



# Un résultat de type "Bang-bang" pour un contrôle ponctuel

L.W. White

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**A BANG - BANG RESULT  
FOR POINT CONTROL**

Institut National  
de Recherche  
en Informatique  
et en Automatique

**Luther W. WHITE**

Domaine de Voluceau  
Rocquencourt  
BP 105  
78153 Le Chesnay Cedex  
France  
Tel 954 90 20

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UN RESULTAT DE TYPE "BANG-BANG" POUR UN CONTROLE PONCTUEL

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par

Luther W. WHITE

Université de l'Oklahoma  
Département de Mathématiques et Centre  
de Ressources Energétiques  
NORMAN OKLAHOMA 73019 (USA)

RESUME :

On considère le contrôle d'un problème parabolique ou pseudo-parabolique en un point  $a$  dans  $\Omega \subset \mathbb{R}^p$ ,  $p=1, 2$  ou  $3$ , soumis à des contraintes.

On obtient une propriété "bang-bang" pour un contrôle optimal qui dépend de l'irrationalité des composantes de  $a$ .

On montre également que ce résultat n'est pas vrai pour le cas parabolique avec  $p = 2$  ou  $3$ .

# A Bang-Bang Result for Point Control

by

L. W. White\*

Department of Mathematics, The University of Oklahoma, Norman, Oklahoma 73019

## Abstract

In this note we consider the control of a pseudo-parabolic or parabolic problem at a point  $a$  in  $\Omega \subset \mathbb{R}^P$ ,  $P = 1, 2, \text{ or } 3$  subject to constraints. We obtain a bang-bang property for an optimal control that depends on whether the components of  $a$  are irrational. We also show that such a result does not hold for the parabolic case  $p = 2$  or  $3$ .

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## A Bang-Bang Result for Point Controls

by

L. W. White

### 1. Introduction.

In this paper we consider a sample problem involving a pseudo-parabolic equation on an interval  $\Omega = (0,1)$  with control at a point  $a \in (0,1)$ .

The underlying equation is given by

$$(1) \quad (1 - \Delta)y_t - \Delta y = v(t)\delta(x - a) \quad \text{in } Q$$

$$(2) \quad y(x,0) = 0 \quad \text{in } \Omega$$

$$(3) \quad y(0,t) = y(1,t) = 0 \quad \text{in } (0,T)$$

where  $Q = \Omega \times (0,T)$ . In addition, we consider the minimization problem

$$(4) \quad \begin{aligned} &\text{minimize } J(v) = \|y(\cdot, T; v) - z(\cdot)\|_0^2 = \int_0^1 (y(x, T; v) - z(x))^2 dx \\ &\text{subject to } v \in U \end{aligned}$$

where the set  $U$  is defined by

$$(5) \quad U = \{v \in L^\infty(0,T) : |v(t)| \leq 1 \text{ a.e. } [0,T]\}.$$

We study here the necessary conditions for  $u$  to be a solution of (1)-(5) and determine certain bang-bang properties that are dependent on whether the number  $a$  is rational or irrational. In section 3 we give some examples

illustrating our results. Finally, in the last section we indicate extensions of these results to higher dimensions as well as which results remain valid for parabolic equations.

## 2. Necessary conditions for a solution.

Let  $\{v_n\}_{n=1}^{\infty}$  be a minimizing sequence for (4). Then there is a subsequence  $\{v_{n_k}\}_{k=1}^{\infty}$  that converges weak star in  $L^{\infty}(0,T)$ , and thus weakly in  $L^2(0,T)$ , to an element  $u$  in  $U$ .

Lemma 1. If  $w_n \rightarrow w$  weakly in  $L^2(0,T)$ , then  $y(\cdot, T; w_n) \rightarrow y(\cdot, T; w)$  weakly in  $L^2(\Omega)$ .

Proof. Introduce the adjoint problem

$$(6) \quad -(1 - \Delta)p_t - \Delta p = 0 \quad \text{in } Q$$

$$(7) \quad p(x, T) = (1 - \Delta)^{-1}\theta(x) \quad \text{in } \Omega$$

$$(8) \quad p(0, t) = p(1, t) = 0 \quad \text{in } (0, T)$$

with  $\theta \in L^2(\Omega)$ . If we multiply (1) by  $p$  and integrate by parts, we obtain

$$(9) \quad (y(\cdot, T; w_n), \theta)_{L^2(\Omega)} = \int_0^T w_n(t) p(a, t) dt .$$

Since  $p \in H^1(0, T; H^2(\Omega))$ , it follows that  $p(a, t) \in L^2(0, T)$ , [2]. Hence, we see that

$$\int_0^T w_n(t) p(a, t) dt \rightarrow \int_0^T w(t) p(a, t) dt ,$$

and the result follows.

