The Russian method for linear inequalities III:bounded integer programming

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ABSTRACT.

We discuss implications of the Russian method for bounded integer programming problems. The Russian method is a polynomially bounded algorithm for optimizing a linear function subject to linear inequalities. We show that the polynomially bounded linear programming algorithm yields a polynomially bounded algorithm for bounded integer linear programming problem if and only if the facet-identification problem (i.e. the problem of finding a facet-defining inequality which chops off a given rational vector $\bar{x}$ or proving that no such inequality exists) can be solved in polynomial time and space. We then combine the results of this paper and those of an earlier paper ("Odd Minimum Cut-Sets and b-Matching") to derive a polynomial algorithm for b-matching problem in undirected finite graphs.

RESUME.

Nous discutons ici de quelques conséquences de l'algorithme de Khachian (appelé ici "Méthode russe") pour la solution des programmes linéaires en variables entières bornées. La méthode russe est un algorithme polynomial pour l'optimisation d'une fonction linéaire sous des contraintes linéaires. Nous montrons que la méthode russe donne un algorithme polynomial pour la solution de programmes linéaires en variables entières bornées si et seulement si le problème d'identification d'une facette, i.e. celui de trouver une inégalité qui définisse une facette du polytope associé et qui élimine un point donné quelconque avec des composants rationnels où celui de démontrer qu'une telle inégalité n'existe pas, peut être résolu en temps et espace polynomial. La combinaison de ce résultat avec les résultats d'un autre article "Odd Minimum Cut-Sets and b-Matchings" donne un algorithme polynomial pour la détermination de c-couplages dans les graphes non-orientés finis.

0. INTRODUCTION.

In this paper we relate the Russian method, see [7], [12], [15], [16], [18] to integer linear programming. We consider the <u>bounded</u> integer programming problem

$$(P1) \qquad \max \{px \mid Mx \leq d,\ 0 \leq x \leq \underline{u},\ x \in \mathcal{D}\}$$

where M is a (possibly vacuous) $\ell \times n$ matrix of integer components, d is a vector with ℓ integer components, p and $\underline{u}$ are vectors with n integer components p_j and u_j, respectively, and $\mathcal{D}$ is any subset of the integer points $x \in \mathbb{R}^n$ satisfying $0 \leq x \leq \underline{u}$. It follows from Weyl's Theorem [19] that (P1) can be solved as a linear programming problem (LP) by appending all facet-defining inequalities to the linear constraint system of (P1) since the resulting <u>linear</u> constraint system has only integer extreme points. The facet-defining inequalities are generally too numerous to be explicitly enumerated. What is worse, in those cases for which various classes of facet-defining inequalities are known, the respective number of inequalities generally depends exponentially (rather than polynomially) upon the number n of variables of the problem (P1). Nevertheless, the Russian method yields a polynomial time and space algorithm for certain integer programming problems of this kind. This follows from the fact that in the Russian method one needs to know only whether or not all linear constraints are satisfied by the current solution. In fact, in order to solve the linear program (LP) associated with (P1), the following facet-identification problem must be solved repeatedly in the Russian method if one wants to find a violated constraint :

<u>Facet Identification Problem</u> : <u>Given any rational vector y find a facet defining inequality fx ≤ fo that is violated by y, i.e. that satisfies fy > fo, or prove that no such inequality exists</u>.

If no violated facet-defining inequality exists, then y satisfies <u>all</u> linear inequalities of the linear program (LP) associated with (P1) and the Russian method continues by modifying the objective function value ; else, we find a violated linear inequality and the updating of the Russian method is carried out as usual, see below where we state a simplified version of the algorithm of [15] for matters of clarity.

Different from many simplex-method based approaches to solving integer programs, in the Russian method there is no need to "remember", i.e. store the violated constraint. It is clear, however, that due to the possibly enormous number of facet-defining inequalities the facet identification problem must itself be carried out by an algorithm. See [1,13] where this facet-identification problem is addressed in the context of the symmetric travelling salesman problem and see [14] where this problem is solved completely for b-matching problems. From these studies -the latter has obvious implications for the symmetric travelling salesman problem as well- we know that we have to distinguish different classes of facet-defining inequalities by the type of algorithm that is required to identify a violated facet-defining inequality. For instance, in the symmetric travelling salesman problem, the first class of constraints are the subtour-elimination constraints : the identification of subtour-elimination constraints is carried out by the polynomially bounded minimum-cut algorithm due to Gomory and Hu see [1], [13]. The second class of constraints in this context are the 2-matching constraints ; here the identification problem is solved by finding an odd minimum-cut set [14] which can be done in polynomial time by a modified Gomory-Hu algorithm. A third class of constraints are the comb-constraints [10], [11] for which the algorithmic identification is not yet resolved. Other possible classes of facets for the travelling salesman problem are discussed in [8]. A necessary condition for the polynomial solvability of integer programming problems is consequently that the number of different classes of facet-defining inequalities itself is bounded by polynomial function.

As we will be concerned here with the solvability of linear programs involving an arbitrary number of constraints we restate a simplified version of the Russian method for linear optimization, see Section 3 of [15]. We consider a linear program of the form

$$\max \{ \sum_{j=1}^{n} c_j x_j \mid Ax \le b\}$$

and denote that the i-th row of (A,b) as (a^i, b_i) for $i=1,\dots,m$ with integer data c, A and b. Denote P the feasible set of the linear program and suppose that we want to maximize cx over the set

$$S = P \cap \{x \in \mathbb{R}^n \mid \sum_{j=1}^{n} x_j^2 \le u^2\}$$

where u is some positive integer which is chosen large enough such that S contains all basic solutions to $Ax \le b$. Define :

$$z_L = -1 + u \sum_{j=1}^{n} \min(0, c_j), \quad \bar{c} = 1 + \max\{|c_j|, \ j=1,\dots,n\}$$

$$h \ge 2\bar{c}\, n^2\, \Delta_A, \ h \text{ integer}$$

$$T = 4n^2 \lceil \log_2 h^2 u\, n^{-1/2} \rceil$$

$$R = 16n \lceil \log_2 h\, u^{1/2} \rceil$$

where $\Delta_A \ge 2$ is an integer upper bound on the absolute value of a determinant of a submatrix of A. The (simplified) Russian method for linear optimization using approximate arithmetic is the following algorithm ALG1(u,h) :

Step 0 : (Initialization) Set $x_j^0 = 0$ for $j=1,\dots,n$; $Q_0 = u\, I_n$ where I_n is the $n \times n$ identity matrix ; set $\bar{z} = z_L$, $k = 0$ and go to Step 1.

Step 1 : (Constraint identification) If $a^i x^k \le b_i + 1/h$ for all $i=1,\dots,m$ go to Step 2. Else let j be any index with $a^j x^k > b_j + 1/h$. Set $r = a^j$ and go to Step 3.

Step 2 : (Objective function as a constraint) : If $c\, x^k > \bar{z}$, replace $\bar{z}$ by $c\, x^k$, set $\bar{x} = x^k$, $r = -c$ and go to Step 3. Else set $r = -c$ and go to Step 3.

Step 3 : (Updating) : If k=T go to Step 4. Else set

$$x^{k+1} \approx x^k - Q_k Q_k^T r^T / (n+1) \|rQ_k\|$$

$$Q_{k+1} \approx 2^{1/8n^2} (n / \sqrt{n^2 - 1}) Q_k (I_n - (1 - \sqrt{\tfrac{n-1}{n+1}}) Q_k^T r^T r Q_k / \|rQ_k\|)$$

where $\approx$ means that for each component the computation is carried out with a precision of R binary positions after the point. Replace k+1 by k and go to Step 1.

