j such that $s_{ij} = 1$ and $s_{kj} = 1$. Otherwise, all of the elements in the column j will be one leading to the contradiction. Therefore, the nodes y_i and y_k can not be connected. This is a contradiction.

5) Suppose the support of the row i is a proper subset of the support of the row k. In other words, $J_i \subset J_k$. Then, there exists $j^* \in J_k - J_i$ such that $s_{ij^*} = 0$ and $s_{kj^*} = 1$. From 2), we know that there exists β-1 plants in $\overline{I_{j^*}}$ including i such that all of the destinations except j* are covered. Thus, if we replace the plant i by the plant k, then the covering number will be β-1 because plant k covers the destination j* and all of the destinations that can be covered by the plant i. This gives a contradiction.

6) Suppose the column j and q are same. Then, we must have $\overline{I_j} = \overline{I_q}$. From 2), we know that, if $s_{ij} = 0$, then there exist β-1 plants among $\overline{I_j}$ including i such that all of the destinations except j are covered. But, if $\overline{I_j} = \overline{I_q}$, then any plant in $\overline{I_j}$ can not cover the destination q. Thus, we have a contradiction.

Suppose the support of the column j is a proper subset of the support of the column q. In other words, $\overline{I_q} \subset \overline{I_j}$. Then, there must exist β-1 plants among $\overline{I_q}$ that cover all of the destinations except q. But, this is impossible because these β-1 plants can not cover j.

7) Suppose $|J_i| = 1$. Then, the support of the row i will be a proper subset of the support of some other row. This is so because for $j \in J_i$ there must be at least one more plant k such that $x_{kj} = 1$ and $x_{kq} = 1$ for some $q \neq j$ in order to have two rows different from row i.

8) Suppose $|I_j| = 1$. Then, the support of the column j will be a proper subset of the support of some other column because each row must have at least two one's. Otherwise, we can not have distinct columns. Thus, $|I_j| \geq 2$ for all $j \in J^S$. Δ

Remark 3.3. A maximal pd-adjacency matrix S may have identical rows.

Theorem 3.4. Let $I^S \subseteq P$ and $J^S \subseteq D$. Then, the inequality

$$\sum_{i\in I^S} \sum_{j\in J^S} s_{ij} x_{ij} + \sum_{i\in I^S} y_i \leq \alpha(G^S) \qquad (3.8)$$

is a non-trivial facet of L_I^{pd} if and only if S is an $|I^S| \times |J^S|$ maximal pd-adjacency matrix.

Proof:

(⇒) This is obvious. Because, if not, then the inequality (3.8) can be lifted.

(⇐) Let S' be a submatrix of S such that all of the rows in S' are distinct. Note that $I^{S'} \subseteq I^S$ and $J^{S'} = J^S$. Since all of the rows in $I^S - I^{S'}$ are replicates of some rows in $I^{S'}$, if we can show that the inequality

$$\sum_{i \in I^{S'}} \sum_{j \in J^{S'}} s'_{ij} x_{ij} + \sum_{i \in I^{S'}} y_i \leq \alpha(G^{S'}) \tag{3.9}$$

is a facet of L_I^{pd} where $\alpha(G^{S'}) = \alpha(G^S)-|I^S|+|I^{S'}|$, then, from the replicating theorem, the inequality (3.8) is a facet of L_I^{pd}.

We first show that all of the edges in $G^{S'} = (N^{S'},E^{S'})$ are critical.

Case 1. Suppose we drop an edge connecting the nodes x_{ij} and y_i where $i \in I^{S'}$ and $j \in J^{S'}$. Then, from 2) of lemma 3.1, there exist β-1 plants among $\overline{I_j}$ ($= I^{S'}-I_j$) different from i such that all of the destinations except j are covered. Thus, if we let $x_{ij} = 1$ and $y_i = 1$, then the left hand side of (3.5) can be equal to $\alpha(G^{S'})+1$.

Case 2. Suppose we drop an edge connecting the nodes x_{ij} and x_{kj} where $i,k \in I^{S'}$ and $j \in J^{S'}$. Then, from 3) of lemma 3.1, there exist β-2 additional plants in $I^{S'}$ different from i and k so that together with i and k cover all destinations. Thus, if we let $x_{ij} = 1$, $x_{kj} = 1$, $y_i = 0$ and $y_k = 0$, then the left hand side of (3.9) can be made equal to $\alpha(G^{S'})+1$.

Thus, from theorem 3.1, the inequality (3.9) is a facet of $L_I^{S'}$.

Now, since S' is also a maximal pd-adjacency matrix, all of the lifting coefficients will be equal to 0 and, from theorem 3.2, the inequality (3.9) is a facet of L_I^{pd}. Δ

4. Two families of facets of (PLP).

In this section, we will derive some facets of (PLP) applying the results obtained in section 2 and 3.

Recently, Cornuejols and Thizy (C-T) [3] derived a particular family of facets of (PLP).

Theorem 4.1. Consider any integer q and t such that $2 \le t < q \le p$ and any subset $J \subseteq D$ such that $|J| = \binom{q}{t} \le d$ and $|I| = q$. Let A^{qt} be a matrix with $|I|$ rows and $|J|$ columns so that columns are all of the combinations of 0-1 vectors with t ones and $q-t$ zeroes. Then, the inequality

$$\sum_{i\in I} \sum_{j\in J} a^{qt}_{ij} x_{ij} + \sum_{i\in I} y_i \le \binom{q}{t} + t - 1 \qquad (4.1)$$

is a facet of L^{pd}_I.

Proof: See [3, theorem 6].

Remark 4.1. A^{qt} is a maximal pd-adjacency matrix with $\beta(G^{A^{qt}}) = q-t+1$ and $\alpha(G^{A^{qt}}) = \binom{q}{t}+t-1$.

In section 3 of [2], we showed that

$$\sum_{i\in I^C} \sum_{j\in J^C} c_{ij}x_{ij} + \sum_{i\in I^C} y_i \le 2k - \left\lceil \frac{k}{t} \right\rceil$$

is a valid inequality of (PLP) where the matrix C is defined to be a $k \times k$ cyclic matrix whose rows are 0-1 vectors in which t consecutive ones are successively moved one position to the right. We show one way of lifting this inequality to a facet of (PLP). For this, we let $k = tm + n$ where m and n are integers satisfying $1 \le n < t$ and $m \ge 1$.

Let $C_s\,(s=1,2,\ldots,m)$ be the submatrix of C obtained by deleting the first $t(s-1)$ rows and the first $t(s-1)$ columns of C. (note: $C_1 = C$ and C_m is a $(t+n) \times (t+n)$ matrix.) The following construction generates a new matrix $\tilde{C}$ whose $\beta(G^{\tilde{C}})$ equals $\beta(G^{C})$.

<u>Construction of $\tilde{C}$.</u>

0) Start with a $(tm+n) \times (tm+n)$ cyclic matrix C with t consecutive ones and define C_m as done above.

1) Add one's into C_m so the C_m becomes a $(t+n) \times (t+n)$ cyclic matrix with $t+n-1$ consecutive one's per row. Call this matrix $\tilde{C}_m$.

2) For $s = m-1, m-2, \ldots, 1$, we generate $\tilde{C}_s$ by appending additional t columns and t rows to $\tilde{C}_{s+1}$. First, we append columns so that the first $(t-1)$ columns of $\tilde{C}_s$ are replicates of the first $(t-1)$ columns of $\tilde{C}_{s+1}$ and the t-th column is a vector of 0's. Next, we append rows so that the first t rows of $\tilde{C}_s$ are replicates of the first t rows of C_s.

