A pertubation study of a jet-like annular free boundary problem and an application to an optimal control problem

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SUMMARY : The NASH-MOSER Generalized Implicit Function Theorem is applied to an annular Free Boundary Problem related to jet or wave phenomena : local existence, uniqueness, and data dependance results are derived, together with an application to an Optimal Control Problem.

RESUME : Nous appliquons le théorème des Fonctions Implicite de J. NASH et J. MOSER à un problème à Frontière Libre voisin d'un problème d'écoulement avec jet posé dans un domaine annulaire du plan. Nous obtenons des résultats d'existence locale, unicité locale, et de régularité par rapport à certaines données ; nous en déduisons une application au Contrôle Optimal.

INTRODUCTION

In this paper we shall be interested by the following class of Free Boundary Problems :

(C) *Find a function* z *and a curve* γ *such that* z *is a harmonic function defined in the subregion* $\Omega_{u,\gamma}$ *of the two-dimensional space limited by a fixed (given) curve* u *together with* γ *;* z *is equal to a given function along* u *and* z *satisfies to :*

$$\left| \begin{array}{l} z = 0 \\ \dfrac{\partial z}{\partial n} = q|_\gamma \end{array} \right.$$

along γ *where* q *is a given function and* $\vec{n}$ *the usual normal vector.*

Such a class is a very important one for the modelization of physical processes, such as

- Flows with *jets* or *wakes* : see for example P.R. GARABEDIAN [1].
- Stationnary *waves* : see for example R.K.C CHAN [1], J.T. BEALE [1,2].
- *Porous flows* with Free Surfaces : seè C. BAIOCCHI et Al. [1].
- *Plasma equilibria* in tokamak machines : see A.S. DEMIDOV [1].

The purpose of our work is to study a *model case* of (C), for which we prove local *existence and uniqueness results together with regularity with respect to some data* : in particular, *first order variational formulas* are derived for both z (distributed solution) and γ (free boundary) ; such formulas can be straightforward *formally* extended to the general class (C) ; this extension is interesting in numerical applications : we give such an application to a simplified Optimal Control Problem.

The main tool of our theoretical study is the NASH-MOSER Generalized Implicit Function Theorem (G.I.F.T.) stated as in E. ZEHNDER [1], and used as we explain in A. DERVIEUX [2]. This kind of G.I.F.T. have been used for a different Free Boundary Problem by D.G. SCHAEFFER [1], and a different kind by J.T. BEALE [1,2] ; J.T. BEALE's problems are not very far from our model problem ; however we propose here a very direct study, in the physical domain of the definition (C).

On the other hand, the differentiability results and variational formulas which we give are, as far as we know, new.

The plan is the following :

.../...

.../...

.../...

1. THE MODEL PROBLEM

We want to consider Free Boundary Problems in annular domains of the plane, i.e. whose geometrical domains are bounded by two curves : a fixed (given) boundary and a free (unknown) one ; this leads to two cases, according as the Free Boundary is "in" or "out from" the fixed one.

1.1. The inner problem

Let u be a sufficiently smooth curve without self-intersection point which lies in the 2-dimensional space ; q is a real-valued non-negative function defined in the open set Ω_u, limited by the curve u (Ω_u *is bounded*) ; the Free Boundary Problem is

(1.1) *Find an other simple closed sufficiently smooth curve* γ, *lying in* Ω_u, *which, together with* u, *limits a doubly connected open set* $\Omega_{u,\gamma}$, *and for which there exists a function* z *such that (see Fig. 1)*

$$(1.1)_1 \quad \begin{cases} \Delta z = 0 & \text{in } \Omega_{u,\gamma} \\ z = 0 & \text{on } u \\ \left.\begin{array}{l} z = 1 \\ |\overrightarrow{\text{grad}}\, z| = q \end{array}\right\} & \text{on } \gamma \end{cases}$$

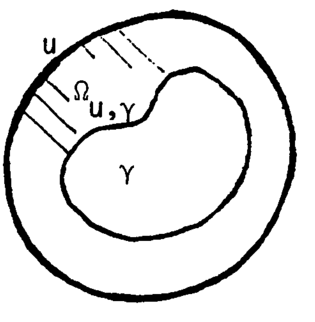


Fig.1

It is easily seen, via Maximum Principle, that $(1.1)_1$ is equivalent to :

$$(1.1)_2 \quad \begin{cases} \Delta z = 0 & \text{in } \Omega_{u,\gamma} \\ z = -1 & \text{on } u \\ \left.\begin{array}{l} z = 0 \\ \dfrac{\partial z}{\partial n} = q \end{array}\right\} & \text{on } \gamma \end{cases}$$

where $\vec{n}$ is the normal vector outward pointing from $\Omega_{u,\gamma}$.

Let us consider the following assumptions ($0<\alpha<1$) :

(1.2) The boundary u is a $C^{2+\alpha}$ simple curve.

(1.3) The function q belongs to $C^{1+\alpha}$ ($\overline{\Omega}_u$).

(1.4) For every $x = (x_1, x_2)$ of $\bar{\Omega}_u$, we have

$$q(x) \geq q_0$$

where q_0 is a positive constant.

(1.5) There exists a couple (γ, z) such that

$(1.5)_1$ γ is a simple curve of Ω_u with a continuous tangential vector and z belongs to $C^1(\bar{\Omega}_{u,\gamma})$ while its second derivatives are piecewise continuous functions on $\bar{\Omega}_{u,\gamma}$

$(1.5)_2$ z is equal to 0 on u and 1 on γ

$(1.5)_3$ $\displaystyle\int_{\Omega_{u,\gamma}} \left\{|\vec{\text{grad}}\, z|^2 + q^2\right\} dx \leq \int_{\Omega_u} q^2\, dx$.

Then we have the following existence result , proved by I.I. DANILJUK [1,3] :

Theorem 1 : *Under assumptions* (1.2) *to* (1.5), *there exists a couple* (γ, z), *satisfying* $(1.5)_1$, *which is a solution of Problem* (1.1). □

Additional regularity property for the Free Boundary γ is stated as follows (I.I. DANILJUK [Ibid]).

Theorem 2 : *Under assumptions* (1.2) *to* (1.5), *assuming too that*

(1.6) $\qquad q \in C^{\ell+\alpha}(\bar{\Omega}_u)$, $\ell \in \mathbb{N}$, $\ell \geq 2$,

every solution of Problem (1.1) *such that* $(1.5)_1$ *holds satisfies*

(1.7) γ *is a* $C^{\ell+1+\alpha}$ *curve.* □

Remark 1 : Theorem 2 can also be derived from a result of D. KINDERLEHRER, L. NIRENBERG [1]. □

1.2. The outer problem

The localisations of the two curves limiting the domain are exchanged (see Fig.2).

More precisely, u is a sufficiently smooth curve without self-intersection point which lies in the 2-dimensional space ; q is a real-valued non-negative continuous function defined in the doubly connected open set D_u, limited by the curve u and the point at infinity ; the Free Boundary Problem is

(1.8) *Find an other simple closed sufficiently smooth curve* γ, *lying in* D_u, *which, together with* u, *limits a doubly connected open set* $\Omega_{u,\gamma}$

and for which there exists a function z *such that*

$$(1.8)_1 \qquad \begin{cases} \Delta z = 0 & \text{in } \Omega_{u,\gamma} \\ z = 0 & \text{on } u \\ \left.\begin{array}{l} z = 1 \\ |\vec{\text{grad}}\ z| = q \end{array}\right\} & \text{on } \gamma \end{cases}$$

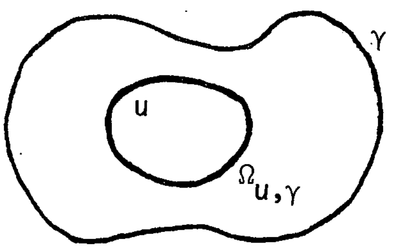


Fig.2

Analogously to $(1.1)_2$, $(1.8)_1$ reduces to

$$(1.8)_2 \qquad \begin{cases} \Delta z = 0 & \text{in } \Omega_{u,\gamma} \\ z = -1 & \text{on } u \\ \left.\begin{array}{l} z = 0 \\ \frac{\partial z}{\partial n} = q \end{array}\right\} & \text{on } \gamma \end{cases}$$

This second family of Problems have been studied by A. BEURLING [1] and the important case where q is a constant by D.E TEPPER [1,2] whose following results will be useful in the sequel :

<u>*Theorem 3*</u> *: Suppose that* q *is a positive constant and that the bounded open set limited by* u *is a starshape (resp. convex) set ; then Problem* (1.8) *has a unique solution ; furthermore, the Free Boundary is an analytic curve and limits a starshape (resp. convex) bounded open set.* □

2. THE DIFFERENTIABILITY PROBLEM

2.1. Boundary parametrization

An important feature in the sequel is the analysis of the *pertubation of boundaries* ; for recent developments in boundary or domain dependence techniques, we refer to J. CEA, A. GIOAN, J. MICHEL [1], D. CHENAIS [1], A.M. MICHELETTI [1], F. MURAT, J. SIMON [1,2], B. PALMERIO, A. DERVIEUX [1,2], O. PIRONNEAU [1], B. ROUSSELET [1,2], J.P. ZOLESIO [1].

In this work, the boundary perturbation analysis is dealt with by equipping a set of domains with a parametrization whichs maps into a Banach space : a *very simple* parametrization applies to a family of *starshape domains*, which we shall first consider ; an extension to a larger family of domains will be explained in the sequel.

Let $\mathbf{T}$ denote the one-dimensional torus :

$$\mathbf{T} = \mathbb{R}/2\pi\mathbf{Z} = \{[0,2\pi],\ 0 = 2\pi\}\ ;$$

let s_1, s_2, s_3 be three real positive numbers and $\mathcal{U}_{ad}$ the subset of $C^{s_1}(\mathbf{T})$ defined by

$$\mathcal{U}_{ad} = \{u \in C^{s_1}(\mathbf{T}),\ u_1 \leq u \leq u_2\}$$

where u_1 and u_2 are two given functions of $C^{s_1}(\mathbf{T})$ such that $0 < u_1 < u_2$;
let Γ_{ad} be the subset of $C^{s_2}(\mathbf{T})$ defined by

$$\Gamma_{ad} = \{\gamma \in C^{s_2}(\mathbf{T}),\ \gamma_1 \leq \gamma \leq \gamma_2\}$$

where γ_1 and γ_2 are two given functions of $C^{s_2}(\mathbf{T})$ such that $u_2 < \gamma_1 < \gamma_2$.

For every couple (u,γ) of $\mathcal{U}_{ad} \times \Gamma_{ad}$, we shall denote by $\Omega_{u,\gamma}$ the open domain defined by (see Fig.3)

(2.1) $$\Omega_{u,\gamma} = \{(r\cos\theta, r\sin\theta) \in \mathbb{R}^2,\ u(\theta) < r < \gamma(\theta)\}$$

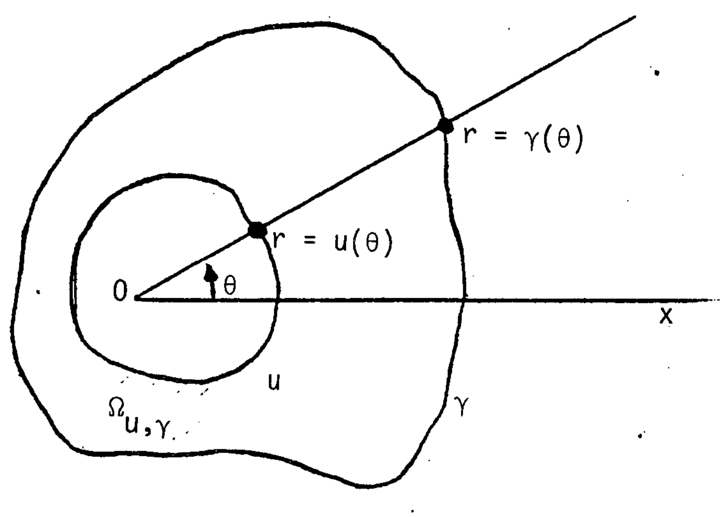


Fig.3

For simplicity of notations, the curves parametrized by u and γ will be also denoted by u and γ (so that $\partial\Omega_{u,\gamma} = u \cup \gamma$).

Let q be a sufficiently regular real valued function defined on $\mathbb{R}^2$; for an arbitrary u in $\mathcal{U}_{ad}$, we consider the following Free Boundary Problem :

(2.2) *Find a couple* (γ, z)

(2.2)$_1$ $$\begin{cases} \gamma \in \Gamma_{ad} \\ z \in C^{s_3}(\overline{\Omega}_{u,\gamma}) \end{cases}$$

such that

$$(2.2)_2 \quad \begin{cases} \Delta z = 0 & \text{in } \Omega_{u,\gamma} \\ z = -1 & \text{on } u \\ \left.\begin{array}{l} z = 0 \\ \dfrac{\partial z}{\partial n} = q \end{array}\right\} & \text{on } \gamma \end{cases}$$

where n is the normal vector on γ*, pointing from* $\Omega_{u,\dot{\gamma}}$.

2.2. Formulation of the mathematical problem

Problem : Given a triple (u_0, γ_0, z_0) *solution* (1) *of* (2.2)*, do there exists two positive numbers* σ_1 *and* σ_2*, a neighbourhood* D_{σ_1} *of* u_0 *in* $C^{\sigma_1}(\mathbf{T})$ *and a mapping*

$$(2.3) \quad \begin{cases} \Gamma : & D_{\sigma_1} \longrightarrow C^{\sigma_2}(\mathbf{T}) \\ & u \longmapsto \Gamma(u) \end{cases}$$

such that

$$(2.4) \quad \Gamma(u_0) = \gamma_0$$

$$(2.5) \quad \begin{cases} \forall u \in D_{\sigma_1}, \exists Z(u) \text{ such that} \\ (u, \Gamma(u), Z(u)) \text{ is a solution of } (2.2)? \end{cases}$$

Is this mapping continuous? Differentiable?

The main purpose of this work is to give *positive answers* to those three questions. For the understanding of the proof, it will be very useful to do *formally* the main computations.

2.3. Sketch of a formal computation

It has been observed in the two preceding papers [1,2] that when a Free Boundary Problem is decomposed to get rid of the distributed dependent variable (namely z in (2.2)), it is useful to look for a *convenient decomposition*. In particular a coupling distributed well-posed problem can be introduced, which satisfies a *stationarity property* with respect to the Free Boundary ; this property is useful to obtain a canonic linearization of the Free Boundary Problem.

