An existence theorem for a class of nonlinear shallow shell problems

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RESUME

Dans ce travail nous analysons une classe de problèmes non linéaires de coques peu profondes. Nous rappelons tout d'abord les équations non linéaires classiques de coques peu profondes de W.T. KOITER basées sur une représentation de la surface moyenne de la coque à l'aide d'un système général de coordonnées curvilignes. Ensuite, nous démontrons un théorème d'existence de solutions de ces équations en utilisant la théorie des opérateurs pseudo-monotones. Enfin, nous montrons l'unicité des solutions lorsque les charges sont suffisamment faibles. Comme cas particulier de cette étude, on retrouve les résultats d'existence et d'unicité pour la théorie non linéaire des plaques.

ABSTRACT

This work is devoted to the analysis of a class of nonlinear shallow shell problems. First we recall the classical nonlinear shallow shell equations of W.T. KOITER based on a representation of the middle surface of the shell by a general system of curvilinear coordinates. Next, we prove an existence theorem for solutions of these equations using the theory of pseudo-monotone operators. Finally, we prove that solutions are unique whenever the loads are sufficiently small. Existence and uniqueness results for the nonlinear theory of plates are reduced as a special case from our theory.

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1. INTRODUCTION

1.1 Introduction. The classical nonlinear equations of shallow shells are derived and discussed in, for example, KOITER [1, paragraph 11]. These equations generalize the classical von Karman equations for large deflections of thin plates, which have been analyzed in some detail by several authors; in particular, we mention the works of BERGER [1], BERGER and FIFE [1], CIARLET [1], CIARLET and DESTUYNDER [1], DO [1], DUVAUT and LIONS [1], FIFE [1], JOHN [1], JOHN and NAUMANN [1], KNIGHTLY [1], KNIGHTLY and SATHER [1], NAUMANN [1,2], and NECAS and NAUMANN [1]. A review of the results can be found in WEINITSCHKE [1].

By contrast, few authors have studied questions of existence and uniqueness for theories governing the geometrically nonlinear behavior of elastic shells. We mention, for example, NAUMANN [3], RUPPRECHT [1], and VOROVICH and LEBEDEV [1]. A review of some results can be found in DIKMEN [1].

For simplicity, we consider the case of a clamped shell so that we emphasize the study of the properties of the nonlinear operator associated to the problem under consideration. More general boundary conditions like, for example, unilateral conditions could be considered by using the techniques developed by DUVAUT and LIONS [1], NAUMANN [1,2], and others, for nonlinear plate problems.

Following this Introduction, we record the equations governing a theory of large deflections of arbitrary shallow elastic shells. We give the associated variational formulation of these equations in paragraph 3. The main results of this investigation are given in paragraph 4 in the form of an existence theorem of solutions for the problem under consideration. We essentially use the abstract theory of pseudo-monotone operators - see LIONS [1, chapter 2, § 2.4]. Similar methods have been employed by NAUMANN [2] in the case of the nonlinear plate theory, and ODEN [1] for certain nonlinear elliptic problems. Finally, we conclude the study with some results on the uniqueness of solutions to the shell equations.

1.2 Hypothesis and Notations. Throughout this study, the geometrical and mechanical hypothesis and notations of KOITER [1] and certain results of BERNADOU and CIARLET [1] will be used. We shall frequently make use of the properties of the Sobolev spaces

$$\left.\begin{array}{l} W^{m,p}(\Omega) = \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega) \quad \text{for} \quad |\alpha| \leq m\} \\ \\ m \geq 1 \quad \text{integer}, \quad 1 \leq p < \infty . \end{array}\right\} \quad (1.2\text{-}1)$$

Here Ω denotes an open bounded region in the plane E^2. When equipped with the norm

$$\|v\|_{m,p} = \left(\int_\Omega \sum_{|\alpha| \leq m} |D^\alpha v|^p \, d\xi^1 d\xi^2 \right)^{1/p} \quad (1.2\text{-}2)$$

$W^{m,p}$ is a Banach space. Here ξ^1, ξ^2 denote a system of orthonormal coordinates of the E^2-plane. Particularly, the space $W^{m,2}(\Omega)$ is a Hilbert space when endowed with the scalar product

$$((u,v))_{m,2} = \sum_{|\alpha| \leq m} \int_\Omega D^\alpha u D^\alpha v \, d\xi^1 d\xi^2 \ . \tag{1.2-3}$$

Moreover, we define

$$W_o^{m,p}(\Omega) = \text{closure of } \mathcal{D}(\Omega) \text{ in } W^{m,p}(\Omega) \ , \tag{1.2-4}$$

and we shall use the following consequence of the Sobolev embedding theorem:

$$\left.\begin{aligned} & W^{m+1,2}(\Omega) \hookrightarrow W^{m,4}(\Omega) \hookrightarrow W^{m,2}(\Omega) \ , \\ & m = 0,1 \ . \end{aligned}\right\} \tag{1.2-5}$$

Here the notation $\hookrightarrow$ represents inclusion with continuous injection.

For more details on Sobolev spaces, we refer to ADAMS [1], LIONS and MAGENES [1], NEČAS [1], and ODEN and REDDY [1], for example.

Throughout this paper, c, c_1, c_2, ..., denote generic positive constants, not necessarily the same at each occurrence, which will be particularly independent of the parameters ε and η that are introduced below.

2. THE SYSTEM OF NONLINEAR EQUATIONS

2.1 Geometrical Considerations. We begin by recording geometrical descriptions of a general thin shell essentially following KOITER [1] and we then introduce some convenient complementary geometrical hypothesis of the type considered in shallow shell theory.

General Definition of the Middle Surface:

The middle surface S of the shell is defined as the image of a connected bounded open set Ω in a plane E^2 by a mapping $\underset{\sim}{\phi} \in C^3(\overline{\Omega})$, i.e.,

$$\underset{\sim}{\phi} : \overline{\Omega} \subset E^2 \to E^3 \tag{2.1-1}$$

where E^3 is the usual three-dimensional Euclidean space; i.e.,

$$S = \{\underset{\sim}{\phi}(\xi) \ , \ \ \xi = (\xi^1, \xi^2) \in \overline{\Omega}\} \ . \tag{2.1-2}$$

Subsequently, we shall assume that all the points of $\overline{S} = \underset{\sim}{\phi}(\overline{\Omega})$ are regular, in the sense that the two vectors

$$\underset{\sim}{a}_\alpha = \underset{\sim}{\phi}_{,\alpha} = \frac{\partial \underset{\sim}{\phi}}{\partial \xi^\alpha} \ , \quad \alpha = 1,2, \tag{2.1-3}$$

are linearly independent for all points $\xi = (\xi^1,\xi^2) \in \overline{\Omega}$. These two vectors define the tangent plane to the surface $\overline{S}$ at the point $\underset{\sim}{\phi}(\xi)$. Next, we define the normal vector

$$\underset{\sim}{a}_3 = \frac{\underset{\sim}{a}_1 \times \underset{\sim}{a}_2}{|\underset{\sim}{a}_1 \times \underset{\sim}{a}_2|} \ , \tag{2.1-4}$$

$|\cdot|$ denoting the euclidean norm in E^3 equipped with its usual scalar product $(\underset{\sim}{a},\underset{\sim}{b}) \to \underset{\sim}{a} \cdot \underset{\sim}{b}$. Then, the point $\underset{\sim}{\phi}(\xi)$ and the three vectors $\underset{\sim}{a}_i$ define a local reference system for the middle surface.

As a rule, we shall use Greek letters $\alpha, \beta, \ldots,$ for indices which take their values in the set $\{1,2\}$, while Latin letters $i, j, \ldots,$ will be used for indices which take their values in the set $\{1,2,3\}$. Also, we shall employ the summation convention for a repeated index, occuring once as a subscript and once as a superscript.

To the vectors $\underset{\sim}{a}_\alpha$, we associate two other vectors $\underset{\sim}{a}^\beta$ of the tangent plane defined by

$$\underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}^\beta = \delta_\alpha^\beta . \qquad (2.1\text{-}5)$$

These vectors are linked to the vectors $\underset{\sim}{a}_\alpha$ by the relations

$$\underset{\sim}{a}_\alpha = a_{\alpha\beta}\, \underset{\sim}{a}^\beta \;,\; \underset{\sim}{a}^\alpha = a^{\alpha\beta}\, \underset{\sim}{a}_\beta \;,\; a^{\alpha\beta} = \underset{\sim}{a}^\alpha \cdot \underset{\sim}{a}^\beta = a^{\beta\alpha} \;, \qquad (2.1\text{-}6)$$

where $(a_{\alpha\beta})$ denotes the components of the <u>first fundamental form</u> of the surface, i.e.,

$$a_{\alpha\beta} = a_{\beta\alpha} = \underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}_\beta = \underset{\sim}{\phi}_{,\alpha} \cdot \underset{\sim}{\phi}_{,\beta} \;, \qquad (2.1\text{-}7)$$

and where the matrix $(a^{\alpha\beta})$ is the inverse of the matrix $(a_{\alpha\beta})$. This inverse matrix is well-defined since all the points of the middle surface S are assumed to be regular, i.e.,

$$\det\,(a_{\alpha\beta}) = a \neq 0\ . \tag{2.1-8}$$

Moreover, the second fundamental form of the surface $(b_{\alpha\beta})$ is defined by

$$b_{\alpha\beta} = b_{\beta\alpha} = -\underset{\sim}{a}_{\alpha} \cdot \underset{\sim}{a}_{3,\beta} = \underset{\sim}{a}_{3} \cdot \underset{\sim}{a}_{\alpha,\beta} = \underset{\sim}{a}_{3} \cdot \underset{\sim}{a}_{\beta,\alpha}\ . \tag{2.1-9}$$

This second form permits us to measure the normal curvatures of the middle surface. In particular, the mean and Gaussian curvatures H and K of the middle surface are defined by

$$\left.\begin{aligned} H &= \frac{1}{2}\left(\frac{1}{R_{N_1}} + \frac{1}{R_{N_2}}\right) = \frac{1}{2}b^{\alpha}_{\alpha}\ , \\ K &= \frac{1}{R_{N_1}}\frac{1}{R_{N_2}} = b^{1}_{1}b^{2}_{2} - b^{1}_{2}b^{2}_{1}\ , \end{aligned}\right\} \tag{2.1-10}$$

where

$$b^{\alpha}_{\beta} = a^{\alpha\lambda}b_{\lambda\beta}\ . \tag{2.1-11}$$

The expressions $\frac{1}{R_{N_1}}$ and $\frac{1}{R_{N_2}}$ denote, respectively, the maximum and the minimum of the normal curvatures of the middle surface.

General Definition of the Shell

In addition to the two curvilinear coordinates ξ^1, ξ^2 defining the middle surface, we introduce a third curvilinear coordinate, ξ^3 , which is measured along the normal $\underset{\sim}{a}_3$ to the surface $\overline{S}$ at point $\underset{\sim}{\phi}(\xi^1,\xi^2)$. The system (ξ^1,ξ^2,ξ^3) of coordinates is, at least locally, a system of curvilinear coordinates in E^3 , generally called a <u>normal coordinates system</u>.