3) Let $\tilde{C}_1$ be $\tilde{C}$.

<u>Remark 4 .2</u>. For $t = 3$, $m = 3$ and $n = 2$, the newly constructed $\tilde{C}$ is of the following form where the circled ones are the new +1 entries added to the original matrix C.

	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1								
2		1	1	1							
3			1	1	1						
4	①	①		1	1	1					
5		①			1	1	1				
6						1	1	1			
7	①	①		①	①		1	1	1	①	
8		①			①			1	1	1	①
9	①			①			①		1	1	1
10	1	①		①	①		①	①		1	1
11	1	1		①	①		①	①	①		1

Remark 4.3. $\tilde{C}$ is non-singular and $|\tilde{C}| = (t+n-1)2^{(t-1)(m-1)}$. (See appendix I.)

To describe the matrix $\tilde{C}$ combinatorially we denote the plant $(r-1)t+k$ to be I_k^r and the destination $(r-1)t+k$ to be J_k^r where $k \le t$ if $r \le m-1$. The plant (destination) $(m-1)t+h$ will be denoted by I_h^m (J_h^m) where $h \le t+n$. This gives a unique representation of the plants and destinations. Let $I^r = \bigcup_{k=1}^{t} \{I_k^r\}$ for $r = 1,2,\dots,m-1$ and $I^m = \bigcup_{h=1}^{t+n} \{I_h^m\}$ (similarly for J^r and J^m).

Finally, let $I_j = \{i|\tilde{c}_{ij}=1\}$, $J_i = \{j|\tilde{c}_{ij}=1\}$, $\overline{I_j} = I^{\tilde{C}}-I_j$ and $\overline{J_i} = J^{\tilde{C}}-J_i$. Note that I_j, $\overline{I_j}$, J_i and $\overline{J_i}$ are non-enpty. Also, we define $I_k^o = \emptyset$ and if $h > t$, $\bigcup_{k=h}^{t} I_k^r = \emptyset$.

I. A destination j can be supplied from the following set of plants.

Case 1. Suppose $j = J_q^s$ for $s = 1,2,\dots,m-1$ where $1 \le q \le t-1$. Then,

$$I_j = \bigcup_{k=q+1}^{t} \{I_k^{s-1}\} \quad \cup \quad \bigcup_{r=s}^{m} \bigcup_{k=1}^{q} \{I_k^r\} \quad \cup \quad \bigcup_{k=q+2}^{t+n} \{I_k^m\}.$$

Case 2. Suppose $j = J_t^s$ for $s = 1,2,\dots,m-1$. Then,

$$I_j = \bigcup_{k=1}^{t} \{I_k^s\}.$$

Case 3. Suppose $j = J_h^m$ where $1 \le h \le t+n$. Then,

$$I_j = \bigcup_{k=h+1}^{t} \{I_k^{m-1}\} \quad \cup \quad \bigcup_{k\in K} \{I_k^m\}$$

where $K = \{k|\ k = 2,3,\dots,t+n$ if $h = t+n$; $k = 1,2,\dots,h,h+2,\dots,t+n$ otherwise$\}$.

II. A plant i can supply the following set of destinations.

Case 1. Suppose $i = I_q^s$ for $s = 1,2,\ldots,m-1$ where $1 \le q \le t-1$. Then,

$$J_i = \bigcup_{r=1}^{s-1} \bigcup_{k=q}^{t-1} \{J_k^r\} \quad \cup \quad \bigcup_{k=q}^{t} \{J_k^s\} \quad \cup \quad \bigcup_{k=1}^{q-1} \{J_k^{s+1}\}.$$

(note: If $q-1 = 0$, then the last set is empty.)

Case 2. Suppose $i = I_t^s$ for $s = 1,2,\ldots,m-1$. Then,

$$J_i = \{J_t^s\} \quad \cup \quad \bigcup_{k=1}^{t-1} \{J_k^{s+1}\}.$$

Case 3. Suppose $i = I_h^m$ where $1 \le h \le t+n$. Then,

$$J_i = \bigcup_{r=1}^{m-1} \bigcup_{k \in K_1} \{J_k^r\} \quad \cup \quad \bigcup_{k \in K_2} \{J_k^m\}$$

where $K_1 = \{k \mid k = 2,3,\ldots,t-1$ if $h = 2$;

$k = 1,2,\ldots,h-2,h,\ldots,t-1$ if $3 \le h \le t+1$

$k = 1,2,\ldots,t-1$ otherwise$\}$,

$K_2 = \{k \mid k = 1,2,\ldots,t+n-1$ if $h = 1$; $k = 2,3,\ldots,t+n$ if $h = 2$;

$k = 1,2,\ldots,h-2,h,\ldots,t+n$ otherwise$\}$.

Lemma 4.1. The matrix $\tilde{C}$ is a maximal pd-adjacency matrix with $\beta = m+1$.

Proof:

First, we can easily see that $\tilde{C}$ is a pd-adjacency matrix. Note that G^C is connected because C is a cyclic matrix, which, in turn, implies that $\tilde{G}^C$ is connected.

Step 1. $\beta = m+1$.

We first show that $\beta \ge m + 1$.

It can be easily seen that in order to have $\tilde{c}_{ij} = 1$ for all j in $\bigcup_{r=1}^{m} \{J_t^r\}$ we need at least one i from each I^r for $r = 1,2,\ldots,m$.

Suppose a plant i chosen from I^m is such that $i = I_{q*}^m$ where q* is as follows:

Case 1. Suppose $q^* = 1$. Then,

$$\tilde{c}_{i^*j} = 0 \quad \text{for} \quad j \in \bigcup_{r=1}^{m-1} \{J_t^r\} \cup \{mt+n\}.$$

Thus, we need at least m more plants because there is no single plant that can cover more than one of the above destinations.

Case 2. Suppose $2 \le q^* \le t$. Then,

$$\tilde{c}_{i^*j} = 0 \quad \text{for} \quad j \in \bigcup_{r=1}^{m-1} \{J_{q^*-1}^r, J_t^r\} \cup J_{q^*-1}^m.$$

The destination $J_{q^*-1}^m$ can be covered by either a plant in I^m different from i^* or the plant $I_{q^*}^{m-1}$ because $q^* \le t$. In the former case, we still have to cover $\bigcup_{r=1}^{m-1} \{J_t^r\}$ destinations which require $m-1$ additional plants. Thus, $\beta \ge m+1$. In the latter case, we can cover J_t^{m-2} but not $J_{q^*-1}^{m-2}$. In other words, there is no single plant that can cover $\{J_{q^*-1}^{r-1}, J_t^{r-1}, J_{q^*-1}^r\}$ for a particular r. Thus, we must have at least m more plants beside i^* to cover all destinations.

Case 3. Suppose $q^* \ge t+1$. Then,

$$\tilde{c}_{i^*j} = 0 \quad \text{for} \quad j \in \bigcup_{r=1}^{m-1} \{J_t^r\} \cup J_{q^*-1}^m.$$

Thus, we need at least m more plants because there is no single plant that can cover more than one of the above destinations.

Therefore, we have shown that $\beta \ge m+1$.

Now, it is easily verified that if we choose any one plant from each I^r for $r = 1,2,\ldots,m-1$ and any two plants from I^m, then all destinations can be covered.

Hence, $\beta = m+1$.

