The vortex method with finite elements

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Résumé :

Ce travail montre que la méthode des caractéristiques est bien adaptée à la résolution numérique des problèmes hyperboliques du type transport lorsque les coefficients des équations sont approximés par des fonctions constantes par morceau sur une triangulation.

On applique la méthode à l'équation d'Euler et on montre que la méthode converge en $O(h+\Delta t)$. Les résultats supposent que les estimations de Kato et Wolibner sont valides (pas de chocs).

La méthode apparaît comme une généralisation de la méthode des vortex étudiée d'habitude dans un cadre différences finies.

Abstract :

This work shows that the method of characteristics is well suited for the numerical solution of first order hyperbolic partial differential equations whose coefficients are approximated by functions piecewise constant on a finite element triangulation of the domain of integration.

We apply this method to the numerical solution of Euler's equation and show that it converges when the time step and the mesh size tend to zero. The proof is based upon the results of regularity given by Kato and Wolibner and on L^∞ estimates for the solution of the Dirichlet problem given by Nitsche. The method obtained appears to be a natural generalization of the vortex method usualy studied in a finite difference context.

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INTRODUCTION

The vortex method is based on an old concept of fluid mechanics which says that for two dimensional non viscous flows the vorticity in the fluid is transported by the flow ; thus if the initial distribution of vorticity consists of a finite number of point vortices, the flow at later times can be found by transport of these point vortices along the stream lines of the flow that they create.

In mathematical terms this means that the two dimensional stream function-vorticity formulation of Euler's equations :

$$(1)\qquad \begin{cases} \dfrac{\partial \omega}{\partial t} + u\,\nabla\omega = 0 \ ; \ -\Delta\Psi = \omega \ ; \ u = \nabla\wedge\Psi \quad \text{in} \quad \Omega \times]0,T[\\ \omega(t=0) = \Sigma\, \omega_i^0\, \delta(x-x_i) \quad ; \quad \Psi|_\Gamma = \Psi_0 \end{cases}$$

where δ is the Dirac function, Γ the boundary of Ω , is integrated by

$$(2)\qquad \begin{cases} \omega(x,t) = \Sigma\, \omega_i^0\, \delta(x-x_i(t)) \\ \dfrac{dx_i}{dt} = \nabla\wedge\Psi(x_i,t) \quad ; \quad x_i(t=0) = x_i \\ -\Delta\Psi = \Sigma\, \omega_i^0\, \delta(x-x_i(t)) \quad ; \quad \Psi|_\Gamma = \Psi_0 \end{cases}$$

This method was first implemented by Christiansen [6] and Chorin [5] and thouroughly tested by Baker [2] on the roll up of vortex sheets. From the theoretical point of view if $\Omega = R^2$ Hald [10] showed that when (2) is discretized explicitly in time, when the Dirichlet problem is approximated by a suitable discretization of the corresponding Green function and when the Dirac functions are smoothed by appropriate convolutions then the method converges. In Baker [2] the Dirichlet operator is discretized with finite differences and as far as we know the convergence is not established in such a case.

The present work is based on the rather straight forward observation that the system (2) is perhaps easier in terms of error analysis to discretized by the finite element method than by the two previously mentioned method because

the equation for the characteristics $x_i(t)$ can be integrated exactly if $\nabla\Lambda\Psi$ is piecewise constant on a triangulation of $\Omega \times]0,T[$. However the error analysis shows that in the finite element context it is no longer feasible to work with Dirac functions ; it is better to use a piecewise constant discretization of $\omega^0(x)$.

Therefore, we shall not work with point vortices but with a piecewise constant approximation of the vortex field :

$$(3) \qquad \omega(x,t) = \sum_{i=1}^{N} \omega^0_{j(i)} I(x-x_i(t))$$

where $I(x-x_i(t))$ equals 1 if x and $x_i(t)$ belong to the same element of the triangulation and zero otherwise, where N is the total number of elements and where $j(i)$ is the index of the element to which $x_i(0)$ belongs to. Therefore we will have to compute certain characteristics backward in t in order to define $\omega(x,t)$ by (3).