The <u>thickness</u> e of the shell is defined through the mapping

$$e: \quad (\xi^1,\xi^2) \in \overline{\Omega} \rightarrow \{x \in \mathbb{R} ; \; x > 0\} \; . \qquad (2.1\text{-}12)$$

Then, <u>the shell</u> S is defined as the closed subset of E^3

$$S = \{M \in E^3 \; ; \; \overrightarrow{OM} = \underset{\sim}{\phi}(\xi^1,\xi^2) + \xi^3 \underset{\sim}{a}_3 \; ,$$

$$(\xi^1,\xi^2) \in \overline{\Omega} \; , \; -\frac{1}{2}\, e(\xi^1,\xi^2) \leq \xi^3 \leq \frac{1}{2}\, e(\xi^1,\xi^2)\} \; . \qquad (2.1\text{-}13)$$

<u>The Geometry of the Shell</u>

Using the conventions and geometrical properties outlined above, we can now describe geometrical features of a wide class of shells by essentially imposing certain restrictions on the normal curvatures and on their variations.

<u>Firstly</u>, let us observe that for a given surface, as defined above, there exists two constants $\rho_1 > 0$ and $\rho_2 > 0$ such that (BERNADOU and CIARLET [1, § 3.1]):

$$\left.\begin{array}{c} |a_{\alpha\beta}|,\ |a^{\alpha\beta}|,\ |a|,\ |b_{\alpha\beta}|,\ |b^{\alpha}_{\beta}|,\ |b^{\alpha\beta}|,\ |a_{\alpha\beta,\lambda}|,\ |a^{\alpha\beta}{}_{,\lambda}| \\ |a_{,\lambda}|,\ |b_{\alpha\beta,\lambda}|,\ |b^{\alpha}_{\beta,\lambda}|,\ |b^{\alpha\beta}{}_{,\lambda}|,\ |\Gamma^{\alpha}_{\beta\gamma}| \end{array}\right\} \leq \rho_1 \quad (2.1\text{-}14)$$

$$\rho_2 \leq a_{11},\ a^{11},\ a_{22},\ a^{22},\ a\ , \qquad (2.1\text{-}15)$$

uniformly on $\overline{S}$, where in (2.1-14), $\Gamma^{\alpha}_{\beta\gamma}$ denote the Christoffel symbols of the middle surface, i.e.,

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta} = \underset{\sim}{a}^{\alpha} \cdot \underset{\sim}{a}_{\gamma,\beta} = \underset{\sim}{a}^{\alpha} \cdot \underset{\sim}{a}_{\beta,\gamma} \ . \qquad (2.1\text{-}16)$$

These coefficients are, of course, used in the definition of the covariant derivatives of vector and tensor fields defined on $\overline{S}$; for example,

$$\left.\begin{array}{l} T_{\alpha|\gamma} = T_{\alpha,\gamma} - \Gamma^{\lambda}_{\alpha\gamma}\, T_{\lambda}\ , \\ T^{\alpha}|_{\gamma} = T^{\alpha}{}_{,\gamma} + \Gamma^{\alpha}_{\lambda\gamma}\, T^{\lambda}\ , \\ T_{\alpha\beta|\gamma} = T_{\alpha\beta,\gamma} - \Gamma^{\lambda}_{\alpha\gamma}\, T_{\lambda\beta} - \Gamma^{\lambda}_{\beta\gamma}\, T_{\alpha\lambda}\ , \ \ldots\ . \end{array}\right\} \qquad (2.1\text{-}17)$$

The tensors T_{α} or T^{α} are the covariant and contravariant components of a vector $\underset{\sim}{T}$ with respect to the basis $(\underset{\sim}{a}^{\beta})$ or $(\underset{\sim}{a}_{\beta})$ of the tangent plane to the middle surface, i.e.,

$$\underset{\sim}{T} = T_{\alpha}\, \underset{\sim}{a}^{\alpha} + T_3\, \underset{\sim}{a}^{3} = T^{\alpha}\, \underset{\sim}{a}_{\alpha} + T^{3}\, \underset{\sim}{a}_{3} \ . \qquad (2.1\text{-}18)$$

For the normal components $T_3 = T^3$, we have

$$T_{3|\alpha} = T_{3,\alpha} , \quad T_{3|\alpha\beta} = T_{3,\alpha\beta} - \Gamma^{\lambda}_{\alpha\beta} T_{3,\lambda} . \tag{2.1-19}$$

Secondly, we note that, for a given shell defined as above, there exists two constants $e_1 > 0$ and $e_2 > 0$ such that the thickness $e(\xi^1,\xi^2)$ satisfies

$$0 < e_1 \leq e(\xi^1,\xi^2) \leq e_2 , \quad \forall (\xi^1,\xi^2) \in \bar{\Omega} . \tag{2.1-20}$$

Finally, in the remainder of this paper, we shall be interested in the set of shells $S_o(\rho_\alpha,e_\alpha,\omega_\alpha)$ and in its subset $S_\varepsilon(\rho_\alpha,e_\alpha,\omega_\alpha)$ defined as follows.

Definition 2.1-1: The set $S_o(\rho_\alpha,e_\alpha,\omega_\alpha)$. Given once and for all, seven constants ρ_1, ρ_2, e_1, e_2, ω_1, ω_2, R_o such that

$$0 < \rho_2 \leq \rho_1 , \quad 0 < e_1 \leq e_2 , \quad 0 < \omega_1 \leq \omega_2 , \quad R_o > 0 , \tag{2.1-21}$$

we denote by $S_o(\rho_\alpha,e_\alpha,\omega_\alpha)$ the set of shells S, defined as in (2.1-13), such that

(i) the inequalities (2.1-14), (2.1-15), and (2.1-20) are satisfied;

(ii) the coefficients of the second fundamental form b^β_α are such that

$$|b_\alpha^\beta| \ , \quad |b_\alpha^\beta|_\lambda| \leq \frac{1}{R_o} \ , \tag{2.1-22}$$

uniformly on $\bar{S}$ for all α, β, λ .

(iii) the reference domains Ω are sufficiently smooth, and, in particular,

$$C(\omega_1) \subset \Omega \subset C(\omega_2) \ , \tag{2.1-23}$$

where the notation $C(\omega_\alpha)$ denotes a square with a side of length ω_α . □

This definition will be useful in obtaining estimates in which constants are independent of the shell under consideration among those in the set $S_o(\rho_\alpha, e_\alpha, \omega_\alpha)$. Moreover, the assumption (2.1-23) combined with the relations (1.2-5) insure the existence of the constants $d_1(\omega_\alpha) > 0$ and $d_2(\omega_\alpha) > 0$ such that $(m \in N)$

$$\left.\begin{aligned} &\|v\|_{m,2} \leq d_1 \ \|v\|_{m,4} \ , \quad \forall\, v \in W^{m,4}(\Omega) \ , \\ &\|v\|_{m,4} \leq d_2 \ \|v\|_{m+1,2} \ , \quad \forall\, v \in W^{m+1,2}(\Omega) \ , \\ &\forall\, \Omega \text{ verifying (2.1-23)} \ . \end{aligned}\right\} \tag{2.1-24}$$

Another family of shells of interest here is introduced next.

<u>Definition</u> 2.1-2. The <u>subset</u> $S_\varepsilon(\rho_\alpha, e_\alpha, \omega_\alpha)$. Let there be given a parameter $\varepsilon > 0$ such that

$$0 < \varepsilon \leq \frac{1}{R_o} \ ; \tag{2.1-25}$$

Then the set $\mathcal{S}_\varepsilon(\rho_\alpha, e_\alpha, \omega_\alpha)$ is the set of all the shells $\mathcal{S} \in \mathcal{S}_o(\rho_\alpha, e_\alpha, \omega_\alpha)$ such that

$$|b_\alpha^\beta| \ , \ |b_\alpha^\beta|_\lambda| \leq \varepsilon \ . \qquad (2.1\text{-}26) \ \square$$

<u>Remark 2.1-1</u>: Comparing condition (2.1-22) with the result (2.1-10), this condition can be interpreted as follows: the normal curvatures and their derivatives are uniformly bounded on Ω for any shell $\mathcal{S} \in \mathcal{S}_o(\rho_\alpha, e_\alpha, \omega_\alpha)$. In particular, when the parameter ε is small with respect to ρ_1, the condition (2.1-26) implies that the normal curvatures and their variations are "small." This appears to be one of the basic assumptions in the usual forms of <u>shallow</u> shell theory. This suggests that the set $\mathcal{S}_\varepsilon(\rho_\alpha, e_\alpha, \omega_\alpha)$ covers a large class of shallow shells. For discussions of the connection between the condition (2.1-26) and the usual hypothesis of shallow shell theory, consult GREEN and ZERNA [1] and RUTTEN [1]. $\square$

<u>Remark 2.1-2</u>: Note that the special case of a flat plate is included in the definition of the set $\mathcal{S}_\varepsilon(\rho_\alpha, e_\alpha, \omega_\alpha)$. $\square$

<u>2.2 Strain and Stress Measures in Nonlinear Shallow Shell Theory</u>. In the present shell theories, the basic strain measures can be defined by introducing the <u>middle surface strain tensor</u> $\gamma_{\alpha\beta}$ and the <u>modified</u> (KOITER'S terminology) <u>change-of-curvature tensor</u> $\rho_{\alpha\beta}$. Consider a displacement field $\tilde{u}$ on points of the middle surface of the shell measured from a fixed reference configuration

$$\underset{\sim}{u} = u_i \cdot \underset{\sim}{a}^i . \tag{2.2-1}$$

For a <u>classical nonlinear shallow shell theory</u>, KOITER [1, (11.43) (11.44)] shows that suitable expressions for the tensors $\gamma_{\alpha\beta}$ and $\rho_{\alpha\beta}$ associated to the displacement $\underset{\sim}{u}$ are given by:

$$\gamma_{\alpha\beta}(\underset{\sim}{u}) = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}\, u_3 + \frac{1}{2}\, u_{3,\alpha}u_{3,\beta} , \tag{2.2-2}$$

$$\rho_{\alpha\beta}(\underset{\sim}{u}) = u_{3|\alpha\beta} . \tag{2.2-3}$$

We note that in KOITER'S general linear theory of shells (see KOITER [1, (3.7) (4.10)] and BERNADOU and CIARLET [1]) the corresponding values of these tensors are

$$\gamma^L_{\alpha\beta}(\underset{\sim}{u}) = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}u_3 , \tag{2.2-4}$$

$$\rho^L_{\alpha\beta}(\underset{\sim}{u}) = u_{3|\alpha\beta} + b^\lambda_{\alpha|\beta}u_\lambda + \frac{1}{4}b^\lambda_\alpha(3u_{\lambda|\beta} - u_{\beta|\lambda}) + \frac{1}{4}b^\lambda_\beta(3u_{\lambda|\alpha} - u_{\alpha|\lambda}) . \tag{2.2-5}$$

For convenience, we set $\underline{u} = (u_1, u_2)$ and

$$f_{\alpha\beta}(\underline{u}) = b^\lambda_{\alpha|\beta}u_\lambda + \frac{1}{4}b^\lambda_\alpha(3u_{\lambda|\beta} - u_{\beta|\lambda}) + \frac{1}{4}b^\lambda_\beta(3u_{\lambda|\alpha} - u_{\alpha|\lambda}) , \tag{2.2-6}$$

so that the expression (2.2-5) can be written

$$\rho^{L}_{\alpha\beta}(\underset{\sim}{u}) = u_{3|\alpha\beta} + f_{\alpha\beta}(\underline{u}) \ . \tag{2.2-7}$$

Thus, the main changes in progressing from the general linear theory to the classical nonlinear shallow shell are

(i) the introduction of a nonlinear term in derivatives of u_3 in the expression of the middle surface strain tensor $\gamma_{\alpha\beta}$;

(ii) the great simplification of the expression of the modified change-of-curvature tensor $\rho_{\alpha\beta}$ due to absence of the term $f_{\alpha\beta}(\underline{u})$ in (2.2-7).