Step 2. $\tilde{C}$ is maximal.

We prove this by showing that, if $\tilde{c}_{i*j*}$ were changed from 0 to 1, then β reduces to m. There are 3 different cases depending on j*.

Let $i* = I_b^u \in \overline{I_{j*}}$.

Case 1. Suppose $j* = J_q^s$ where $1 \le s \le m-1$ and $1 \le q \le t-1$. Then,

$$\overline{I_{j*}} = \bigcup_{r=1}^{s-2} I^r \cup \bigcup_{k=1}^{q} \{I_k^{s-1}\} \cup \bigcup_{r=s}^{m-1} \bigcup_{k=q+1}^{t} \{I_k^r\} \cup I_{q+1}^m .$$

Thus, if we choose m plants to be

$$\{i*\} \cup \bigcup_{\substack{r=1 \\ r\neq u}}^{s-1} \{I_q^r\} \cup \bigcup_{\substack{r=s \\ r\neq u}}^{m} \{I_{q+1}^r\},$$

then all destinations can be covered. This can be seen as follows.

Case 1a. If $u \ge s$ and $u \neq m$, then $\bigcup_{r=1}^{s-1} \{I_q^r\} \cup \bigcup_{\substack{r=s \\ r\neq u}}^{m} \{I_{q+1}^r\}$ cover all destinations except $\{J_q^s, J_q^{u+1}, J_t^u\}$. But, all of these can be covered by i* because $u \ge s$ implies that $b \ge q+1$.

Case 1b. If $u \ge s$ and $u = m$, then i* covers all destinations except $\{J_q^r | r=1,2,\ldots,s-1,s+1,\ldots,m\}$ and $\{J_t^r | r = 1,2,\ldots,m-1\}$ all of which can be covered by $\bigcup_{r=1}^{s-1} \{I_q^r\} \cup \bigcup_{r=s}^{m-1} \{I_{q+1}^r\}$.

Case 1c. If $u < s$, then $\bigcup_{\substack{r=1 \\ r\neq u}}^{s-1} \{I_q^r\} \cup \bigcup_{r=s}^{m} \{I_{q+1}^r\}$ cover all destinations except $\{J_q^s, J_q^u, J_t^u\}$. But, if $b \le q$, then all of these are covered by i*. If $b \ge q+1$, then J_q^u is already covered by I_q^{u+1} because $b \ge q+1$ implies that $u < s-1$ in order for $i* \in \overline{I_{j*}}$ and J_t^u and J_q^s are covered by i*.

Case 2. Suppose $j^* = J_t^s$ where $1 \le s \le m-1$. Then $\overline{I_{j^*}} = I - I^s$. Clearly, $u \neq s$. If we choose m plants to be

$$\{i^*\} \cup \bigcup_{\substack{r=1 \\ r\neq u,s}}^{m} \{I_b^r\} \cup I_{b+1}^m ,$$

then all destinations can be covered. This is so because I_b^m and I_{b+1}^m cover all destinations except $\{J_t^r | \ r = 1,2,\ldots,m-1\}$ all of which except J_t^s can be covered by $\{I_b^r | \ r = 1,2,\ldots,s-1,s+1,\ldots,m-1\}$. But, i^* covers J_t^s.

Case 3. Suppose $j^* = J_h^m$ where $1 \le h \le t+n$. Then,

$$\overline{I_{j^*}} = \bigcup_{r=1}^{m-2} I^r \cup \bigcup_{k=1}^{\min\{h,t\}} \{I_k^{m-1}\} \cup I_{h^*}^m \text{ where } h^* = \begin{cases} 1 & \text{if } h = t+n \\ h+1 & \text{otherwise} \end{cases} .$$

If we choose m plants to be

$$\{i^*\} \cup \bigcup_{\substack{r=1 \\ r\neq u}}^{m-1} \{I_h^r\} \cup I_{h^*}^m ,$$

then all destinations can be covered. This can be seen as follows.

Case 3a. If $h \le t$, then $I_{h^*}^m$ covers all destinations except $\{J_t^r | \ r = 1,2,\ldots,m-1\}$ and $\{J_h^r | \ r = 1,2,\ldots,m\}$ all of which except $\{J_t^u, J_h^m\}$ and possibly J_h^u can be covered by $\bigcup_{\substack{r=1 \\ r\neq u}}^{m-1} I_h^r$. Now, i^* covers J_t^u and J_h^m. If $u < m-1$, then I_h^{u+1} covers J_h^u. If $u = m-1$, since $i^* \in \overline{I_{j^*}}$ we must have $b \le h$ and i^* covers J_h^u. Finally, if $u = m$, then we have $J_h^u = J_h^m$ which is covered by i^*.

Case 3b. If $h > t$, then $I_{h^*}^m$ covers all destinations except $\{J_t^r | \ r = 1,2,\ldots,m-1\}$ and $\{J_h^m\}$ all of which except J_h^m can be covered by the other m-1 plants. But, i^* covers J_h^m.

Therefore, the matrix $\tilde{C}$ is a maximal pd-adjacency matrix with $\beta = m+1$. Δ

Since $\alpha(G^{\tilde{C}}) = 2(tm+n)-(m+1)$, from lemma 4.1 and theorem 3.4, we have proven the following theorem.

<u>Theorem 4.2.</u> <u>Let $\tilde{C}$ be a matrix defined as above. Then, the inequality</u>

$$\sum_{i\in I^{\tilde{C}}} \sum_{j\in J^{\tilde{C}}} \tilde{c}_{ij}x_{ij} + \sum_{i\in I^{\tilde{C}}} y_i \le 2(tm+n) - (m+1) \tag{4.2}$$

<u>is a facet of L_I^{pd}.</u>

<u>Remark 4.4.</u> If $t = 2$ and $n = 1$, then (4.2) is a lifting of an odd-cycle inequality. In particular, Theorem 4.2 shows that odd-cycle inequalities must in general be lifted in order to get facets for (PLP).

<u>Remark 4.5.</u> The maximal pd-adjacency matrix $\tilde{C}$ in theorem 4.2 was constructed by appending t columns and t rows to $\tilde{C}_m$ in a particular way and repeating the process $m-1$ times. Since $\tilde{C}_m$ corresponds to a facet studied by Guignard [4], (4.2) is a lifting of Guignard's facet to a higher dimensional polyhedron. In the above construction, t was fixed to be the same at each step. However, we may allow t to vary, i.e. for $s = m-1, m-2,\ldots,1$, we may replace t by t_s in the construction of $\tilde{C}$ provided t_s satisfies $2 \le t_s \le \min(t_{s+1}+1, t+n)$ for $s = 1,2,\ldots,m-1$ while $t_m = t+n-1$. The same arguments as before are valid with the right hand side of (4.2) becoming $2|I^{\tilde{C}}|-(m+1)$. (see appendix II).

<u>Remark 4.6.</u> There are other ways to generate a different maximal pd-adjacency matrix without altering β, and hence giving a different facet from the one constructed in Theorem 4.2. For example, if $t = 3$, $m = 3$ and $n = 2$, then the following matrix does not change β.

	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1								
2		1	1	1							
3	(1)		1	1	1						
4	(1)	(1)		1	1	1					
5					1	1	1				
6	(1)			(1)		1	1	1			
7	(1)	(1)		(1)	(1)		1	1	1	(1)	
8								1	1	1	(1)
9	(1)			(1)			(1)		1	1	1
10	1	(1)		(1)	(1)		(1)	(1)		1	1
11	1	1		(1)	(1)		(1)	(1)	(1)		1

The next theorem extends Guignard's facet [4] to the case where plants are added without altering the number of destinations.