Step 4 : (Termination) If $\bar{z} = z_L$, stop ; the problem has no feasible solution. Else stop ; $\bar{x}$ approximately solves the linear programming problem.

By combining the results of Section 3 of [15]and the analysis in [16] it follows that ALG1(u,h) is a valid algorithm for linear programs of the above kind whose running time and workspace requirements are a polynomial function of n, $\log_2 \bar{c}$, $\log_2(u+1)$ and the number m of constraints of the linear program. The latter dependence is linear in m and follows from the fact that in the statement of Step 1 of ALG1(u,h) we assume that all constraints of the problem are simply listed and checked off one by one.

In Section 1 of this paper we give a characterization of the facets of arbitrary bounded integer programs by way of basic solutions to a certain set of linear inequalities. In particular, it follows that for any such problem every facet can be described by a linear inequality whose coefficients grow moderately in n and the logarithm of the u_j, j=1,...,n. In Section 2, we address the polynomial solvability of bounded integer programs, while in Section 3 we address the polynomial solvability of the facet-identification problem. It is shown that an integer program of the form (P1) is solvable in polynomial time and space if and only if the facet identification problem is solvable in polynomial space and time. Finally, in Section 4 we combine the results of this paper and those of [14] to give a polynomial algorithm for b-matching problems in undirected graphs.

This paper is a revised version of our earlier paper [17]. Like in the earlier paper we do <u>not</u> make the assumption that the feasible set of the bounded integer linear programming problem is full-dimensional. Thus our results apply to a wider class of integer programs than those contained in [9]. In particular, our results apply to the travelling salesman problem whose associated polytope is not full-dimensional. For this problem the results of our paper can be summarized as follows : The travelling salesman problem is solvable in polynomial time and space if and only if (i) the number of different classes of facet-defining inequalities is polynomial in the number of cities and (ii) if for each class of facet-defining inequalities the facet identification problem is solvable in polynomial time and space. It is remarkable that for the two classes of facet defining inequalities for which the facet-identification has been solved to date the respective algorithms are polynomial. However, the results of [8] concerning more complicated forms of facets of the travelling salesman problem strongly suggest that not all facets for this problem can be found in polynomial time. Thus, in view of the equivalence of the facet-identification problem and the travelling salesman problem vis-a-vis polynomiality the results of this paper open new avenues for settling the question NP = ?P.

1. FACETS OF BOUNDED INTEGER PROGRAMS.

We denote :

(1.1) $$P_I = \text{conv}\{x \in \mathbb{R}^n \mid Mx \leq d,\ 0 \leq x \leq \underline{u},\ x \in \mathcal{B}\}$$

the (convex bounded) polytope given by the convex hull of feasible solutions to the constraint system of (P1). Let $r \leq n$ denote the dimension of P_I. A facet defining inequality for P_I is defined as a valid linear inequality which is satisfied as equality by at least r affinely independent integer points of P_I. Our definition differs slightly from the usual definition where "at least" is replaced by "exactly". Since we are dealing with polytopes whose dimension r may be smaller than n it is more convenient to work with the above definition. In the case of full dimensional polytopes our definition coïncides with the standard one.

We denote B the $t \times n$ matrix whose rows are given by all integer points in P_I, i.e. B is a list of all feasible solutions to (P1). Since P_I is bounded, B is a finite matrix. Since dim $P_I = r$ it follows that B has exactly r+1 affinely independent rows and thus B contains either r or r+1 linearly independent rows, i.e. the rank r(B) satisfies r(B) = r or r+1. Further more, if $0 \in P_I$ holds, then we have r(B) = r. Since $0 \in P_I$ need not hold it is convenient to translate the rows of B so that the resulting matrix contains a zero row. To this end let v be any row of B if $0 \notin P_I$ and let v=0 if $0 \in P_I$. Let V be the $t \times n$ matrix whose rows are all equal to v and define :

(1.2) $$W = B - V.$$

Since B contains exactly r+1 affinely independent rows it follows that r(W)=r holds. Consider now the system of linear inequalities

(1.3) $$W\pi \leq e_t$$

where e_t is the vector with t components equal to 1.

Proposition 1.1. : If $\pi^T x \leq \pi_0$ is a valid inequality for P_I, then there exists a scalar $t > 0$ such that $t\pi$ is a feasible solution to (1.3). On the other hand, if π is a feasible solution to (1.3), then $\pi^T x \leq \pi_0$ is a valid inequality for P_I where $\pi_0 = 1 + v\pi$.

The proof of proposition 1.1 is straightforward and left as an exercise. In order to characterize P_I by way of linear inequalities it is thus sufficient to look at feasible solutions to the linear system (1.3).

Proposition 1.2. : If $\pi^T x \leq \pi_0$ is a facet-defining inequality for P_I, then there exists a scalar $t > 0$ such that $t\pi$ is a feasible solution to (1.3) and either
(i) there exists a $r \times n$ submatrix W_1 of W with $r(W_1) = r$ such that $W_1(t\pi) = e_r$ holds, or,
(ii) $W(t\pi) \leq 0$ holds and there exists a $r \times n$ submatrix W_1 of W with r affinely independent rows such that $W_1(t\pi) = 0$ holds.

Proof : Since $\pi^T x \leq \pi_0$ is a facet-defining inequality for P_I there exists s affinely independent rows b_i of B satisfying $b_i\pi = \pi_0$ for $i=1,\dots,s$ where $s=r$ or $s = r+1$. If $v\pi < \pi_0$ holds then it follows that v is affinely independent of b_i, $i=1,\dots,s$. For suppose not ; then there exists λ_i such that :

$$\lambda_0 v + \sum_{i=1}^{s} \lambda_i b_i = 0 \qquad \text{and} \qquad \sum_{i=0}^{s} \lambda_i = 0$$

and such that not all λ_i are zero. Hence, we have :

$$\lambda_0 v\pi + \sum_{i=1}^{s} \lambda_i b_i \pi = \lambda_0(v\pi - \pi_0) = 0$$

and consequently $\lambda_0 = 0$. It follows that b_i, $i=1,\dots,s$ are affinely dependent, contradiction . Consequently, $s=r$ holds and the matrix W_1 with rows $b_i - v$ for $i=1,\dots,r$ has rank r. Hence we can choose $t = \pi_0 - v\pi > 0$ and (i) follows. If $v\pi = \pi_0$ holds, then we can choose $t=1$ and (ii) follows.

Remark 1.3. : (Trivial facets) If W_1 is a $r \times n$ submatrix of W satisfying $r(W_1) = r$, then W_1 has zeros only in column k, say, if and only if column k of the entire matrix W is a zero column. This follows immediately from $r(W_1) = r(W)$. If W_1 has a zero column it follows that the two inequalities $x_k \geq v_k$ and $x_k \leq v_k$ are facet-defining inequalities for P_I where v_k is the k-th component of the

row v used to define W, i.e. P_I is contained in the hyperplane $x_k = v_k$. However, as we do not wish to distinguish between equations and inequalities we do not make explicit use of the latter fact, rather we will work with the two inequalities. More generally, facet defining inequalities of the form $x_k \geq \alpha$ or $x_k \leq \beta$ are called trivial where $0 \leq \alpha \leq \beta \leq u_k$ are any integer numbers. Trivial facet-defining inequalities are easily characterized :
An equality of the form $x_k \geq \alpha$ ($x_k \leq \beta$, respectively) is facet-defining for P_I if and only if W contains a $r \times n$ submatrix W_1 with r affinely independent rows such that the entries of column k in W_1 are all equal to some non-negative integer δ (to some non-positive integer δ, respectively) and the remaining entires of column k in W are all greater-or-equal to δ (less than or equal to δ, respectively).