From Lemma 1 and the weak lower semicontinuity of the  $L^2(\Omega)$  norm, we have the following.

Proposition 2. There exists a solution  $u$  of (1)-(5).

By taking the first variation of the functional in (4), we may obtain the necessary condition

$$(10) \quad (y(T;u) - z, y(T;v) - y(T;u))_{L^2(\Omega)} \geq 0$$

for all  $v$  in  $U$ . Now, setting  $\theta = y(T;u) - z$  in equation (7), the condition in (10) implies the following.

Proposition 3. A solution of problem (1)-(5) satisfies the inequality

$$(11) \quad \int_0^T p(a,t)(v(t) - u(t))dt \geq 0$$

for all  $v \in U$ .

We now study the consequences of the condition in (11). Suppose  $|u(t)| < 1$  for  $t$  in a set  $E \subset [0,1]$  with  $\text{meas } E > 0$ . If  $\theta = y(T;u) - z$  in equation (7), the solution  $p$  of (6)-(8) has the property that  $p(a,t)$  is a continuous function of  $t$  in  $(-\infty, \infty)$ . Hence, there is a number  $M > 0$  such that  $|p(a,t)| \leq M$  on  $[0,T]$ . Now set

$$E_n = \{t \in E : 1 - |u(t)| \geq \frac{1}{n}\}.$$

By assumption there exists  $k$  such that  $\text{meas } E_k \neq 0$ . Define the function on  $[0,T]$

$$v(t) = \begin{cases} u(t) \pm \delta_k p(a,t) & \text{on } E_k \\ u(t) & \text{on } [0,T] - E_k \end{cases}$$

with the number  $\delta_k \in (0, \frac{1}{kM})$  such that

$$\begin{aligned} |v(t)| &= |u(t) \pm \delta_k p(a,t)| \\ &\leq |u(t) + \delta_k M| \\ &\leq 1 - \frac{1}{k} + \delta_k M. \end{aligned}$$

$$|v(t)| \leq 1 \text{ on } E_k.$$

Thus, we see that  $v \in U$ . From the inequality (11), it follows that

$$(12) \quad \int_{E_k} p^2(a,t) dt = 0,$$

and  $p(a,t) = 0$  a.e. in  $E_k$ . Since  $E$  is the countable union of countably many such sets, we conclude

$$(13) \quad p(a,t) = 0 \text{ a.e. in } E.$$

Hence, we have shown the following

Theorem 4. If  $u$  is a solution of (1)-(5) and if  $p$  is the solution of (6)-(8) with  $\theta = y(T;u) - z$ , then

$$(14) \quad (|u(t)| - 1)p(a,t) = 0$$

holds a.e. in  $[0,T]$ .

Now we consider the implications of equation (14). Note that if  $z \in L^2(0,T)$  and if we set  $y(T;u) - z = \sum_{n=1}^{\infty} h_n \sin n\pi x$ , then the solution of (6)-(8) with  $\theta = y(T;u) - z$  is given by



$$(15) \quad p(x,t) = \sum_{n=1}^{\infty} \frac{\zeta_n}{1 + n^2 \Pi^2} e^{-(t-\tau)\mu_n} \sin(n\Pi x)$$

where  $\mu_n = \frac{n^2 \Pi^2}{1 + n^2 \Pi^2}$ . For ease set  $\tau = T - t$  and set

$$(16) \quad q(x,\tau) = \sum_{n=1}^{\infty} \frac{\zeta_n}{1 + n^2 \Pi^2} e^{-\tau\mu_n} \sin(n\Pi x).$$

Clearly, the series in (16) is uniformly convergent on  $[0,1] \times [0,\infty)$ .

Thus, we have

$$(17) \quad q(a,\tau) = \sum_{n=1}^{\infty} \frac{\zeta_n}{1 + n^2 \Pi^2} e^{-\tau\mu_n} \sin(n\Pi a)$$

in  $[0,\infty)$ , and taking the Laplace transform, we obtain

$$(18) \quad \hat{q}(a,s) = \sum_{n=1}^{\infty} \frac{\zeta_n}{1 + n^2 \Pi^2} \frac{1}{s + \mu_n} \sin(n\Pi a).$$

Define the meromorphic function

$$(19) \quad f(z) = \sum_{n=1}^{\infty} \left( \frac{\zeta_n \sin(n\Pi a)}{1 + n^2 \Pi^2} \right) \frac{1}{z + \mu_n}.$$