For the present case, let us introduce the following *coupling problem :*

---

(1) u_0 is the data.

$$(2.6)\quad \begin{cases} \text{For } (u,\gamma)\in \mathcal{U}_{ad}\times\Gamma_{ad},\ z_F(u,\gamma)\ (^1) \text{ is defined on } \Omega_{u,\gamma} \text{ by} \\ \Delta z_F(u,\gamma) = 0 \quad \text{in } \Omega_{u,\gamma} \\ z_F(u,\gamma) = -1 \quad \text{on } u \\ \left(\frac{\partial}{\partial n} + b\right) z_F(u,\gamma) = q \quad \text{on } \gamma. \end{cases}$$

It is clear that if the Free Boundary is known, we need only to solve (2.6) to obtain the corresponding distributed solution (this justify the term "coupling"). Two particular cases of (2.6) : the Neumann Problem ($b = 0$) and the Dirichlet Problem ($b = +\infty$; the last line of (2.6) becomes : $z_F(u,\gamma) = 0$ on γ) seem to be introduced more naturally ; however, none of these two choices yields a stationary coupling. On the other hand, if we define the function b as follows :

$$(2.7)\qquad b = H + \frac{1}{q} \langle \vec{n}, \vec{\text{grad}}\, q \rangle$$

in which H denotes the *algebrical curvature* of γ, positive where $\Omega_{u,\gamma}$ is convex,

then, we have the following result (stated here formally ; we precise and prove it in the sequel) :

Proposition 1 : Let (u_0,γ_0) *be a couple of* $\mathcal{U}_{ad}\times\Gamma_{ad}$, *solution of Problem* (2.2) *and such that*

(2.8) *Problem* (2.6) (2.7) *is well posed for* $(u,\gamma) = (u_0,\gamma_0)$.
Then the mapping z_F *defined by* (2.6) (2.7) *satisfies the following stationarity property*

$$(2.9)\qquad \frac{\partial z_F}{\partial \gamma}(u_0,\gamma_0) = 0 \quad \textit{on } \Omega_{u_0,\gamma_0}.$$ □

The choice (2.7) and property (2.9) have a *physical interpretation* : in the case where q is a constant, b is exactly the curvature and the last line of (2.6) becomes

$$(2.10)\qquad \left(\frac{\partial}{\partial n} + H\right) z_F(u,\gamma) = q \ ;$$

now it can be shown for a lot of simple potential jet flows that (2.10) (with $z_F(u,\gamma)$ denoting the stream function) holds along *every streamline*.

(^{1}) The index "F" means that a FOURIER-type boundary condition is chosen along γ.

Condition (2.10) have been first introduced by P.R. GARABEDIAN [1] and generalized to (2.7) by C.W. CRYER [1] to construct fastly convergent iterative methods for solving jet problems analogous to (2.2).

We shall now use the coupling problem to construct a new formulation of the Free Boundary Problem : let (u,γ) be a couple of $\mathcal{U}_{ad}\times\Gamma_{ad}$; then the corresponding solution $z_F(u,\gamma)$ of (2.6) admits a trace along γ ; this trace can be defined as a function of the polar angle θ ; thus we may write

$$z_F(u,\gamma)|_\gamma \in C^0(\mathbf{T}) ;$$

now we define the following mapping

$$(2.11) \qquad \begin{cases} \Psi : & \mathcal{U}_{ad}\times\Gamma_{ad} \longrightarrow C^0(\mathbf{T}) \\ & (u,\gamma) \longrightarrow z_F(u,\gamma)|_\gamma . \end{cases}$$

Then the Free Boundary Problem (2.2) may be represented by the mapping Ψ :

$$(2.12) \qquad \begin{cases} \text{For a couple } (u,\gamma) \text{ of } \mathcal{U}_{ad}\times\Gamma_{ad}, \\ \{(u,\gamma) \text{ is a solution of } (2.2)\} \Longleftrightarrow \Psi(u,\gamma) = 0 ; \end{cases}$$

and the mapping Γ defined in (2.3) is *implicitly defined* by (2.12).

A natural question is whether this new definition is well-posed ; the answer is the main consequence of Proposition 1 and is stated as follows :

__Proposition 2__ : Let (u_0,γ_0) *be a couple solution of Problem* (2.2) *which satisfies* (2.8) *; then the partial derivative of the mapping* Ψ *with respect to its second argument may be at least* ([1]) *formally computed as follows :*

$$(2.13) \qquad \frac{\partial\Psi}{\partial\gamma}(u_0,\gamma_0).\delta\gamma = \langle\vec{n},\vec{r}\rangle \, q|_{\gamma_0} \times \delta\gamma$$

($\vec{r} = (\cos\theta,\sin\theta)$; $\delta\gamma$ *is an arbitrary increment of* γ_0*).*

__Sketch of the proof__ : Let us consider Definition (2.11) : the argument γ occurs in two manners in Ψ : firstly via the *coupling problem* (2.6), secondly via the trace operator on γ ; thus the derivative consists of two terms :

$$\frac{\partial\Psi}{\partial\gamma}(u_0,\gamma_0).\delta\gamma = \left[\frac{\partial z_F}{\partial\gamma}(u_0,\gamma_0).\delta\gamma\right]\Big|_{\gamma_0} + \left[\frac{\partial}{\partial\gamma}\left(\phi|_\gamma\right).\delta\gamma\right]_{\substack{\gamma = \gamma_0 \\ \phi = z_F(u_0,\gamma_0)}} .$$

([1]) The rigorous proof is contained by the sequel.

Now, from Proposition 1, the first term of the sum vanishes ; the second term becomes :

$$\frac{\partial\Psi}{\partial\gamma}(u_0,\gamma_0).\delta\gamma = \frac{\partial z_F(u_0,\gamma_0)}{\partial r}\Big|_{\gamma_0} \times \delta\gamma$$

which, using the boundary conditions on γ_0, becomes (2.13). □

Statement (2.13) means that the Jacobian derivative $\frac{\partial\Psi}{\partial\gamma}(u_0,\gamma_0)$ for equation (2.12) is the simple product by a regular function ; under the natural condition $q \neq 0$ this function does not vanish ; therefore that operator is *inversible* ; moreover, we shall prove in the sequel the two following regularity properties :

Proposition 3 : (i) Let (u_0,γ_0) *a couple of* $\mathcal{U}_{ad}\times\Gamma_{ad}$*, solution of* (2.2) *satisfying* (2.8) *; we assume moreover that* q*,* u_0 *and* γ_0 *are smooth ; then*

$$\frac{\partial\Psi}{\partial\gamma}(u_0,\gamma_0) \in \mathrm{Aut}\left[C^{\ell+\alpha}(\mathbf{T})\right] \quad (^1)$$

(ii) *The mapping* Ψ *is* C^∞ *on* $\mathcal{U}_{ad}\times\Gamma_{ad}$ *for the norms :*

$$C^{\ell+\alpha}(\mathbf{T}) \times C^{\ell+\alpha}(\mathbf{T}) \longrightarrow C^{\ell+\alpha}(\mathbf{T}) \quad \ell \in \mathbb{N},\ \ell\geq 2,\ \alpha\in]0,1[\ ;$$

at the opposite, if γ_0 *does not belong to* $C^{\ell+\alpha}(\mathbf{T})$*, then generally* $\Psi(u_0,\gamma_0)$ *does not belong to* $C^{\ell+\alpha}(\mathbf{T})$.

Sketch of the proof : The point (i) is trivially derived from Proposition 2 ; the first part of (ii) is proved later and the second part is a consequence of the optimality of Schauder estimates with respect of degree of regularity. □

Conclusion : We can see, from Proposition 3 that we cannot apply the classical Implicit Function Theorem ; we cannot show either that the Jacobian derivative remains inversible when the arguments u and γ are slightly perturbed and thus the early generalized Implicit Funtion Theorems of J. NASH [1] and J. MOSER [1,2] do not apply any more and we need to use a variant of NASH-MOSER theorem taking in account this additional difficulty.

(^{1}) Aut (**X**) is the set of one-to-one linear continuous operators from **X** into **X** with continuous inverses.

3. THE MAIN THEOREM

The purpose of this section is to apply the Generalized Implicit Function Theorem to Problem (2.2) under its (2.12) variant ; we shall first show that the mapping Ψ satisfies the assumptions of this G.I.F.T.

We assume *henceforward* that condition (1.4) is fulfilled ; the arguments u and γ of mapping Ψ are assumed to be in $\mathcal{U}_{ad}$ and Γ_{ad}, with $s_1 = \ell+\alpha$, $s_2 = \ell+1+\alpha$ ($\ell \in \mathbb{N}$, $\ell \geq 2$, $\alpha \in]0,1[$), and satisfy (2.8).

3.1. Regularity properties of mapping Ψ

These properties are precised in the following proposition :

Proposition 4 : Let us assume that for an integer k (k≥1)

$$q \in C^{\ell+k+\alpha}(\mathbb{R}^2) \tag{3.1}$$

then Ψ *is a* C^k *mapping for the following norms*

$$C^{\ell+\alpha}(\mathbf{T}) \times C^{\ell+1+\alpha}(\mathbf{T}) \longrightarrow C^{\ell+\alpha}(\mathbf{T}) .$$

This proposition is proved by means of the *method of interior variations* (due to P.R. GARABEDIAN and M. SCHIFFER) ; see for details A. DERVIEUX [1] Sec. 1.1.3. □

3.2. Hadamard-type variational formula

The variations of the solution of the coupling problem with respect to the boundary γ are precised as follows :

Proposition 5 : Let $\mathcal{B}$ *be a ball containing* $\Omega_{u,\gamma}$ *for every* (u,γ) *in* $\mathcal{U}_{ad} \times \Gamma_{ad}$ *; let us assume that* (3.1) *is fulfilled.*

(i) *Then there exists an extension* $\overline{z}_F$ *of* z_F*, defined on* $\overline{\mathcal{B}}$*, which is a* C^k *mapping with respect to* (u,γ) *for the norms*

$$C^{\ell+\alpha}(\mathbf{T}) \times C^{\ell+\alpha+1}(\mathbf{T}) \longrightarrow C^{\ell-k+\alpha}(\overline{\mathcal{B}})$$

(ii) *The first derivative of* $\overline{z}_F$ *with respect to* γ *at* (u,γ) *has its restriction over* $\Omega_{u,\gamma}$ *defined by the following system :*

$$
(3.2)\qquad \begin{cases}
\Delta\left[\dfrac{\partial \bar{z}_F}{\partial \gamma}\,.\delta\gamma\right] = 0 \quad in\ \Omega_{u,\gamma} \\[2mm]
\dfrac{\partial \bar{z}_F}{\partial \gamma}.\delta\gamma = 0 \qquad on\ u \\[2mm]
\left[\dfrac{\partial}{\partial n} + b\right]\left[\dfrac{\partial \bar{z}_F}{\partial \gamma}\,.\delta\gamma\right] = \dfrac{\partial z_F}{\partial s}\left[\delta\gamma\,\dfrac{\partial}{\partial s}\,(\langle \vec{n},\vec{r}\rangle) + \langle \vec{n},\vec{r}\rangle\,\dfrac{d\theta}{ds}\,\delta\gamma'\right] \\[2mm]
- z_F\left[\dfrac{\partial b}{\partial \gamma}\,. + \dfrac{\partial b}{\partial n}\,\langle \vec{n},\vec{r}\rangle - \dfrac{1}{q}\,\dfrac{\partial q}{\partial n}\,b\,\langle \vec{n},\vec{r}\rangle\right]\delta\gamma \\[2mm]
+ \dfrac{\partial^2 z_F}{\partial s^2}\,\langle \vec{n},\vec{r}\rangle\,\delta\gamma \quad on\ \gamma,
\end{cases}
$$

where s *denotes the curvilinear abscissa along* γ, *and* $\delta\gamma'$ *the derivative of the increment* $\delta\gamma$ *with respect to the polar angle* θ.

Proof : (i) The existence of such regular extension $\bar{z}_F$ is obtained by the application of a general method due to B. PALMERIO, A. DERVIEUX ; see A. DERVIEUX [1] for details.

(ii) Let us introduce the following mapping :

$$
(3.3)\qquad \begin{cases}
\Phi\ :\ \mathcal{U}_{ad}\times\Gamma_{ad}\times C^{\ell+\alpha}(\bar{B})\times\left[C^{\ell+\alpha}(\bar{B})\right]^3 \longrightarrow \mathbb{R} \\[2mm]
\Phi(u,\gamma,z,\phi_1,\phi_2,\phi_3) = \displaystyle\int_{\Omega_{u,\gamma}} \phi_1\ \Delta z\ dx \\[2mm]
+\displaystyle\int_u \phi_2(z+1)\ d\sigma + \int_\gamma \phi_3\left[\dfrac{\partial z}{\partial n} + bz - q\right] d\sigma\ ;
\end{cases}
$$

for every couple (u,γ) in $\mathcal{U}_{ad}\times\Gamma_{ad}$, for every triple (ϕ_1,ϕ_2,ϕ_3) of $\left[C^{\ell+\alpha}(\bar{B})\right]^3$ we have, since $\bar{z}_F(u,\gamma)$ is an extension of the solution of system (2.6),

$$
(3.4)\qquad \Phi(u,\gamma,\bar{z}_F(u,\gamma),\phi_1,\phi_2,\phi_3) = 0.
$$

From (i), we may differentiate (3.4) with respect to γ, to obtain

$$(3.5)\quad \begin{cases} \forall(u,\gamma)\in\mathcal{U}_{ad}\times\Gamma_{ad},\quad \forall\delta\gamma\in C^{\ell+\alpha+1}(\mathbf{T}), \\ \forall\phi = (\phi_1,\phi_2,\phi_3)\in\left[C^{\ell+\alpha}(\overline{B})\right]^3, \\ \dfrac{\partial\Phi}{\partial z}(u,\gamma,\overline{z}_F(u,\gamma),\phi).\dfrac{\partial\overline{z}_F}{\partial\gamma}(u,\gamma).\delta\gamma = \\ \qquad -\dfrac{\partial\Phi}{\partial\gamma}(u,\gamma,\overline{z}_F(u,\gamma),\phi).\delta\gamma\ ; \end{cases}$$

Let us compute the right-hand side of (3.5) :

$$(3.6)\quad \begin{cases} \dfrac{\partial\Phi}{\partial\gamma}.\delta\gamma = \dfrac{\partial}{\partial\gamma}\left[\displaystyle\int_{\Omega_{u,\gamma}}\phi_1\ \Delta z\ dx\right].\delta\gamma \\ \qquad + \dfrac{\partial}{\partial\gamma}\left[\displaystyle\int_{\gamma}\phi_3\left[\dfrac{\partial z}{\partial n} + bz - q\right]d\sigma\right].\delta\gamma. \end{cases}$$

We shall use the following differentiation formula (see F. MURAT, J. SIMON [1,2]) :

$$(3.7)\qquad \frac{\partial}{\partial\gamma}\left[\int_{\Omega_{u,\gamma}} C(x)\ dx\right].\delta\gamma = \int_{\gamma} C(x)\ \langle\vec{n},\vec{r}\rangle\ \delta\gamma\ d\sigma\ ;$$

thus

$$\frac{\partial}{\partial\gamma}\left[\int_{\Omega_{u,\gamma}}\phi_1\ \Delta z\ dx\right].\delta\gamma = \int_{\gamma}\phi_1\ \Delta z\ \langle\vec{n},\vec{r}\rangle\ \delta\gamma\ d\sigma$$

and this term vanishes when z is substituted by $\overline{z}_F(u,\gamma)$ which is harmonic from (2.6).