For any vector π denote $S(\pi) = \{j \in N \mid \pi_j \neq 0\}$ the support of π where $N = \{1,\ldots,n\}$ is the column set of W. A solution π to the linear system $W\pi \leq e_t$ is called r-basic if (i) there exists a $r \times n$ submatrix W_1 of W with $r(W_1) = r$ such that $W_1\pi = e_r$ holds and (ii) the $r \times s$ submatrix W_{11} of W_1 formed by the columns with index in $S(\pi)$ satisfies $r(W_{11}) = s$ where $s = |S(\pi)|$. A solution π to the linear system $W\pi \leq 0$ is called r-extreme if (i) there exists a $r \times n$ submatrix W_1 of W with r affinely-independent rows such that $W_1\pi = 0$ holds and (ii) the $r \times s$ submatrix W_{11} of W_1 formed by the columns with index in $S(\pi)$ satisfies $r(W_{11}) = s-1$ where $s = |S(\pi)|$. Clearly, r-basic solutions as well as r-extreme solutions yield facet-defining inequalities for P_I.

Theorem 1.4. : Let π^i for $i=1,\ldots,q$ be the set of all r-basic solution to $W\pi \leq e_t$ and let π^i for $i=q+1,\ldots,p$ be the set of all r-extreme solutions to $W\pi \leq 0$. Then :

$$(1.4) \qquad P_I = \{x \in \mathbb{R}^n / \pi^i x \leq 1 + \pi^i v,\ i=1,\ldots,q\ ,\ \pi^i x \leq \pi^i v,\ \ i=q+1,\ldots,p\}.$$

Proof : Denote P^* the polytope on the right-hand side of (1.4). By construction $P_I \subseteq P^*$ holds. Suppose $P_I \neq P^*$ and let $\bar{x} \in P^*$ such that $\bar{x} \notin P_I$. Then, by Weyl's theorem there exists a facet-defining inequality $\pi^T x \leq \pi_0$ for P_I such that $\pi^T\bar{x} > \pi_0$ holds. It follows $\pi^T(\bar{x} - v^T) > \pi_0 - v\pi$. If $\pi_0 = v\pi$, let $t = 1$ and $\delta = 0$; if $\pi_0 > v\pi$ let $t = 1 / (\pi_0 - v\pi)$ and $\delta = 1$. It follows from Proposition 1.2 that the following linear program has an objective function value greater than δ :

(1.5) $\max \{(\bar{x}^T - v)\pi \mid W_1\pi = \delta e_r, W_2\pi \leq \delta e_{t-r}\} > \delta$

where W_1 is the submatrix identified in Proposition 1.2 part (i) if $\delta=1$, part (ii) if $\delta=0$. Using a standard device of linear programming, i.e. replacement of the unconstrained variables π by the difference of two non negative variables, we can apply the fundamental theorem of linear programming and conclude that the maximum of (1.5) is either attained at some extreme point or that the objective function is unbounded along some extreme ray of the feasible set. In the first case we find that the inequality corresponding to some r-basic solution is violated ; in the second case we find that an inequality corresponding to some r-extreme solution is violated. Thus, $P^* \subseteq P_I$ and the theorem follows.

Remark 1.5. : Let π be an r-extreme solution to $W\pi \leq 0$. By definition, an extreme solution is unique up to a scalar and has at least one non zero component. Thus, we can scale any one component and characterize r-extreme solutions as basic solutions to the 2n systems of linear inequalities

$$W\pi \leq 0$$

(1.6) $$\pi_j = \delta \quad \text{for some } j \in \{1,\ldots,n\}$$

where $\delta=+1$ or $\delta=-1$.

Theorem 1.6. : There exists a system of facet-defining inequalities for P_I such that the absolute value of each component is an integer number less than or equal to :

(1.7) $$\Xi = n^{n/2} \Delta^n$$

where $\Delta = \max \{u_j \mid j=1,\ldots,n\}$.

Proof : By Theorem 1.4 we know that a facet-defining system of inequalities for P_I is given by the r-basic solutions to $W\pi \leq e_t$ plus the r-extreme solutions to $W\pi \leq 0$. Il follows that there exists a system of facet-defining inequalities among the basic solutions to the following 2n+2 systems of linear inequalities.

(1.8) $B\pi \leq e_t$

(1.9) $B\pi \leq -e_t$

(1.10) $$\begin{cases} B\pi \leq 0 & \\ \pi_j = \delta & \text{for some } j \in \{1,\ldots,n\} \end{cases}$$

where δ=+1 or -1. Since each element of B is between 0 and Δ it follows from Hadamard's inequality that the largest absolute value of the determinant of a submatrix of the two matrices :

$$\begin{pmatrix} B, e_t \end{pmatrix} \quad , \quad \begin{pmatrix} B & 0 \\ \underline{u}_j & \underline{+}1 \end{pmatrix}$$

where 0 is the zero vector with t components and $\underline{u}_j$ is the j-th unit vector is less than or equal to Ξ as defined in (1.7). Since B consists of integers only, it follows by Cramer's rule that each component π^*_j of a basic solution π^* to (1.8), (1.9) or (1.10) is given by :

(1.11) $\pi^*_j = I_j/I$

where $I \neq 0$ and I_j are integer numbers whose absolute values are less than or equal to Ξ .

A facet-defining inequalility (in rational form) is thus given by $\pi^* x \leq \pi_0$ where π_0 = 0 or $\underline{+}1$. Multiplying the inequality by the absolute value of I we have that the assertion of Theorem 1.6 follows.

Remark 1.6. : For further reference we note that Ξ defined in (1.7) constitutes also a bound on the largest absolute value of the determinant of a submatrix of the matrix (W, e_t) since every entry of W has an absolute value less than or equal to Δ . As before this follows from Hadamard's inequality.

Remark 1.7. : In the case that the integer points of P_I form an independence system, i.e. if $x \in P_I$ and $0 \le y \le x$ imply that $y \in P_I$, the characterization of facet-defining inequalities given here is essentially the same as the one given by Fulkerson [6]. In this case one can show that all facet-defining inequalities have non-negative components and thus it suffices to consider the non negative basic solutions to (1.3). The (bounded convex) polytope :

(1.12) $$A(P_I) = \{\pi \in \mathbb{R}^n \mid W\pi \le e_t, \pi \ge 0\}$$

is called the anti-blocker of P_I, see [6], in this case if dim $P_I = n$.

Remark 1.8. : It follows from Theorem 1.6 that for a suitably chosen system of facet-defining inequalities for P_I the logarithm (base two) of the absolute value of a non-zero component is bounded by a polynomial in the number n of variables and $\log_2 \Delta$ where Δ is the maximum of the upper bounds on the variables. If in lieu of assuming explicit upper bounds on the variables we assume that the remaining constraints define a bounded polytope, then by Lemma 2.1 of [15] we get an alternate expression for Δ. The assumption that P_I is bounded is, however, crucial since the convex hull of an unbounded discrete set need not to be closed and since we are thus not even guaranteed to have a linear description of the corresponding convex hull. In certain cases e.g. if P_I is such that $x \in P_I$, $y \ge x$ imply $y \in P_I$, a similar characterization of the facets of the convex hull P_I by way of a finite matrix has been given, see e.g. the theory of blocking polyhedra developed by Fulkerson [6]. The same is the case if we consider all non negative integer solutions to a finite set of linear inequalities with rational data.