Thus although in spirit identical, in practice the present method is substantially different from the point vortex method of [2] and [5]. Both have the advantage of being non dissipative, ours is conservative in a statistical sens only in terms of ω^0. On the other hand we do not have to insert new vortices in some regions of the flow as in [6] and the method is more appropriate to smooth flows. But most of all an error analysis will be given and the method is unconditionaly stable in time. This, by the way, may also be true of the cloud and cell vortex method [5], [2] as was observed by Baker.

The proofs are involved and difficult in their details but the guide-lines are simple : we assume that the regularity obtained by Wolibner [14] and Kato [11] for the solution holds. Thus to measure the error between the exact solution and the approximate solution we measure the distance between two particules, one transported by the exact flow and the other by the approximate flow. In the process L^∞ estimates of the finite element solution of the Dirichlet problem for $-\Delta$ will have to be established following the arguments of Nitsche [12].

For the sake of clarity and also because the method of characteristics in the finite element context can be usefull on other hyperbolic systems, we begin by a presentation of the method on the transport equation. Then, in paragraph 2 the method and the error analysis is explained for the two dimensional Euler equation.

Finally the numerical implementation and some numerical tests are presented in paragraph 3.

1. Finite elements and characteristics for the transport equation

1.a. Statement of the problem

Let Ω be a bounded open set of R^n, let $Q = \Omega \times]0,T[$, and u a divergence free $(\nabla.u = 0)$ vector of $(H^1(Q) \cap L^\infty(Q))^n$. Let f and ρ_0 be two functions of $L^\infty(Q)$ and consider the problem :

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + u\,\nabla\rho = f \quad \text{in} \quad \Omega \times]0,T[= Q \\ &\rho(x,t) = \rho_0(x,t) \text{ for all } \{x,t\} \in S = \Omega \times \{0\} \cup \Sigma^- \end{aligned} \tag{1.1}$$

where Γ is the boundary of Ω, ν is outward normal and $\Sigma^- = \{\{x,t\} : u(x,t).\nu \leq 0 \; x \in \Gamma\}$.

Σ^- represents the part of the boundary of Q where the flow enters into Ω ;thus the boundary conditions for ρ are given at initial time and when the velocity u enters into Ω.

Several physical phenomenons are governed by this equation known as the transport equation. On rectangular domains Ω (1.1) is easily discretized by any upwind finite difference scheme but if Ω is complicated there are no simple non dissipative finite element scheme and the method of characteristics is usually considered as an expensive numerical method. Let us show that for first order accurate discretizations this is not so costly.

The method of characteristics is based upon the following observation : given $\{x,t\} \in Q$ define $\{X^{x,t}(\tau),\tau\}$ by

$$(1.2)\qquad \frac{dX}{d\tau} = \begin{cases} u(X(\tau),\tau) & \text{if } X(\tau) \in \Omega \\ 0 & \text{if } X(\tau) \notin \Omega \end{cases} \qquad \forall \tau \in]0,t[$$

$$X(t) = x$$

If u is uniformly lipchitz continuous with respect to x, (1.2) has a unique solution on $]0,t[$; then we define

$$(1.3)\qquad \hat{\rho}(x,t) = \rho_0(X^{x,t}(0),0) + \int_0^t f(X^{x,t}(\tau),\tau)d\tau$$

and claim that $\hat{\rho}$ is a solution of (1.1). If the data u, ρ_0, f are not smooth the proof is difficult [15] but if ρ_0, f, u are in $C^1(Q)$, then $\hat{\rho}$ is differentiable with respect to $\{x,t\}$ and

$$(1.4)\qquad \rho(X^{x,t}(\tau),\tau) - \rho(x,t) = \frac{\partial \rho}{\partial t}(\tau-t) + \nabla_x \rho(X^{x,t}(\tau)-x) + o(\tau-t) + o(X-x)$$

$$= \int_t^\tau f(X^{x,t}(\sigma),\sigma)d\sigma = \left(\frac{\partial \hat{\rho}}{\partial t} + u\nabla_x \hat{\rho}\right)(\tau-t) + o(\tau-t)$$

thus we have the following result :

Proposition 1 :
When u is uniformly lipchitz continuous in x, divergence free and ρ_0, u and f are in $L^\infty(Q)$ then the solution of (1.1) is given by (1.2), (1.3).