<u>Stress Measures</u>

The stress measures are now derived from the above strain measures and an appropriate constitutive equation for the shell material. We will restrict our study to the simplest possible case, based on the following assumptions: (a) the material of the shell is elastic, homogeneous and isotropic; (b) the strains are small everywhere in the shell; (c) the shell is in a state of stress in which all nonzero stress components are developed on surfaces parallel to the middle surface. Then, we have the following relations between the tensors $\gamma_{\lambda\mu}(\underset{\sim}{u})$ and $\rho_{\lambda\mu}(\underset{\sim}{u})$, on the one hand, and the symmetric tensors of tangential (membrane) stress resultants $n^{\alpha\beta}(\underset{\sim}{u})$ and moments $m^{\alpha\beta}(\underset{\sim}{u})$, on the other hand:

$$n^{\alpha\beta}(\underset{\sim}{u}) = e\ E^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}(\underset{\sim}{u}) \tag{2.2-8}$$

$$m^{\alpha\beta}(\underset{\sim}{u}) = \frac{e^3}{12} E^{\alpha\beta\lambda\mu} \rho_{\lambda\mu}(\underset{\sim}{u}) \ , \tag{2.2-9}$$

where e denotes the thickness of the shell and $E^{\alpha\beta\lambda\mu}$ denotes the tensor of elastic moduli for plane stress at the middle surface; i.e.

$$E^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)} [a^{\alpha\lambda} a^{\beta\mu} + a^{\alpha\mu} a^{\beta\lambda} + \frac{2\nu}{1-\nu} a^{\alpha\beta} a^{\lambda\mu}] \tag{2.2-10}$$

where E is Young's modulus and ν is Poisson's ratio for the material.

2.3 The System of Nonlinear Equations. For simplicity, we consider the case of a clamped shell. Let

$$\underset{\sim}{p} = p^i \underset{\sim}{a}_i \tag{2.3-1}$$

denotes the resultant external applied forces per unit surface area defined on the middle surface of the shell. Then, following KOITER [1, p. 48], one can show that the principle of virtual work assumes the form

$$\int_S \{n^{\alpha\beta}(\underset{\sim}{u})\ \delta\gamma_{\alpha\beta}(\underset{\sim}{u}) + m^{\alpha\beta}(\underset{\sim}{u})\ \delta\rho_{\alpha\beta}(\underset{\sim}{u})\}\ ds = \int_S p^i \delta u_i\ ds \tag{2.3-2}$$

We obtain the corresponding equations of equilibrium by means of the divergence theorem,

$$n^{\beta\alpha}(\underset{\sim}{u})|_\beta + p^\alpha = 0 \ , \tag{2.3-3}$$

$$m^{\alpha\beta}(\underset{\sim}{u})|_{\alpha\beta} - b_{\alpha\beta}n^{\alpha\beta}(\underset{\sim}{u}) - (u_{3|\alpha}n^{\alpha\beta}(\underset{\sim}{u}))|_{\beta} - p^3 = 0 \qquad (2.3\text{-}4)$$

with the usual conditions of a clamped boundary, i.e.,

$$\underset{\sim}{u}|_{\partial S} = \underset{\sim}{0} \ , \quad \frac{\partial u_3}{\partial n}|_{\partial S} = 0 \ . \qquad (2.3\text{-}5)$$

Remark 2.3-1. It is important to note that the equations (2.3-3) (2.3-4) represent a slight extension of the nonlinear equations of shallow shells considered by KOITER [1, (11.43) (11.44)]. This extension is based on the introduction of tangential surface loads in order to obtain, as a particular special case, the von Karman equations for large deflection of thin plates. In the same way, more general boundary conditions can be considered. For example, unilateral conditions of the type studied by DUVAUT and LIONS [1] and NAUMANN [1,2] for nonlinear plate theory could be easily incorporated into our theory. □

3. VARIATIONAL FORMULATION

3.1 Variational Formulation of the Problem. We now consider a more specific variational formulation of the problem defined by (2.3-3) and (2.3-4). We begin with the case in which $\underset{\sim}{u} = u_i \underset{\sim}{a}^i \in (C^2(\overline{S}))^2 \times C^4(\overline{S})$, where $C^m(\overline{S})$ denotes the set of functions m-times continuously differentiable on $\overline{S}$. Then, we multiply (2.3-3) by $v_\alpha \in C^1(\overline{S})$ and (2.3-4) by $v_3 \in C^2(\overline{S})$ and we integrate on $\overline{S}$ to obtain

$$\int_S \left(n^{\beta\alpha}(\underset{\sim}{u})|_\beta + p^\alpha\right) v_\alpha \, ds = 0 \ , \tag{3.1-1}$$

$$\int_S \{m^{\alpha\beta}(\underset{\sim}{u})|_{\alpha\beta} - b_{\alpha\beta}n^{\alpha\beta}(\underset{\sim}{u}) - (u_{3|\alpha}n^{\alpha\beta}(\underset{\sim}{u}))|_\beta - p^3\} \, v_3 \, ds = 0 \, . \tag{3.1-2}$$

Assuming that the functions $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are such that

$$\left.\begin{aligned} \underset{\sim}{u}|_{\partial S} = \underset{\sim}{v}|_{\partial S} &= \underset{\sim}{0} \ , & (3.1\text{-}3) \\ \left.\frac{\partial u_3}{\partial n}\right|_{\partial S} = \left.\frac{\partial v_3}{\partial n}\right|_{\partial S} &= 0 \ , & (3.1\text{-}4) \end{aligned}\right\}$$

we can easily integrate (3.1-1) (3.1-2) by parts (see GREEN and ZERNA [1, (1.13.61)]). If we set

$$\underline{u} = (u_1, u_2) \ , \quad \underline{v} = (v_1, v_2) \tag{3.1-5}$$

then, using the mapping $\underset{\sim}{\phi}$ defined by (2.1-1) and $ds = \sqrt{a}\, d\xi^1 d\xi^2$, $a = a_{11}\, a_{22} - (a_{12})^2$., we derive the following set of equations formulated on the domain Ω :

$$\int_\Omega \{n^{\beta\alpha}(\underset{\sim}{u}) \ [\tfrac{1}{2}(v_{\alpha|\beta} + v_{\beta|\alpha})] - p^\alpha v_\alpha\} \sqrt{a}\, d\xi^1 d\xi^2 = 0 \tag{3.1-6}$$

$$\begin{aligned} \int_\Omega \{m^{\alpha\beta}(\underset{\sim}{u}) \ v_{3|\alpha\beta} + n^{\alpha\beta}(\underset{\sim}{u}) \ [u_{3|\alpha} \ v_{3|\beta} - b_{\alpha\beta} \ v_3] \\ - p^3 \ v_3\} \ \sqrt{a}\, d\xi^1 d\xi^2 = 0. \end{aligned} \tag{3.1-7}$$

By continuity, the equations (3.1-6)(3.1-7) are still valid for $\underset{\sim}{u}$, $\underset{\sim}{v} \in \underset{\sim}{V}$, $p^i \in L^2(\Omega)$, $i = 1, 2, 3$, where

$$\underset{\sim}{V} = (W_o^{1,2}(\Omega))^2 \times W_o^{2,2}(\Omega) \ . \tag{3.1-8}$$

The norm $\|\cdot\|$ on the space $\underset{\sim}{V}$ is

$$\underset{\sim}{v} \in \underset{\sim}{V} \to \|\underset{\sim}{v}\| = \left(\sum_{\alpha=1}^{2} \|v_\alpha\|^2_{1,2} + \|v_3\|^2_{2,2} \right)^{1/2} \ . \tag{3.1-9}$$

Then, from equations (2.2-2), (2.2-3),(2.2-8), (2.2-9), (2.2-10), (3.1-6), and (3.1-7), we obtain the following <u>variational</u> <u>formulation</u> of the problem: Find $\underset{\sim}{u} \in \underset{\sim}{V}$ such that

$$B(\underline{u},\underline{v}) + b(u_3,\underline{v}) = \int_\Omega p^\alpha v_\alpha \sqrt{a}\, d\xi^1 d\xi^2 \ , \quad \forall\, \underline{v} \in (W_o^{1,2}(\Omega))^2 \tag{3.1-10}$$

$$A(u_3,v_3) + a(\underline{u},u_3;v_3) = \int_\Omega p^3 v_3 \sqrt{a}\, d\xi^1 d\xi^2 \ , \quad \forall\, v_3 \in W_o^{2,2}(\Omega) \ . \tag{3.1-11}$$

In the previous expressions, we have used the notations,

$$B(\underline{u},\underline{v}) = \frac{1}{4} \int_\Omega e E^{\alpha\beta\lambda\mu} (u_{\lambda|\mu} + u_{\mu|\lambda})(v_{\alpha|\beta} + v_{\beta|\alpha}) \sqrt{a}\, d\xi^1 d\xi^2 \tag{3.1-12}$$

$$b(u_3,\underline{v}) = \frac{1}{2} \int_\Omega e\, E^{\alpha\beta\lambda\mu} [-b_{\alpha\beta} u_3 + \frac{1}{2} u_{3,\alpha} u_{3,\beta}]$$

$$[v_{\lambda|\mu} + v_{\mu|\lambda}] \sqrt{a}\, d\xi^1 d\xi^2 \tag{3.1-13}$$

$$\left.\begin{aligned} A(u_3,v_3) &= \int_\Omega e\; E^{\alpha\beta\lambda\mu} b_{\alpha\beta} b_{\lambda\mu} u_3 v_3 \sqrt{a}\, d\xi^1 d\xi^2 \\ &+ \int_\Omega \frac{e^3}{12} E^{\alpha\beta\lambda\mu} u_{3|\alpha\beta} v_{3|\lambda\mu} \sqrt{a}\, d\xi^1 d\xi^2 , \end{aligned}\right\} \quad (3.1\text{-}14)$$

$$\left.\begin{aligned} a(\underline{u},u_3;v_3) &= \int_\Omega e\; E^{\alpha\beta\lambda\mu} \{-b_{\alpha\beta} u_3\; u_{3,\mu} v_{3,\lambda} \\ &+ \tfrac{1}{2}[(u_{\alpha|\beta} + u_{\beta|\alpha}) + u_{3,\alpha} u_{3,\beta}] \\ &\times [u_{3,\mu} v_{3,\lambda} - b_{\lambda\mu} v_3]\} \sqrt{a}\, d\xi^1 d\xi^2 . \end{aligned}\right\} \quad (3.1\text{-}15)$$

We note that in the particular case of a flat plate, these equations reduce to the classical equations of a nonlinear clamped plate. Indeed, for a representation of the plate through an orthonormal system of coordinates, we need only substitute $b_{\alpha\beta} = 0$ and replace covariant derivatives by usual derivatives in equations (3.1-12) to (3.1-15).