For the computation of the second term of the sum (3.6), we need the two following differentiation formulas (see F. MURAT, J. SIMON [Ibid], B. PALMERIO, A. DERVIEUX [2]) :

$$(3.8)\quad \begin{cases} \dfrac{\partial}{\partial\gamma}\left[\displaystyle\int_{\gamma} C(x,\gamma)\ d\sigma\right].\delta\gamma = \displaystyle\int_{\gamma}\dfrac{\partial C}{\partial\gamma}.\delta\gamma\ d\sigma \\ \qquad + \displaystyle\int_{\gamma}\left\{\langle\vec{\mathrm{grad}}_x C,\vec{n}\rangle + H\ C\right\}\langle\vec{n},\vec{r}\rangle\ \delta\gamma\ d\sigma \end{cases}$$

$$(3.9)\quad \begin{cases} \dfrac{\partial}{\partial\gamma}\left[\displaystyle\int_\gamma \psi \frac{\partial C(x,\gamma)}{\partial n}\, d\sigma\right].\delta\gamma = \displaystyle\int_\gamma \psi \frac{\partial}{\partial n}\left[\frac{\partial C}{\partial\gamma}\,.\delta\gamma\right] d\sigma \\ \qquad + \displaystyle\int_\gamma \left[<\vec{grad}_x C, \vec{grad}_x \psi> + \psi\, \Delta_x C\right] <\vec{n},\vec{r}>\, \delta\gamma\, d\sigma\ ; \end{cases}$$

thus

$$\frac{\partial\Phi}{\partial\gamma}(u,\gamma,z,\phi).\delta\gamma = \sum_{i=1}^{8} A_i$$

with

$$A_1 = \int_\gamma \left[<\vec{grad}\,\phi_3, \vec{grad}\, z> + \phi_3\, \Delta z\right] <\vec{n},\vec{r}>\, \delta\gamma\, d\sigma$$

$$A_2 = \int_\gamma \phi_3\, z\, \frac{\partial b}{\partial\gamma}\, .\delta\gamma\, d\sigma$$

$$A_3 = \int_\gamma \frac{\partial\phi_3}{\partial n}\, (bz - q) <\vec{n},\vec{r}>\, \delta\gamma\, d\sigma$$

$$A_4 = \int_\gamma \phi_3\, z\, \frac{\partial b}{\partial n} <\vec{n},\vec{r}>\, \delta\gamma\, d\sigma$$

$$A_5 = \int_\gamma \phi_3\, b\, \frac{\partial z}{\partial n} <\vec{n},\vec{r}>\, \delta\gamma\, d\sigma$$

$$A_6 = -\int_\gamma \phi_3\, \frac{\partial q}{\partial n} <\vec{n},\vec{r}>\, \delta\gamma\, d\sigma$$

$$A_7 = \int_\gamma H\, \phi_3\, b\, z <\vec{n},\vec{r}>\, \delta\gamma\, d\sigma$$

$$A_8 = -\int_\gamma H\, \phi_3\, q <\vec{n},\vec{r}>\, \delta\gamma\, d\sigma\ ;$$

in those expressions, the curvature H and the normal vector $\vec{n}$ which occur in the coefficient $b = H + \frac{1}{q} <\vec{n},\vec{grad}\, q>$,and which are a priori defined only among the boundary γ are assumed to be extended as constant functions along every polar radius θ = const.

Now we plug $z = z_F(u,\gamma)$: this yields the following simplifications :

$$A_1 = \int_\gamma <\vec{grad}\,\phi_3, \vec{grad}\, z_F> <\vec{n},\vec{r}>\, \delta\gamma\, d\sigma\ ;$$

introducing the curvilinear abscissa s, we obtain :

$$A_1 = \int_\gamma \left(\frac{\partial \phi_3}{\partial s} \frac{\partial z_F}{\partial s} + \frac{\partial \phi_3}{\partial n} \frac{\partial z_F}{\partial n} \right) <\vec{n},\vec{r}> \delta\gamma \, d\sigma$$

thus (denoting by $L(\gamma)$ the length of γ)

$$A_1 + A_3 = \int_0^{L(\gamma)} \frac{\partial \phi_3}{\partial s} \left(\frac{\partial z_F}{\partial s} <\vec{n},\vec{r}> \delta\gamma \right) ds$$

and, integrating by parts, we obtain :

$$\text{(3.10)} \qquad A_1 + A_3 = -\int_0^{L(\gamma)} \phi_3 \left\{ \frac{\partial^2 z_F}{\partial s^2} <\vec{n},\vec{r}> \delta\gamma + \frac{\partial z_F}{\partial s} \left[\delta\gamma \frac{\partial}{\partial s} <\vec{n},\vec{r}> + <\vec{n},\vec{r}> \delta\gamma' \frac{d\theta}{ds} \right] \right\} ds.$$

The four last terms become :

$$\text{(3.11)} \qquad A_5 + A_6 + A_7 + A_8 = \int_\gamma \phi_3 <\vec{n},\vec{r}> \left\{ H \left[\frac{\partial z_F}{\partial n} + b z_F - q \right] + \frac{1}{q} \frac{\partial q}{\partial n} \left[\frac{\partial z_F}{\partial n} - q \right] \right\} \delta\gamma \, d\sigma$$

$$= -\int_\gamma \phi_3 <\vec{n},\vec{r}> \frac{b}{q} z_F \frac{\partial q}{\partial n} \delta\gamma \, d\sigma.$$

which, jointly with (3.10) gives (3.2). □

3.3. Properties of the Jacobian derivative

We assume now that for an integer ℓ, a real α, $\ell \geq 2$, $\alpha \in]0,1[$

$$\text{(3.12)} \qquad \begin{cases} q \in C^{\ell+2+\alpha}(\overline{B}) \\ q > 0 \quad \text{on } \overline{B}. \end{cases}$$

The couple (u_0, γ_0) is assumed to be a solution in $\mathcal{U}_{ad} \times \Gamma_{ad}$ (with $s_1 = \ell+1+\alpha$, $s_2 = \ell+2+\alpha$) of Problem (2.2), satisfying (2.8).

We shall show first some preliminary results :

Lemme 1 : There exists two positive constants K *and* M *such that :*

$$\text{(3.13)} \qquad \begin{cases} |u - u_0|_{C^{\ell+1+\alpha}(\mathbf{T})} < M \\ |\gamma - \gamma_0|_{C^{\ell+2+\alpha}(\mathbf{T})} < M \end{cases}$$

implies, for every element $\delta\gamma$ *of* $C^{\ell+1+\alpha}(\mathbf{T})$,

$$\left| \left\{ \frac{\partial}{\partial\gamma} \left[\phi|_\gamma \right]_{\phi = z_F(u,\gamma)} - q|_\gamma \langle \vec{n},\vec{r}\rangle \right\} . \delta\gamma \right|_{C^{\ell+\alpha}(\mathbf{T})}$$

$$\leq K \, |\Psi(u,\gamma)|_{C^{\ell+1+\alpha}(\mathbf{T})} \, |\delta\gamma|_{C^{\ell+1+\alpha}(\mathbf{T})} .$$

Proof : Let $\vec{\tau}$ be a unitary vector, tangent to γ ; the first differential is computed as follows (A. DERVIEUX [1], Annexe 2)

$$\frac{\partial}{\partial\gamma} \left[\phi|_\gamma \right]_{\phi = z_F} . \delta\gamma = \left\{ \langle \vec{n},\vec{r}\rangle \frac{\partial z_F}{\partial n} + \langle \vec{\tau},\vec{r}\rangle \frac{\partial z_F}{\partial \tau} \right\} \times \delta\gamma .$$

Then the conclusion follows from the following estimates :

$$\left| \langle \vec{\tau},\vec{r}\rangle \frac{\partial z_F}{\partial \tau}\Big|_\gamma \delta\gamma \right|_{C^{\ell+\alpha}(\mathbf{T})} \leq K \left| z_F|_\gamma \right|_{C^{\ell+1+\alpha}(\mathbf{T})} \left| \delta\gamma \right|_{C^{\ell+\alpha}(\mathbf{T})}$$

$$\left| \langle \vec{n},\vec{r}\rangle \left[\frac{\partial z_F}{\partial n} - q \right]\Big|_\gamma \right|_{C^{\ell+\alpha}(\mathbf{T})} = \left| \langle \vec{n},\vec{r}\rangle \, H \, z_F|_\gamma \, \delta\gamma \right|_{C^{\ell+\alpha}(\mathbf{T})}$$

$$\leq K \left| z_F|_\gamma \right|_{C^{\ell+\alpha}(\mathbf{T})} \left| \delta\gamma \right|_{C^{\ell+\alpha}(\mathbf{T})} . \qquad \square$$

Lemma 2 : *Let* $p = p(u,\gamma,w)$ *be the solution of the following system :*

$$(3.14) \qquad \begin{cases} \Delta p = 0 & \text{in } \Omega_{u,\gamma} \\ p = 0 & \text{on } u \\ \left(\dfrac{\partial}{\partial n} + b \right) p = w & \text{on } \gamma \end{cases}$$

where γ *is a sufficiently regular function of the variable* θ ; *then there exists two positive constants* K' *and* K'' *such that :*

$$(3.15) \qquad \begin{cases} |u - u_0|_{C^{\ell+1+\alpha}(\mathbf{T})} \leq K' \\ |\gamma - \gamma_0|_{C^{\ell+2+\alpha}(\mathbf{T})} \leq K'' \end{cases}$$

implies

$$(3.16) \qquad |p(u,\gamma,w)|_{C^{\ell+\alpha}(\Omega_{u,\gamma})} \leq K'' \, |w|_{C^{\ell-1+\alpha}(\mathbf{T})} .$$

Proof : This lemma asserts the local *uniformness* of the estimate (3.16) for the solution p of Problem (3.14) with respect to the boundaries u and γ ; using again the *method of local variations*, we shall see that it is enough to consider a so-called "transported" solution $\tilde{p}(u,\gamma,w)$:

$$\tilde{p}(u,\gamma,w) = p(u,\gamma,w) \circ \Theta(u,\gamma) \tag{3.17}$$

where $\Theta(u,\gamma)$ is a $C^{\ell+\alpha}$ one-to-one mapping from $\overline{B}$ on $\overline{B}$, depending smoothly on u and γ and such that :

$$\Theta(u,\gamma)(\Omega_{u_0,\gamma_0}) = \Omega_{u,\gamma}$$

$$\Theta(u,\gamma)(u(\theta)\cos\theta, u(\theta)\sin\theta) = (u_0(\theta)\cos\theta, u_0(\theta)\sin\theta)$$

$$\Theta(u,\gamma)(\gamma(\theta)\cos\theta, \gamma(\theta)\sin\theta) = (\gamma_0(\theta)\cos\theta, \gamma_0(\theta)\sin\theta).$$

It is clear that $\tilde{p}(u,\gamma)$ is the solution of a transported system, say

$$\begin{cases} A(u,\gamma)\,\tilde{p}(u,\gamma) = 0 & \text{in } \Omega_{u_0,\gamma_0} \\ \tilde{p}(u,\gamma) = 0 & \text{on } u_0 \\ \left[\dfrac{\partial}{\partial\nu_{A(u,\gamma)}} + b\circ\Theta(u,\gamma)\right]\tilde{p}(u,\gamma) = w & \text{on } \gamma_0. \end{cases} \tag{3.18}$$

In system (3.18) the domain Ω_{u_0,γ_0} does not depend on (u,γ), which operates in the distributed and boundary coefficients ; using S. AGMON, A. DOUGLIS, L. NIRENBERG [1]'s Theorem 7.3 (p. 668) and Remark 1, we have the following *uniform* (i.e. assuming (3.15)) estimate

$$|\tilde{p}(u,\gamma,w)|_{C^{\ell+\alpha}\left(\overline{\Omega}_{u_0,\gamma_0}\right)} \le K\left\{|w|_{C^{\ell-1+\alpha}(\mathbf{T})} + |\tilde{p}(u,\gamma,w)|_{C^0\left(\overline{\Omega}_{u_0,\gamma_0}\right)}\right\} \tag{3.19}$$

Now, by an easy application of the classical Implicit Function Theorem, we know that the solution $\tilde{p}(u,\gamma)$ of system (3.18) depends smoothly on the data (u,γ,w) with respect to the following norms :

$$C^{\ell+1+\alpha}(\mathbf{T})\times C^{\ell+2+\alpha}(\mathbf{T})\times C^{\ell-1+\alpha}(\mathbf{T}) \longrightarrow C^{\ell+\alpha}\left(\overline{\Omega}_{u_0,\gamma_0}\right);$$

This remark allows us to get rid of the C^0 norm in (3.19). Then (3.16) follows from (3.17) and from the uniform boundedness of the mapping Θ. □