1.b. Discretization

To discretize (1.1) we choose a triangulation $\mathcal{T}_h$ of Ω made of non overlapping triangles if $n = 2$ or tetraedra if $n = 3$ with the usual properties [4] :

$$(1.5)\qquad \begin{array}{l} \mathcal{T}_h = \{\tau_i\}_{i=1}^N \\ \tau_i \cap \tau_j = \emptyset \text{ or 1 vertex, or one side (or face), (or one edge),} \\ \bigcup_1^N \tau_i = \Omega_h \subset \Omega \text{ , distance } (\Omega_h,\Omega) = O(h^2), \text{ h=size of largest side} \\ \qquad\qquad\qquad\qquad \text{of } \mathcal{T}_h. \end{array}$$

If Δt denote a time step then Q is approximated by

$$(1.6)\qquad Q_h = \bigcup P_{ij} \quad ; \quad P_{ij} = \tau_i \times]j\Delta t,(j+1)\Delta t[\quad j = 1,\ldots,M = E(T/\Delta t).$$

Instead of u we shall approximate the stream function Ψ of u, i.e. the function such that

$$u = \nabla\Lambda\Psi = \left(\frac{\partial\Psi_3}{\partial x_2} - \frac{\partial\Psi_2}{\partial x_1},\ \frac{\partial\Psi_1}{\partial x_3} - \frac{\partial\Psi_3}{\partial x_1},\ \frac{\partial\Psi_2}{\partial x_1} - \frac{\partial\Psi_1}{\partial x_2}\right) \quad \text{if } n = 3$$

(1.7)

$$= \left(\frac{\partial\Psi}{\partial x_2}, -\frac{\partial\Psi}{\partial x_1}\right) \qquad \text{if } n \neq 2$$

Let Ψ_h be an approximation of Ψ in the space H_h :

(1.8) $H_h = \{\varphi_h : \varphi_h$ is continuous in x ; $\varphi_h|_{P_{ij}}$ is linear in x and constant in t, $\forall ij\ \}$

Then we approximate u by a function which is constant on all P_{ij} :

(1.9) $u_h = \nabla\Lambda\Psi_h$

and we shall denote by $\rho^h(x,t)$ the solution of

(1.10) $$\frac{\partial\rho^h}{\partial t} + u_h\,\nabla\rho^h = f_h \quad \text{in} \quad \Omega_h \times]0,T[$$

$$\rho^h(x,t) = \rho_{0h}(x,t) \quad \text{on} \quad S_h$$

where f_h and ρ_{0h} are piecewise constant approximations of f and ρ_0.

Even though ρ^h, given by (1.10), belongs to a finite dimensional function space we approximate it further by choosing a point $\{\xi^{ij},t^j\}$ in each P_{ij} and set

(1.11) $\rho_h(x,t) = \rho^h(\xi^{ij},t^j) \quad \forall\{x,t\} \in P_{ij} \quad \forall i = 1,\ldots,N\ ;\ j = 1,\ldots,M.$

Proposition 2 :
For almost all choices of $\{\xi^{ij},t^j\} \in P_{ij}$, ρ_h is uniquely defined by (1.10), (1.11) and computable in a finite number of operations by the following algorithm :

Algorithm 1 : computes the solution of (1.10) for one x and t.
1. Find the prism P_{ij} which contains $\{x,t\}$; set $\{x^0,t^0\} = \{x,t\}$ and m = 0

2. Compute $\lambda \in R^+$ such that $\{x^m - \lambda u_h(x^m,t^m), t^m - \lambda\} \in \partial P_{ij}$, the boundary of P_{ij}.