Theorem 4.3. Let I and J be a subset of P and D, respectively, such that $|I| = |J| = m$ where $m \geq 3$. Consider a facet given by

$$\sum_{i\in I} \sum_{j\in J_i} \pi_{ij} x_{ij} + \sum_{i\in I} \mu_i y_i \leq 1$$

where $\mu_i = \frac{1}{2m-2}$, $\pi_{ij} = \frac{1}{2m-2}$ or 0 and the sets, J_i's are all distinct sets with $|J_i| = m-1$. Suppose we add $|S|+|T|$ plants and no additional destinations so that a plant in S can supply all but one of the destinations in J and a plant in T can supply all destinations in J. Let $|S| = s$ and $|T| = t$. Then, $\pi x + \mu y \leq 1$ is a facet of L_I^{pd} where

$$\pi_{ij} = \mu_i = \frac{m-1}{(2m+s-2)(m-1)+t(m-2)} \quad \text{for } i \in I \cup S,\ j \in J_i, \text{ and}$$

$$\pi_{ij} = \mu_i = \frac{m-2}{(2m+s-2)(m-1)+t(m-2)} \quad \text{for } i \in T,\ j \in J_i.$$

Proof:

Since each plant in S is a replicate of a plant in I, by the replicating theorem (theorem 3.3),

$$\sum_{i\in I\cup S} \sum_{j\in J_i} \pi_{ij} x_{ij} + \sum_{i\in I\cup S} \mu_i y_i \leq 1$$

is a facet of $L_I^{(m+s)m}$ and $\pi_{ij} = \mu_i = \frac{1}{2m+s-2}$ for $i\in I\cup S$ and $j\in J_i$. Thus, the theorem holds when $T = \emptyset$.

Suppose we add one plant $q \in T$. Then, the new corresponding intersection graph is such that there are edges connecting each new node x_{qj}, with all the nodes in the corresponding C_j for $j = 1,2,\dots,m$ where $C_j = \{x_{ij} | \pi_{ij} > 0 \text{ for } i \in IUS\}$. Furthermore, there is an edge connecting the new node y_q with each of the nodes x_{qj} for $j = 1,2,\dots,m$.

We now, apply theorem 2.4. First, we note that $c = \frac{m+s}{2m+s-2} < 1$ and for $k < m$

$$c^k = \frac{2m+s-k-1}{2m+s-2} = 1 - \frac{k-1}{2m+s-2} < 1 - \frac{m-2}{m-1}\,\frac{k-1}{2m+s-2} = 1 - \frac{(k-1)(1-c)}{m-1}.$$ Thus, all of the conditions in the theorem 2.4 are satisfied.

Since $1-c = \frac{m-2}{2m+s-2}$ and $m-c = (m-1)+(1-c) = \frac{(2m+s-2)(m-1)+m-2}{2m+s-2}$, if we let

$$\pi_{ij} = \mu_i = \frac{m-1}{m-c}\quad\frac{1}{2m+s-2} = \frac{m-1}{(2m+s-2)(m-1)+m-2} \text{ for } i \in IUS,\ j \in J_i \text{ and}$$

$$\pi_{qj} = \mu_q = \frac{1-c}{m-c} = \frac{m-2}{(2m+s-2)(m-1)+m-2} \text{ for } j \in J_q = \{1,2,\dots,m\}, \text{ then}$$

$$\sum_{i \in IUSU\{q\}} \sum_{j \in J_i} \pi_{ij} x_{ij} + \sum_{i \in IUSU\{q\}} \mu_i y_i \le 1$$

is a facet of the appropriate pd-subgraph.

We note that the sequential lifting coefficients for x_{ij}, $i \in IUS$ and $j \notin J_i$ continue to be zero. This is seen as follows. Suppose $\pi_{i^*j^*} > 0$ for $i^* \in IUS$, $j^* \notin J_i$. Then the plant i^* covers all destinations and the left hand side of the inequality becomes $1 + \pi_{i^*j^*} > 1$, leading to a contradiction.

Thus, the theorem holds when $|T| = 1$.

Suppose the theorem holds when $|T| = t-1$ and consider the case when $|T| = t$.

Then, $c = \frac{(m+s)(m-1)+(t-1)(m-2)}{(2m+s-2)(m-1)+(t-1)(m-2)} < 1$ and for $k < m$

$$c^k = \max\left\{\frac{(m+s)(m-1)+(t-2)(m-2)+(m-k)(m-2)}{(2m+s-2)(m-1)+(t-1)(m-2)},\ \frac{(m+s-1)(m-1)+(m-k)(m-1)+(t-1)(m-2)}{(2m+s-2)(m-1)+(t-1)(m-2)}\right\}$$

$$= \frac{(m+s-1)(m-1)+(m-k)(m-1)+(t-1)(m-2)}{(2m+s-2)(m-1)+(t-1)(m-2)}$$

$$= \frac{(2m+s-2)(m-1)+(t-k)(m-2)+1-k}{(2m+s-2)(m-1)+(t-1)(m-2)} .$$

Since $1-c = \frac{(m-2)(m-1)}{(2m+s-2)(m-1)+(t-1)(m-2)}$, we have

$$1 - \frac{(k-1)(1-c)}{m-1} = 1 - \frac{(k-1)(m-2)}{(2m+s-2)(m-1)+(t-1)(m-2)} = \frac{(2m+s-2)(m-1)+(t-k)(m-2)}{(2m+s-2)(m-1)+(t-1)(m-2)} .$$

Thus, $c^k \le 1 - \frac{(k-1)(1-c)}{m-1}$ and all of the conditions of the theorem 2.4 are satisfied. Since $m-c = \frac{(2m+s-2)(m-1)^2+(t-1)(m-2)(m-1)+(m-2)(m-1)}{(2m+s-2)(m-1)+(t-1)(m-2)}$

$$= \frac{(2m+s-2)(m-1)^2+t(m-2)(m-1)}{(2m+s-2)(m-1)+(t-1)(m-2)} ,$$ if we let

$$\pi_{ij} = \mu_i = \frac{m-1}{m-c} \times \frac{m-1}{(2m+s-2)(m-1)+(t-1)(m-2)} = \frac{(m-1)^2}{(2m+s-2)(m-1)^2+t(m-2)(m-1)}$$

$$= \frac{m-1}{(2m+s-2)(m-1)+t(m-2)}$$ for $i \in I \cup S$, $j \in J_i$ and

$$\pi_{ij} = \mu_i = \frac{1-c}{m-c} = \frac{m-2}{(2m+s-2)(m-1)+t(m-2)}$$ for $i \in T$, $j \in J_i$, then

$\sum_{i \in I \cup S \cup T} \sum_{j \in J_i} \pi_{ij} x_{ij} + \sum_{i \in I \cup S \cup T} \mu_i y_i \le 1$ is a facet of the appropriate pd-subgraph.

We note that the sequential lifting coefficients for x_{ij}, $i \in I \cup S$ and $j \notin J_i$ continue to be zero. This follows by the same argument as in the case where $|T| = 1$.

Thus, the theorem also holds when $|T| = t$. Δ

We note that the family of facets given by Theorem 4.3 has coefficients different from zero and one when brought into the standard form with integer coefficients.