2. POLYNOMIAL SOLVABILITY OF BOUNDED INTEGER PROGRAMS.

Denote $Ax \leq b$ a complete system of facet-defining inequalities for the polytope P_I. Then problem (P1) is equivalent to the linear program :

$$\text{(LP)} \qquad \max \{ \sum_{j=1}^{n} p_j x_j \mid Ax \leq b \}.$$

In order to get a unique optimum solution to (LP) if (LP) has any feasible solution at all, we perturb the objective function coefficients. Let :

$$\text{(2.1)} \qquad \Delta = \max \{u_j,\ j=1,\ldots,n\}$$

$$\text{(2.2)} \qquad c_j = (1 + \Delta)^n\, p_j + (1 + \Delta)^{n-j} \quad \text{for } j=1,\ldots,n$$

and denote (P2) the corresponding linear program with integer data :

$$\text{(P2)} \qquad \max \{ \sum_{j=1}^{n} c_j x_j \mid Ax \leq b \}.$$

Since $0 \leq x_j \leq \Delta$ holds for all $x \in P_I$ and since $\Delta \geq 1$ it follows that (P2) has a unique optimum solution and that the optimum solution to (P2) is optimum for (P1) as well if (P1) has any solution at all.

By Theorem 1.6 we know that there exists a matrix A and vector b for (LP) such that every component of A and b has absolute value less than or equal to

$$\text{(2.3)} \qquad \Xi = n^{n/2}\, \Delta^n$$

In the following we use the term facet-defining inequality for P_I only to mean facet-defining inequalities with integer components whose absolute values are less than or equal to Ξ. Let $\Delta_A \geq 2$ be an integer upper bound on the absolute value of a determinant of a submatrix of A. With the above convention it follows from Hadamard's inequality that :

$$\text{(2.4)} \qquad \Delta_A \leq n^{n/2}\, \Xi^n = n^{(n^2+n)/2}\, \Delta^{n^2}$$

holds.

We denote a^i the rows of A, b_i the components of b for i=1,...,m define the following quantities :

(2.5) $$u = \sqrt{n}(\Delta + 1) , \quad h = 2 \bar{c} n^2 \Delta_A$$

and run the Russian method ALG1(u,h) with the parameters defined in (2.5). Noting that η as used in Lemma 3.1 of [15] can be chosen integer (rather than rational) it follows that the conclusion of Lemma 3.1 holds with h as defined above. Furthermore, using an argument parallel to the one used to prove Theorem 3.1 of [15] it follows that Step 4 of ALG1(u,h) can be replaced by the following :

*Step 4** : (Termination) : If $\bar{z} = z_L$, stop : (P2) has no feasible solution. Else set $x_j^* = \lceil \bar{x}_j + 0.5 \rceil$ for j=1,...,n and stop : x^* solves (P2).

Remark 2.1. : Like in Section 3 of [15] we can initialize ALG1(u,h) differently using the fact that we are optimizing in the non-negative orthant. The changes are left to the reader. While the application of the exact results obtained in [16] permit one to derive slightly smaller values for the running time T and the required precision R of the algorithm than the following ones :

(2.6) $$T = 8n^2 \lceil \log_2 h\sqrt{\Delta + 1} \rceil$$
$$R = 16n \lceil \log_2 \; h \sqrt{n\Delta + n} \rceil$$

we prefer (2.6) for reasons of notational simplicity.(Note that we have increased R somewhat as compared to the introduction ; which we are, of course, perfectly free to do). It follows from the application of the results of [16] that O(R) bits suffice for the calculation of each component of the arrays to be updated. Thus, the algorithm requires $O(n^2R)$ total workspace for all computations and thus has a total workspace requirement which is polynomial in n, $\log_2 \bar{p}$ and $\log_2(\Delta + 1)$.

Remark 2.2. : The algorithm as stated implies polynomial solvability of bounded integer programs only if the number m of constraints of (P2) is a polynomial function of n and/or $\log_2(\Delta + 1)$. Thus, for instance, maximum flow calculations using the above algorithm require a polynomial total computational effort. This follows from the fact that Step 1 is stated with the implicit assumption that all constraints of (P2) are listed and checked off one by one. We note furthermore that in this case the above algorithm solves problem (P1) in polynomial time and space because :

(2.6) $$\bar{c} \leq 2(\Delta + 1)^n \bar{p}$$

holds. Thus, using :

(2.7) $$h = 4\bar{p}\, n^2(\Delta + 1)^n \Delta_A$$

we have that the running time as well as the required precision of the algorithm to solve problem (P1) are polynomial functions of n, $\log_2\bar{p}$ and $\log_2(\Delta + 1)$. This implies that (P1) is solvable in polynomial time and space if the number m of constraints is polynomial in n, $\log_2\bar{p}$ and $\log_2(\Delta + 1)$.

The following theorem summarizes the results of this section :

Theorem 2.3. : If for a bounded integer programming problem the facet-identification problem can be solved in time and space which depend polynomially upon n and $\log_2(\Delta + 1)$, then the bounded integer programming problem can be solved in time and space which depend polynomially upon n, $\log_2\bar{p}$ and $\log_2(\Delta + 1)$.

The assumption of Theorem 2.3 assures that the work of Step 1 can be done in polynomial space and time where the polynomial is in the same of parameters as the time and space requirements for the other steps of the algorithm. Hence Theorem 2.3 follows.

Remark 2.4. : In view of the introductory remarks to this paper we know that a necessary condition for the constraint identification to be solvable in polynomial time is that the number of different classes of facet-defining inequalities be bounded by a polynomial in n and $\log_2(\Delta + 1)$. This follows because different algorithms are required to identify members of different classes of facet-defining inequalities. We have chosen to formulate Theorem 2.3 so as to reflect the (trivial) fact that a concatenation of polynomially many problems which are all solvable in polynomial time is solvable in polynomial time where polynomiality is with respect to the same set of parameters.

Remark 2.5. : Suppose the facet-defining inequalities are replaced by any set of valid inequalities with integer coefficients such that the expanded problem has either only integral extreme points or a unique optimum solution which is integral. Suppose further that the logarithm of the absolute value of a non-zero component of such valid inequalities is bounded by a polynomial $V(n,\Delta)$ in the number of variables n and the parameter $\log_2(\Delta + 1)$. If the corresponding constraint identification problem, (i.e. given any rational vector y, find a valid inequality that is violated by y or prove that no such inequality exists) can be solved in $O(P(n,\Delta))$ steps, then the bounded integer problem can be solved in $O(n^3 L(n,\Delta,\bar{p})\, P(n,\Delta))$ steps where :

$$L(n,\Delta,\bar{p}) = \max \{\tfrac{1}{2} \log_2 n,\ \log_2 \Delta ,\tfrac{1}{n} \log_2 \bar{p}, V(n,\Delta)\}. \tag{2.8}$$

The difference of this remark to the formulation of Theorem 2.3 is that in order to prove the theorem, in lieu of a global description of the linear programming problem (P1) by linear inequalities, we only need a local description of the integer programming problem in the "neighborhood" of the optimizing point by way of linear inequalities to conclude polynomial solvability. Of course, whereas facet-defining inequalities for (P1) a priori (see Theorem 1.6) can be chosen to have bounded integer coefficients we have to assume integrality and boundedness if we work with the arbitrary valid systems of inequalities for (P1).