3. Set $x^{m+1} = x^m - \lambda u_h(x^m,t^m)$, $t^{m+1} = t^m - \lambda$. If $t^{m+1} = 0$ or $x^{m+1} \in \partial\Omega_h$ go to 4, else find $P_{k\ell}$ such that

$$(1.12) \qquad \{x^{m+1},t^{m+1}\} \in P_{k\ell}$$

$$u_h(x^{m+1},t^{m+1})\big|_{P_{k\ell}} \quad \text{points inside } P_{k\ell},$$

and go back to 2 with $m = m+1$, $i = k$, $j = \ell$.

4. Set $\rho^h(x,t) = \rho_{0h}(x^{m+1},t^{m+1}) + \sum_{i=0}^{m} f_h(x^i,t^i)(t^{i+1}-t^i)$

Proof :

Since u_h is piecewise constant on Q_h, the characteristic that passes through a given point $\{x,t\}$ is a broken straight line ; thus to compute it it suffices to compute the nodes of the broken line. These are located at the discontinuities of u_h, therefore on the boundaries of the P_{ij}.

Condition (1.9) insures that a $P_{k\ell}$ satisfying the condition of step 3 can be found. In fact in the sense of distribution u_h is divergence free. Therefore the normal components of u_h are continuous across the sides of the triangles thus (1.12) can be fulfilled. However this procedure does not give a unique solution, whenever the broken straight line passes through a vertex. This is why proposition 2 is stated for almost all $\{\xi_j,t_j\}$. ∎

Algorithm 1 simply states that the characteristic to be computed can be found by starting from $\{x,t\}$ and following the directions u_h from one P_{ij} to the next.

Remark : Construction (1.9) may proove to be expensive because it cost a Dirichlet problem at each time step. In most cases u_h can be discretize directly but then the algorithm may get stuck in one element.

1.c. Implementation

In practice one usually computes ρ_h for all $\{x,t\} \in Q_h$ therefore it would be too costly to compute N × M characteristics. It is more feasible to proceed as follows : Pick one point in each element and compute the characteristic that begin at these points forward in time. If at some time step an element is not crossed by one of these characteristic, pick a new point inside the element and compute the characteristic backward in time that ends at this new point. At all time step make sure that no more than N characteristics are stored, i.e. discard some characteristics if they end up in the same element. This gives the following algorithm :

Algorithm 2 : Computes $\rho_h(x,t)$ solution of (1.10)-(1.11) for all $\{x,t\} \in Q_h$.

0. Choose $\xi^{i0} \in \tau_i$, i = 1,...,N. Set $t^0 = 0$, j = 0 ;

1. For all i = 1, to M do

With Procedure $F(\xi^{ij}, t^j, \Delta t, u_h, 1)$ compute all the nodes $\{x_m^{ij}, t_m^{ij}\}_{m=1}^{m_{ij}}$ of the characteristics solution of

$$(1.13) \qquad \frac{dx}{d\tau} = \begin{cases} u_h(x,\tau) & \text{if } x \in \Omega_h \\ 0 & \text{otherwise} \end{cases}$$

for all $\tau \in]t^j, t^j + \Delta t]$ with $X(t^j) = \xi^{ij}$.

Set

$$(1.14) \qquad \xi^{k(i)j+1} = x_{m_{ij}}^{ij} \quad , \quad t^{j+1} = t^j + \Delta t \quad ,$$

$$(1.15) \qquad \rho_h(\xi^{k(i),j+1}, t^{j+1}) = \rho_h(\xi^{ij}, t^j) + \sum_{1}^{m_{ij}-1} f_h(x_m^{ij}, t_m^{ij})(t_{m+1}^{ij} - t_m^{ij})$$

where k(i) is such that

$$(1.16) \qquad \xi^{k(i)j+1} \in P_{k(i)j+1}$$

2. For all $i' \leq N$ such that there are no $k(i) = i'$ do with procedure $F(\xi^{i'0}, t^{j+1}, t^{j+1}, -u_h, -1)$ compute all the nodes $\{x_m^{ij}, t_m^{ij}\}_1^{m_{ij}}$ of the characteristic solution of (1.13) for all $\tau \in]0, t^{j+1}]$ with $X(t^{j+1}) = \xi^{i'0}$,

and set

$$(1.17) \qquad \xi^{i'j+1} = \xi^{i'0}$$

$$(1.18) \qquad \rho_h(\xi^{i'j+1}, t^{j+1}) = \rho_{0h}(X(o)) + \sum_1^{m_{ij}} f_h(x_m^{ij}, t_m^{ij})(t_m - t_{m+1})$$

3. Replace j by j+1 and stop if $j > M$, else keep only one ξ^{ij} per τ_i and go back to 1.

Remark : The nodes $\{x_m^{ij}, t_m^i\}$ of the characteristics are the vertices of these broken lines.