Now  $q(a,\tau)$  is an analytic function of  $\tau$ . If  $|u(t)| < 1$  for  $t \in E$  where  $\text{meas } E > 0$ , then  $q(a,\tau) = 0$  for  $\tau \in \{T\} - E$ . But the set  $\{T\} - E$  has a cluster point since it has positive measure. Thus,  $q(a,\tau) \equiv 0$  in  $[0,+\infty)$  from analyticity. Accordingly, we see that  $\hat{q}(a,s) = 0$  for all  $s > 0$  so that  $f(z) = 0$  for all  $z$  with  $\text{Re } z > 0$ . Hence, we have  $f(z) \equiv 0$ . However, this implies that the residues of  $f$  are zero, and we see that

$$(20) \quad \zeta_n \sin(n\Pi a) = 0$$

for each  $n$ . It is from equation (20) that we may deduce properties of the sequence  $\{\zeta_n\}_{n=1}^{\infty}$ .

In the case that the number  $a$  is irrational, equation (20) implies

$\zeta_n = 0$  for each  $n$ . Hence, we conclude that  $p(x,t) \equiv 0$  so that  $y(T;u) = z$  in  $L^2(\Omega)$ .

Theorem 5. If  $a \in (0,1)$  is an irrational number, then either

$$(21) \quad y(\cdot, T; u) = z(\cdot) \text{ in } L^2(\Omega)$$

or

$$(22) \quad |u(t)| = 1 \text{ a.e. in } [0, T].$$

### 3. Examples.

We consider the problem with  $T = 1$ .

$$(23) \quad \left(1 - \frac{\partial^2}{\partial x^2}\right) y_t - \frac{\partial^2}{\partial x^2} y = v(t)\delta(x-a) \text{ in } (0,1) \times (0,1)$$

$$(24) \quad y(x,0) = 0 \text{ in } (0,1)$$

$$(25) \quad y(0,t) = y(1,t) = 0 \text{ in } (0,1).$$

First, we note that in terms of a Fourier series the Green's function of the operator  $M = 1 - \frac{\partial^2}{\partial x^2}$  on  $(0,1)$  with the Dirichlet boundary conditions is given by

$$g(x|\xi) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 \pi^2} \sin(n\pi\xi)\sin(n\pi x).$$

Hence, the solution of (23)-(25) may be given in terms of a Fourier series by

$$(26) \quad y(x,t;v) = \sum_{n=1}^{\infty} \left( \int_0^t e^{-\mu_n(t-s)} v(s) ds \right) \frac{\sin(n\pi a)}{1 + n^2 \pi^2} \sin(n\pi x)$$

where  $\mu_n = \frac{n^2 \pi^2}{1 + n^2 \pi^2}$ . Setting  $z = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$  we see that

$$(27) \quad J(v) = \frac{1}{2} \sum_{n=1}^{\infty} \left[ \left( \int_0^1 e^{-\mu_n(1-t)} v(t) dt \right) \frac{\sin(n\pi a)}{1 + n^2 \pi^2} - c_n \right]^2.$$

Example 1. Let  $a$  be irrational and  $z$  be defined by a Fourier series

$$z(x) = \sum_{n=1}^{\infty} \frac{\sin(n\pi a)}{n^2 \pi^2} \left(1 - e^{-\frac{\mu_n}{2}}\right) \sin(n\pi x).$$

Take  $u(t) \equiv 0$  on  $[0, \frac{1}{2})$  and  $u(t) \equiv 1$  on  $[\frac{1}{2}, 1]$ , then

$$\begin{aligned} \int_0^1 e^{-\mu_n(1-t)} u(t) dt &= \int_{\frac{1}{2}}^1 e^{-\mu_n(1-t)} dt \\ &= \frac{1}{\mu_n} \left(1 - e^{-\frac{\mu_n}{2}}\right), \end{aligned}$$

and we see that  $J(u) = 0$ . In this case we see have  $y(T; u) = z$  in  $L^2(0, 1)$  and  $|u(t)| = 0 < 1$  for  $t \in [0, \frac{1}{2})$ .

Example 2. Again let  $a$  be irrational and choose  $a \in (0, \frac{1}{4})$  so that  $\sin(2\pi a) > 0$ . Let  $z$  be given in terms of a Fourier series

$$z(x) = c_2 \sin 2\pi x + \sum_{n \neq 2}^{\infty} \frac{\sin(n\pi a)}{n^2 \pi^2} \left(1 - e^{-\mu_n}\right) \sin(n\pi x).$$

Note that for  $u(t) \equiv 1$  on  $[0, 1]$ ,

$$(28) \quad J(u) = \frac{1}{2} \left[ \int_0^1 e^{\frac{-4\pi^2}{1+4\pi^2}} u(t) dt \frac{\sin(2\pi a)}{1 + 4\pi^2} - c_2 \right]^2.$$