3. POLYNOMIAL SOLVABILITY OF THE FACET-IDENTIFICATION PROBLEM.

To show that the reverse of Theorem 2.3 holds as well we consider the following pair of dual linear programs :

$$\text{(P3)} \qquad \min \{ \sum_{i=1}^{t} \lambda_i \ / \ \lambda W = y-v \ , \ \lambda \geq 0\}$$

$$\text{(P4)} \qquad \max \{(y-v)\pi \ / \ W\pi \leq e_t\}$$

where y is a rational (row-) vector for which we want to decide whether or not y^T is contained in P_I. In order to obtain a linear program with integer data we clear the fractions in the objective function of (P4).

Denote $\underline{a}$ the vector thus obtained, i.e. $\underline{a} = D(y-v)$ where D is least common multiple of the denominators of the rationals y_j for j=1,...,n and note that (P4) is then equivalent to (P5) where :

$$\text{(P5)} \qquad \max \{\underline{a}\pi \ / \ W\pi \leq e_t\}.$$

For matters of notational simplicity we say that (P4) can be solved by a polynomial algorithm if there exists an algorithm for (P5) whose time and space requirements depend polynomially upon the parameters n, $\log_2 \bar{a}$ and $\log_2 (\Delta + 1)$ of (P5), where :

$$\text{(3.1)} \qquad \bar{a} = \max \{|a_j|, \ j=1,\dots,n\}.$$

Likewise, we use the phrase for (P5) and for (P1). In the latter case $\log_2 \bar{p}$ is used in lieu of $\log_2 \bar{a}$ because of the choice of the objective function in (P1).

Lemma 3.1. : If (P1) can be solved by a polynomial algorithm, then for rational y the problem (P4) can be solved by a polynomial algorithm.

Proof : In order to distinguish the case where (P4) has an unbounded optimum from the bounded case we run the Russian method ALG1 (u,h) on problem (P5) twice. The first time we initialize :

(3.2) $u = 2n^{3/2}\ \bar{a}\ \Xi^3$, $h = 2\bar{a}\ n^2\ \Xi^2$

where Ξ is defined in (1.7). It follows that the initial ellipsoid contains all points satisfying :

(3.3) $-2n\ \bar{a}\ \Xi^3 \leq \pi_j \leq 2n\ \bar{a}\ \Xi^3$ for j=1,...,n

and that the unbounded case arrises if and only if the objective function value ξ obtained at the end of the calculations satisfies $\xi \geq n\ \bar{a}\ \Xi$. Suppose now that we can carry out the necessary calculations and that we arrive at the conclusion that (P4) has a bounded optimum. We run ALG1(u,h) on (P5) with additional upper and lower bounds, i.e. over the constraint system :

(3.4) $W\pi \leq e_t$, $-\Xi \leq \pi_j \leq \Xi$ for j=1,...,n,

and the following parameters :

(3.5) $u = \sqrt{n}(\Xi + 1)$, $h = 2\bar{a}\ n^2\ \Xi^3$.

Assuming for simplicity that (P5) has a unique optimum over the feasible set of (3.4) -alternatively, we perturb the objective function of (P5) as we did in Section 2, see (2.2) with Δ replaced by Ξ ,- it follows from Theorem 3.1 of [15] that we get an optimal solution to (P5) which is basic to (3.4) by rounding the final trial solution $\bar{x}$ using the process of continued functions, see Section 4 of [15]. All that remains to be shown to prove the lemma is to show that the constraint-identification (Step 1) in ALG1(u,h) can be carried out in time and space which depend polynomially upon n, $\log_2 \bar{a}$ and $\log_2(\Delta + 1)$. Denote π^k the trial solution at iteration k of ALG1(u,h). Then π^k is a rational vector which can be represented using O(R) binary positions where R is given by :

(3.6) $R = 16n \lceil \log_2 h\ u^{1/2} \rceil$

where u, h respectively, are defined in (3.2) in the first run, in (3.5) in the second run of ALG1(u,h).

In order to check all constraints of (P5) we need to find a row w of W such that $w\pi^k > 1 + 1/h$ holds ; in the second run we also check the upper and lower bounds which can clearly be done in polynomial time and space in the desired parameters. Recalling the definition of W it follows that the constraint identification can be carried out by solving the integer program :

(3.7) $\max \{(\pi^k)^T x \ / \ x \in P_I\}$.

Setting $p = D(\pi^k)^T$ where D is the least common multiple of the denominators of the rational π_j^k for $j=1,\ldots,n$ we thus have to solve the integer program :

(3.8) $\max \{px \ / \ x \in P_I\}$

with an integer objective function satisfying $\log_2 \bar{p} \le (n+1)R$. By assumption (3.8) can be solved by a polynomial algorithm and thus the constraint identification problem can be solved in time and space which depend polynomially upon n, $\log_2 \bar{a}$ and $\log_2(\Delta + 1)$. Lemma 3.1 follows.

Remark 3.2 : If the points of P_I form an independence system, see Remark 1.7, then it follows from Lemma 3.1 that polynomial solvability of (P1) implies that the linear program over the anti-blocker $A(P_I)$, see (1.12), is solvable in polynomial time and space. Since $A(A(P_I)) = P_I$, see Fulkerson [6], it follows that the following statement is true : The problem (P1) can be solved by a polynomial algorithm for all integer vectors p if and only if the problem over the anti-blocker $A(P_I)$

(3.9) $\max \{qx \ / \ x \in A(P_I)\}$

can be solved by a polynomial algorithm for all integer vectors q, see also [9]. A similar statement is true for pairs of blocking polyhedra.

The solution of (P5) provides a valid inequality for P_I and permits one to conclude whether or not the rational vector y of problem (P3) belongs to P_I. By Lemma 3.1 this can be done in polynomial space and time if (P1) can be solved by a polynomial algorithm. We next address the problem of finding a facet-defining inequality of P_I which chops off y or proving that no such inequality exists. To this end we distinguish two cases : If (P5) has a finite objective function value ξ we consider the problem :

(P6) $\left\{\begin{array}{l} \text{Find a basic solution } \pi \text{ to the system} \\ W\pi \le e_t \text{ such that} \\ \underline{a}\pi = \xi \\ w_{i(j)}\pi = 1 \quad \text{for } j=1,\dots,r \\ \text{where } w_{i(j)},\ j=1,\dots,r \text{ are } r \text{ linearly independent rows of } W. \end{array}\right.$

If (P5) has an unbounded solution we consider instead the problem :

(P7) $\left\{\begin{array}{l} \text{Find a basic solution } \pi \text{ to the system} \\ W\pi \le 0\ ,\ \pi_j = \delta \text{ for some } j,\ 1 \le j \le n, \text{ such that} \\ \underline{a}\pi > 0 \\ w_{i(j)}\pi = 0 \quad \text{for } j=1,\dots,r \\ \text{where } w_{i(j)},\ j=1,\dots,r\ , \text{ are } r \text{ affinely independent rows of} \\ W \text{ and } \delta = +1 \text{ or } -1. \end{array}\right.$

Remark 3.3. : Note that we make no assumption about the rank r of the matrix W. If the system $W\pi \le e_t$ defines a bounded polytope, then the rank of W satisfies $r=n$. It is easily verified that in this case the unique optimum solution obtained by perturbing the objective function as stated in the proof of Lemma 3.1 solves (P6), see also [9].

In the general case we do not explicitly know the rank of W and thus the proof is more complicated.

Lemma 3.4. : *If (P1) can be solved by a polynomial algorithm and if (P5) is bounded, then (P6) can be solved in time and space which depend polynomially upon n,* $\log_2 \bar{a}$ *and* $\log_2 (\Delta + 1)$.