Procedure $F(\xi, \tau, \Delta t, v_h, \varepsilon)$

Computes with the method of Algorithm 1 the intersections $\{x_m, t_m\}$ with the boundaries ∂P_{ij} of the elements of the triangulation of Ω_h, of the characteristic $X(t)$ solution of :

$$\frac{dX}{dt} = v_h(X,t), \text{ or } 0 \text{ if } X \notin \Omega_h \quad \begin{cases} \text{on }]\tau, \tau+\Delta\tau] & \text{if } \varepsilon = +1 \\ \text{on } [\tau-\Delta\tau, t[& \text{if } \varepsilon = -1 \end{cases}$$

$$(1.19) \qquad \text{with } X(\tau) = \xi.$$

Alternatively we may choose to use

Algorithm 3 : Computes the solution $\rho_h(x,t)$ of (1.10), (1.11).

0. Choose $\theta = T/r$, $r \in \mathbb{N}$

1. For $i = 1$ to r compute with algorithm 2 the solution of (1.10), (1.11). over $t \in [(i-1)\theta, i\theta[$ with $\rho_h(.,(i-1)\theta)$ for initial condition.

Comments : For regular fields u_h most elements are expected to contain one end point of the characteristics that are computed forward in time. For those elements which have no such characteristic at a given time we pick any point,

here ξ^{i0}, and compute the characteristic that end at ξ^{i0}, backward time ; last, note that it is not absolutly necessary to go back till t = 0 ; the user may pick up a time θ and $\rho(t)$ can be computed from $\rho_h(t-\theta)$ according to (1.18) ; this gives algorithm 3. These schemes could be made conservative by putting appropriate weights in (1.14) (1.15) according to the number of characteristics that ends in the same elements and starts from the same elements , but the error analysis below would no longer be valid.

The number of operations, NX, of algorithm 3 is of the order of

$$(1.20) \qquad NX \leq \left(\frac{D^2 T}{h^2 \Delta t} \max\left(\frac{u_h \Delta t}{h}, 1\right) + \nu \frac{u_h \theta}{\Delta t}\right) C_1$$

where C_1 is some constant of the order of 10, ν is of an upperbound on the number of backward characteristic computed at each time step, D is the diameter of Ω_h.
We have found that the best choice for Δt is the one that makes the characteristics cross one element, i.e. :

$$(1.21) \qquad \Delta t \simeq h / u_h$$

Then even for $\theta = T$ assuming that ν is of the order of D/h the method is in $O(1/h^3)$ that is comparable to the work necessary to solve a Dirichlet problem with a good finite element method.

1.d. Error Analysis

We shall denote by $|.|_{p,\Omega}$ the $L^p(\Omega)$ norm.

Theorem 1 :
If ρ_h is computed by algorithm 2 then the L^∞ error is in $O(h+\Delta t)$ for smooth data. More precisely

$$(1.22) \qquad |\rho_h(.,t)-\rho(.,t)|_{\infty,\Omega} \leq |f_h - f|_{\infty,Q} + |\rho_{0h}-\rho_0|_{\infty,\Omega} + C_2 |\nabla \Lambda \Psi_h - \nabla \Lambda \Psi|_{\infty,Q} \exp(t|\Psi''|_{\infty,\Omega})$$