3.2 Some Ellipticity Properties. Here we establish several fundamental properties of the bilinear forms $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$ which will be useful in the next paragraph. We start with three lemmas.

For $\underset{\sim}{v} \in (W^{1,2}(\Omega))^2 \times W^{2,2}(\Omega)$, we set

$$\Phi(\underset{\sim}{v}) = \{|\gamma^L_{11}(\underset{\sim}{v})|^2 + |\gamma^L_{12}(\underset{\sim}{v})|^2 + |\gamma^L_{22}(\underset{\sim}{v})|^2 + |\rho^L_{11}(\underset{\sim}{v})|^2$$

$$+ |\rho^L_{12}(\underset{\sim}{v})|^2 + |\rho^L_{22}(\underset{\sim}{v})|^2\}^{1/2} \qquad (3.2\text{-}1)$$

where $|\cdot|$ denotes the $L^2(\Omega)$-norm and where the linearized strain tensor $\gamma^L_{\alpha\beta}$ and the change-of-curvature tensor $\rho^L_{\alpha\beta}$ are given by the relations (2.2-4), (2.2-5), i.e.,

$$\gamma^L_{\alpha\beta}(\underset{\sim}{v}) = \frac{1}{2}(v_{\alpha|\beta} + v_{\beta|\alpha}) - b_{\alpha\beta}v_3 \ , \qquad (3.2\text{-}2)$$

$$\begin{aligned}\rho^L_{\alpha\beta}(\underset{\sim}{v}) = v_{3|\alpha\beta} + b^\lambda_{\alpha|\beta}v_\lambda + \frac{1}{4}b^\lambda_\alpha(3v_{\lambda|\beta} - v_{\beta|\lambda}) \\ + \frac{1}{4}b^\lambda_\beta(3v_{\lambda|\alpha} - v_{\alpha|\lambda}) \ .\end{aligned} \qquad (3.2\text{-}3)$$

Our first lemma is as follows:

Lemma 3.2-1 (Infinitesimal rigid body motion lemma). For any shell S defined by (2.1-13), the following conditions (i), (ii) are equivalent :

(i) $\Phi(\underset{\sim}{v}) = 0 \ , \quad \underset{\sim}{v} \in (W^{1,2}(\Omega))^2 \times W^{2,2}(\Omega)$

where Φ *is given by (3.2-1);* (3.2-4)

(ii) $\underset{\sim}{v} = \underset{\sim}{A} + \underset{\sim}{B} \times \underset{\sim}{\phi}$

where $\underset{\sim}{A}, \underset{\sim}{B}$ *are constant vectors in* E^3 *and*

where $\times$ *denotes the vector product in* E^3 *.* (3.2-5)

Proof: See BERNADOU and CIARLET [1, Theorem 5.1-1 and Remark 5.1-2]. □

As a consequence of the previous lemma, we have:

Lemma 3.2-2: Let $\underset{\sim}{V}$ be the space defined by (3.1-8) and let Φ be the functional defined by (3.2-1). Then,

$$\{\Phi(\underset{\sim}{v}) = 0 \ , \ \ \underset{\sim}{v} \in \underset{\sim}{V}, \} \Rightarrow \underset{\sim}{v} = \underset{\sim}{0} \ . \tag{3.2-6}$$

Proof: See BERNADOU and CIARLET [1, Theorem 5.2-1]. □

Finally, we have the following fundamental result on the equivalence of norms:

Lemma 3.2-3: There exist constants $c_1 >$ and $c_2 > 0$ such that

$$\left.\begin{aligned} & c_1 \|\underset{\sim}{v}\| \leq \Phi(\underset{\sim}{v}) \leq c_2 \|\underset{\sim}{v}\| \ , \\ & \forall \underset{\sim}{v} \in \underset{\sim}{V} \ , \quad \forall S \in S_o(\rho_\alpha, e_\alpha, \omega_\alpha) \ , \end{aligned}\right\} \tag{3.2-7}$$

where $\|\underset{\sim}{v}\|$ is defined by (3.1-9).

Proof: For a given shell S the equivalence of norms (3.2-7) is a consequence of BERNADOU and CIARLET [1, Lemma 3.4-2 and Theorem 6.1-1]. Since the set $S_o(\rho_\alpha, e_\alpha, \omega_\alpha)$ is defined through the compact sets $[0, \rho_1]$, $[\rho_2, \rho_1]$, $[e_1, e_2]$ (see (2.1-14), (2.1-15), (2.1-20), respectively), proof by contradiction reveals immediately the existence of constants $c_1 > 0$ and $c_2 > 0$ such that the inequalities (3.2-7) are valid for any shell $S \in S_o(\rho_\alpha, e_\alpha, \omega_\alpha)$. □

Now we turn to some convenient properties of ellipticity of the bilinear forms in (3.1-12) and (3.1-14).

Theorem 3.2-1: *There exist* $\varepsilon > 0$ *and two constants* $c_3 > 0$ *and* $c_4 > 0$ *(*c_3, c_4 *independent of* ε*) such that*

$$\left.\begin{aligned} & B(\underline{v},\underline{v}) \geq c_3 \|\underline{v}\|^2_{1,2} \\ & \forall\ \underline{v} \in (W^{1,2}_o(\Omega))^2\ ,\quad \forall\ S \in S_\varepsilon(\rho_\alpha, e_\alpha, \omega_\alpha) \end{aligned}\right\} \qquad (3.2\text{-}8)$$

$$\left.\begin{aligned} & A(v_3,v_3) \geq c_4 \|v_3\|^2_{2,2}\ , \\ & \forall\ v_3 \in W^{2,2}_o(\Omega)\ ,\quad \forall\ S \in S_o(\rho_\alpha, e_\alpha, \omega_\alpha)\ , \end{aligned}\right\} \qquad (3.2\text{-}9)$$

where the bilinear forms $B(\cdot,\cdot)$ *and* $A(\cdot,\cdot)$ *are respectively defined by (3.1-12) and (3.1-14), and where* $\|\underline{v}\|^2_{1,2} = \sum_{\alpha=1}^{2} \|v_\alpha\|^2_{1,2}$.

Proof: The definition of the middle surface as an image of a compact set $\overline{\Omega}$ through a mapping $\underset{\sim}{\phi} \in (C^3(\overline{\Omega}))^3$, the assumption $\underset{\sim}{a}_1 \times \underset{\sim}{a}_2 \neq \underset{\sim}{0}$ for all the points of $\overline{\Omega}$ and the definition (2.2-10) of the tensor $E^{\alpha\beta\lambda\mu}$ involve the existence of a constant $c_5 > 0$ such that

$$\left.\begin{aligned} & B(\underline{v},\underline{v}) \geq c_5 \sum_{\alpha,\beta=1}^{2} \|v_{\alpha|\beta} + v_{\beta|\alpha}\|^2_{0,2} \\ & \forall\ \underline{v} \in (W^{1,2}_o(\Omega))^2\ ,\quad \forall\ S \in S_o(\rho_\alpha, e_\alpha, w_\alpha)\ . \end{aligned}\right\} \qquad (3.2\text{-}10)$$

But, for all $\underset{\sim}{v} \in \underset{\sim}{V}$ such that $v_3 = 0$, we have, from (2.2-4), (2.2-7), (3.2-1),

$$(\Phi(\underline{v},0))^2 = \sum_{\underline{1<\alpha<\beta<2}} (\frac{1}{4}\|v_{\alpha|\beta} + v_{\beta|\alpha}\|^2_{0,2} + \|f_{\alpha\beta}(\underline{v})\|^2_{0,2}) \ . \qquad (3.2\text{-}11)$$

The definition 2.1-2 of the subset $S_\varepsilon(\rho_\alpha, e_\alpha, \omega_\alpha)$ and the relations (2.2-6) immediately lead to

$$\|f_{\alpha\beta}(\underline{v})\|_{0,2} \leq c\varepsilon \|\underline{v}\|_{1,2} \ . \qquad (3.2\text{-}12)$$

Then, the equivalence (3.2-7) and the relations (3.2-11)(3.2-12) give

$$\sum_{\underline{1<\alpha,\beta<2}} \|v_{\alpha|\beta} + v_{\beta|\alpha}\|^2_{0,2} \geq 4(c_1^2 - c^2\varepsilon^2)\ \|\underline{v}\|^2_{1,2} \ . \qquad (3.2\text{-}13)$$

For any $\varepsilon > 0$ such that $\varepsilon < \frac{c_1}{\sqrt{2}c}$, the relations (3.2-10) (3.2-13) give the result (3.2-8) with $c_3 = 2c_1^2 c_5$.

Similarly we prove (3.2-9). From (3.1-14) we obtain

$$\left.\begin{aligned} & A(v_3,v_3) \geq c_6 \sum_{\alpha,\beta=1}^{2} (\|b_{\alpha\beta} v_3\|^2_{0,2} + \|v_{3|\alpha\beta}\|^2_{0,2}) , \\ & \forall\, v_3 \in \mathring{W}^{2,2}(\Omega) \ , \quad \forall\, S \in S_o(\rho_\alpha \cdot e_\alpha, \omega_\alpha) \ . \end{aligned}\right\} \qquad (3.2\text{-}14)$$

Moreover, for any $\underset{\sim}{v} \in \underset{\sim}{V}$ such that $\underline{v} = \underline{0}$, we have, from (2.2-4) (2.2-7)(3.2-1)

$$(\Phi(\underline{0},v_3))^2 = \sum_{\underline{1<\alpha<\beta<2}} (\|b_{\alpha\beta} v_3\|^2_{0,2} + \|v_{3|\alpha\beta}\|^2_{0,2}) \ . \qquad (3.2\text{-}15)$$

The result now follows from Lemma 3.2-3. □

4. EXISTENCE THEOREM - REMARKS ON UNIQUENESS

4.1 The Abstract Setting. The goal of this paragraph is to prove that the nonlinear system of variational equations (3.1-10) and (3.1-11) has at least one solution $\underset{\sim}{u} \in \underset{\sim}{V}$. The main arguments of the proof will be to use the properties of pseudo-monotonicity and coerciveness of certain operators that we define later. For clarity we recall the following results:

Definition 4.1-1 (LIONS [1, chapter 2, definition 2.1). Let V be a separable, reflexive Banach space. An operator A from $V \to V'$ is said pseudo-monotone if:

(i) A is bounded, i.e.,

$$\|Au\|_* \leq M \text{ for any } u \in V \text{ such that } \|u\| < c ; \tag{4.1-1}$$

(ii) If $u_j \rightharpoonup u$ weakly in V and $\limsup_j \langle A(u_j), u_j - u\rangle \leq 0$

then

$$\liminf_j \langle A(u_j), u_j - v\rangle \geq \langle A(u), u - v\rangle, \quad \forall v \in V. \tag{4.1-2}$$ □

In the following lemma, we give a convenient sufficient condition to check the property (4.1-2).