5. Non-trivial facets for the case of several plants and 3 destinations ($p \geq 3$).

In section 6 of [2], we presented all the facets of L_I^{pd} when $p = 3$ and $d \geq 3$. There, we showed that the facet defining graphs are odd-cycles of length 9. In this section, we will derive all the facets of L_I^{pd} when $p \geq 3$ and $d = 3$, and show that the facet defining graph, in general, will not be just a chordless odd-cycle of length 9 but must contain an odd-cycle of length 9. In fact, we give the necessary and sufficient conditions for non-trivial facets of L_I^{pd} when $p \geq 3$ and $d = 3$.

For our discussions below, we let $K_2 = \{i| \ |J_i| = 2\}$, $K_3 = \{i| \ |J_i| = 3\}$ and $D = \{1,2,3\}$. Note that $K_1 = \{i| \ |J_i| = 1\}$ should be empty because we must have either $|J_i| = 0$ or $|J_i| \geq 2$ as shown by theorem 5.1 in [2].

Lemma 5.1. If $\pi x + \mu y \leq 1$ is a non-trivial facet of L_I^{pd} with $p \geq 3$ and $d = 3$, then the facet defining graph must contain an odd-cycle of length 9.

Proof: By remark 5.2 of [2], if $\pi x + \mu y \leq 1$ is a non-trivial facet, then for each destination j there must exist three plants i such that $\mu_i > 0$ with at least one plant satisfying $\pi_{ij} = 0$. Let the plants i_1, i_2, i_3 be such that $\mu_{i_t} > 0$ for $t = 1,2,3$ and $\pi_{i_1 1} = 0$, $\pi_{i_2 2} = 0$ and $\pi_{i_3 3} = 0$. Then, we must have $J_{i_1} = \{2,3\}$, $J_{i_2} = \{1,3\}$ and $J_{i_3} = \{1,2\}$ because $|J_i|$ must be equal to 2. Thus, i_1, i_2 and i_3 are distinct. Now, the subgraph induced by the nodes $\{y_i | i = i_1, i_2, i_3\}$ and $\{x_{ij} | j \in J_i \text{ for } i = i_1, i_2, i_3\}$ is an odd-cycle of legth 9.

Lemma 5.2. If $\pi x + \mu y \leq 1$ is a non-trivial facet of L_I^{pd} with $p \geq 3$ and $d = 3$, then the following statements are true.

1) $\mu_i = \pi_{ij}$ for all $j \in J_i$, $i \in K_2 \cup K_3$.

2) $\mu_i = \mu_k$ if $i,k \in K_2$ or $i,k \in K_3$.

$\mu_i = 2\mu_g$ if $i \in K_2$ and $g \in K_3$.

Proof: 1) Let plant $1 \in K_2$ and $J_1 = \{2,3\}$. Since $\pi x + \mu y \leq 1$ is a non-trivial facet, there exists a solution vector $(x^*,y^*) \in H_{\pi,\mu}$ such that, $x^*_{12} = 0$ and $y^*_1 = 0$. This, in turn, implies that $x^*_{13} = 1$ because, if not (i.e. $x^*_{13} = 0$), then we can let $(\bar{x},\bar{y})$ be same with (x^*,y^*) except $\bar{y}_1 = 1$ and $\bar{x}_{11} = 0$ so that $(\bar{x},\bar{y})$ is feasbile but $\pi\bar{x} + \mu\bar{y} = \pi x^* + \mu y^* + \mu_1 > 1$, which leads to a contradiction. Furthermore, we must have $\mu_1 = \pi_{13}$ because, if not (i.e. $\mu_1 > \pi_{13}$.See theorem 5.1 in [2]), then we can let $(\hat{x},\hat{y})$ be same with (x^*,y^*) except $\hat{x}_{13} = 0$, $\hat{y}_1 = 1$ and $\hat{x}_{11} = 0$ so that $(\hat{x},\hat{y})$ is feasible but $\pi\hat{x} + \mu\hat{y} = \pi x^* + \mu y^* - \pi_{13} + \mu_1 > 1$ which leads to a contradiction. Thus, $\mu_1 = \pi_{13}$.

Similarly, by noting the fact that there exists a solution vector $(x^*,y^*) \in H_{\pi,\mu}$ such that $x^*_{13} = 0$ and $y^*_1 = 0$, we can show that $\mu_1 = \pi_{12}$. Since plant 1 was arbitrarily chosen from K_2, we have proved that $\mu_i = \pi_{ij}$ for all $j \in J_i$ if $i \in K_2$.

Let plant $4 \in K_3$. Then, there exists a solution vector $(x^*,y^*) \in H_{\pi,\mu}$ such that $x^*_{41} = 0$ and $y^*_4 = 0$. This, in turn, implies that $x^*_{42} = 1$ or $x^*_{43} = 1$ but not both. This is so because if $x^*_{42} = 0$ and $x^*_{43} = 0$ then we can let $(\bar{x},\bar{y})$ be same with (x^*,y^*) except $\bar{y}_4 = 1$ so that $(\bar{x},\bar{y})$ is feasible but $\pi\bar{x} + \mu\bar{y} = \pi x^* + \mu y^* + \mu_4 > 1$ which leads to a contradiction. On the other hand, if $x^*_{42} = 1$ and $x^*_{43} = 1$, then there must exist a plant $k \in K_2 \cup K_3$ such that $y^*_k = 0$ and $x^*_{k1} = 1$ because, if not (i.e. $x^*_{i1} = 0$ for all i), then we can let $(\tilde{x},\tilde{y})$ be same with (x^*,y^*) except $\tilde{x}_{41} = 1$ so that $(\tilde{x},\tilde{y})$ is feasible but $\pi\tilde{x} + \mu\tilde{y} = \pi x^* + \mu y^* + \pi_{41} > 1$ which leads to a contradiction. However, if we let $(\hat{x},\hat{y})$ be same with (x^*,y^*) except $\hat{x}_{41} = 1$, $\hat{y}_k = 1$ and $\hat{x}_{k1} = 0$, then $(\hat{x},\hat{y})$ is feasible but $\pi\hat{x} + \mu\hat{y} = \pi x^* + \mu y^* + \pi_{41} + \mu_k - \pi_{k1} > 1$ which leads to a contradiction.

Thus, we have proved that if a solution vector (x^*,y^*) is such that $y^*_4 = 0$ and $x^*_{41} = 0$, then we must have either $x^*_{42} = 1$ or $x^*_{43} = 1$ but not both. Without loss of generality, let $x^*_{42} = 1$ and $x^*_{43} = 0$. Then, we must have $\mu_4 = \pi_{42}$ because, if not (i.e. $\mu_4 > \pi_{42}$), then we can let $(\bar{x},\bar{y})$ be same with

(x^*,y^*) except $\bar{y}_4 = 1$ and $\bar{x}_{42} = 0$ so that $(\bar{x},\bar{y})$ is feasible but $\pi\bar{x} + \mu\bar{y} = \pi x^* + \mu y^* - \pi_{42} + \mu_4 > 1$ which leads to a contradiction.

Note that this solution vector (x^*,y^*) satisfies the condition that there must exist a solution vector $(x,y) \in H_{\pi,\mu}$ such that $y_4 = 0$ and $x_{43} = 0$. However, there must also exist a solution vector $(x^*,y^*) \in H_{\pi,\mu}$ such that $y_4^* = 0$ and $x_{42}^* = 0$ in which case we must have either $x_{41}^* = 1$ or $x_{43}^* = 1$ but not both. Without loss of generality, let $x_{43}^* = 1$ and $x_{41}^* = 0$. This implies that $\mu_4 = \pi_{43}$.