<u>Lemma 3</u> : *There exist two positive constants* K *and* M *such that* (3.13) *implies :*

$$(3.20)\qquad \begin{cases} \forall \delta\gamma \in C^{\ell+1+\alpha}(\mathbf{T}) \\ \left| \left[\dfrac{\partial \bar{z}_F}{\partial \gamma}\,(u,\gamma).\delta\gamma \right]_{|\gamma} \right|_{C^{\ell+\alpha}(\mathbf{T})} \leq K\, |\Psi(u,\gamma)|_{C^{\ell+1+\alpha}(\mathbf{T})} |\delta\gamma|_{C^{\ell+1+\alpha}(\mathbf{T})} . \end{cases}$$

<u>Proof</u> : We need to estimate the right-hand side (along the boundary) of system (3.2) :

$$(3.21)\qquad \begin{cases} E.\delta\gamma = z_F(u,\gamma)\left[E_1 + E_2 + E_3\right].\delta\gamma \\ \quad + \dfrac{\partial z_F(u,\gamma)}{\partial \gamma}\left[E_4\delta\gamma + E_5\delta\gamma'\right] \\ \quad + \dfrac{\partial^2 z_F(u,\gamma)}{\partial s^2}\, E_6\delta\gamma \end{cases}$$

with

$$E_1 = \frac{\partial b}{\partial \gamma}$$

$$E_2 = \frac{\partial b}{\partial n}\, \langle \vec{n},\vec{r}\rangle$$

$$E_3 = -\frac{1}{q}\, b\, \langle \vec{n},\vec{r}\rangle$$

$$E_4 = -\frac{\partial}{\partial s}\, \langle \vec{n},\vec{r}\rangle$$

$$E_5 = -\langle \vec{n},\vec{r}\rangle \frac{d\theta}{ds}$$

$$E_6 = -\langle \vec{n},\vec{r}\rangle .$$

The curvilinear derivatives of $z_F(u,\gamma)_{|\gamma}$ are estimated as follows :

$$(3.22)\qquad \begin{cases} |z_F(u,\gamma)|_{C^{\ell+\alpha}(\mathbf{T})} + \left|\dfrac{\partial z_F(u,\gamma)}{\partial s}\right|_{C^{\ell+\alpha}(\mathbf{T})} \\ + \left|\dfrac{\partial^2 z_F(u,\gamma)}{\partial s^2}\right|_{C^{\ell-1+\alpha}(\mathbf{T})} \leq K \left| z_F(u,\gamma)_{|\gamma} \right|_{C^{\ell+1+\alpha}(\mathbf{T})} . \end{cases}$$

Let us study expression E_1 :

$$E_1.\delta\gamma = \frac{\partial H}{\partial \gamma}\,.\delta\gamma + \frac{1}{q} \sum_{i=1,2} \left\{ \frac{\partial q}{\partial x_i}\, \langle \vec{e}_i, \frac{\partial \vec{n}}{\partial \gamma}\,.\delta\gamma \rangle \right.$$

$$\left. + \langle \vec{n},\vec{e}_i\rangle\, \delta\gamma\, \frac{\partial}{\partial r}\left[\frac{1}{q}\,\frac{\partial q}{\partial x_i}\right] \right\}$$

with $\vec{e}_1 = (1,0)$, $\vec{e}_2 = (0,1)$; The curvature H can be expressed by means of derivatives of γ with respect to θ, the order being less or equal to 2 ; vector $\vec{n}$ can likewise be expressed by derivatives of γ with order less or equal to 1 ; therefore

(3.23) $$|E_1.\delta\gamma|_{C^{\ell-1+\alpha}(\mathbf{T})} \leq K\ |\delta\gamma|_{C^{\ell+1+\alpha}(\mathbf{T})} .$$

Let us decompose expression E_2 :

$$E_2 = \langle\vec{n},\vec{r}\rangle \frac{\partial}{\partial n}\left\{H + \frac{1}{q}\sum_{i=1,2}\langle\vec{n},\vec{e}_i\rangle\frac{\partial q}{\partial x_i}\right\} ;$$

since functions H and $\vec{n}$ are assumed to be extended by constant functions along polar radius, we may write :

$$\frac{\partial}{\partial n} H(x,\gamma) = \langle\vec{n},\vec{r}\rangle \frac{\partial H}{\partial r} + \langle\vec{n},\vec{e}_\theta\rangle \frac{1}{\gamma}\frac{\partial H}{\partial\theta}$$

$$= \langle\vec{n},\vec{e}_\theta\rangle \frac{1}{\gamma}\frac{\partial H}{\partial\theta}$$

$$\frac{\partial}{\partial n}\vec{n}(x,\gamma) = \langle\vec{n},\vec{e}_\theta\rangle \frac{1}{\gamma}\frac{\partial\vec{n}}{\partial\theta} .$$

with

$$\vec{e}_\theta = (\sin\theta, -\cos\theta) ;$$

thus

$$E_2 = \langle\vec{n},\vec{r}\rangle\left\{\frac{\partial q}{\partial n}\frac{\partial}{\partial n}\left(\frac{1}{q}\right) + \frac{1}{\gamma}\langle\vec{n},\vec{e}_\theta\rangle\left[\frac{\partial H}{\partial\theta} + \left\langle\frac{\partial\vec{n}}{\partial\theta},\vec{\text{grad}}\ q\right\rangle\right]\right\} ;$$

now the derivate $\frac{\partial H}{\partial\theta}$ can be expressed by means of the θ-derivatives γ', γ'', γ''', which belong to $C^{\ell-1+\alpha}(\mathbf{T})$, and the derivative $\frac{\partial\vec{n}}{\partial\theta}$ by means of γ , γ', γ'' ; then we have the estimate

(3.24) $$|E_2.\delta\gamma|_{C^{\ell-1+\alpha}(\mathbf{T})} \leq K\ |\delta\gamma|_{C^{\ell-1+\alpha}(\mathbf{T})} .$$

We obtain analogously

(3.25) $$|(E_3 + E_4 + E_6)\delta\gamma + E_5\ \delta\gamma'|_{C^{\ell-1+\alpha}(\mathbf{T})} \leq K\ |\delta\gamma|_{C^{\ell+\alpha}(\mathbf{T})} ;$$

from (3.22) to (3.25) we derive

(3.26) $$|E.\delta\gamma|_{C^{\ell-1+\alpha}(\mathbf{T})} \leq K\ |\delta\gamma|_{C^{\ell+\alpha}}\ |\Psi(u,\gamma)|_{C^{\ell+1+\alpha}(\mathbf{T})} .$$

Using Lemma 2 with

$$p = z_F(u,\gamma)$$

$$w = E.\delta\gamma$$

we get finally the conclusion of Lemma 3. □

We remember the decomposition of the Jacobian derivative :

$$\frac{\partial\Psi}{\partial\gamma}(u,\gamma).\delta\gamma = \left[\frac{\partial\bar{z}_F}{\partial\gamma}(u,\gamma).\delta\gamma\right]\Big|_\gamma + \frac{\partial}{\partial\gamma}\left[\bar{\phi}\big|_\gamma\right]_{\phi = z_F(u,\gamma)}.\delta\gamma \ ;$$

due to Lemmas 1 and 3, we may estimate both two terms of the right-hand sum, thus (3.13) implies :

$$\left|\left(\frac{\partial\Psi}{\partial\gamma}(u,\gamma) - q\big|_\gamma \langle\vec{n},\vec{r}\rangle\right).\delta\gamma\right|_{C^{\ell+\alpha}(\mathbf{T})}$$

$$\leq K\ |\Psi(u,\gamma)|_{C^{\ell+1+\alpha}(\mathbf{T})}\ |\delta\gamma|_{C^{\ell+1+\alpha}(\mathbf{T})} .$$

Now we have also (trivially) :

$$|\eta(u,\gamma)|_{C^{\ell+1+\alpha}(\mathbf{T})} = \left|\frac{1}{\langle\vec{n},\vec{r}\rangle\ q\big|_\gamma}\right|_{C^{\ell+1+\alpha}(\mathbf{T})} \leq K \ ;$$

then, setting

$$\delta\gamma = \eta(u,\gamma)\ w,$$

we have finally proved the following proposition :

Proposition 6 : Let q *satisfy* (3.12) *and* (u_0,γ_0) *be a couple solution in* $\mathcal{U}_{ad}\times\Gamma_{ad}$, *with* $s_1 = \ell+1+\alpha$, $s_2 = \ell+2+\alpha$, *of Problem* (2.2), *satisfying* (2.8) *; then there exists two positive constants* K *and* M *such that* (3.13) *implies ;*

$$(3.27)\quad \begin{cases} \forall w \in C^{\ell+1+\alpha}(\mathbf{T}) \\ \left|\left[\frac{\partial\Psi}{\partial\gamma}(u,\gamma)\ \eta(u,\gamma) - 1\right] w\right|_{C^{\ell+\alpha}(\mathbf{T})} \\ \qquad \leq K\ |\Psi(u,\gamma)|_{C^{\ell+1+\alpha}(\mathbf{T})}\ |w|_{C^{\ell+1+\alpha}(\mathbf{T})} . \end{cases}$$

□

3.4. Application of E. ZEHNDER's theorem

For the sake of simplicity, we consider first the following assumption :

(3.28) *We assume that the function* q *does not depend on the polar coordinate* θ.

Lemma 4 : Under assumption (3.28)*, the mappings* Ψ*,* η *and* $\eta\Psi$ *are translation-invariant.*

Proof: Similarly to A. DERVIEUX [2] Lemma 4, we note that when a rotation around the origin of the axis is applied on the data u and γ, then the same rotation follows for the function $z_F(u,\gamma)$, thus for $\Psi(u,\gamma)$, and also for $\eta(u,\gamma)$, thus for $\eta(u,\gamma)\ \Psi(u,\gamma)$. □

The *main results* of this work are the following :

Theorem 4 : Let α *and* ε *be two real numbers with* $0<\varepsilon<\alpha<1$, $\alpha+\varepsilon \neq 1$ *and* ℓ *be an integer greater or equal to 2. We assume that* q *is a strictly positive function of* $C^{s_0}(\overline{B})$*, with*

$$s_0 = \ell + \alpha + 8(\varepsilon + 1),$$

and such that (u_0,γ_0) *is a solution in* $\mathcal{U}_{ad}\times\Gamma_{ad}$ *with* $s_1 = s_0$, $s_2 = s_0 + 1$ *of Problem* (2.2) *satisfying* (2.8).

Then there exist a neighborhood D_{σ_1} *of* u_0 *in* $C^{\sigma_1}(\mathbf{T})$ *with* $\sigma_1 = \ell + \alpha + \frac{15}{4}(\varepsilon + 1)$ *and a mapping* $\Gamma : u \mapsto \Gamma(u)$*, defined on* D_{σ_1}*, satisfying* (2.4) (2.5)*, continuous on* D_{σ_1} *and differentiable at* u_0 *for the norms :*

$$C^{\sigma_1}(\mathbf{T}) \longrightarrow C^{\ell+\alpha+1}(\mathbf{T}) \quad ;$$

its derivative at u_0 *is given by :*

$$(3.29)\qquad \begin{cases} \forall\,\delta u \in C^{\sigma_1}(\mathbf{T}) \\ \dfrac{d\Gamma}{du}(u_0).\delta u = -\dfrac{1}{\langle\vec{n},\vec{r}\rangle\ q_{|\gamma_0}}\ \dot{z}_F.\delta u \end{cases}$$

with

$$(3.30)\qquad \begin{cases} \Delta\dot{z}_F.\delta u = 0 \quad \text{in } \Omega_{u_0,\gamma_0} \\ \left(\dfrac{\partial}{\partial n} + b\right)\dot{z}_F.\delta u = 0 \quad \text{on } \gamma_0 \\ \dot{z}_F.\delta u = \langle\vec{n},\vec{r}\rangle\ \dfrac{\partial z_F(u_0,\gamma_0)}{\partial n}\ \delta u \quad \text{on } u. \end{cases}$$

□

Theorem 5 : Under the conditions of Theorem 4, we assume moreover that q, u_0 *and* γ_0 *are* C^∞ *; then, for every real number* ε' *with* $0<\varepsilon'<1$ *there exists a neighborhood* $D_{\sigma_1'}$ *of* u_0 *in* $C^{\sigma_1'}(\mathbf{T})$ *with*

$$\sigma_1' = \ell + \alpha + 2(1+\varepsilon) + \varepsilon'$$

such that the mapping Γ *is continuous on* $D_{\sigma_1'}$ *and differentiable at* u_0 *for the norms :*

$$C^{\sigma_1'}(\mathbf{T}) \longrightarrow C^{\ell+\alpha+1}(\mathbf{T}) .$$

□

Remark 2 : For $\ell = 2$, Theorem 4 implies : if q belongs to $C^{10+\alpha+8\varepsilon}(\overline{B})$, u_0 to $C^{10+\alpha+8\varepsilon}(\mathbf{T})$, and γ_0 to $C^{11+\alpha+8\varepsilon}(\mathbf{T})$, then the mapping Γ is differentiable at u_0 for the norms :

$$C^6(\mathbf{T}) \longrightarrow C^{3+\alpha}(\mathbf{T}) .$$

Theorem 5 implies : if q, u_0 and γ_0 are C^∞, then Γ is differentiable for the norms :

$$C^{4+\beta}(\mathbf{T}) \longrightarrow C^{3+\alpha}(\mathbf{T}) \text{ with } 0<\alpha<\beta, \ \alpha,\beta \text{ small.}$$

□

Proof of Theorem 4 : We apply E. ZEHNDER's theorem (see [1],Theorem 3.1) as it is formulated in A. DERVIEUX [2] (Theorem 1 and Corollaries 1 and 2) and the present proof is identical to the proof of theorem 9 of that paper, except the following features :

In a first step, (3.28) *is assumed to be satisfied* and, by reference to the notation of [Ibid.],we introduce

$$X_\sigma = C^{\ell+\alpha+\sigma}(\mathbf{T})$$
$$Y_\sigma = C^{\ell+\alpha+1+\sigma}(\mathbf{T})$$
$$Z_\sigma = C^{\ell+\alpha-\varepsilon+\sigma}(\mathbf{T})$$

and

$$\text{(3.31)} \quad \begin{cases} \overline{\pi} = 1 + \varepsilon \\ s = 8(1+\varepsilon) \\ \lambda = \dfrac{15}{4}(1+\varepsilon) ; \end{cases}$$

then Proposition 4 makes it easy to verify that assumptions (1.12), (1.13) and $(1.14)_2$ of [Ibid.] are satisfied.