Lemma 4.1-1: Let U, V be two reflexive Banach spaces such that

$$V \hookrightarrow\hookrightarrow U: \quad (V \text{ is compactly embedded in } U). \tag{4.1-3}$$

The norms in U *and* V *are respectively denoted* $\|\cdot\|_U$ *and* $\|\cdot\|$.

Let A *an operator from* V *into* V' *such that*

$$\left.\begin{array}{l} \langle A(u) - A(v), u - v\rangle \geq -g(\|u\|, \|v\|)\,\|u - v\|_U^\alpha \\ \forall\, u,v \in V,\ \alpha > 1,\ g \text{ a continuous function;} \end{array}\right\} \quad (4.1\text{-}4)$$

$$\phi(t) = \langle A(u + tv), w\rangle \text{ is continuous in } t\ ,\quad \forall\, u,v,w \in V\ . \quad (4.1\text{-}5)$$

Then, A *satisfies property (4.1-2).*

Proof: This result is essentially the same as a theorem given by ODEN [1]. For completeness, we will outline some of the details of the proof. Let

$$\left.\begin{array}{l} u_j \rightharpoonup u \quad \text{(weakly in } V) \\ \limsup\limits_j \langle A(u_j), u_j - u\rangle \leq 0\ . \end{array}\right\} \quad (4.1\text{-}6)$$

We must show that

$$\liminf_j \langle A(u_j), u_j - w\rangle \geq \langle A(u), u - w\rangle\ ,\quad \forall\, w \in V\ .$$

Perform the following steps:

Step 1: For all $v \in V$,

$$\langle A(u_j) - A(v), u_j - v \rangle \geq -g(\|u_j\|, \|v\|)\, \|u_j - v\|_U^{\alpha}$$

hence

$$\langle A(u_j), u - v \rangle \geq -\langle A(u_j), u_j - u \rangle + \langle A(v), u_j - v \rangle$$

$$- g(\|u_j\|, \|v\|)\, \|u_j - v\|_U^{\alpha} .$$

Step 2: Since $u_j \rightharpoonup u$ in V (weakly), $u_j \to u$ in U (strongly). Thus, using the fact that $(u_j \rightharpoonup u)$ implies that $\|u_j\|$ is bounded,

$$\liminf_j \langle A(u_j), u - v \rangle \geq \liminf_j [-\langle A(u_j), u_j - u \rangle]$$

$$+ \langle A(v), u - v \rangle - c\, \|u - v\|_U^{\alpha} .$$

But

$$\liminf_j \langle A(u_j), u - v \rangle \geq - \limsup_j \langle A(u_j), u_j - u \rangle$$

$$+ \langle A(v), u - v \rangle - c\, \|u - v\|_U^{\alpha} ,$$

and then, using (4.1-6)

$$\liminf_j \langle A(u_j), u - v \rangle \geq \langle A(v), u - v \rangle - c\, \|u - v\|_U^{\alpha} . \qquad (4.1\text{-}7)$$

Step 3: Set $v = u - \theta(u - w)$, $\theta > 0$, w arbitrary. Then,

$$\theta \liminf_j \langle A(u_j), u - w\rangle \geq \theta \langle A(u - \theta(u - w)), u - w\rangle - c\theta^{\alpha} \|u - w\|_U^{\alpha} .$$

Divide by θ and take the limit as $\theta \to 0$:

$$\liminf_j \langle A(u_j), u - w\rangle \geq \langle A(u), u - w\rangle , \quad \forall\, w \in V . \tag{4.1-8}$$

Step 4: Next, observe that

$$\langle A(u_j) - A(u), u_j - u\rangle \geq - g(\|u_j\| , \|u\|) \|u_j - u\|_U^{\alpha} .$$

Hence

$$\langle A(u_j), u_j - u\rangle \geq \langle A(u), u_j - u\rangle - c \|u_j - u\|_U^{\alpha} .$$

Thus,

$$\liminf_j \langle A(u_j), u_j - u\rangle \geq 0 \tag{4.1-9}$$

Step 5: Combining (4.1-8) and (4.1-9) gives

$$\liminf_j \langle A(u_j), u_j - w\rangle \equiv \liminf_j \langle A(u_j), u_j - u + u - w\rangle \Big)$$

$$\left.\begin{aligned} &\geq \liminf_{j} \langle A(u_j), u_j - u\rangle \\ &\quad + \liminf_{j} \langle A(u_j), u - w\rangle \\ &\geq \langle A(u), u - w\rangle , \quad \forall\, w \in V . \;\Box \end{aligned}\right\} \qquad (4.1\text{-}10)$$

<u>*Lemma 4.1-2*</u> *(LIONS [1, chapter 2, theorem 2.7]). Let V be a separable, reflexive Banach space. Let A be an operator from $V \to V'$ satisfying the following properties:*

(i) A is pseudo-monotone; *(4.1-11)*

(ii) A is coercive in the sense that

$$\frac{\langle A(v), v\rangle}{\|v\|} \to +\infty \quad \text{if} \quad \|v\| \to +\infty \; . \qquad (4.1\text{-}12)$$

Then, for all $f \in V'$, there exists at least one $u \in V$ such that

$$A(u) = f \; . \qquad (4.1\text{-}13)$$

In other words, A is surjective from V onto V'. $\Box$

<u>Remark 4.1-1</u>: In the particular case of a plate, the method of pseudomonotonicity was used by NAUMANN [2]. For orthotropic inhomogeneous shallow shells parameterized with the help of a curvilinear coordinates system giving the principal lines of curvature (i.e., $\underset{\sim}{a}_1 \cdot \underset{\sim}{a}_2 = 0$), VOROVICH and LEBEDEV [1] proved an existence theorem using the theory of the topological degree of a map. Unfortunately this method is not constructive and, with respect to subsequent studies of approximation

(e.g. by finite element methods) and of bifurcation problems, the pseudo-monotonicity method is more attractive. □

4.2 The Existence Theorem:

Theorem 4.2-1: There exist real numbers ε, $0 < \varepsilon \leq \frac{1}{R_0}$ *and* $p_0 > 0$ *such that for any* $p^i \in L^2(\Omega)$, $i = 1, 2, 3$, *verifying* $\|p\|_{0,2} \leq p_0$, *for any shell* $S \in S_\varepsilon(\rho_\alpha, e_\alpha, \omega_\alpha)$ *which particularly satisfy (by definition)*

$$\underset{\overline{\Omega}}{Max}\ \{|b_\alpha^\beta|, |b_{\alpha|\lambda}^\beta|\} \leq \varepsilon \ , \qquad (4.2\text{-}1)$$

The variational system (3.1-10)(3.1-11) has (at least) one solution $\underset{\sim}{u} \in \underset{\sim}{V}$. *This result is independent of the value of the third component* $p^3 \in L^2(\Omega)$ *of the load resultant* $\underset{\sim}{p}$.

Proof: For clarity, the proof is subdivided into six steps.

Step 1: For each fixed $u_3 \in W_0^{2,2}(\Omega)$, the equation (3.1-10) has one and only one solution $\underline{u} \in (W_0^{1,2}(\Omega))^2$:

From Theorem 3.2-1, the bilinear form $B(\cdot,\cdot)$ is $(W_0^{1,2}(\Omega))^2$-elliptic, i.e.,

$$c_3 \|\underline{v}\|_{1,2}^2 \leq B(\underline{v},\underline{v}) \ , \quad \forall\ \underline{v} \in (W_0^{1,2}(\Omega))^2 \ ,$$

and the constant c_3 is independent of the shell S under consideration in the set $S_\varepsilon(\rho_\alpha, e_\alpha, w_\alpha)$. Next, from (2.1-14), (2.1-20), (3.1-13), we derive the following estimate:

$$|b(u_3,\underline{v})| \leq c\ \{\underset{\overline{\Omega}}{Max}(|b_{\alpha\beta}|)\ \|u_3\|_{0,2} + \|u_3\|_{1,4}^2\}\|\underline{v}\|_{1,2} \ ,$$

or, with the relations (2.1-24)(4.2-1):

$$|b(u_3,\underline{v})| \leq c\,\|\underline{v}\|_{1,2}\,\|u_3\|_{1,4}\,(\varepsilon + \|u_3\|_{1,4}) .$$

Moreover, setting $\|\underline{p}\|_{0,2} = \left(\sum_{\alpha=1}^{2} \|p^\alpha\|_{0,2}^2\right)^{1/2}$, we get

$$\left|\int_\Omega p^\alpha v_\alpha \sqrt{a}\, d\xi^1 d\xi^2\right| \leq c\,\|\underline{v}\|_{1,2}\,\|\underline{p}\|_{0,2} .$$

Then, it remains only to apply the Lax-Milgram lemma.

Step 2: The map $G : u_3 \in W_o^{2,2}(\Omega) \mapsto \underline{u} = G(u_3) \in (W_o^{1,2}(\Omega))^2$ is continuous.

Substituting the estimates of Step 1, written for $\underline{v} = \underline{u}$, into the variational equation (3.1-10) and taking into account the definition of $\underline{u} = G(u_3)$, we finally get

$$\|\underline{u}\|_{1,2} = \|G(u_3)\|_{1,2} \leq c[\,\|u_3\|_{1,4}\,(\varepsilon + \|u_3\|_{1,4}) + \|\underline{p}\|_{0,2}\,] . \qquad (4.2\text{-}2)$$

Now, let us consider solutions $\underline{u}$ and $\underline{v}$ of the equations

$$B(\underline{u},\underline{w}) + b(u_3,\underline{w}) = \int_\Omega p^\alpha w_\alpha \sqrt{a}\, d\xi^1 d\xi^2 , \quad \forall\, \underline{w} \in (W_o^{1,2}(\Omega))^2 ,$$

$$B(\underline{v},\underline{w}) + b(v_3,\underline{w}) = \int_\Omega p^\alpha w_\alpha \sqrt{a}\, d\xi^1 d\xi^2 , \quad \forall\, \underline{w} \in (W_o^{1,2}(\Omega))^2 ,$$

so that

$$B(\underline{u} - \underline{v},\underline{w}) + b(u_3,\underline{w}) - b(v_3,\underline{w}) = 0 , \quad \forall \underline{w} \in (W_o^{1,2}(\Omega))^2$$

We set $\underline{w} = \underline{u} - \underline{v}$ and, by analogy with the derivation of the estimate (4.2-2), obtain:

$$\left.\begin{aligned} \|\underline{u} - \underline{v}\|_{1,2} &= \|G(u_3) - G(v_3)\|_{1,2} \\ &\leq c\,\|u_3 - v_3\|_{1,4}\,(\varepsilon + \|u_3\|_{1,4} + \|v_3\|_{1,4}) \end{aligned}\right\} \quad (4.2\text{-}3)$$

Step 3: We substitute $\underline{u} = G(u_3)$ into the equation (3.1-11). For every $u_3 \in W_o^{2,2}(\Omega)$, the relations (3.1-14)(3.1-15) reveal that the mapping

$$v_3 \in W_o^{2,2}(\Omega) \mapsto A(u_3,v_3) + a(G(u_3),u_3;v_3)$$

defines an element $A(u_3) \in (W_o^{2,2}(\Omega))'$ such that

$$\langle A(u_3),v_3\rangle = A(u_3,v_3) + a(G(u_3),u_3;v_3) , \quad \forall v_3 \in W_o^{2,2}(\Omega) \qquad (4.2\text{-}4)$$

Moreover, the application $v_3 \in W_o^{2,2}(\Omega) \rightarrow \int_\Omega p^3 v_3\sqrt{a}\, d\xi^1 d\xi^2$ defines a linear and continuous form $f \in (W_o^{2,2}(\Omega))'$ such that

$$\langle f,v_3\rangle = \int_\Omega p^3 v_3\sqrt{a}\, d\xi^1 d\xi^2 , \quad \forall v_3 \in W_o^{2,2}(\Omega) . \qquad (4.2\text{-}5)$$

Thus, solving the equation (3.1-11) is equivalent to finding a solution $u_3 \in W_o^{2,2}(\Omega)$ of the equation:

$$A(u_3) = f \tag{4.2-6}$$

To conclude, we check that we are allowed to apply the Lemma 4.1-2 to the equation (4.2-6). Briefly, we must show that the operator A is (i) pseudomonotone, (ii) coercive. This is the objective of the next two steps.