Proof : Let $X_h^{x,t}$ be a characteristic computed by algorithm 1. Then

$$(1.23)\quad |\rho^h(x,t)-\rho(x,t)| \le \int_0^t f_h\,|(X_h^{x,t}(\tau),\tau)-f(X^{x,t}(\tau),\tau)|\,d\tau +$$

$$+ |\rho_{0h}(X^{x,t}(0) - \rho_0(X_h^{x,t}(0)|$$

$$\le |\nabla f|_{\infty,Q} \int_0^t |X_h^{x,t}(\tau) - X^{x,t}(\tau)|\,d\tau + |f_h - f|_{1,Q}$$

$$+ |\nabla\rho_0|_{\infty,\Omega}|X_h^{x,t}(0) - X^{x,t}(0)| + |\rho_{0h}-\rho_0|_{\infty,\Omega}$$

On the other hand

$$(1.24)\qquad \delta(\tau) = X_h^{x,t}(\tau) - X^{x,t}(\tau)$$

satisfies

$$(1.25)\qquad |\dot{\delta}| = |\nabla\Lambda\Psi_h(X_h,\tau) - \nabla\Lambda\Psi(x,\tau)| \le |\nabla\Lambda\Psi_h - \nabla\Lambda\Psi|_{\infty,Q} + |\Psi''|_{\infty,Q}|\delta|$$

and from the Bellman-Gronwall lemma this yields

$$(1.26)\quad |\delta(\tau)| \le \frac{|\nabla\Lambda\Psi_h - \nabla\Lambda\Psi|_{\infty,Q}}{|\Psi''|_{\infty,Q}}\ (-1+\exp(|\Psi''|_{\infty,Q}(t-\tau))$$

Therefore

$$(1.27)\quad |\rho^h(x,t)-\rho(x,t)| \le |f_h-f|_{1,Q} + |\rho_{0h}-\rho_0|_{\infty,\Omega} + C_2|\nabla\Lambda\Psi_h - \nabla\Lambda\Psi|_{\infty,Q}\exp(|\Psi''|_{\infty,Q}\ t)$$

Now we make the same analysis with the characteristics computed forward in time and we get the result.

The fact that (1.22) implies an error of order h and Δt is classical (see Ciarlet [4] for example). ∎

It may prove to be too memory consuming to take $\theta = T$.

Theorem 2 :

For smooth data algorithm 3 yields also

$$(1.28)\quad |\rho_h(.,t)-\rho(.,t)|_{\infty,\Omega} \le C_3\ (h+\Delta t) \qquad \forall t\ ;\ \forall\theta = p\Delta t\ ;\ p \ge 1.$$

Proof :

As for theorem 1

$$\xi_h^j = |\rho_h(\xi^{ij},j\Delta t)-\rho(\xi^{ij},j\Delta t)| \le |\rho_h(X_h^{\xi^{ij},j\Delta t}((j-p)\Delta t))-\rho(X^{\xi^{ij},j\Delta t}(j-p)\Delta t))|$$

$$+\int_{(j-p)\Delta t}^{j\Delta t} |f_h(X_h^{\xi^{ij},j\Delta t}(\tau),\tau)-f(X^{\xi^{ij},j\Delta t}(\tau),\tau)|\,d\tau$$

$$\le |\rho_h(.,(j-p)\Delta t)-\rho(.,(j-p)\Delta t)|_\infty + |\nabla\rho|_\infty \, |X_h^{\xi^{ij},j\Delta t}((j-p)\Delta t)-X^{\xi^{ij},j\Delta t}((j-p)\Delta t)|$$

$$+ |f-f_h|_\infty \, p\Delta t$$

Now we make use of (1.26) with $\tau = (j-p)\Delta t$ and $t = j\Delta t$ to find that

$$\varepsilon_h^j \le \varepsilon_h^{j-p} + |\nabla\rho|_\infty \frac{|\nabla\Lambda\Psi_h - \nabla\Lambda\Psi|_{\infty,Q}}{|\Psi''|_{\infty,Q}} (\exp(|\Psi''|_{\infty,Q} P\Delta t) - 1)$$

$$+ |f-f_h|_\infty \, p\Delta t.$$

It remains to sum up the above inequalities from $j = t/\Delta t$ to $j = p$; it yields

$$|\rho_h(.,t)- \rho(.,t)|_{\infty,\Omega} \le |\nabla\rho|_\infty (h+\Delta t) + |\rho_{oh}-\rho_o|_\infty + |f-f_h|_\infty$$

$$+ |\nabla\rho|_\infty |\nabla\Lambda\Psi_h - \nabla\Lambda\Psi|_{\infty,Q} \frac{\exp(|\Psi''|_{\infty,Q} p\Delta t)-1}{|\Psi''|_{\infty,Q} p\Delta t} . \quad \blacksquare$$

Remark :

Thus from the point of view of error estimates it does not seem necessary to go back very far in time along the characteristics to compute ρ_h. From the point of view of numerical test however there is much less dissipation when θ is large.