Using Lemma 4, jointly with Propositions 4 and 6, we see that Corollary 3 of [Ibid.] applies so that conditions $(1.14)_3$ and $(1.15)_4$ of [Ibid] are satisfied.

Conditions $(1.14)_1$ and $(1.15)_3$ are easily directly verified while $(1.15)_2$ is a consequence of Proposition 6.

It remains then to apply Corollary 1 [Ibid.] and the proof of Theorem 4 is complete under the assumption (3.28).

Second step : Condition (3.28) is no more assumed ; the method to extend the result is the same that in [Ibid.]. □

Proof of Theorem 5 : The two steps of the above proof are available, the Corollary 2 of [Ibid.] being applied in Step 1 instead of Corollary 1. □

3.5. Uniqueness and local smoothness

We do not know if the "solution" Γ which is proved to exist by Theorem 4 is always unique. However, if that theorem applies for both ℓ and $\ell+3$, then uniqueness holds for a continuous solution (with respect to $\ell+3$) ; more precisely, we introduce the following conditions :

(3.32) $$\begin{cases} 0<\varepsilon<\frac{1}{8} \\ 2\varepsilon<\alpha<1 \end{cases}$$

(3.33) $$\begin{cases} q\in C^{5+\alpha+8(\varepsilon+1)}(\overline{B}) \\ q>0 \end{cases}$$

(3.34) The couple (u_0,γ_0) is a solution of Problem (2.2), satisfying (2.8), with

$$\left| \begin{array}{l} u_0\in C^{5+\alpha+8(\varepsilon+1)}(\mathbf{T}) \\ \gamma_0\in C^{6+\alpha+8(\varepsilon+1)}(\mathbf{T}) \ ; \end{array} \right.$$

then by the same argument as in [2] (section 2.5) we get the following result :

Corollary 1 (Uniqueness) : We assume that condition (3.32) *to* (3.34) *are fulfilled ; let* $\hat{\Gamma}$ *be a continuous mapping from a neighborhood* V *of* u_0 *in* $C^{5+\alpha+\frac{15}{4}(\alpha+1)}(\mathbf{T})$ *into* $C^{3+\alpha+\overline{\pi}}(\mathbf{T})$ *such that :*

$$\left| \begin{array}{l} \hat{\Gamma}(u_0) = \gamma_0 \\ \Psi[u,\hat{\Gamma}(u)] = 0 \quad \forall u\in V \\ |\hat{\Gamma}(u)|_{C^{6+\alpha}(\mathbf{T})} \le M \quad \forall u\in V \\ (M \text{ indep. of } u) \end{array} \right.$$

then $\hat{\Gamma}$ *is identical to every solution mapping* Γ *whose existence is proved by the application of Theorem 4 with* $\ell = 5$. □

Corollary 7 of [Ibid.] is trivially transposed as follows :

Corollary 2 (Smoothness) : Let us assume that (u_0,γ_0) *is a solution of Problem* (2.2)*, satisfying condition* (2.8)*, with*

$$(3.35) \qquad \begin{cases} u_0,\gamma_0 \in C^\infty(\mathbf{T}) \\ q \in C^\infty(\overline{B}) \\ q>0 \end{cases}$$

then, for every number λ *greater than* 2*, there exists a neighborhood* D_λ *of* u_0 *in* $C^{2+\alpha+\lambda}(\mathbf{T})$ *and a solution* Γ *defined on* D_λ *which is* C^∞ *from* $D_\lambda \cap C^\infty(\mathbf{T})$ *into* $C^\infty(\mathbf{T})$. □

4. EXTENSIONS

4.1. General annular regions

The above results still hold when the curves u_0 and γ_0 are *the (regular) boundaries of two simply connected domains* with one in the other one.

This extension can be decomposed in three steps which we shall sum up here.

Step 1 : We must firstly define an *extended parametrization of the boundaries* : according to A. DERVIEUX [1,2], we assume that the points of both two boundaries are moved along two smooth vectors $\vec{V}$ and $\vec{W}$ defined respectively on γ_0 and u_0 and sufficiently near of the corresponding normal vector field.

The Fourier-type coupling problem is introduced mutatis mutandis and the *stationarity property* is similarly verified, jointly with the regularity properties with respect to u and γ ; unfortunately, the translation invariance property of the starshaped case seems to have no straightforward extension to the present case.

Step 2 : There exists a one-to-one smooth mapping Θ from $\overline{B}$ into $\overline{B}$ such that for boundaries u, γ sufficiently near of u_0 and γ_0, the two bounded domains limited by (respectively) the curves $\Theta(u)$ and $\Theta(\gamma)$ are starshaped with respect to the same point.

Transporting the initial problem with Θ, we get a new starshaped Free Boundary Problem, with a stationary coupling system with variable coefficients. Now, using the remarks of A. DERVIEUX [2] (Section 2.7.1.), we see that the above Section 3 still applies to this new Free Boundary Problem ; then the conclusion of the G.I.F.T. holds.

Step 3 : Since, from Step 2 and returning to the initial geometry, existence, continuity and (punctual) differentiability hold, the gradient of the mapping Γ can be directly identified and is given by formulas (3.29) except the taking in account of the extended parametrization of the boundaries :

$$(4.1) \quad \begin{cases} \forall \delta u \; C^{\sigma_1}(u_0) \; (\sigma_1 \text{ is defined in Theorem 4}) \\ \dfrac{d\Gamma}{du}(u_0).\delta u = -\dfrac{1}{\langle\vec{n},\vec{V}\rangle \; q|_{\gamma_0}} \; \dot{z}_F.\delta u|_{\gamma_0} \end{cases}$$

with

$$(4.2) \quad \begin{cases} -\Delta(\dot{z}_F.\delta u) = 0 \quad \text{in } \Omega_{u_0,\gamma_0} \\ \dot{z}_F.\delta u = \langle\vec{n},\vec{W}\rangle \dfrac{\partial z_F(u_0,\gamma_0)}{\partial n} \delta u \quad \text{on } u_0 \\ \left(-\dfrac{\partial}{\partial n} + b\right)\left(\dot{z}_F.\delta u\right) = 0 \quad \text{on } \gamma_0 \end{cases}$$

4.2. Choice of the perturbed data

Similarly to A. DERVIEUX [2], the above results are easily extended to the study of the perturbation of a lot of boundary or distributed data with the restriction that we only obtain a punctual Gateaux-derivative in the case where the perturbed data is a distributed one.

Let us consider, for example, that the boundary u_0 is fixed but that the function q is perturbed from an "initial" value q_0 ; then the Gateaux-derivative of the Free Boundary with respect to q at point q_0 is given by (general annular case) :

$$(4.3) \quad \begin{cases} \forall \delta q \in C^{\sigma_1}(\overline{B}) \quad (\sigma_1 \text{ defined on Theorem 4}) \\ \dfrac{d\Gamma}{dq}(q_0).\delta q = -\dfrac{1}{\langle\vec{n},\vec{V}\rangle \; q_0|_{\gamma_0}} \; \dot{z}_F.\delta q|_{\gamma_0} \end{cases}$$

with

$$(4.4)\qquad \begin{cases} -\Delta(\dot{z}_F.\delta q) = 0 & \text{in } \Omega_{u_0,\gamma_0} \\ \dot{z}_F.\delta q = 0 & \text{on } u_0 \\ \left(\frac{\partial}{\partial n} + b\right)\left(\dot{z}_F.\delta q\right) = \delta q & \text{on } \gamma_0. \end{cases}$$

4.3. Application to monotonicity properties

The sign study of derivatives (4.1) and (4.3) is straightforward derived when Maximum Principle applies to the linearized coupling system occuring in (4.2) and (4.4) : let us consider the following eigenvalue problem :

$$(4.5)\qquad \begin{cases} -\Delta\phi = \lambda\phi & \text{in } \Omega_{u_0,\gamma_0} \\ \phi = 0 & \text{on } u_0 \\ \left(\frac{\partial}{\partial n} + b\right)\phi = 0 & \text{on } \gamma_0 \end{cases}$$

and let us assume that

(4.6) Every eigenvalue λ of (4.5) is negative.

Then this implies; via Rayleigh quotient that coerciveness holds for the corresponding bilinear Functionnal

$$a(\phi,\psi) = \int_{\Omega_{u_0,\gamma_0}} \langle \vec{\text{grad}}\ \phi, \vec{\text{grad}}\ \psi\rangle\, dx + \int_{\gamma_0} b\ \phi\psi\ d\sigma.$$

Therefore the weak Maximum Principle (see for example G. STAMPACCHIA [1]) applies and we have the proposition :

Proposition 7 : We assume that (4.6) *holds ;*

(i) *if the increment* δu *is non positive, then the expression defined by* (4.1) (4.2) *satisfies*

$$\frac{d\Gamma}{du}(u_0).\delta u \geq 0$$

(ii) *if the increment* δq *is non negative along* γ_0, *then the expression defined by* (4.3) (4.4) *satisfies*

$$\frac{d\Gamma}{dq}(q_0).\delta q \geq 0.$$

□

Consequences : Since smoothness holds *locally* via Corollary 2, the sign property of the Frechet-derivtive (resp. Gateaux-) (4.1) (resp. (4.3)) implies the corresponding monotonicity property.

Remark 3 : In the particular case where q_0 is a real constant and if u_0 limits a convex bounded domain, then the coefficent b is identical to the curvature H which is non-negative ; therefore condition (4.6) is trivially satisfied and Proposition 7 applies and we get some monotonicity properties shown by E. TEPPER [1]. □

4.4. Variation of the distributed solution

The continuity and differentiability of an extension $\overline{Z}(u)$ of the distributed solution Z(u) of Problem (2.2) (corresponding to the data u) is obtained from Theorems 4 and 5 via the mapping $\overline{z}_F$ defined in Proposition 5.

$$\overline{Z}(u) = \overline{z}_F(u,\Gamma(u)). \tag{4.7}$$

The composite differentiation gives

$$\frac{d\overline{Z}}{du}(u_0).\delta u = \frac{\partial \overline{z}_F}{\partial u}(u_0,\gamma_0).\delta u + \frac{\partial \overline{z}_F}{\partial \gamma}(u_0,\gamma_0).\frac{d\Gamma}{du}(u_0).\delta u. \tag{4.8}$$

Now, we remember that the coupling problem satisfies the stationary condition (2.9) ; thus the second term of the right-hand sum is identically zero ; the first term is easily computed via Hadamard's formula and we get the

Corollary 3 : *Under the assumption of Theorem 4, there exists a mapping* $\overline{Z} : u \mapsto \overline{Z}(u)$ *defined on* D_{σ_1} *such that*

$$\begin{cases} \overline{Z}(u_0)\big|_{\Omega_{u_0,\gamma_0}} & \textit{is the distributed solution of Problem (2.2)} \\ \textit{associated to } \gamma_0 & \end{cases} \tag{4.9}$$

$$\forall u \; D_{\sigma_1},\ \left(u,\Gamma(u),\overline{Z}(u)\big|_{\Omega_{u,\Gamma(u)}}\right) \textit{ is a solution of Problem (2.2) ;} \tag{4.10}$$

this mapping is continuous on D_{σ_1} *and differentiable at* u_0 *for the norms*

$$C^{\sigma_1}(\mathbf{T}) \longrightarrow C^{\ell-1+\alpha}(\overline{B}) \ ;$$

the restriction to Ω_{u_0,γ_0} *of its derivative at* u_0 *is uniquely defined by*

$$(4.11)\quad \begin{cases} \forall \delta u \in C^{\sigma_1}(\mathbf{T}) \\ \Delta\left[\frac{d\overline{Z}}{du}(u_0).\delta u\right] = 0 \quad on\ \Omega_{u_0,\gamma_0} \\ \frac{d\overline{Z}}{du}(u_0).\delta u = \langle\vec{n},\vec{r}\rangle\ \frac{\partial Z(u)}{\partial n}\ \delta u \quad on\ u_0 \\ \left(\frac{\partial}{\partial n} + b\right)\left[\frac{d\overline{Z}}{du}(u_0).\delta u\right] = 0 \quad on\ \gamma_0 \end{cases}$$

Comment : Let us consider -just for this comment- that the distributed solution Z is the main unknown of Problem (2.2) ([1]) ; then the operator of system (4.11) is the *linearized variant* of the non-linear (because of the Free Boundary) one acting in (2.2). In particular, not enough this operator is (as noted above) useful for the construction of iterative solvers for (2.2) but it can be also used for the construction of the adjoint system in Optimal Control (see Section 6). Furthermore, we see that the stationarity property implies a first order decoupling between Γ and Z. □

5. STATIONARITY CONDITION : ADDITIONAL REMARKS

5.1. Non-zero distributed right-hand side

As we told in Section 2.3., several authors looked for stationary problems to construct superlinearly fast iterative resolution methods for Free Boundary Problems. In particular, it is interesting, for both iterative methods and perturbation study, to precise the abstract sufficient condition of stationarity given by P.R. GARABEDIAN

$$(5.1)\qquad \{\mathcal{B}(z)\}_\nu = 0$$

where $\mathcal{B}$ denotes the boundary operator of what we call the "coupling system", and the index ν represents a normal derivative (see P.R. GARABEDIAN [2] p. 222 and C.W. CRYER [1] Stat.(5.5)).

We want to point out that, for this analysis, wariness is necessary to deal with the formal computation of variations. Let us consider the following Free Boundary Problem

---

([1]) We cannot say "the only one unknown" as for the Obstacle Problem because in the present case, the Free Boundary seems necessary to define Z(u).

$$(5.2)\qquad \begin{cases} \Delta z = f \quad \text{in } \Omega_{u,\gamma} \\ \left.\begin{array}{l} z = h \\ \dfrac{\partial z}{\partial n} = g \end{array}\right\} \text{on the Free Boundary } \gamma \\ + \text{ a boundary condition on } u \ ; \end{cases}$$

the condition along the fixed boundary u is not important for the sequel.