Step 4: The operator A is pseudomonotone.

First, we show that A is bounded in the sense (4.1-1). From the definition 2.1-1, (2.2-10), and (3.1-14) we readily obtain the bound

$$|A(u_3,v_3)| \leq c \, \|u_3\|_{2,2} \, \|v_3\|_{2,2} \quad .$$

From (2.1-24), (2.2-10), (3.1-15), and (4.2-2), we derive

$$|a(G(u_3),u_3;v_3)| \leq c(1 + \|u_3\|_{1,4} + \|u_3\|_{1,4}^2)$$

$$\times \, (1 + \|u_3\|_{1,4}) \, \|v_3\|_{2,2} \quad .$$

Hence, with (2.1-24) and (4.2-4),

$$\|A(u_3)\|_* \leq c(1 + \|u_3\|_{2,2} + \|u_3\|_{2,2}^2)(1 + \|u_3\|_{2,2}) \tag{4.2-7}$$

Next, we use lemma 4.1-1 with $V = W^{2,2}(\Omega)$ and $U = W^{1,4}(\Omega)$. From Kondrasov's theorem (see ADAMS [1] and LIONS and MAGENES [1], for example) the injection from V into U is compact. We have, with (3.1-14), (3.2-9), and (4.2-4),

$$\langle A(u_3) - A(v_3), u_3 - v_3 \rangle \geq c_4 \|u_3 - v_3\|^2_{2,2}$$

$$+ a(G(u_3), u_3; u_3 - v_3) - a(G(v_3), v_3; u_3 - v_3) \quad (4.2\text{-}8)$$

where the constant c_4 is independent of the shell S under consideration in the set $S_o(\rho_\alpha, e_\alpha, \omega_\alpha)$. But, for any $\underline{u}, \underline{v} \in (W_o^{1,2}(\Omega))^2$, $u_3, v_3, w_3 \in W_o^{2,2}(\Omega)$, we have from (2.2-2), (2.2-8), and (3.1-15),

$$\left.\begin{aligned} a(\underline{u}, u_3; w_3) - a(\underline{v}, v_3; w_3) = \int_\Omega &\{n^{\lambda\mu}(\underset{\sim}{u}) u_{3,\mu} \\ &- n^{\lambda\mu}(\underset{\sim}{v}) v_{3,\mu}\} w_{3,\lambda} \sqrt{a}\, d\xi^1 d\xi^2 \\ &- \frac{1}{2} \int_\Omega e\, E^{\alpha\beta\lambda\mu} b_{\lambda\mu} w_3 \{u_{\alpha|\beta} + u_{\beta|\alpha} \\ &+ u_{3,\alpha} u_{3,\beta} - v_{\alpha|\beta} - v_{\beta|\alpha} - v_{3,\alpha} v_{3,\beta}\} \\ &\sqrt{a}\, d\xi^1 d\xi^2 \end{aligned}\right\} \quad (4.2\text{-}9)$$

so that with (4.2-1)

$$\left.\begin{aligned}|a(\underline{u},u_3;w_3) - a(\underline{v},v_3;w_3)| &\leq c\,\{\|n^{\lambda\mu}(\tilde{u}) - n^{\lambda\mu}(\tilde{v})\|_{0,2}\,\|u_{3,\mu}\|_{0,4}\\ &+ \|n^{\lambda\mu}(\tilde{v})\|_{0,2}\,\|u_{3,\mu} - v_{3,\mu}\|_{0,4}\}\,\|w_{3,\lambda}\|_{0,4}\\ &+ c\,\varepsilon\|w_3\|_{0,2}\,\|\underline{u} - \underline{v}\|_{1,2} + c\,\varepsilon\|u_3 - v_3\|_{1,4}\\ &\times (\|u_3\|_{1,4} + \|v_3\|_{1,4})\,\|w_3\|_{0,2}\,.\end{aligned}\right\} \quad (4.2\text{-}10)$$

But, the expressions (2.2-2) and (2.2-8) and the hypothesis (4.2-1) reveal that

$$\|n^{\lambda\mu}(\tilde{v})\|_{0,2} \leq c(\|\underline{v}\|_{1,2} + \varepsilon\,\|v_3\|_{0,2} + \|v_3\|^2_{1,4}) \qquad (4.2\text{-}11)$$

Moreover, with (2.2-4) and (2.2-8), we have

$$\begin{aligned}n^{\lambda\mu}(\tilde{u}) - n^{\lambda\mu}(\tilde{v}) = e\,E^{\lambda\mu\alpha\beta}\,\{&\gamma^L_{\alpha\beta}(\tilde{u} - \tilde{v})\\ &+ \frac{1}{2}[u_{3,\alpha}(u_{3,\beta} - v_{3,\beta})\\ &+ v_{3,\beta}(u_{3,\alpha} - v_{3,\alpha})]\}\end{aligned}$$

and then,

$$\|n^{\lambda\mu}(\tilde{u}) - n^{\lambda\mu}(\tilde{v})\|_{0,2} \leq c\,\{\|\underline{u} - \underline{v}\|_{1,2} + \varepsilon\,\|u_3 - v_3\|_{0,2}$$

$$+ \|u_3 - v_3\|_{1,4} (\|u_3\|_{1,4} + \|v_3\|_{1,4})\} \ . \qquad (4.2\text{-}12)$$

We now substitute the estimates (4.2-11) and (4.2-12) into the estimate (4.2-10), use properties (2.1-24) and the hypothesis (4.2-1), and obtain

$$\left.\begin{aligned} |a(\underline{u},u_3;w_3) - a(\underline{v},v_3;w_3)| &\le c\,\|w_3\|_{1,4}\{\|\underline{u} - \underline{v}\|_{1,2} \\ &\times (\varepsilon + \|u_3\|_{1,4}) + \|u_3 - v_3\|_{1,4}[\|\underline{v}\|_{1,2} \\ &+ (\|u_3\|_{1,4} + \|v_3\|_{1,4})(\varepsilon + \|u_3\|_{1,4} \\ &+ \|v_3\|_{1,4})]\} \end{aligned}\right\} \qquad (4.2\text{-}13)$$

Next, we replace $\underline{u}$ by $G(u_3)$, $\underline{v}$ by $G(v_3)$, w_3 by $u_3 - v_3$ and use the estimates (4.2-2) and (4.2-3) to obtain

$$\left.\begin{aligned} &|a(G(u_3),u_3;u_3 - v_3) - a(G(v_3),v_3;u_3 - v_3)| \\ &\le c\,\|u_3 - v_3\|_{1,4}^2\,[\|\underline{p}\|_{0,2} + (\varepsilon + \|u_3\|_{1,4} \\ &+ \|v_3\|_{1,4})^2] \ . \end{aligned}\right\} \qquad (4.2\text{-}14)$$

Then, the estimate (4.2-8) becomes

$$\left.\begin{array}{l}\langle A(u_3) - A(v_3), u_3 - v_3\rangle \geq c_4 \|u_3 - v_3\|^2_{2,2} \\ \qquad - c\|u_3 - v_3\|^2_{1,4}\; g(\|u_3\|_{1,4}, \|v_3\|_{1,4}) \;,\end{array}\right\} \quad (4.2\text{-}15)$$

so that we get an inequality of the form (4.1-4).

To apply Lemma 4.1-1 and to conclude this step, it remains to be proven that property (4.1-5) holds. Let us consider the expression (4.1-5), i.e.,

$$\phi(t) = \langle A(u_3 + tv_3), w_3\rangle, u_3, v_3, w_3 \text{ fixed in } \overset{\circ}{W}{}^{2,2}(\Omega) \;. \qquad (4.2\text{-}16)$$

From (4.2-4), we get

$$\phi(t) = A(u_3, w_3) + t\,A(v_3, w_3) + a(G(u_3 + tv_3), u_3 + tv_3; w_3) \qquad (4.2\text{-}17)$$

For a fixed $t_o \in \mathbb{R}$, the estimates (4.2-3) and (4.2-13) involve the existence of a continuous function $t \mapsto \Theta(t)$ such that

$$\left.\begin{array}{l}|a(G(u_3 + tv_3), u_3 + tv_3; w_3) - a(G(u_3 + t_o v_3), u_3 + t_o v_3; w_3)| \\ \qquad \leq |t - t_o|\,\Theta(t)\end{array}\right\}$$

This estimate implies immediately the continuity of the function $t \to \phi(t)$ defined by (4.2-16) or (4.2-17).

Step 5: The operator A is coercive.