It is interesting to note that the present frame work is well suited to show the convergence of the classical method of characteristic. To obtain a scheme in $O(h+\Delta t)$ it is sufficient to use Euler's method to compute $X_h^{x,t}(.)$:

$$(1.29) \qquad X_h^{x,t}(t-\Delta t) = x - u_h(x,t)\Delta t$$

However ρ_h must be interpolated from its values at the vertices of $\mathcal{T}_h$, piecewise linearly in x. Also X_h might not be in Ω_h thus u_h must be extended (smoothely) outside Ω_h. For simplicity but without loss of generality we assume $u.n|_\Gamma = 0$.

Proposition 3 :

Under the above assumptions, if $\rho_h(.,t)$ is piecewise linear on $\mathcal{T}_h$ and $\forall x$ vertex of $\mathcal{T}_h$, $\forall t \in](j-1)\Delta t, j\Delta t[$

$$\rho_h(x,t) = \rho_h(X_h^{x,t^j}(t^{j-1}),t^{j-1}) + f_h(x,t)\Delta t$$

where X_h is computed by (1.29) then for smooth data (u and its extension and f in $L^\infty(Q)$, $\rho(.,t)$ in $W^{1,\infty}(\Omega)$)

$$|\rho_h - \rho|_\infty \leq c(h+\Delta t+h^2/\Delta t)$$

Proof :

We proceed as before :

$$|\rho_h(\xi,j\Delta t)-\rho(\xi,j\Delta t)| \leq |\rho_h(.,(j-1)\Delta t)-\rho(.,(j-1)\Delta t)|_\infty + |\nabla\rho|_\infty |X_h^{\xi,j\Delta t}((j-1)\Delta t) - X^{\xi,j\Delta t}((j-1)\Delta t)| + |f_h - f|_\infty \Delta t$$

From (1.29) we get

$$|X_h^{\xi,j\Delta t}((j-1)\Delta t) - X^{\xi,j\Delta t}(j-1)\Delta t \leq |u_h - u|_\infty \Delta t$$

which yields the result, because

$$|\rho_h(.,j\Delta t)-\rho(.,j\Delta t)|_\infty \leq \max_{\xi \text{ vertex}} |\rho_h(\xi,j\Delta t) - \rho(\xi,j\Delta t)| + |\nabla\rho|_\infty h^2$$

Remark :

Therefore from the point of view of error analysis it is not necessary to compute the characteristics with the accurency chosen but from the numerical point of view (1.29) (as well as $\theta = \Delta t$ in algorithm 3) introduces a lot of discretization errors.

Theoretically the method is non-dissipative ; for example if Ω is a duct and $\rho_0 = 1$ for $x < 0$ and $\rho_0 = 0$ for $x > 0$ and if $u = (1,0)$ then ρ_h can take only the values 1 or 0. But nevertheless the method will distort the shock so in a statistical sens it is dissipative.
Let us give a euristic argument to estimate the statistical dissipativity of the method.

Equation (1.10) can be viewed as

$$\begin{aligned} &\frac{\partial \rho^h}{\partial t} + (u+\eta_h)\nabla\rho^h = f + \eta_h' \quad \text{in} \quad \Omega_h \times]0,T[. \\ &\rho^h(x,t) = \rho_0(x,t) + \eta_h''(x,t) \quad \text{on} \quad S_h^- \end{aligned} \tag{1.30}$$

where η_h, η_h', η_h'' are the errors due to the discretization.
Assume that each triangulation is taken with a given probability ; then the errors become random variables or process; ρ^h can be seen as a random process solution of a stochastic partial differential equation.