We introduce the following Fourier-type coupling system

$$(5.3)\qquad \begin{cases} \Delta z_F(u,\gamma) = f \quad \text{in } \Omega_{u,\gamma} \\ \mathcal{B}[z_F(u,\gamma)] = 0 \quad \text{on } \gamma \\ + \text{ a boundary condition on } u \end{cases}$$

with

$$(5.4)\qquad \mathcal{B}[\phi] = \frac{\partial \phi}{\partial n} - g + b(\phi - h).$$

We look for a function b such that (5.3) will be stationary with respect to γ when γ is the Free Boundary solution of (5.2).

Using again the above domain dependance techniques, we introduce the following functional

$$\Phi(u,\gamma,z,\phi_1,\phi_2) = \int_{\Omega_{u,\gamma}} \phi_1(\Delta z - f)\, dx + \int_\gamma \phi_2 \left[\frac{\partial z}{\partial n} + b(z - h) - g\right] d\sigma.$$

The variation of $z_F(u,\gamma)$ with respect to γ is defined by (excepted for the bounbary condition along u) :

$$(5.5)\qquad \begin{cases} \forall \phi_1,\phi_2 \in C^\infty(\overline{\Omega}_{u,\gamma}) \\ \dfrac{\partial \Phi}{\partial z}(u,\gamma,z_F(u,\gamma),\phi_1,\phi_2) . \dfrac{\partial z_F}{\partial \gamma}(u,\gamma) = \\ \qquad - \dfrac{\partial \Phi}{\partial \gamma}(u,\gamma,z_F(u,\gamma),\phi_1,\phi_2) . \delta\gamma . \end{cases}$$

Let us compute the right-hand side of (5.5)

$$\frac{\partial \Phi}{\partial \gamma}(u,\gamma,z_F(u,\gamma),\phi_1,\phi_2) . \delta\gamma = \sum_{i=1}^{9} A_i$$

with (using the notations of Sec. 4.1.) :

$$A_1 = \int_\gamma \phi_1 \ (\Delta z - f) \ <\vec{n},\vec{V}> \ \delta\gamma \ d\sigma$$

$$A_2 = \int_\gamma \{<\vec{\text{grad}} \ \phi_2, \vec{\text{grad}} \ z> + \phi_2 \ \Delta z\} \ <\vec{n},\vec{V}> \ \delta\gamma \ d\sigma$$

$$A_3 = \int_\gamma \phi_2 \ (z - h) \ \frac{\partial b}{\partial \gamma} \ .\delta\gamma \ d\sigma$$

$$A_4 = \int_\gamma \frac{\partial \phi_2}{\partial n} \left[(z - h) \ b - g\right] <\vec{n},\vec{V}> \ \delta\gamma \ d\sigma$$

$$A_5 = \int_\gamma \phi_2 \ (z - h) \ \frac{\partial b}{\partial n} \ <\vec{n},\vec{V}> \ \delta\gamma \ d\sigma$$

$$A_6 = \int_\gamma \phi_2 \ b \ \frac{\partial}{\partial n} \ (z - h) \ <\vec{n},\vec{V}> \ \delta\gamma \ d\sigma$$

$$A_7 = -\int_\gamma \phi_2 \ \frac{\partial g}{\partial n} \ <\vec{n},\vec{V}> \ \delta\gamma \ d\sigma$$

$$A_8 = \int_\gamma \phi_2 \ H \ (z - h) \ b \ <\vec{n},\vec{V}> \ \delta\gamma . d\sigma$$

$$A_9 = \int_\gamma H \ \phi_2 \ g \ <\vec{n},\vec{V}> \ \delta\gamma \ d\sigma .$$

If the couple $(\gamma, z_F(u,\gamma))$ is a solution of the Free Boundary Problem (5.2), we have

(5.6)
$$\begin{cases} A_1 = A_3 = A_5 = A_8 = 0 \\ A_2 + A_4 + A_6 + A_7 + A_9 = \int_\gamma \frac{\partial \phi_2}{\partial \tau} \ \frac{\partial h}{\partial \tau} \ <\vec{n},\vec{V}> \ \delta\gamma \ d\sigma \\ \qquad + \int_\gamma \phi_2 \left\{ b \ \frac{\partial}{\partial n} \ (z - h) + f - Hg - \frac{\partial g}{\partial n} \right\} <\vec{n},\vec{V}> \ \delta\gamma \ d\sigma . \end{cases}$$

Integrated by parts, the first term of the right-hand sum yields

$$\int_\gamma \frac{\partial \phi_2}{\partial \tau} \ \frac{\partial h}{\partial \tau} \ <\vec{n},\vec{V}> \ \delta\gamma \ d\sigma =$$

$$- \int_\gamma \phi_2 \ \frac{\partial h}{\partial \tau} \ <\vec{n},\vec{V}> \ \frac{\partial(\delta\gamma)}{\partial \tau} \ \delta\gamma$$

$$- \int_\gamma \phi_2 \ \delta\gamma \ \frac{\partial}{\partial \tau} \left[\frac{\partial h}{\partial \tau} \ <\vec{n},\vec{V}> \ \frac{d\sigma}{ds} \right] \frac{ds}{d\sigma} \ d\sigma .$$

Returning to (5.5) we get

$$(5.7)_1 \quad \begin{cases} \Delta \left[\dfrac{\partial z_F}{\partial \gamma} (u,\gamma).\delta\gamma \right] = 0 \quad \text{on } \Omega_{u,\gamma} \\ \left(\dfrac{\partial}{\partial n} + b \right) \dfrac{\partial z_F}{\partial \gamma} (u,\gamma).\delta\gamma = C(f,g,h). \\ + \text{ a homogeneous boundary condition on } u \end{cases}$$

with

$$(5.7)_2 \quad \begin{cases} C(f,g,h).\delta\gamma = - \dfrac{\partial h}{\partial \tau} <\vec{n},\vec{V}> \dfrac{\partial(\delta\gamma)}{\partial \tau} + \Big\{ b \dfrac{\partial}{\partial n} (z - h) + f \\ - Hg - \dfrac{\partial g}{\partial n} - \dfrac{1}{<\vec{n},\vec{V}>} \dfrac{ds}{d\sigma} \dfrac{\partial}{\partial \tau} \left[\dfrac{\partial h}{\partial \tau} <\vec{n},\vec{V}> \dfrac{d\sigma}{ds} \right] \Big\} \times <\vec{n},\vec{V}> \delta\gamma. \end{cases}$$

We want that $C(f,g,h)$ identically vanishes ; this is possible only if

$$(5.8) \qquad \frac{\partial h}{\partial \tau} \equiv 0 \quad \text{on } \gamma$$

and by choosing

$$(5.9) \qquad b = \frac{f - Hg - \dfrac{\partial g}{\partial n}}{h - \dfrac{\partial g}{\partial n}}$$

Now, an easy computation shows that, even if (5.8) does not hold, (5.9) implies (5.1) ; then we have the following result :

Proposition 8 : We assume that expression (5.9) *is defined and that the corresponding coupling system is well-posed :*

(i) *Let* γ_0 *be a solution of Problem* (5.2) *such that* (5.8) *holds, then the coupling problem* (5.3) (5.9) *is stationary at* γ_0.

(ii) *If condition* (5.8) *does not hold for a solution* γ_1 *of Problem* (5.2), *then the coupling system defined by* (5.3) (5.9) *satisfies* (5.1) *but it is not stationary at* γ_1. □

Remark 4 : Considering Problem (5.2), we distinguish two particular cases :

Case 1 : The trace $\left(g - \dfrac{\partial h}{\partial n} \right)\Big|_\gamma$ is strictly positive and $\dfrac{\partial h}{\partial \tau}$ is equal to zero ; then if the coupling system is well-posed we have a stationary one ; we study in A. DERVIEUX [3] a case where it is not well-posed.

Case 2 : The trace $\left(g - \frac{\partial h}{\partial n}\right)\Big|_\gamma$ is identically zero : this is the Obstacle Problem studied in D.G. SCHAEFFER [1], A. DERVIEUX [2].

For the remaining cases, the problem of existence of stationary coupling systems seems to be widely open. Now, this is not an insuperable impediment as we shall show in the following section. □

5.2. A non-stationary coupling

The above stationary condition brings some basic simplifications but it is not indispensable : to prove this assertion, we shall consider again the Free Boundary Problem (5.2), and introduce, as the coupling system, the Dirichlet one

$$(5.10) \qquad \begin{cases} \Delta z_D(u,\gamma) = f \quad \text{on } \Omega_{u,\gamma} \\ z_D(u,\gamma) = h \quad \text{on } \gamma \\ + \text{ B.C. on } u \end{cases}$$

and we define the mapping

$$(5.11) \qquad \Psi(u,\gamma) = \left(\frac{\partial z_D(\gamma)}{\partial n} - g\right)\Big|_\gamma$$

Then Ψ trivially represents Problem (5.2).

Our purpose is to find a sufficient condition for the (at least formal) inversibility of the Jacobian derivative $\frac{\partial \Psi}{\partial \gamma}(u,\gamma)$ for a solution of (5.2).

Let us compute this derivative : differentiating formally (1) $z_D(u,\gamma)$ we obtain

$$(5.12) \qquad \begin{cases} \Delta\left[\frac{\partial z_D}{\partial \gamma}(u,\gamma).\delta\gamma\right] = 0 \quad \text{on } \Omega_{u,\gamma} \\ \frac{\partial z_D}{\partial \gamma}(u,\gamma).\delta\gamma = \left[g - \frac{\partial h}{\partial n}\right] \langle\vec{n},\vec{V}\rangle \, \delta\gamma \, d\sigma \\ + \text{ homogeneous B.C. on } u, \end{cases}$$

and, by a composite differentiation, we have

(1) But this can be done rigorously as hinted in Section 3.2.

(5.13) $$\frac{\partial\Psi}{\partial\gamma}(u,\gamma).\delta\gamma = \frac{\partial}{\partial n}\left[\frac{\partial z_D}{\partial\gamma}(u,\gamma).\delta\gamma\right]\Big|_\gamma$$

$$+ \sum_{i=1,2} \frac{\partial n_i}{\partial\gamma}(\gamma).\delta\gamma \frac{\partial z_D}{\partial x_i}\Big|_\gamma$$

$$+ \sum_{i,j=1,2} n_i(\gamma)\, V_j(\gamma) \frac{\partial^2 z_D}{\partial x_i \partial x_j}\Big|_\gamma \delta\gamma.$$

It can be shown that the derivative $\frac{\partial n_i}{\partial\gamma}(\gamma)$ admits the following decomposition

$$\frac{\partial n_i}{\partial\gamma}(\gamma).\delta\gamma = E_0^i \times \delta\gamma + E_1^i \times \frac{\partial\delta\gamma}{\partial s}$$

where E_0^i, E_1^i do not depend on $\delta\gamma$, and where s is a curvilinear abscissa ; let

$$E_0 = \sum_{i=1,2} E_0^i \frac{\partial z_d}{\partial x_i}\Big|_\gamma + \sum_{i,j=1,2.} n_i(\gamma)\, V_j(\gamma) \frac{\partial^2 z_D}{\partial x_i \partial x_j}$$

$$E_1 = \sum_{i=1,2} E_1^i \frac{\partial z_D}{\partial x_i}\Big|_\gamma$$

then

$$\frac{\partial\Psi}{\partial\gamma}(u,\gamma).\delta\gamma = E_0 \times \delta\gamma + E_1 \times \frac{\partial\delta\gamma}{\partial s}$$

Now we introduce the following function

(5.14) $$\psi(s) = \left[g - \frac{\partial h}{\partial n}\right]\Big|_\gamma <\vec{n},\vec{V}>$$

and,for ψ positive (resp. negative), we consider, for a given function v defined on γ, the system

(5.15) $$\begin{cases} \Delta w = 0 \quad \text{in } \Omega_{u,\gamma} \\ \frac{\partial w}{\partial n} + E_1\, \psi \frac{\partial w}{\partial\theta} - E_1 \frac{\psi'}{\psi^2} w + E_0 \frac{1}{\psi} w = v \quad \text{on } \gamma \\ + \text{ homogeneous B.C. on } u. \end{cases}$$

Proposition 9 : If, for a solution γ_0 *of Problem* (5.2) *corresponding to the data* u_0, γ_0, *the function* ψ *defined by* (5.14) *is positive (resp. negative) and if the corresponding system* (5.15) *is well-posed in convenient spaces, then the Jacobian derivative* $\Psi(u_0,\gamma_0)$ *is at least formally inversible and its inverse is given by :*

$$(5.16) \qquad \begin{cases} \forall v \in C^\infty(\gamma_0) \\ \left[\frac{\partial \Psi}{\partial \gamma}(u_0,\gamma_0)\right]^{-1} . v = \frac{1}{\psi} w\big|_\gamma \\ w \textit{ given by } (5.15). \end{cases}$$

□

Remark 5 : Problem (5.15) have an oblique-derivative boundary condition along γ (see e.g. AGMON - DOUGLIS - NIRENBERG [1], J.L. LIONS, E. MAGENES [1]).

Although it is farther from the physical context, the present point of view brings important simplifications in the proofs, which we want to discuss in a work in preparation. □

Remark 6 : The main difference with the conditions of Case 1 in Remark 4 is that the condition $\frac{\partial h}{\partial \tau} \equiv 0$ disappears ; the variation formulas obtained are definitely more complicated : in particular, in the composite differentiation corresponding to (4.8), the second term of the right-hand sum is generally not zero. □

6. APPLICATION TO OPTIMAL CONTROL

The purpose of this section is to show how the above results can be applied to a very simplified but concrete problem. Our point of view is now definitely different of the above one since most of the following computations will be formal ; rigorous proofs seem to be out of reach.

6.1. The physical problem

We shall be interested by the two-dimensional irrotational stationary flow of an inviscid incompressible fluid getting out from an exhaust outlet. Thought the flow is physically assumed to be extended to infinity at right and left horizontal directions,we shall consider a lengthened bounded domain with parallel streamlines at extremities (Fig.4).

For the sake of simplicity, the Kutta-Joukovsky condition at the separation point will not be taken in account (this point is discussed in a remark later).