We must show the property (4.1-12), i.e.,

$$\frac{\langle A(v_3),v_3\rangle}{\|v_3\|_{2,2}} \to +\infty \quad \text{if} \quad \|v_3\|_{2,2} \to +\infty \; . \tag{4.2-18}$$

By definition (4.2-4),

$$\langle A(v_3),v_3\rangle = A(v_3,v_3) + a(G(v_3),v_3;v_3) \; . \tag{4.2-19}$$

For simplicity, we write $\underline{v}$ instead of $G(v_3)$. We obtain from (3.1-10), (3.1-12), (3.1-13), and (3.1-15),

$$\left.\begin{aligned}
& a(\underline{v},v_3;v_3) + 2\int_\Omega p^\alpha v_\alpha \sqrt{a}\, d\xi^1 d\xi^2 \\
& = a(\underline{v},v_3;v_3) + 2\,B(\underline{v},\underline{v}) + 2\,b(v_3,\underline{v}) \\
& = \frac{1}{2}\int_\Omega e\, E^{\alpha\beta\lambda\mu}[(v_{\lambda|\mu} + v_{\mu|\lambda}) + v_{3,\lambda}v_{3,\mu} \\
& \quad - \frac{3}{2}\, b_{\lambda\mu}v_3] \times [(v_{\alpha|\beta} + v_{\beta|\alpha}) + v_{3,\alpha}v_{3,\beta} \\
& \quad - \frac{3}{2}\, b_{\alpha\beta}v_3] \sqrt{a}\, d\xi^1 d\xi^2 \\
& \quad - \frac{9}{8}\int_\Omega e\, E^{\alpha\beta\lambda\mu}\, b_{\alpha\beta}b_{\lambda\mu}v_3^2 \sqrt{a}\, d\xi^1 d\xi^2 \; .
\end{aligned}\right\} \tag{4.2-20}$$

The definition (2.2-10) reveals immediately that the first integral in the last member is positive. Next, we get

$$\left| \int_\Omega p^\alpha v_\alpha \sqrt{a}\, d\xi^1 d\xi^2 \right| \leq c \, \|\underline{p}\|_{0,2} \, \|\underline{v}\|_{0,2} \; .$$

The hypothesis (4.2-1) implies

$$\left| \int_\Omega e \; E^{\alpha\beta\lambda\mu} \; b_{\alpha\beta} b_{\lambda\mu} v_3^2 \sqrt{a}\, d\xi^1 d\xi^2 \right| \leq c \; \varepsilon^2 \, \|v_3\|^2_{0,2}$$

so that, with (2.1-24) and (4.2-2), we obtain

$$\begin{aligned} a(G(v_3),v_3;v_3) &\geq -c[\, \|\underline{p}\|_{0,2} \, \|G(v_3)\|_{0,2} + \varepsilon^2 \|v_3\|^2_{0,2} \,] \\ &\geq -c[\, \|\underline{p}\|^2_{0,2} + (\varepsilon^2 + \|\underline{p}\|_{0,2}) \|v_3\|^2_{2,2} \\ &\qquad + \varepsilon \, \|\underline{p}\|_{0,2} \, \|v_3\|_{1,4} \,] \; . \end{aligned}$$

Then, this estimate, the property (3.2-9) and the relation (4.2-19), yield

$$\begin{aligned} \langle A(v_3),v_3 \rangle &\geq (c_4 - c\varepsilon^2 - c\,\|\underline{p}\|_{0,2}) \, \|v_3\|^2_{2,2} \\ &\quad - c \, \|\underline{p}\|_{0,2} \, (\|\underline{p}\|_{0,2} + \varepsilon \, \|v_3\|_{1,4}) \; . \qquad (4.2\text{-}21) \end{aligned}$$

Hence, for ε and $\|\underline{p}\|_{0,2}$ sufficiently small, we obtain the property (4.2-18).

Step 6: There exists at least one solution $u_3 \in W_o^{2,2}(\Omega)$ for the equation (4.2-6)

In the last two steps, we have checked that the operator A is pseudomonotone and coercive. Then, according to Lemma 4.1-2, A is surjective. Hence, the system (3.1-10)(3.1-11) has at least one solution $\underset{\sim}{u} \in \underset{\sim}{V}$. $\square$

Remark 4.2-1: According to the Sobolev embedding theorem, we have a continuous injection $W^{2,2}(\Omega) \hookrightarrow C^o(\Omega)$. We are thus able to assume only $p^3 \in L^1(\Omega)$ in the statement of Theorem 4.2-1. $\square$

4.3 Remarks on the Uniqueness. In the next theorem, we prove that under the hypothesis of the Theorem 4.2-1, the variational problem (3.1-10) (3.1-11) has one and only one solution $\underset{\sim}{u} \in \underset{\sim}{V}$ whenever the normal load p^3 is sufficiently small.

By analogy with DUVAUT and LIONS [1, §6], a convenient way to obtain such results is to introduce a fixed load resultant $\hat{\underset{\sim}{p}}$ and a small parameter $\eta > 0$, $\eta \to 0$ such that

$$\underset{\sim}{p} = \eta \hat{\underset{\sim}{p}} , \eta > 0, \eta \to 0 . \qquad (4.3\text{-}1)$$

To every solution $\underset{\sim}{u}$ of the problem (3.1-10) (3.1-11) we associate the function $\hat{\underset{\sim}{u}}_\eta$, defined by the relation

$$\underset{\sim}{u} = \eta \hat{\underset{\sim}{u}}_\eta . \qquad (4.3\text{-}2)$$

Through a substitution of (4.3-1) (4.3-2) into the variational system (3.1-10) (3.1-11), we get the following problem:

Find $\hat{\underset{\sim}{u}}_\eta \in \underset{\sim}{V}$ such that

$$B(\underline{\hat{u}}_\eta,\underline{v}) + \frac{1}{2}\int_\Omega e\, E^{\alpha\beta\lambda\mu}[-b_{\alpha\beta}\hat{u}_{3\eta} + \frac{\eta}{2}\hat{u}_{3\eta,\alpha}\hat{u}_{3\eta,\beta}](v_{\lambda|\mu} + v_{\mu|\lambda})\sqrt{a}\,d\xi^1 d\xi^2 = \int_\Omega \hat{p}^\alpha v_\alpha \sqrt{a}\,d\xi^1 d\xi^2 ,\quad \forall\, \underline{v} \in (W_o^{1,2}(\Omega))^2 , \tag{4.3-3}$$

$$A(\hat{u}_{3\eta},v_3) + \int_\Omega e\, E^{\alpha\beta\lambda\mu}\{-\eta b_{\alpha\beta}\hat{u}_{3\eta}\hat{u}_{3\eta,\mu}v_{3,\lambda} + \frac{1}{2}[(\hat{u}_{\alpha\eta|\beta} + \hat{u}_{\beta\eta|\alpha}) + \eta\hat{u}_{3\eta,\alpha}\hat{u}_{3\eta,\beta}] \times [\eta\hat{u}_{3\eta,\mu}v_{3,\lambda} - b_{\lambda\mu}v_3]\}\sqrt{a}\,d\xi^1 d\xi^2 = \int_\Omega \hat{p}^3 v_3 \sqrt{a}\,d\xi^1 d\xi^2,\quad \forall\, v_3 \in W_o^{2,2}(\Omega) . \tag{4.3-4}$$

Theorem 4.2-1 implies that for ε and η sufficiently small (ε,η are respectively defined by (4.2-1) (4.3-1)) the problem (4.3-3) (4.3-4) has at least one solution $\hat{\underset{\sim}{u}}_\eta$.

Under suitable hypothesis, we will prove in the Theorem 4.3-1 the uniqueness of $\hat{\underset{\sim}{u}}_\eta$ and, hence, the uniqueness of $\underset{\sim}{u}$. Moreover, in Theorem 4.3-2, we shall study the asymptotic behavior of $\hat{\underset{\sim}{u}}_\eta$ as $\eta \to 0$.

Theorem 4.3-1: Let $\hat{\underset{\sim}{p}}$ *be a fixed load resultant and* η *a small parameter such that* $\eta > 0$ *. Then there exist* $\varepsilon > 0 (\varepsilon \leq \frac{1}{R_o})$ *and* $\eta_o > 0$ *such that the variational system (4.3-3)(4.3-4) on the one hand, and the variational system (3.1-10)(3.1-11) associated with*

$$\underset{\sim}{p} = \eta \, \hat{\underset{\sim}{p}} \tag{4.3-5}$$

on the other hand, have respectively unique solutions $\hat{\underset{\sim}{u}}_\eta$ *and* $\underset{\sim}{u}$ *for any given shell* $S \in S_\varepsilon(\rho_\alpha, e_\alpha, \omega_\alpha)$ *, and for any* η *such that* $0 < \eta \leq \eta_o$ *. Moreover, the solutions* $\hat{\underset{\sim}{u}}_\eta$ *and* $\underset{\sim}{u}$ *are linked through the relation*

$$\underset{\sim}{u} = \eta \, \hat{\underset{\sim}{u}}_\eta \, . \tag{4.3-6}$$

Proof:

Step 1: The solutions $\underset{\sim}{u}$ (respectively $\hat{\underset{\sim}{u}}_\eta$) of the variational system (3.1-10)(3.1-11) (respectively (4.3-3)(4.3-4)) are bounded.

From (4.2-4)(4.2-5) the nonlinear equation (3.1-11) is equivalent to

$$\langle A(u_3), v_3 \rangle = \langle f, v_3 \rangle, \quad \forall \, v_3 \in W_0^{2,2}(\Omega) \, .$$

The definition (4.2-5) implies immediately

$$|\langle f, v_3 \rangle| \leq c \, \|p^3\|_{0,2} \, \|v_3\|_{0,2}$$

On the other hand, the inequality (4.2-21) gives

$$\langle A(u_3),u_3\rangle \geq (c_4 - c\varepsilon^2 - c\,\|\underline{p}\|_{0,2})\|u_3\|^2_{2,2}$$

$$- c\,\|\underline{p}\|_{0,2}(\|\underline{p}\|_{0,2} + \varepsilon\,\|u_3\|_{1,4}) ,$$

so that, for ε and $\|\underline{p}\|_{0,2}$ sufficiently small, Theorem 4.2-1 insures the existence of a solution u_3 such that (observe that η sufficiently small implies $\|\underline{p}\|_{0,2}$ sufficiently small):

$$(c_4 - c\varepsilon^2 - c\,\|\underline{p}\|_{0,2})\,\|u_3\|^2_{2,2} - (c\varepsilon\,\|\underline{p}\|_{0,2}$$

$$+ c\,\|p^3\|_{0,2})\,\|u_3\|_{2,2} - c\,\|\underline{p}\|^2_{0,2} \leq 0 \qquad (4.3\text{-}7)$$

Thus, for ε and $\|\underline{p}\|_{0,2}$ sufficiently small, i.e.,

$$c\varepsilon^2 + c\,\|\underline{p}\|_{0,2} \leq \frac{c_4}{2} ,$$

the inequality (4.3-7) involves the existence of constants $c > 0$ independent of ε and $\underset{\sim}{p}$ such that

$$\|u_3\|_{2,2} \leq c(\|\underline{p}\|_{0,2} + \|p^3\|_{0,2}) \leq c\,\|\underset{\sim}{p}\|_{0,2} . \qquad (4.3\text{-}8)$$

Then, the estimate (4.2-2) gives

$$\|\underline{u}\|_{1,2} \leq c\, \|\underset{\sim}{p}\|_{0,2} (1 + \|\underset{\sim}{p}\|_{0,2}) \ . \qquad (4.3\text{-}9)$$

Moreover, the estimates (4.3-8)(4.3-9) involve the following estimates on $\hat{\underset{\sim}{u}}_\eta$:

$$\|\hat{u}_{3\eta}\|_{2,2} \leq c\, \|\hat{\underset{\sim}{p}}\|_{0,2} \ , \qquad (4.3\text{-}10)$$

$$\|\hat{\underline{u}}_\eta\|_{1,2} \leq c\, \|\hat{\underset{\sim}{p}}\|_{0,2} (1 + \eta\, \|\hat{\underset{\sim}{p}}\|_{0,2}) \ . \qquad (4.3\text{-}11)$$

Step 2: For $\varepsilon > 0$ and $\eta > 0$ sufficiently small, the problem (4.3-3)(4.3-4) has one and only one solution $\hat{\underset{\sim}{u}}_\eta \in \underset{\sim}{V}$:

Let $\hat{\underset{\sim}{u}}_\eta$ and $\hat{\underset{\sim}{u}}^*_\eta$ two different solutions of the system (4.3-3)(4.3-4). For simplicity we shall write $\hat{\underset{\sim}{u}}$ and $\hat{\underset{\sim}{u}}^*$ instead of $\hat{\underset{\sim}{u}}_\eta$ and $\hat{\underset{\sim}{u}}^*_\eta$ throughout this step in our proof.