Proof : Since (P5) is bounded, we can find in time and space which depend polynomially upon n, $\log_2 \bar{a}$ and $\log_2(\Delta + 1)$ a rational solution π^* to the system

(3.10) $W\pi \le e_t + 1/h \quad , \quad -\Xi - 1/h \le \pi_j \le \Xi + 1/h$ for $j=1,\dots,n$

using the algorithm ALG1(u,h) with u and h as defined by :

(3.11) $u = \sqrt{n}(\Xi + 1) \quad , \quad h = 2\bar{a}\, n^2\, \Xi^3$

We can assume without loss of generality, i.e. possibly after perturbing the objective function of (P5), that the optimum is unique. It follows that π^* can be rounded to a basic solution $\bar{\pi}$ of (3.4) and that every component of $\bar{\pi}$ is a ratio of two integers which are both less than or equal to Ξ in absolute value. The proof of the lemma consists in three steps : Using $\bar{\pi}$ we first construct a submatrix W_1 of maximal rank of W such that every row of W_1 is satisfied by $\bar{\pi}$ at equality. It follows that the vector $\underline{a}$ of (P5) can be written as a linear combination of the rows of W_1. The second step of the proof consists in finding a basic solution to $W\pi \le e_t$ which is optimal and satisfies the rows of W_1 at equality. Finally, in the third step the rank of W is determined and the solution $\bar{\pi}$ is modified further so as to solve (P6).

Since the optimum is bounded it follows that $\bar{\pi}$ satisfies at least one of the inequalities $W\pi \le e_t$ as equality. We solve (3.7) with $\pi^k = \bar{\pi}$ and the corresponding row w of W for which the maximum is attained satisfies $w\bar{\pi} = 1$. Denote

$$N_1^* = \{j \in \{1,\dots,n\} \ / \ \bar{\pi}_j \ne 0 \ , \ -\Xi < \pi_j < +\Xi\}$$

$$N_2 = \{j \in \{1,\dots,n\} \ / \ \bar{\pi}_j = +\Xi \ \text{ or } \ \bar{\pi}_j = -\Xi\}$$

and let g denote the cardinality of N_1^*. Since (P5) has a bounded optimum it follows that $g \ge 1$; from $\Xi \ge 2$ it follows $w_j \ne 0$ for at least one $j \in N_1^*$. Let W_1 be a subset of q linearly independent rows of W such that :

(i) each row of W_1 is satisfied by $\bar{\pi}$ at equality,

(ii) W_1 contains a $q \times q$ submatrix W_{11} with column set N_1 which is non-singular,

(iii) $N_1 \subseteq N_1^*$ holds.

If $q < g$ holds, it follows that there exists a row of W which is linearly independent of the rows W_1 and which is satisfied by $\bar{\pi}$ at equality since $\bar{\pi}$ is a basic solution to (3.4). To find such a row let $k \in N_1^*$ denote the index of a column of W_1 and let α^k be the solution vector to

$$W_1\alpha = 0$$

$$(3.12) \qquad \alpha_k = -1 \quad \text{for some } k \notin N_1$$

$$\alpha_j = 0 \quad \text{for all } j \notin N_1,\ j \neq k.$$

It follows that problem (3.7) with :

$$(3.13) \qquad \pi^k = \bar{\pi} + \varepsilon\,\alpha^k$$

for $\varepsilon = 1/h^2$ or $\varepsilon = -1/h^2$ has an optimal solution such that the corresponding row w of W for which the maximum is attained satisfies $w(\bar{\pi} + \varepsilon\,\alpha^k) > 1$. It follows that row w is linearly independent of the rows of W_1 and hence by adjoining w to W_1 we have a $(q + 1) \times n$ matrix satisfying (i), (ii) and (iii). Part (i) follows from the choice of ε , the integrality of the matrix W and the fact that $\bar{\pi}$ is a rational vector all of whose components have a denominator less than or equal to Ξ in absolute value. Hence, we can assume that we have a $g \times n$ submatrix W satisfying (i), (ii) and (iii). Consider next any column with index $k \in N_2$ and let α^k be the solution vector to (3.12). As before we solve (3.7) with π^k as defined in (3.13) and $\varepsilon = 1/h^2$, $\varepsilon = -1/h^2$ respectively. If the maximum of the two objective function values is greater than 1, we proceed as before and enlarge the submatrix W_1. Furthermore, we redefine N_1 to be $N_1 \cup \{k\}$ and N_2 to be $N_2 - \{k\}$. The matrix W_1 satisfies (i) and (ii). If the maximum of the two objective function values is less than or equal to 1, then it follows that $w\alpha^k = 0$ holds for every row of W for which $w\bar{\pi} = 1$ holds. We check column k as having been examined and repeat the process with the remaining (unchecked) columns of N_2. After having examined all columns in N_2 we proceed likewise with the columns with index not in $N_1^* \cup N_2$. It follows that after at most n steps, the solution of at most n systems of equations (3.12) and at most 2n solutions of (3.7) with rational objective functions we obtain a submatrix W_1 of W which is of maximal rank among all sub-matrices of W satisfying (i) and for which (ii) holds. Since (P5) has a bounded optimum solution it follows that :

(3.14) $\underline{a} = \lambda W_1$

holds for some rational vector λ.

In the second step we modify the solution $\bar{\pi}$ which is a basic solution to (3.4) so as to get a basic solution to :

(3.15) $W\pi \leq e_t$

If N_2 is empty, then $\bar{\pi}$ is a basic solution to (3.15). Else, let $k \in N_2$ and denote α^k the solution to (3.12). We solve (3.7) with $\pi^k = \alpha^k$ and denote w the corresponding row of W for which the maximum is attained. Since W contains a zero row, we have $w\alpha^k \geq 0$ and distinguish two cases.

Case 1 : $w\alpha^k > 0$. Define $\bar{\varepsilon}$ to be

(3.16) $\bar{\varepsilon} = (1 - w\bar{\pi})/w\alpha^k$

and let $\pi(\varepsilon) = \bar{\pi} + \varepsilon\alpha^k$ for $0 \leq \varepsilon \leq \bar{\varepsilon}$. It follows that row w is linearly independent of the rows of W_1 and satisfies $w\pi(\bar{\varepsilon}) = 1$. We solve (3.7) with $\pi^k = \pi(\bar{\varepsilon})$ to check if $\pi(\bar{\varepsilon})$ is feasible to (3.15). If $\pi(\bar{\varepsilon})$ is feasible to (3.15) then by construction it is a basic solution to (3.15) with the additional constraints $-\Xi \leq \pi_j \leq +\Xi$ for all $j \in N_2$, $j \neq k$, and we adjoin row w to W_1, replace $\bar{\pi}$ by $\pi(\bar{\varepsilon})$, redefine N_1 to be $N_1 \cup \{k\}$ and N_2 to $N_2 - \{k\}$. It follows that (i) and (ii) are satisfied and we repeat the process until N_2 is empty or the rank of W_1 equals n or until we have the case $w\alpha^k = 0$ below. If $\pi(\bar{\varepsilon})$ is not feasible to (3.15) then by interval halving and solving (3.7) each time with $\pi^k = \pi(\varepsilon)$ we find in at most $2 \log_2 h + \log_2 \Xi_2$ steps two values ε_1 and ε_2 such that $0 \leq \varepsilon_1 < \varepsilon_2 \leq \bar{\varepsilon}$ and $\varepsilon_2 - \varepsilon_1 \leq 1/h^2$ hold. Furthermore, $\pi(\varepsilon_1)$ is feasible to (3.15) and for some row w^* of W