Let  $|v(t)| \leq 1$  a.e. in  $[0, 1]$ . Then we have

$$(29) \quad J(v) = \int_0^1 y^2(1; v - u) dx + 2 \int_0^1 y(1; v - u)(y(1; u) - z) dx + J(u).$$

From equation (26) the second term of (29) becomes

$$\begin{aligned}
 & \int_0^1 y(1;v-u)(y(1;u)-z)dx = \\
 (30) \quad & = \left[ \int_0^1 e^{-\mu_2(1-t)} (v(t)-1)dt \right] \left( \frac{\sin(2\pi a)}{1+4\pi^2} \right) \times \\
 & \quad \left[ \frac{\sin(2\pi a)}{4\pi^2} (1 - e^{-\mu_2}) - c_2 \right].
 \end{aligned}$$

Since  $|v(t)| \leq 1$  a.e., the integral is nonpositive. Hence, if  $c_2 \geq \frac{1}{4\pi^2}(1 - e^{-\mu_2})\sin(2\pi a)$  then  $\int_0^1 y(1;v-u)(y(1;u)-z)dx \geq 0$  for all  $v$  satisfying  $|v(t)| \leq 1$  a.e. in  $[0,1]$ . As a consequence of equation (29), we have that  $J(v) \geq J(u)$  for all such  $v$ . Similarly, if  $c_2 \leq -\frac{1}{4}(1 - e^{-\mu_2})\sin(2\pi a)$ , then  $u(t) \equiv -1$  is a solution.

Example 3. Let  $a = \frac{1}{2}$ . Set  $z(x) = \sum_{n=1}^{\infty} c_n \sin(n\pi x)$  where  $c_2 \neq 0$ ,  $c_{2n} = 0$  for  $n \geq 2$ , and if  $n$  is odd  $c_n = \frac{1}{2n^2\pi^2}(1 - e^{-\mu_n})\sin(\frac{n\pi}{2})$ . Thus, from equation (27) we see that  $J(\frac{1}{2}) = \frac{1}{2}c_2^2$ . Clearly,  $u(t) \equiv \frac{1}{2}$  for  $[0,1]$  is a solution. Here, however, we have  $y(x,1;\frac{1}{2}) - z(x) = c_2 \sin(2\pi x)$ . Thus, we note that

$$p(x,t) = \frac{c_2}{1+4\pi^2} e^{-\mu_2(1-t)} \sin(2\pi x)$$

and  $p(a,t) = 0$ . Hence, this example shows that if the number  $a$  is rational Theorem 5 does not hold although Theorem 4 applies.

#### 4. Higher dimensions and the Parabolic Case.

By inspecting the results in section 2, we see that Theorem 5 remains true for problem (1)-(3) with  $(0,1)$  replaced by  $\Omega$  a rectangle or parallelepiped with sides parallel to the coordinate axes. Hence,  $\Omega$  may

contained in  $R^p$  for  $p = 2$  or  $3$  and convergence of the series in (16)-(19) is still uniform. This leads to the following.

Theorem 6. If  $a \in \Omega \subset R^p$ ,  $p = 2$  or  $3$ , with  $\Omega$  a rectangle or parallelepiped with sides parallel to the coordinate axes and if the components of the  $p$ -tuple  $a$  are all irrational, then either (21) or (22) holds.

For the parabolic case equation (17) becomes

$$(31) \quad q(a, \tau) = \sum_{n=1}^{\infty} \zeta_n e^{-\tau \lambda_n^2} \sin(n\pi a).$$

By the results of Muntz-Szasz [1, 3] the functions  $\{e^{-\tau \lambda_n^2}\}_{n=1}^{\infty}$  are independent if and only if  $\sum \frac{1}{\lambda_n} < \infty$ . Hence,  $q(a, \tau) = 0$  for equation (31) implies that equation (20) holds for the one-dimensional case. Here, however, if  $\Omega \subset R^p$  for  $p = 2$  or  $3$  by the Muntz-Szasz theorem  $\{e^{-\tau \lambda_n^2}\}_{n=1}^{\infty}$  is no longer independent and (20) does not follow.

Theorem 7. For the problem

$$\begin{aligned} y_t - \Delta y &= v(t)\delta(x - a) \quad \text{in } Q \\ y(x, 0) &= 0 \quad \text{in } \Omega \\ y(x, t)|_{\partial\Omega} &= 0 \quad \text{in } (0, T), \end{aligned}$$

the results of Theorem 5 hold if  $\Omega \subset R$ . If  $\Omega \subset R^p$  for  $p = 2$  or  $3$ , Theorem 5 no longer remains valid.

Remark 8. Even though Theorem 5 does not go through, Theorem 4, hence equation (14), still holds.

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