We begin noting that (i) $\hat{\underset{\sim}{u}}$ is solution of (4.3-3) with $\underline{v} = \hat{\underline{u}} - \hat{\underline{u}}^*$ and (ii) $\hat{\underset{\sim}{u}}^*$ is solution of (4.3-3) with $\underline{v} = \hat{\underline{u}}^* - \hat{\underline{u}}$. By adding the equations derived in (i), (ii), we obtain

$$\left.\begin{aligned} & B(\hat{\underline{u}} - \hat{\underline{u}}^*, \hat{\underline{u}} - \hat{\underline{u}}^*) + \frac{1}{2}\int_\Omega e\, E^{\alpha\beta\lambda\mu}[-b_{\alpha\beta}(\hat{u}_3 - \hat{u}^*_3) \\ & + \frac{\eta}{2}(\hat{u}_{3,\alpha}\hat{u}_{3,\beta} - \hat{u}^*_{3,\alpha}\hat{u}^*_{3,\beta})][(\hat{u} - \hat{u}^*)_{\lambda|\mu} \\ & + (\hat{u} - \hat{u}^*)_{\mu|\lambda}]\, \sqrt{a}\, d\xi^1 d\xi^2 = 0 \end{aligned}\right\} \qquad (4.3\text{-}12)$$

On the other hand, we write that (i)' $\underset{\sim}{\hat{u}}$ is solution of (4.3-4) with $v_3 = \hat{u}_3 - \hat{u}_3^*$; (ii)' $\underset{\sim}{\hat{u}}^*$ is solution of (4.3-4) with $v_3 = \hat{u}_3^* - \hat{u}_3$. By then adding the equations derived in (i)' and (ii)', we obtain

$$\begin{aligned}
& A(\hat{u}_3 - \hat{u}_3^*, \hat{u}_3 - \hat{u}_3^*) + \int_\Omega e\, E^{\alpha\beta\lambda\mu} \{-\eta b_{\alpha\beta}(\hat{u}_3\hat{u}_{3,\mu} \\
& - \hat{u}_3^*\hat{u}_{3,\mu}^*)(\hat{u}_{3,\lambda} - \hat{u}_{3,\lambda}^*) - \tfrac{1}{2} b_{\lambda\mu}(\hat{u}_3 - \hat{u}_3^*)[(\hat{u} - \hat{u}^*)_{\alpha|\beta} \\
& + (\hat{u} - \hat{u}^*)_{\beta|\alpha}] - \tfrac{\eta}{2} b_{\lambda\mu}(\hat{u}_3 - \hat{u}_3^*)[\hat{u}_{3,\alpha}\hat{u}_{3,\beta} \\
& - \hat{u}_{3,\alpha}^*\hat{u}_{3,\beta}^*] + \tfrac{\eta}{2}(\hat{u}_{3,\lambda} - \hat{u}_{3,\lambda}^*)[(\hat{u}_{\alpha|\beta} + \hat{u}_{\beta|\alpha})\hat{u}_{3,\mu} \\
& - (\hat{u}_{\alpha|\beta}^* + \hat{u}_{\beta|\alpha}^*)\hat{u}_{3,\mu}^*] + \tfrac{\eta^2}{2}(\hat{u}_{3,\lambda} - \hat{u}_{3,\lambda}^*)[\hat{u}_{3,\alpha}\hat{u}_{3,\beta}\hat{u}_{3,\mu} \\
& - \hat{u}_{3,\alpha}^*\hat{u}_{3,\beta}^*\hat{u}_{3,\mu}^*]\}\sqrt{a}\, d\xi^1 d\xi^2 = 0 \, .
\end{aligned} \tag{4.3-13}$$

The ellipticity properties (3.2-8), (3.2-9), the assumption (4.2-1), and the estimates (4.3-10), (4.3-11), valid for all the solutions of the variational system (4.3-3)(4.3-4) (particularly for $\underset{\sim}{\hat{u}}$ and $\underset{\sim}{\hat{u}}^*$), permit to derive from (4.3-12)(4.3-13) the estimates,

$$c_3 \|\underline{\hat{u}} - \underline{\hat{u}}^*\|_{1,2}^2 \leq c(\varepsilon + \eta) \|\underline{\hat{u}} - \underline{\hat{u}}^*\|_{1,2} \|\hat{u}_3 - \hat{u}_3^*\|_{2,2} \, ,$$

$$c_4 \|\hat{u}_3 - \hat{u}_3^*\|_{2,2}^2 \leq c[(\varepsilon + \eta) \|\underline{\hat{u}} - \underline{\hat{u}}^*\|_{1,2}$$

$$+ \eta \|\hat{u}_3 - \hat{u}_3^*\|_{2,2}] \|\hat{u}_3 - \hat{u}_3^*\|_{2,2} ,$$

or, after simplification,

$$\|\underline{\hat{u}} - \underline{\hat{u}}^*\|_{1,2} \leq c(\varepsilon + \eta) \|\hat{u}_3 - \hat{u}_3^*\|_{2,2} \quad (4.3\text{-}14)$$

$$\|\hat{u}_3 - \hat{u}_3^*\|_{2,2} \leq c[(\varepsilon + \eta) \|\underline{\hat{u}} - \underline{\hat{u}}^*\|_{1,2} + \eta \|\hat{u}_3 - \hat{u}_3^*\|_{2,2}] . \quad (4.3\text{-}15)$$

Then, for ε and η sufficiently small, the estimates (4.3-14), (4.3-15) reveal immediately that $\underline{\hat{u}} = \underline{\hat{u}}^*$ and $\hat{u}_3 = \hat{u}_3^*$, i.e., $\tilde{\hat{u}} = \tilde{\hat{u}}^*$.

Step 3: For $\varepsilon > 0$ and $\eta > 0$ sufficiently small, the problem (3.1-10)(3.1-11) has one and only one solution $\tilde{u} \in \tilde{V}$.

This is an immediate consequence of Step 2 and of the derivation of the system (4.3-3)(4.3-4) from the variational equations (3.1-10), (3.1-11) through the relations (4.3-5), (4.3-6). □

To study the behavior of the unique solution $\tilde{\hat{u}}_\eta$ of the system (4.3-3), (4.3-4) when $\eta \to 0$, we introduce the following problem:

Find $\tilde{u}^L \in \tilde{V}$ such that

$$\left.\begin{aligned} &B(\underline{u}^L, \underline{v}) - \frac{1}{2} \int_\Omega e\, E^{\alpha\beta\lambda\mu} b_{\alpha\beta} u_3^L (v_{\lambda|\mu} + v_{\mu|\lambda}) \sqrt{a}\, d\xi^1 d\xi^2 \\ &= \int_\Omega \hat{p}^\alpha v_\alpha \sqrt{a}\, d\xi^1 d\xi^2 , \quad \forall\, \underline{v} \in (W_0^{1,2}(\Omega))^2 , \end{aligned}\right\} \quad (4.3\text{-}16)$$

$$\left.\begin{aligned} &A(u_3^L, v_3) - \frac{1}{2}\int_\Omega e\, E^{\alpha\beta\lambda\mu} b_{\lambda\mu}(u^L_{\alpha|\beta} + u^L_{\beta|\alpha}) v_3 \sqrt{a}\, d\xi^1 d\xi^2 \\ &= \int_\Omega \hat{p}^3 v_3 \sqrt{a}\, d\xi^1 d\xi^2 \;, \quad \forall\, v_3 \in W_o^{2,2}(\Omega) \;. \end{aligned}\right\} \qquad (4.3\text{-}17)$$

We then have the following result:

Theorem 4.3-2: Under the hypothesis of Theorem 4.3-1 we have

$$\hat{\underset{\sim}{u}}_\eta \rightharpoonup \underset{\sim}{u}^L \text{ weakly in } \underset{\sim}{V} \text{ when } \eta \to 0 \;, \qquad (4.3\text{-}18)$$

where $\underset{\sim}{u}^L$ is the unique solution of the problem (4.3-16)(4.3-17).

Proof: Firstly, for ε sufficiently small, we use the Lax-Milgram lemma to establish the existence and the uniqueness of a solution $\underset{\sim}{u}^L \in \underset{\sim}{V}$ for the linear variational system (4.3-16)(4.3-17).

Secondly, for ε and η sufficiently small, Theorem 4.3-1 verifies that the problem (4.3-3) (4.3-4) has one and only one solution $\hat{\underset{\sim}{u}}_\eta \in \underset{\sim}{V}$. The estimates (4.3-10), (4.3-11) prove that the solution $\hat{\underset{\sim}{u}}_\eta$ is uniformly bounded in $\underset{\sim}{V}$ with respect to η . Then, the EBERLEIN-SHMULYAN theorem (see YOSIDA [1, page 141]) and the compactness of the injection of $(W^{1,2}(\Omega))^2 \times W^{2,2}(\Omega)$ into $(W^{0,4}(\Omega))^2 \times W^{1,4}(\Omega)$ imply the existence of a subsequence, still denoted $\hat{\underset{\sim}{u}}_\eta$ and the existence of a function $\underset{\sim}{w} \in \underset{\sim}{V}$ such that, as $\eta \to 0$,

$$\left.\begin{aligned} &\hat{\underset{\sim}{u}}_\eta \rightharpoonup \underset{\sim}{w} \text{ weakly in } \underset{\sim}{V} \;, \\ &\hat{\underset{\sim}{u}}_\eta \to \underset{\sim}{w} \text{ strongly in } (W^{0,4}(\Omega))^2 \times W^{1,4}(\Omega). \end{aligned}\right\}$$

Then, when $\eta \to 0$, the system (4.3-3)(4.3-4) becomes

$$B(\underline{w},\underline{v}) - \frac{1}{2}\int_\Omega e\, E^{\alpha\beta\lambda\mu} b_{\alpha\beta} w_3 (v_{\lambda|\mu} + v_{\mu|\lambda}) \sqrt{a}\, d\xi^1 d\xi^2$$

$$= \int_\Omega \hat{p}^\alpha v_\alpha \sqrt{a}\, d\xi^1 d\xi^2 , \quad \forall\, \underline{v} \in (W_o^{1,2}(\Omega))^2$$

$$A(w_3,v_3) - \frac{1}{2}\int_\Omega e\, E^{\alpha\beta\lambda\mu} b_{\lambda\mu} (w_{\alpha|\beta} + w_{\beta|\alpha}) v_3 \sqrt{a}\, d\xi^1 d\xi^2$$

$$= \int_\Omega \hat{p}^3 v_3 \sqrt{a}\, d\xi^1 d\xi^2 , \quad \forall\, v_3 \in W_o^{2,2}(\Omega) .$$

If we compare these equations with (4.3-16), (4.3-17) we obtain uniqueness: $\underset{\sim}{w} = \underset{\sim}{u}^L$. Thus, the limit $\underset{\sim}{w}$ is independent of the subsequence considered. Hence, the assertion follows. □

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