With respect to these assumptions and simplifications, the mathematical modelization is defined as follows :

The geometrical data are two curves ABC and FE parametrized by two continuous functions b and u :

$$(x,y)\in ABC \Leftrightarrow \{x\in]0,2[,\ y = b(x)\}$$

$$(x,y)\in FE \Leftrightarrow \{x\in]0,1[,\ y = u(x)\}$$

$$\forall x\in[0,1]\quad u(x)>b(x)$$

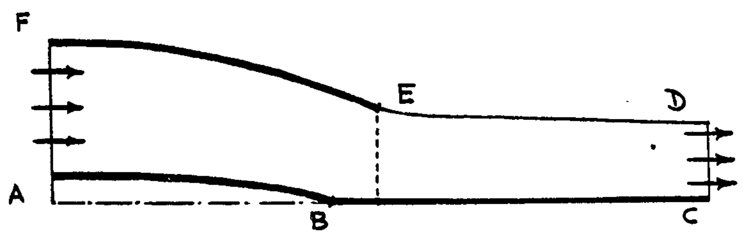


Fig.4

Let d be a negative real number ; *the Free Boundary Problem* is

(6.1) *Find a continuous curve* γ *:*

$(6.1)_1$ $$(x,y)\in\gamma \Leftrightarrow \{x\in[1,2],\ y = \gamma(x)\}$$

such that

$(6.1)_2$ $$\begin{cases}\gamma(1) = u(1) \\ \gamma(x)>b(x) \quad \forall x\in[1,2]\end{cases}$$

and a function z *defined in the domain* $\Omega_{u,\gamma}$

$(6.1)_3$ $$\begin{cases}\Omega_{u,\gamma} = \{(x,y)\in\mathbb{R}^2 \text{ such that} \\ \qquad x\in]0,1] \text{ and } b(x)<y<u(x) \text{ or} \\ \qquad x\in[1,2[\text{ and } b(x)<y<\gamma(x)\}\end{cases}$$

and satisfying

$$(6.1)_4 \quad \begin{cases} \Delta z = 0 & \text{in } \Omega_{u,\gamma} \\ z = d & \text{on ABC} \\ \frac{\partial z}{\partial x} = 0 & \text{on AF} \cup \text{CD} \\ z = 0 & \text{on FD} \\ \frac{\partial z}{\partial n} = 1 & \text{on ED.} \end{cases}$$

The function z is a stream function and the number d is a parameter related to the output.

For existence and uniqueness results for problems of this type, we refer to P.R. GARABEDIAN, H. LEWY, M. SCHIFFER [1], P.R. GARABEDIAN [1], G. BIRKHOFF, E.H. ZARANTONELLO [1], P.R. GARABEDIAN, D.C. SPENCER [1].

The Optimal Control Problem which we consider is the following :

(6.2) *Find in an "admissible" set* U_{ad} *the function* u *for which the following cost functional will be minimal*

$$(6.2)_1 \qquad j(u) = \int_{AB} \left| \frac{\partial Z(u)}{\partial y} - \frac{\partial z_d}{\partial y} \right|^2 d\sigma$$

where Z(u) *is the solution of Problem* (6.1) *for the data* u, *and* z_d *a prescribed smooth enough stream function.*

This problem have a (much more complicated) industrial origin, viz the optimal structural design of an airplane exhaust outlet, which has been indicated to us by A. GENINET (S.N.E.C.M.A.).

<u>*Remark 7*</u> *: (Kutta-Joukovsky condition)* Problem (6.1) is physically likely only if the Free Boundary ED separates tangentially to the wall FE at point E or equivalently if the solution satisfies $\frac{\partial Z(u)}{\partial n} = 1$ *at* E (see for example Ph. MORICE [1] for computations including this condition) ; in the present case, to take in account this condition, we could for example introduce d as an additional control variable and consider Kutta-Joukovsky condition as an additional constraint (operating on the state variables). □

6.2. Application of the above sections

Let us introduce the Fourier-type coupling system : for every convenient couple (u,γ) we denote by $z_F(u,\gamma)$ the solution of

$$
(6.3)\qquad \begin{cases} \Delta z_F(u,\gamma) = 0 \quad \text{in } \Omega_{u,\gamma} \\ z_F(u,\gamma) = \begin{cases} d & \text{on } AC \\ 0 & \text{on } FE \end{cases} \\ \dfrac{\partial}{\partial x}\left[z_F(u,\gamma)\right] = 0 \quad \text{on } AF\cup CD \\ \left(\dfrac{\partial}{\partial n} + H\right) z_F(u,\gamma) = 1 \quad \text{on } ED \end{cases}
$$

with

$$
(6.4)\qquad H = \frac{\gamma''}{\left[1+\gamma'^2\right]^{3/2}} \; ;
$$

The curve γ is likely analytic (cf. H. LEWY [1], or more recently D. KINDERLEHRER, L. NIRENBERG [1]) except at the separation point E where a singularity occurs.

The coupling problem has two *uses* :

Firstly both P.R. GARABEDIAN's work (see [1]) and Sections 2 and 3 suggest to introduce the following quasi-Newton iterative algorithm to solve Problem (6.1) (¹)

$$
(6.5)\qquad \begin{cases} \gamma^{m+1} = \gamma^m - \eta(\gamma^m)\,\Psi(\gamma^m) \text{ with} \\ \Psi(\gamma) = z_F(\gamma)\big|_\gamma \\ \eta(\gamma) = \dfrac{1}{\langle\vec{n},\vec{e}_2\rangle\big|_\gamma} \end{cases} .
$$

Secondly, following Section 4.4., we can choose, as a descent direction the functional D

$$
(6.6)\qquad \begin{cases} \forall\delta u\in C^\infty[1,2] \quad \delta u(1) = 0, \\ D(u).\delta u = \displaystyle\int_{FE} \frac{\partial z_F[u,\Gamma(u)]}{\partial n}\,\frac{\partial p}{\partial n}\,\langle\vec{n},\vec{e}_2\rangle\,\delta u\, d\sigma \end{cases}
$$

where $\Gamma(u)$ is the corresponding Free Boundary and p the distributed adjoint state, solution of

(¹) For a discussion concerning the choice of the solver of a Free Boundary Problem, see A. DERVIEUX [2].

$$(6.7)\qquad \begin{cases} \Delta p = 0 \quad \text{on } \Omega_{u,\Gamma(u)} \\ p = 0 \quad \text{on EF} \\ p = \langle \vec{n},\vec{e}_2 \rangle \left\{ \dfrac{\partial z_F[u,\Gamma(u)]}{\partial y} - \dfrac{\partial z_d}{dy} \right\} \quad \text{on AB} \\ \dfrac{\partial p}{\partial x} = 0 \quad \text{on AF} \cup \text{DC} \\ \left(\dfrac{\partial}{\partial n} + H \right) p = 0 \quad \text{on ED} \end{cases}$$

6.3. Discrete approximation and solving methods

The coupling problem (6.3) is approximated by a classical P_1-conforming finite element with a variable grid of 400 triangles, depending on the boundaries,with fixed vertical rows (see Fig.5) ; the curvature is approximated from formula (6.4) via a three point parabolic interpolation ; the corresponding linear system is solved by Cholesky's factorization method.

The *discretized Free Boundary Problem* is the following :

$$(6.8)\qquad \Psi^h(\gamma^h) = z_F^h(u^h,\gamma^h)_{|\gamma^h} = 0 \quad .$$

where $z_F^h(u^h,\gamma^h)$ denotes the discretized coupling system.

We have not showed the existence of a solution to (6.8) ; we shall just give a very simple local result :

Let us assume that

$$(6.9)\qquad b_0 \equiv 0 \ ; \ u_0 \equiv 1 \ ; \ d_0 = -1 \ ;$$

then, both continuous and discrete Free Boundary Problems have the following trivial solution

$$(6.10)\qquad \Gamma(u_0) \equiv 1 \ ; \ Z(u_0) = x - 1.$$

We introduce now the assumption

$$(6.11)\qquad \begin{cases} \text{for the data (6.8) and solution (6.9)} \\ \dfrac{\partial \Psi^h}{\partial \gamma^h}(u_0,\gamma_0) \text{ is an inversible matrix} \end{cases}$$

We showed in the above sections that the continuous variant of (6.11) is satisfied in the sense that the product by a non vanishing function, viz the constant 1 in the present case, is inversible. Thus it is reasonable to hope that this inversibility still holds for the discrete variant. We verified it in numerical experiments.

The Implicit Function Theorem implies :

Proposition 10 : Under condition (6.11), *Problem* (6.8) *has a solution* $\Gamma^h(u^h)$ *for every* u^h *in a neighborhood* V^h *of* $u_0 \equiv 1$ *; this solution is* C^∞ *with respect to* u^h *and locally unique as a continuous branch of solutions.* □

The existence of a solution to Problem (6.8) have been always *numerically* verified ($\Psi_h(u^h,\gamma^h)<10^{-16}$ with extended precision) while never several solutions have been obtained.

The algorithm for solving (6.8) is directly transposed from (6.5).

The discrete cost functional is chosen as follows :

$$j^h(u_h) = J^h\left\{z_F^h\left[u^h,\Gamma^h(u^h)\right]\right\} \tag{6.12}$$

with

$$J^h(z) = \sum_i \alpha_i \left[\delta_{i,N-1}(z - z_d^h)\right]^2$$

where the sum is taken for the points of AB (with α_i = 0.5 for extremities, α_i = 1. elsewhere) and with the difference operator

$$\delta_{i,j}\, w = \frac{w_{i,j+1} - w_{i,j}}{y_{i,j+1} - y_{i,j}} ,$$

$y_{i,j}$ denoting the ordinate of vertex i,j.

The *descent direction* $D^h(u^h)$ is not a rough discretization of (6.6) (6.7), but an approximation of the discrete gradient

$$\frac{dj^h}{du^h}(u^h) = \frac{\partial J^h}{\partial z} \cdot \left\{\frac{\partial z_F^h}{\partial u^h} - \frac{\partial z_F^h}{\partial \gamma^h}\left[\frac{\partial \Psi^h}{\partial \gamma^h}\right]^{-1} \frac{\partial \Psi^h}{\partial u^h}\right\}$$

obtained by neglecting the second term of the right-hand difference (this term is zero in the continuous case)

$$D^h(u^h) = \frac{\partial J^h}{\partial z} \cdot \frac{\partial z_F^h}{\partial u^h} \quad ; \tag{6.13}$$

the right-hand side is computed *exactly*, as in D. BEGIS, R. GLOWINSKI [1], A. MARROCCO, O. PIRONNEAU [1], with the introduction of an adjoint system which is a discrete variant of (6.7).

The optimization algorithm which we used in an unconstrained FLETCHER-REEVES conjugate gradient algorithm.

6.4. Numerical results

The algorithm for solving the Free Boundary Problem, although it is rigorously only a Picard iteration for the discrete problem, was proved to be an efficient one ; we compared it with the Picard iteration obtained by substituting the curvative Fourier coefficient by 0 : then we observed that the present algorithm is definitely *faster* ($\mathrm{Max}|\Psi_h|$ decreases from 10^{-1} to 10^{-17} within 20 iterations for the good cases of Fig.6, 7 and 8), and more *reliable*. However, we had to use a 0.5 - relaxation factor for very bad cases as in Fig.5. A physically likely computation, with the Kutta-Joukovski condition satisfied by moving the separation point, is presented in Fig.6 : this flow past a symmetric profile can be compared with J.F. BOURGAT, G. DUVAUT [1] (case $\sigma_5 = 0$).

The descent direction was observed to be near enough of the real gradient : we verified this by means of slight perturbations of every control components ; moreover, we verified also that the conjugate gradient algorithm was more efficient than the pure gradient one.

Let us present now two (academic) global tests : the prescribed stream function z_d is constructed from a solution of the Free Boundary Problem, so that the minimum of the cost functional is zero.

First experiment : The data are

$$AC = \{(x,0),\ x\in[0,2]\}$$
$$d = -\ 0.95\ ;$$

the optimal structure is

$$FE_{opt} = \{(x,y),\ x\in[0,1], y = 1.1 - 0.1x\}\ ;$$

the initial one is

$$FE_{init} = \{(x,y),\ x\in[0,1],\ y = 1.\}.$$

Then the cost functional j_h decreases as follows :

optimization iteration	cost functional
0	$0.25\ 10^{-2}$
5	$0.92\ 10^{-4}$
10	$0.19\ 10^{-5}$
15	$0.81\ 10^{-8}$
20	$0.25\ 10^{-10}$

within 20 iterations (6.5 minutes of CII-IRIS 80), the maximum deviation between the control and the optimal one decreases from 0.1 to $5.\ 10^{-6}$.

Second experiment : The data are

$$AB = \{(x,y),\ x\in[0,1],\ y = 0.2 \cos\left(\frac{\Pi x}{2}\right)\}$$

$$BC = \{(x,0),\ x\in[1,2]\}$$

$$d = -\ 1.\ ;$$

the optimal structure is (Fig.7)

$$FE_{opt} = \{(x,y),\ x\in[0,1],\ y = 0.2 \cos\left(\frac{\Pi x}{2}\right) + 1.\}\ ;$$

and the initial one is (Fig.8)

$$FE_{init} = \{(x,1),\ x\in[0,1]\}.$$

Then the cost functionnal decreases as follows :

optimization iteration	cost functional
0	$0.25\ 10^{-1}$
5	$0.48\ 10^{-4}$
10	$0.20\ 10^{-5}$
15	$0.17\ 10^{-6}$
20	$0.25\ 10^{-7}$

within 20 iterations (6.0 minutes of CII-IRIS 80), the maximum deviation between the control and the optimal ones decreases from 0.2 to $2.\ 10^{-4}$.

6.5 Comments

The optimal control is clearly well-conditioned in the above experiments.

The introduction of the curvature is definitely useful, even when the Free Boundary seems to be not much bent.

Most feature of this section, in particular stationary property, extend formally to the compressible case, so that we think that this study could lead to an industrial application.

7. CONCLUSION

The general method which we introduce in [2] applies here to a Free Boundary Problem which generally cannot be transformed in a Variational Inequality (this question is discussed in C. BAIOCCHI [1] ; a particular counter-example is described in H. BREZIS, G. DUVAUT [1]).

Interesting answers are brought concerning the *local well-posedness* together with *stability* with respect to some data, with natural assumptions introduced in the physical plane.

We tried to make clearer the way to introduce the *stationary coupling problem*, or to deal without ; we hope to study in a future work the case of several two-phase Free Boundary Problems.

For the numerical point of view, we emphasize the fact that the variation formulas derived in the continuous case are proved to be useful in the discrete context, namely for the construction of fastly convergent iterative methods, and of a good approximated descent direction.