Now, we prove that $\mu_4 = \pi_{41}$. Note that, for each $j \in D$, there exists a solution vector $(x^*,y^*) \in H_{\pi,\mu}$ such that $y_4^* = 0$ and $x_{4j}^* = 1$. Otherwise, the facet must be a trivial facet of the form $x_{4j} \geq 0$. Thus, if a solution vector $(x^*,y^*) \in H_{\pi,\mu}$ is such that $y_4^* = 0$ and $x_{41}^* = 1$, then we must have either $x_{42}^* = 0$ and $x_{43}^* = 0$ or $x_{42}^* = 1$ and $x_{43}^* = 1$.

Suppose $\mu_4 > \pi_{41}$. Then, there exists no solution vector $(x^*,y^*) \in H_{\pi,\mu}$ such that $y_4^* = 0$, $x_{41}^* = 1$, $x_{42}^* = 0$ and $x_{43}^* = 0$. Otherwise, we can let $(\bar{x},\bar{y})$ be same with (x^*,y^*) except $\bar{y}_4 = 1$ and $\bar{x}_{41} = 0$ so that $(\bar{x},\bar{y})$ is feasible but $\pi\bar{x} + \mu\bar{y} = \pi x^* + \mu y^* + \mu_4 - \pi_{41} > 1$ which leads to a contradiction. Thus, any solution vector $(x,y) \in H_{\pi,\mu}$ must satisfy $y_4 - x_{41} + x_{42} + x_{43} = 1$. This is a contradiction to the fact that $\pi x + \mu y \leq 1$ is a facet. Therefore, $\mu_4 = \pi_{41}$.

Since plant 4 was arbitrarily chosen from K_3, we have proved that $\mu_i = \pi_{ij}$ for all $j \in J_i$ if $i \in K_3$.

2) We first show that $\sum_{t \in K_2 \cup K_3} \mu_t + \mu_k = 1$ where $\mu_k = \max_{i \in K_2} \mu_i$.

Let $k \in K_2$ be such that $\mu_k = \max_{i \in K_2} \mu_i$. Without loss of generality, let $J_k = \{1,2\}$. Then, there must exist a solution vector $(x^*,y^*) \in H_{\pi,\mu}$ such that $y_k^* = 0$, $x_{k1}^* = 1$ and $x_{k2}^* = 1$. Otherwise, we must have $y_k + x_{k1} + x_{k2} = 1$ for all solutions in $H_{\pi,\mu}$. Now, we may or may not have $x_{i3}^* = 1$ for some i. But, if $x_{i3}^* = 1$, then $y_i^* = 0$. In any case, noting from 1) that $\mu_i = \pi_{i3}$, we have that $\pi x^* + \mu y^* = \sum_{t \in K_2 \cup K_3} \mu_t + \mu_k = 1$.

Let plant $1 \in K_2$ and $J_1 = \{2,3\}$. We now show that $\mu_1 = \mu_k$, i.e. $\mu_q = \mu_k$ if $q \in K_2$. Suppose $\mu_1 < \mu_k$. Then, there exists no solution vector $(x^*,y^*) \in H_{\pi,\mu}$ such that $y_1^* = 0$, $x_{12}^* = 1$ and $x_{13}^* = 1$. This is so because, if $y^*_1 = 0$, $x_{12}^* = 1$ and $x_{13}^* = 1$, then

$$\pi x^* + \mu y^* = \sum_{\substack{t \in K_2 \cup K_3 \\ t \neq 1}} \mu_t + \pi_{12} + \pi_{13} = \sum_{t \in K_2 \cup K_3} \mu_t + \mu_1 < \sum_{t \in K_2 \cup K_3} \mu_t + \mu_k = 1.$$

In other words, any solution vector $(x,y) \in H_{\pi,\mu}$ must be on the hyperplane $y_1 + x_{12} + x_{13} = 1$ which implies that $y_1 + x_{12} + x_{13} \leq 1$ is a facet. This is a contradiction because $y_1 + x_{12} + x_{13} \leq 1$ is not even a valid inequality.

Finally, let plant $4 \in K_3$. We now show that $\mu_4 = \frac{1}{2}\mu_k$, i.e. $\mu_q = \frac{1}{2}\mu_k$ if $q \in K_3$. Suppose $\mu_4 \neq \frac{1}{2}\mu_k$. Then, we must have $\mu_4 \leq \frac{1}{2}\mu_k$ because a vector (x,y) such that $y_4 = 0$, $x_{41} = 1$, $x_{42} = 1$, $x_{43} = 1$ and $y_i = 1$ for all $i \neq 4$ is feasible, and we thus have $\pi x + \mu y = \sum_{t \in K_2 \cup K_3} \mu_t + 2\mu_4$. But, $\sum_{t \in K_2 \cup K_3} \mu_t + \mu_k = 1$, which implies that $\mu_4 \leq \frac{1}{2}\mu_k$. Suppose now that $\mu_4 < \frac{1}{2}\mu_k$. Then, it is obvious that there exists no solution vector $(x^*,y^*) \in H_{\pi,\mu}$ such that $y_4^* = 0$, $x_{41}^* = 1$, $x_{42}^* = 1$ and $x_{43}^* = 1$. Now, recall that if $(x^*,y^*) \in H_{\pi,\mu}$ is such that $y_4^* = 0$ then $x_{4j}^* = 1$ for at least one j, $1 \leq j \leq 3$, but not for exactly 2 different values of j. Hence, any solution vector $(x,y) \in H_{\pi,\mu}$ must satisfy $y_4 + x_{41} + x_{42} + x_{43} = 1$ which again leads to a contradiction. Thus, $\mu_4 = \frac{1}{2}\mu_k$. Δ

Now, we are in a position to prove the necessary and sufficient conditions for the non-trivial facets of L_I^{pd} where $p \geq 3$ and $d = 3$.

<u>Theorem 5.1. $\pi x + \mu y \leq 1$ is a non-trivial facet of L_I^{pd} where $P = K_2 \cup K_3$ and $D = \{1,2,3\}$ if and only if i) $|K_2| \geq 3$ and there exists an odd-cycle of length 9 and ii) the coefficients are as follows.</u>

$$\pi_{ij} = \mu_i = \frac{2}{2|K_2|+|K_3|+2} \quad \text{for all } i \in K_2,\ j \in J_i; \text{ and if } K_3 \neq \emptyset$$

$$\pi_{kj} = \mu_k = \frac{1}{2|K_2|+|K_3|+2} \quad \text{for all } k \in K_3,\ j \in J_k = \{1,2,3\} \qquad (5.1)$$

Proof:

(⇒)From lemma 5.1, $|K_2| \geq 3$ and the existence of an odd-cycle of length 9 must be satisfied in order for $\pi x + \mu y \leq 1$ to be a non-trivial facet of L_I^{p3} where $p \geq 3$.

Using lemma 5.2 and letting $a = \mu_k > 0$ where $k \in K_2$ we can express $\pi x + \mu y \leq 1$ as

$$a\Big(\sum_{i\in K_2}\sum_{j\in J_i} x_{ij} + \sum_{i\in K_2} y_i\Big) + \frac{a}{2}\Big(\sum_{k\in K_3}\sum_{j\in J_k} x_{kj} + \sum_{k\in K_3} y_k\Big) \leq 1$$

or

$$2\Big(\sum_{i\in K_2}\sum_{j\in J_i} x_{ij} + \sum_{i\in K_2} y_i\Big) + \sum_{k\in K_3}\sum_{j\in J_k} x_{kj} + \sum_{k\in K_3} y_k \leq \frac{2}{a}.$$

However, it is obvious that the maximum possible value of the left hand side is equal to $2|K_2|+|K_3|+2$. Thus, $a = \dfrac{2}{2|K_2|+|K_3|+2}$ and the result holds.