(3.17) $w^*\pi(\varepsilon_2) > 1$

holds for $\pi(\varepsilon_2)$. Consequently, $w^*\alpha^k > 0$ and thus row w^* is linearly independent of the rows of W_1. Define ε^* as in (3.16) with w replaced by w^* and note that $\varepsilon_1 \leq \varepsilon^* < \varepsilon_2$. It follows that $\pi(\varepsilon^*)$ is feasible to (3.15) and hence by construction a basic solution to (3.15) with the additional constraints $-\Xi \leq \pi_j \leq +\Xi$ for all $j \in N_2$, $j \neq k$. For suppose that $\pi(\varepsilon^*)$ is not feasible ;

then we have for some row $\tilde{w}$ of W that :

$$\tilde{w}\bar{\pi} + \varepsilon^* \tilde{w}\alpha^k \geq 1 + 1/\Xi \qquad (3.18)$$

holds. Since $\varepsilon^* - \varepsilon_1 \leq 1/h^2$ and $\tilde{w}\alpha^k \leq h$ it follows that :

$$\tilde{w}\,\pi(\varepsilon_1) = \tilde{w}\pi(\varepsilon^*) - (\varepsilon^* - \varepsilon_1)\tilde{w}\alpha^k \geq 1 + 1/\Xi - 1/h > 1 , \qquad (3.19)$$

contradicting the feasibility of $\pi(\varepsilon_1)$. Thus, we can replace $\bar{\pi}$ by $\pi(\varepsilon^*)$, adjoin w^* to W_1, redefine N_1 to be $N_1 \cup \{k\}$ and $N_2 = N_2 - \{k\}$. It follows that the new matrix satisfies (i) and (ii) and we repeat the process until N_2 is empty or the rank of W_1 equals n or we have the case $w\alpha^k = 0$ below. If the rank of W_1 equals n then $\bar{\pi}$ solves (P6) and the lemma follows. Otherwise, we have Case 2 below or we continue in Step 3 of the proof.

Case 2 : $w\alpha^k = 0$. In this case we set $\bar{\varepsilon} = \bar{\pi}_k$. It follows that $\pi(\bar{\varepsilon})$ is a basic solution to (3.15) with the additional constraints $-\Xi \leq \pi_j \leq +\Xi$ for all $j \in N_2$, $j \neq k$. Hence we replace $\bar{\pi}$ by $\pi(\bar{\varepsilon})$, redefine N_2 to be $N_2 - \{k\}$ and repeat the process until N_2 is empty.

Note that all computations in Case 1 and Case 2 can be done in time and space which depend polynomially upon the parameters n, $\log_2 \bar{a}$ and $\log_2(\Delta + 1)$. This follows because the interval halving in Case 1 stops in at most $2 \log_2 h + \log_2 \Xi$ steps and because the logarithm (base two) of the components of the objective functions of the problems of the form (P1) to be solved stay bounded polynomially in the desired parameters and because, by assumption, (P1) can be solved by a polynomial algorithm. Hence we find after at most n steps a basic solution $\bar{\pi}$ to (3.15) since N_2 is empty and we are now ready for the third and last step of the proof of Lemma 3.4. This step -to find both the rank r of the entire matrix W and r linearly independent rows of W satisfied by a (possibly modified) solution $\bar{\pi}$ to (3.15) at equality- is carried out essentially as before. For $k \notin N_1$ let α^k denote the solution to (3.12) and solve (3.7) with $\pi^k = \alpha^k$. As before we are either in Case 1 and proceed as done here. Or we are in Case 2. In this case we solve (3.7) with $\pi^k = -\alpha^k$ and denote w^* the corresponding row of W for which the maximum is attained. As before, we have

$w^*(-\alpha^k) \geq 0$. If $w^*(-\alpha^k) > 0$ holds we apply the same reasoning as in Case 1 except that $-\alpha^k$ replaces α^k and w^* replaces w. Thus, we are left with the case $w^*(-\alpha^k) = 0$. Note that by construction it follows that :

$$(3.20) \qquad \sum_{i \in N_1} w^i \alpha_i^k - w^k = 0$$

holds where w^i denotes the i-th column of the entire matrix W, i.e. column w^k is linearly dependent upon the columns of W with index in N_1. If $|N_1| = n-1$ holds, we are done. Else, we check off column k and process the next column with index not in N_1, i.e. the process is continued. It follows that after at most n-1 iterations we find that $|N_1| = r$ holds where r is the rank of W and the corresponding $\bar{\pi}$ solves (P6). This completes the proof of Lemma 3.4.

The next lemma corresponds to the case in which (P5) has an unbounded optimum.

Lemma 3.5. : If (P1) can be solved by a polynomial algorithm and if (P5) has an unbounded optimum, then (P7) can be solved in time and space which depend polynomially upon n, $\log_2 \bar{a}$ and $\log_2(\Delta + 1)$.

The proof of Lemma 3.5 is a minor modification of the proof of Lemma 3.4 and the details are omitted here.

Theorem 3.6. : The bounded integer programming problem (P1) can be solved by a polynomial algorithm for any integer vector p if and only if the facet-identification problem can be solved for any rational vector y in time and space which depend polynomially upon n, $\log_2 \bar{a}$ and $\log_2(\Delta + 1)$.

Proof : By Theorem 2.3 we have the sufficiency of the condition. To show the necessity let y be the given rational vector for which the facet-identification problem is to be solved. Let $\underline{a}$ be the integer vector obtained after clearing the fractions in y-v where v is defined as in Section 1 to be any integer point of P_I which we can choose to be zero if the origin is contained in P_I. We solve (P5) by the Russian method ALG1(u,h) as in Lemma 3.3. If (P5) has an unbounded optimum, by Lemma 3.5, we find a basic solution π^* to (P7) in time and space which depend polynomially upon the desired parameters. We then have :

(3.21) $y\pi^* > v\pi^*$ and $B\pi^* \leq (v\pi^*)e_t$

Furthermore, we have among the r affinely independent rows of W which are satisfied by π^* at equality at least r-1 linearly independent ones and thus we have at least r affinely independent rows of B which are satisfied at equality. Thus clearing the fractions we obtain a facet-defining inequality for P_I with integer coefficients which chops off y.

On the other hand if (P5) has a finite optimum objective function value ξ we find a basic solution π^* to (P6). If π^* satisfies $(y-v)\pi^* > 1$, we have

(3.22) $y\pi^* > 1 + v\pi^*$ and $B\pi^* \leq (1 + v\pi^*)e_t$.

Furthermore, we have r linearly independent rows of W which are satisfied as equalities by π^*. Consequently, at least r affinely rows b of B satisfy $b\pi^* = 1 + v\pi^*$. Clearing fractions in π^* we thus obtain a facet-defining inequality for P_I with integer coefficients which chops off y. If π^* satisfies $(y-v)\pi^* \leq 1$, then by equivalence of (P5) and (P4) it follows from (P3) that y-v is a convex combination of the rows of W since W contains a zero row. Hence, y is a convex combination of the rows of B, i.e. $y^T \in P_I$ holds. By Lemmata 3.3, 3.4 and 3.5 the necessary computation can be done in polynomial time and space and Theorem 3.6 follows.