Fig.5

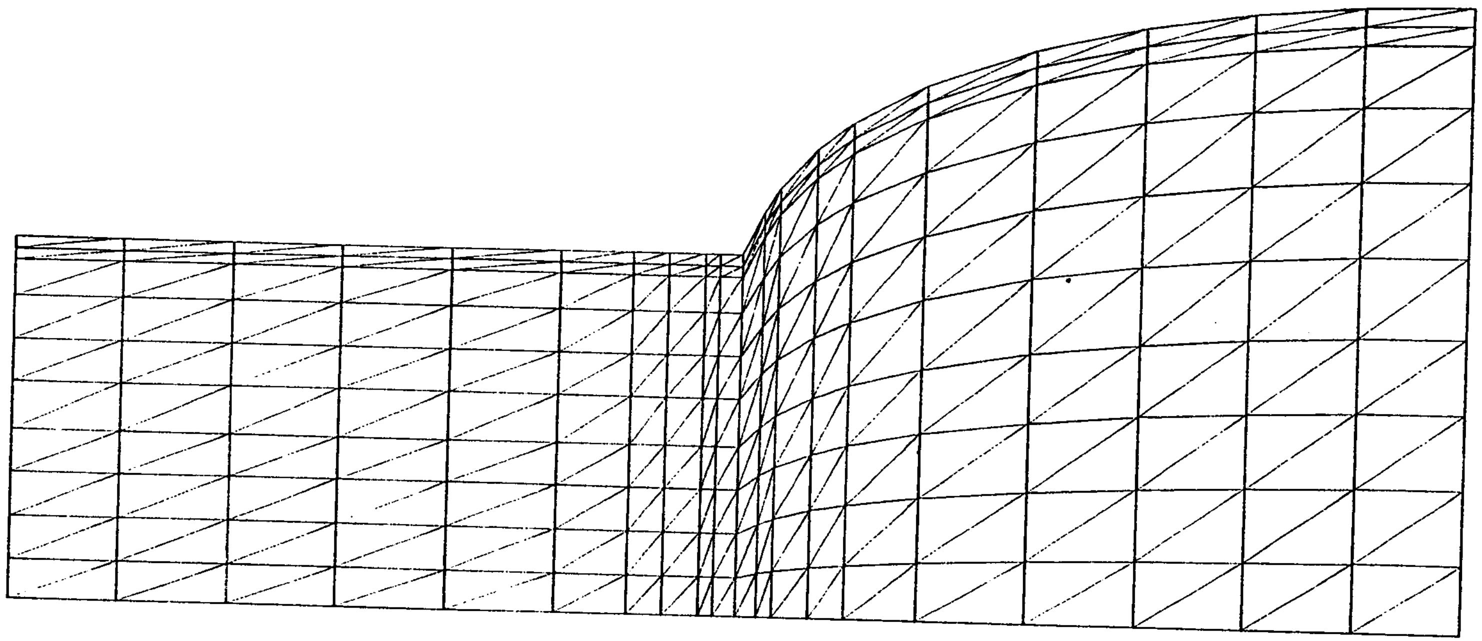


Fig.6

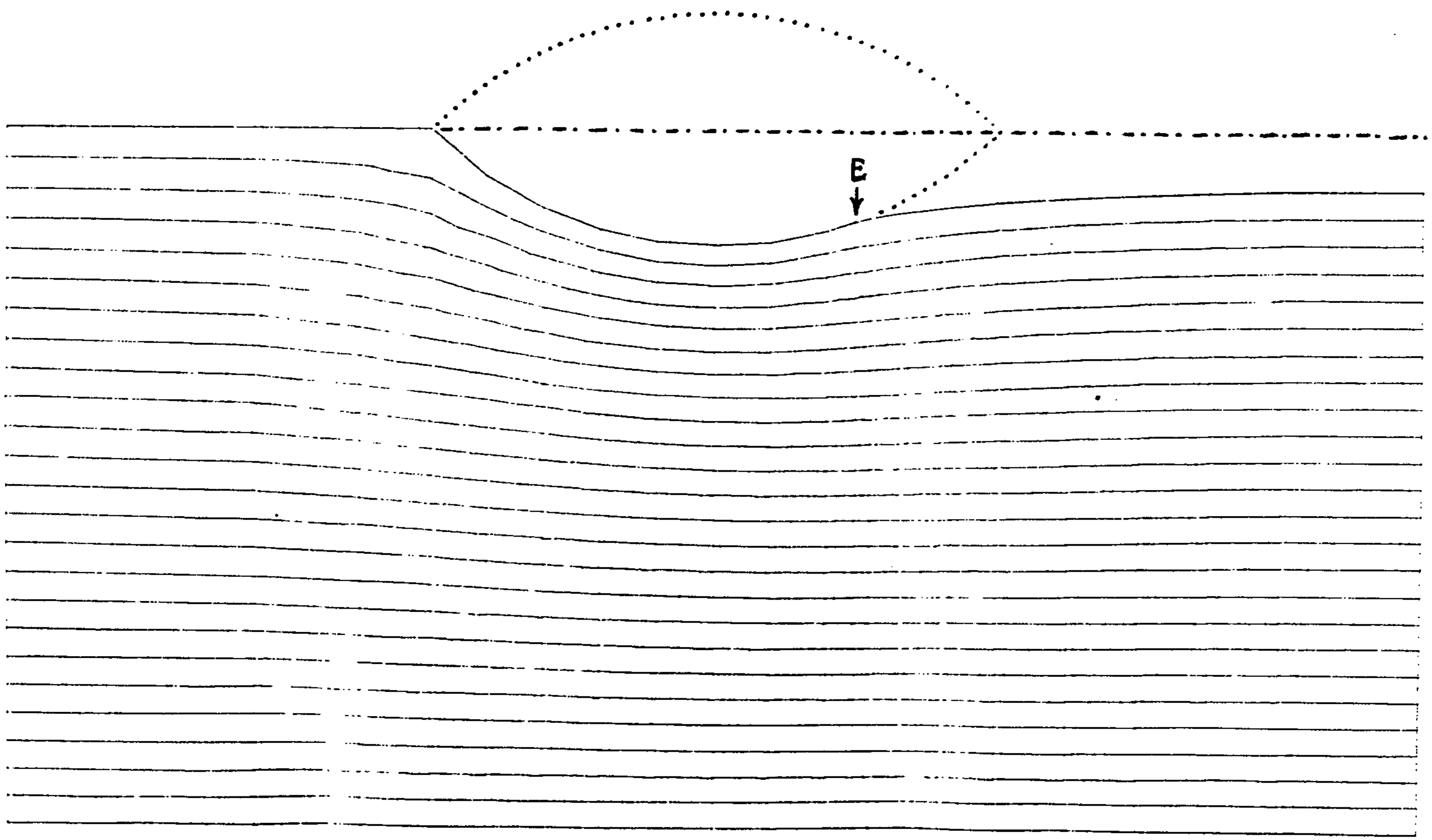


Fig.7

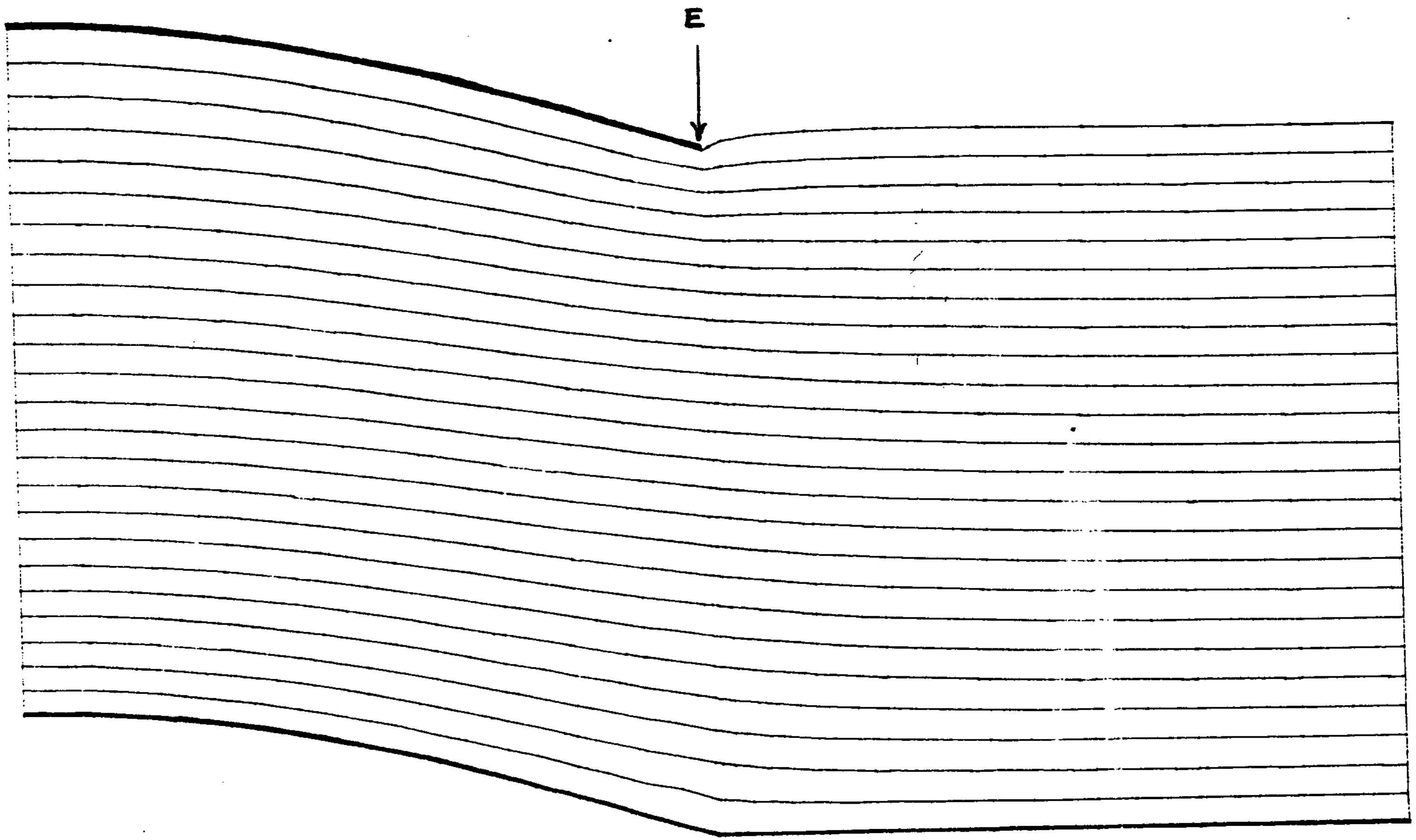
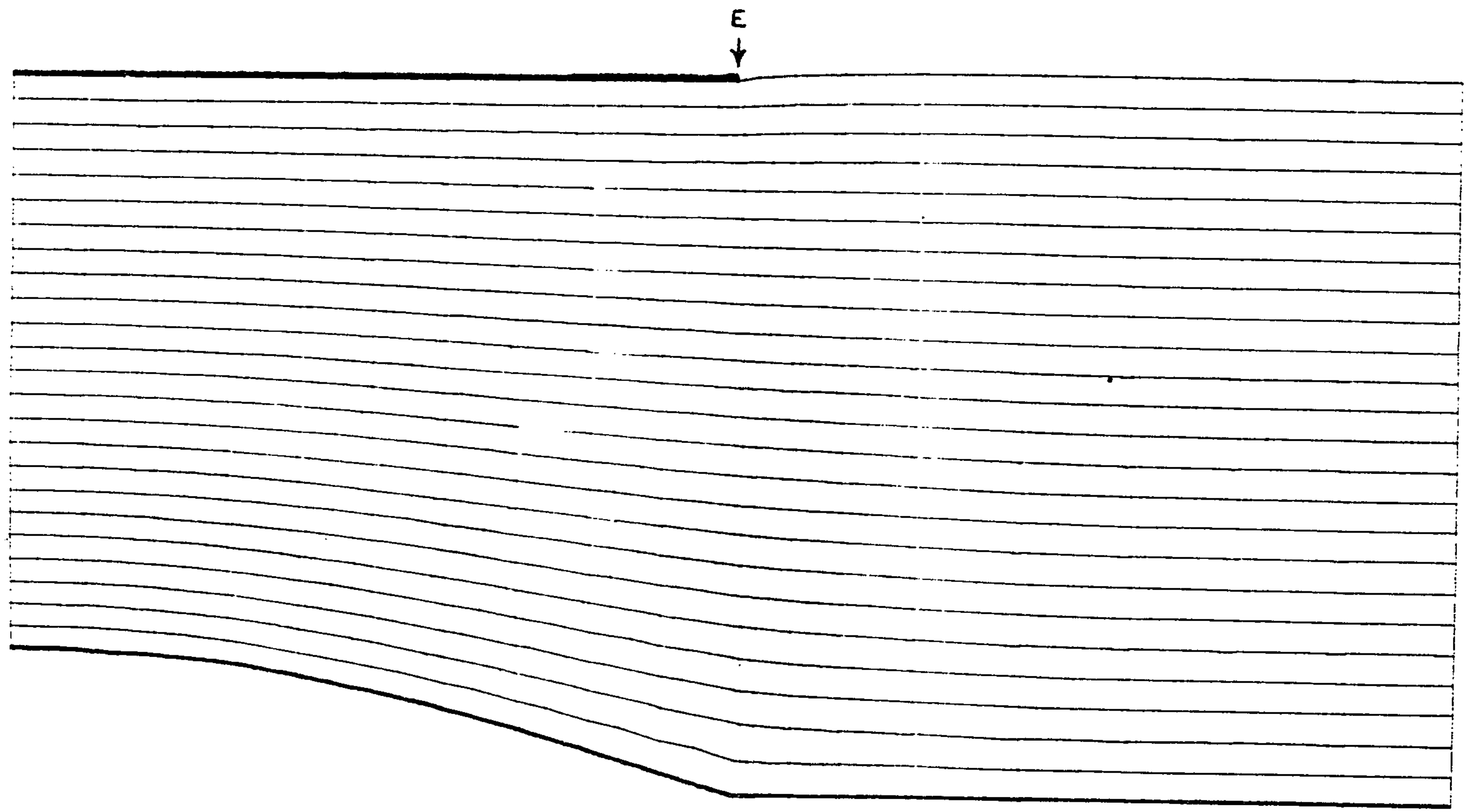


Fig.8



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