(⇐)Sufficiency follows directly from theorem 4.3. In other words, if we let $m = 3$, $s = |K_2|-3$ and $t = |K_3|$, then $(2m+s-2)(m-1)+t(m-2) = 2|K_2|+|K_3|+2$ and we get (5.1). □

Appendix I.

$|\tilde{C}| = (t+n-1)2^{(t-1)(m-1)}$.

Proof:

We first note that $\tilde{C}_s$ is constructed as follows.

$\tilde{C}_m$ is a matrix obtained by interchanging the adjacent rows of $(E-I)_{(t+n)x(t+n)}$ for (t+n-1) times where E is a matrix of one's and I is an identity matrix.

In other words,

$$\tilde{C}_m = \begin{pmatrix} 1&1&1&1&1&1&0 \\ 0&1&.&.&.&.&1 \\ 1&0&1&.&.&.&1 \\ 1&1&0&1&.&.&1 \\ 1&.&1&0&1&.&1 \\ 1&.&.&1&0&1&1 \\ 1&1&1&1&1&0&1 \end{pmatrix}_{(t+n)x(t+n)}$$

For s = m-1,m-2,...,1,

$$C_s = \begin{pmatrix} T_1 & T_2 \\ T_3 & \tilde{C}_{s+1} \end{pmatrix}_{(t(m-s+1)+n)x(t(m-s+1)+n)}$$

where

$$T_1 = \begin{pmatrix} 1&1&1&1&1&1 \\ 0&1&.&.&.&1 \\ 0&0&1&.&.&1 \\ 0&.&0&1&.&1 \\ 0&.&.&0&1&1 \\ 0&0&0&0&0&1 \end{pmatrix}_{txt}$$

$$T_2 = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0 \\ 1&0&.&.&.&.&.&.&.&.&.&0 \\ 1&1&0&.&.&.&.&.&.&.&.&0 \\ 1&.&1&0&.&.&.&.&.&.&.&0 \\ 1&.&.&1&0&.&.&.&.&.&.&0 \\ 1&1&1&1&1&0&0&0&0&0&0&0 \end{pmatrix}_{tx(t(m-s)+n)}$$

$$T_3 = \begin{pmatrix} \begin{matrix} \text{First } (t-1) \\ \text{columns of} \\ \tilde{C}_{s+1} \end{matrix} & \begin{matrix} 0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0\\0 \end{matrix} \end{pmatrix}_{(t(m-s)+n) x t}$$

Thus,

$$\tilde{C}_{s+1}^{-1}\, T_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & 0 & . & 0 \\ 0 & . & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & . & . & . & . & 0 \\ 0 & . & . & . & . & 0 \\ 0 & . & . & . & . & 0 \\ 0 & . & . & . & . & 0 \\ 0 & . & . & . & . & 0 \\ 0 & . & . & . & . & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{(t(m-s)+n) x t}$$

$$T_2\tilde{C}_{s+1}^{-1}T_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & . & . & . & 0 \\ 1 & 1 & 0 & . & . & 0 \\ 1 & . & 1 & 0 & . & 0 \\ 1 & . & . & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}_{t x t}$$

$$T_1 - T_2\tilde{C}_{s+1}^{-1}T_3 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 1 & . & . & . & 1 \\ -1 & -1 & 1 & . & . & 1 \\ -1 & . & -1 & 1 & . & 1 \\ -1 & . & . & -1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & 1 \end{pmatrix}_{t x t}$$

But, $|\tilde{C}_s| = |\tilde{C}_{s+1}|\,|T_1 - T_2\tilde{C}_{s+1}^{-1}T_3|$ and $|\tilde{C}_m| = [(-1)^{t+n-1}]^2(t+n-1)$.

Therefore, since $|T_1 - T_2\tilde{C}_{s+1}^{-1}T_3| = 2^{t-1}$, $|\tilde{C}_s| = |\tilde{C}_{s+1}|\; 2^{t-1}$ for $s = m-1, m-2, \ldots, 1$

and $|\tilde{C}| = (t+n-1)(2^{t-1})^{m-1}$. Δ

Appendix II: Lifting of the Guignard's facet.

1. (Lifting of odd-cycle valid inequalities, i.e. t=2 and n=1.)

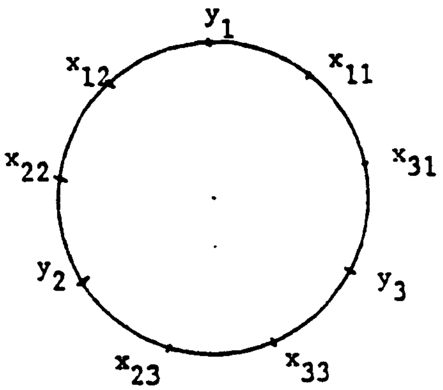
a.

$\tilde{C}$ =

	1	2	3
1	1	1	0
2	0	1	1
3	1	0	1

$\alpha(G^{\tilde{C}}) = 4$

$m = 1,\ t_1 = 2$



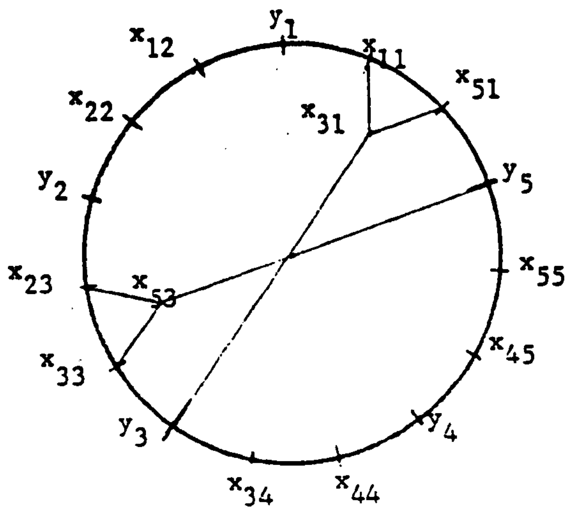
b.

$\tilde{C}$ =

	1	2	3	4	5
1	1	1	0	0	0
2	0	1	1	0	0
3	(1)	0	1	1	0
4	0	0	0	1	1
5	1	0	(1)	0	1

$\alpha(G^{\tilde{C}}) = 7$

$m = 2,\ t_2 = 2,\ t_1 = 2$



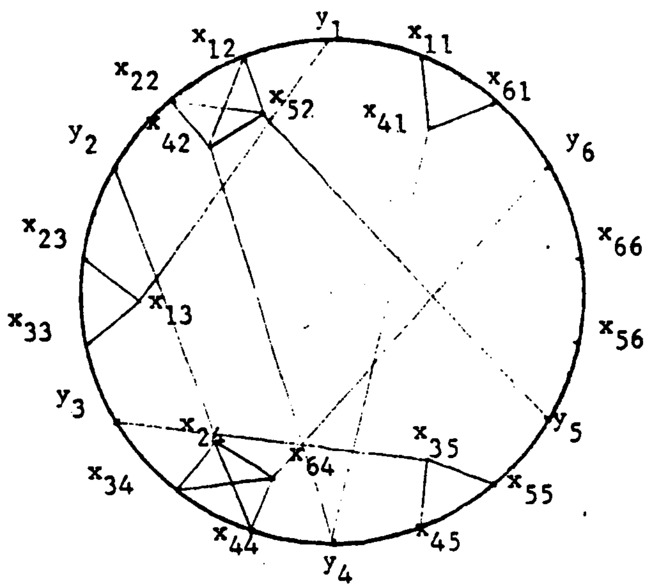
c.