4. A POLYNOMIAL ALGORITHM FOR B-MATCHING.

The b-matching problem with upper bounds is the following combinatorial optimization problem :

$$\text{(P8)} \qquad \max \{px \ / \ Mx \leq d \ ; \ 0 \leq x \leq \underline{u}, \ x \text{ integer}\}$$

where M is the mxn incidence matrix of an undirected graph having m nodes and n edges ; d is a vector of m positive integers, p and $\underline{u}$ are vectors with n positive integers.

This problem has been treated in the more general context of bidirected networks by Edmonds and Johnson [4] ; see also [2,3] for the matching problem without upper bounds and [5] for a computer code implementation of a polynomially bounded procedure.

Let G = (V,E) be the undirected graph whose incidence matrix is given by M. For $W \subseteq V$, let (W : V-W) denote a cut set of G. Let $T \subseteq (W : V-W)$ be any subset (possibly empty) of edges of G which are in the cut-set defined by W (if W = V, T is empty). Define :

$$d(W) = \sum_{i \in W} d_i, \qquad u(T) = \sum_{e \in T} u_e \ ,$$

$$x(W) = \sum_{e \in E(W)} x_e \ , x(T) = \sum_{e \in T} x_e \ ,$$

where $E(W) \subseteq E$ is the set of edges of G having both ends in W. Denote

$$I = \{(W,T) \mid W \subseteq V \ ; \ \ T \subseteq (W : V-W) \ ; \ d(W) + u(T) \text{ is odd}\}$$

$$v(W,T) = [d(W) + u(T) - 1] \ / \ 2 \quad \text{for } (W,T) \in I.$$

Edmonds has shown that (P8) together with the valid inequalities

$$\text{(4.1)} \qquad x(W) + x(T) \leq v(W,T) \quad \text{for } (W,T) \in I$$

defines a polyhedron whose extreme points are all integer. Edmond's proof is algorithmic and his algorithm is a prime example of a technically good algorithm for a genuine combinatorial optimization problem. In [14] we have given a different algorithm for this problem which is based on linear programming and which uses a modified Gomory-Hu algorithm [14] for the identification of the evidently exponentially many constraints of the form (4.1). We will show here how the results of [14] together with the results of this paper can be combined to give a polynomial bounded algorithm for matching problems in order to demonstrate on a specific example that the results of this paper can indeed to be used to prove the existence of polynomial algorithms.

We define :

$$\Delta = 1 + \max \{u_j, j=1,\dots,n\},$$

$$c_j = \Delta^n p_j + \Delta^{n-j} \qquad \text{for } j=1,\dots,n ,$$

$$\bar{c} = 1 + \max \{|c_j|, j=1,2,\dots,n\} ,$$

and consider the linear program :

(P9) $\max \{cx/Mx \le d \; ; \; 0 \le x \le u \; ; \; x(W) + x(T) \le v(W,T) \quad \text{for } (W,T) \in I\}$

and for any integer $h \ge 1$ denote by (P9h) the problem :

$$\max cx$$

(4.2) $Mx \le d + \underline{h}^{-1}$

(4.3) $x \le u + \underline{h}^{-1}$

(4.4) $-x \le \underline{h}^{-1}$

(4.5) $x(W) + x(T) \le v(W,T) + 1/h$ for $(W,T) \in I$

where the vector $\underline{h}^{-1}$ has all components equal to $1/h$.

Let A denote the constraint matrix of (P9) and $\Delta_A \geq 2$ be an integer upper bound on the absolute value of the largest determinant of a non singular submatrix of A. Note that since A has n columns with all the elements 0 or 1, we have that $\Delta_A \leq n^{n/2}$.

Denote $W(m,n,\Delta,\bar{p})$ a function which is polynomial in $\log_2 m$, $\log_2 n$, $\log_2 \Delta$ and $\log_2 \bar{p}$ where m is the number of nodes and n the number of edges of the graph G. Let P(m,n) denote number of steps required for the modified Gomory-Hu procedure, see [14]. P(m,n) is a polynomial function of m and n.

Theorem 4.1. : The Russian method using the modified Gomory-Hu procedure to identify a violated constraint solves the b-matching problem in $O(n^3 W(m,n,\Delta,\bar{p})P(m,n))$ steps.

To prove Theorem 4.1 we run the Russian method ALG1(u,h) with the following parameters :

$$(4.6) \qquad u = \sqrt{n}\,\Delta \qquad h = 2(n+m)n^2\bar{c}\Delta_A$$

where n is the number of edges and m the number of nodes of the graph G and where we use Step 4^* as stated in Section 2. We have to show that we can find a violated constraint for the current iterate x^k of ALG1(u,h). This evident for the constraints (4.2), (4.3) and (4.4) which are simply listed and checked off one by one. To prove that we can find a violated constraint (4.5) if there exists any without enumerating all of them we proceed as follows :

Adding the slack vectors s^k and t^k in (4.2) and (4.3) respectively, we have :

$$(4.7) \qquad \begin{aligned} M x^k + s^k &= d + \underline{h}^{-1} \\ x^k + t^k &= u + \underline{h}^{-1} \end{aligned}$$

Now, for any $(W,T) \in I$, adding the rows of the first set of constraints of (4.7) with index in W and the rows of the second set with index in T, we have :

$$(4.8) \qquad 2x^k(W) + x^k(W:V-W) + x^k(T) + s^k(W) + t^k(T) =$$

$$= d(W) + u(T) + (|W| + |T|)/h$$

where $|W|$ and $|T|$ denote the cardinality of the sets W and T respectively. Now x^k violates a constraint (4.5) iff

(4.9) $$x^k(W:V-W) + u(T) + |T|/h - 2x^k(T) + s^k(W) < 1 + (|W| + |T| - 2)/h$$

for some $(W,T) \in I$.

Note that in (4.9) all edges in (W:V-W)-T appear with "correct" edge values while the edge values of all edges in T appear in "complemented" form with reference to (4.7). We construct the auxiliary graph $G(\bar{x},u)$ of [14] with reference to $\bar{x} = x^k$ labelling the nodes with reference to u while carrying out the complementing of the variables with respect to $u + \underline{h}^{-1}$. Then any negative edge values are set equal to 0. Note that the negative values can not be less than -1/h. We apply the modified Gomory-Hu procedure and suppose that we find an odd minimum cut-set, with capacity less than 1. It follows that we have identified the corresponding $(W,T) \in I$ such that (4.9) is satisfied or equivalently (4.5) is violated for the current solution x^k. This follows from the fact $|W| + |T| \geq 2$ and the edge values which have been set to zero from negative values only reduce the left hand side of the inequality (4.9).

Now suppose the odd minimum cut set has capacity greater than or equal to 1. This implies that :

(4.10) $$x^k(W:V-W) + u(T) + |T|/h - 2x^k(T) + s^k(W) \geq 1 - n/h \quad \text{for } (W,T) \in I$$

since the maximum error in setting some edge values to zero from negative values is n/h. Now combining (4.8) and (4.10) and noting that $n + |T| + |W| < (n+m)$ we have :

(4.11) $$x^k(W) + x^k(T) \leq v(W,T) + (n+m)/h \quad \text{for } (W,T) \in I$$

Furthermore, since x^k satisfies (4.2) to (4.4), we have from (4.11) that x^k is feasible to (P9h) for

$$h = 2n^2 \bar{c} \Delta_A$$

which by the discussion in Section 2 suffices to guarantee that Step 4^* can be carried out. Thus we continue in this case by using the objective function as a constraint. Since ALG1(u,h) stops after

$$T = 8n^2 \lceil \log_2 h\sqrt{\Delta + 1} \rceil$$

steps where h is given by (4.6), Theorem 4.1 follows.

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