$\tilde{C}$ =

	1	2	3	4	5	6
1	1	1	(1)	0	0	0
2	0	1	1	(1)	0	0
3	0	0	1	1	(1)	0
4	(1)	(1)	0	1	1	0
5	0	(1)	0	0	1	1
6	1	0	0	(1)	0	1

$\alpha(G^{\tilde{C}}) = 9$

$m = 2,\ t_2 = 2,\ t_1 = 3$



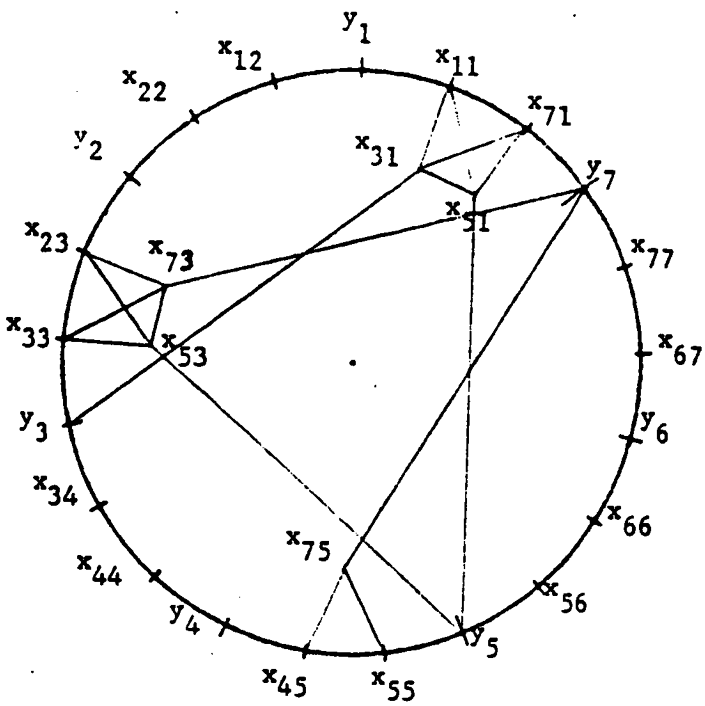
d.

	1	2	3	4	5	6	7
1	1	1	0	0	0	0	0
2	0	1	1	0	0	0	0
3	(1)	0	1	1	0	0	0
4	0	0	0	1	1	0	0
5	(1)	0	(1)	0	1	1	0
6	0	0	0	0	0	1	1
7	1	0	(1)	0	(1)	0	1

$\tilde{C}$ = (matrix above)

$\alpha(G^{\tilde{C}}) = 10$

$m = 3,\ t_3 = 2,\ t_2 = 2,\ t_1 = 2$



e.

	1	2	3	4	5	6	7	8
1	1	1	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0
3	(1)	0	1	1	0	0	0	0
4	0	0	0	1	1	0	0	0
5	(1)	0	(1)	0	1	1	(1)	0
6	0	0	0	0	0	1	1	(1)
7	(1)	0	(1)	0	(1)	0	1	1
8	1	0	(1)	0	(1)	(1)	0	1

$\alpha = 12$

$m = 3,\ t_3 = 3,\ t_2 = 2,\ t_1 = 2$

f.

	1	2	3	4	5	6	7	8
1	1	1	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0
3	(1)	0	1	1	(1)	0	0	0
4	0	0	0	1	1	(1)	0	0
5	0	0	0	0	1	1	(1)	0
6	(1)	0	(1)	(1)	0	1	1	0
7	0	0	0	(1)	0	0	1	1
8	1	0	(1)	0	0	(1)	0	1

$\alpha = 12$

$m = 3,\ t_3 = 2,\ t_2 = 3,\ t_1 = 2$

g.

	1	2	3	4	5	6	7	8	9
1	1	1	0	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0	0
3	(1)	0	1	1	(1)	0	0	0	0
4	0	0	0	1	1	(1)	0	0	0
5	0	0	0	0	1	1	(1)	0	0
6	(1)	0	(1)	(1)	0	1	1	(1)	0
7	0	0	0	(1)	0	0	1	1	(1)
8	(1)	0	(1)	0	0	(1)	0	1	1
9	1	0	(1)	(1)	0	(1)	(1)	0	1

$\alpha = 14$

$m = 3,\ t_3 = 3,\ t_2 = 3,\ t_1 = 2$

2. (Lifting of the Guignard's facet when $t = 3$ and $n = 1$.)

a.

	1	2	3	4
1	1	1	1	0
2	0	1	1	1
3	1	0	1	1
4	1	1	0	1

$\alpha = 6$

$m = 1,\ t_1 = 3$

b.

	1	2	3	4	5	6
1	1	1	0	0	0	0
2	0	1	1	0	0	0
3	1	0	1	1	1	0
4	0	0	0	1	1	1
5	1	0	1	0	1	1
6	1	0	1	1	0	1

$\alpha = 9$

$m = 2,\ t_2 = 3,\ t_1 = 2$

c.

	1	2	3	4	5	6	7
1	1	1	1	0	0	0	0
2	0	1	1	1	0	0	0
3	0	0	1	1	1	0	0
4	1	1	0	1	1	1	0
5	0	1	0	0	1	1	1
6	1	0	0	1	0	1	1
7	1	1	0	1	1	0	1

$\alpha = 11$

$m = 2,\ t_2 = 3,\ t_1 = 3$

d.

	1	2	3	4	5	6	7	8
1	1	1	1	1	0	0	0	0
2	0	1	1	1	1	0	0	0
3	0	0	1	1	1	1	0	0
4	0	0	0	1	1	1	1	0
5	1	1	1	0	1	1	1	0
6	0	1	1	0	0	1	1	1
7	1	0	1	0	1	0	1	1
8	1	1	0	0	1	1	0	1

$\alpha = 13$

$m = 2,\ t_2 = 3,\ t_1 = 4$

REFERENCES

1. V. Chvátal, On Certain Polytopes Associated with Graph, J. Comb. Theory B 18 (1975), 138-154.

2. D.C. Cho, E.L. Johnson, M.W. Padberg and M.R. Rao, On the Uncapacitated Plant Location Problem I: Valid inequalities and Facets, GBA Working Paper, New York University, New York, December 1980.

3. G. Cornuejols and J.M. Thizy, Some Facets of the Simple Plant Location Polytope, Working Paper, Carnegie-Mellon University.

4. M. Guignard, Fractional Vertices, Cuts and Facets of the Simple Plant Location Problem, Mathematical Programming Study 12 (1980), 150-162.

5. M.W. Padberg, On the Facial Structure of Set Packing Polyhedra, Math. Programming 5 (1973 199-215.

6. M.W. Padberg, On the complexity of Set Packing Polyhedra, Annals of Discrete Math. 1 (1977), 421-434.

Imprimé en France

par
l'Institut National de Recherche en Informatique et en Automatique