It is known that if η, η', η'' are Gaussian processes with zero mean then the expected value $\bar{\rho}$ of ρ^h will satisfy :

$$\begin{aligned} &\frac{\partial \bar{\rho}}{\partial t} + u\nabla\bar{\rho} - \nabla.(\sigma\nabla\bar{\rho}) = f \quad \text{in} \quad \Omega \times]0,T[\\ &\bar{\rho} = \rho_0 \quad \text{on} \quad S^- \end{aligned} \tag{1.31}$$

where σ is the variance of η_h.

In our case of course it may be difficult to show that η_h is Gaussian when the triangulations $\mathcal{T}_h$ are choosen with equal probability but if it were then σ would be proportional to h^2 so that in all likelyhood there exists C_4 such that $\bar{\rho}$ satisfies (1.31) with $\sigma = C_4 h^2$.
Then we may says that <u>statistically the order of dissipation is $O(h^2)$</u>.

2. Application to the Euler Equation in two dimensions : construction of the approximate solution and Error Bounds.

2.a. Construction of the approximate solution

Ω will denote a simply connected smooth convex and bounded open set of $\mathbb{R}^2$. In this open set we consider the solution of the Euler equation :

$$(2.1) \qquad \frac{\partial u}{\partial t} + u.\nabla u = - \nabla p \ , \ \nabla.u = 0 \ \text{ in } \ \Omega \times]0,T[$$

$$(2.2) \qquad u(x,t).n(x)\big|_{\partial\Omega} = 0 \ ,$$

$n(x)$ denotes the outward normal to $\partial\Omega$, boundary of Ω and (2.2) means that the fluid is tangent to the boundary of Ω. Non homogeneous boundary conditions can be handle but we restrict ourselves to this case for clarity. When the initial data $u_0(x)$ or $\omega_0(x) = \nabla\Lambda u_0(x)$ are known the solution of (2.1) and (2.2) is completely determined. Indeed using the stream function ψ or the vorticity $\omega = \nabla\Lambda u$, one can show that (2.1) and (2.2) are equivalent to the equations

$$(2.3) \qquad \frac{\partial \omega}{\partial t} + u\nabla\omega = 0 \ \text{ in } \ \Omega \times \mathbb{R}_t$$

$$(2.4) \qquad u = \nabla\Lambda\psi = \left(\frac{\partial \psi}{\partial x_2} , - \frac{\partial \psi}{\partial x_1}\right)$$

$$(2.5) \qquad - \Delta\psi = \omega \ , \ \psi\big|_{\partial\Omega} = 0$$

We will introduce the P1 approximation of (2.4) and (2.5). We will denote by Ω_h a convex polygonal approximation of Ω, constructed with a regular triangulation $\mathcal{T}_h$ as in § 1.b. We will assume that the boundary vertices of $\mathcal{T}_h$ belong to $\partial\Omega_h \cap \partial\Omega$. We will denote by V_h the space of continuous functions on Ω_h which are linear affine on each triangle $\mathcal{T}_h$ and which vanish on the boundary $\partial\Omega_h$.

The discretized problem :

$u_h(x,t)$ is choosen constant on every prism $P_{ik} = \tau_i \times [k\Delta t, (k+1)\Delta t]$; it is defined by the relations :

$$(2.6) \qquad u_h(x,t) = \nabla\Lambda\psi_h^k \ \text{ for } \ t \in [k\Delta t, (k+1)\Delta t[$$

$$(2.7)\qquad \psi_h^k \in V_h \quad \text{and} \quad \int_{\Omega_h} \nabla\psi_h^k \nabla\theta_h = \sum_{\tau_i \in \mathcal{J}_h} \omega_h(\xi^{ik}, k\Delta t) \int_{\tau_i} \theta_h(x)dx, \ \forall\theta_h \in V_h \ ;$$

ω_h satisfies the transport equation :

$$(2.8)\qquad \frac{\partial\omega_h}{\partial t} + u_h \nabla\omega_h = 0 \ , \ \omega(x,0) = (\nabla\Lambda u_0)(x)$$

and $\xi^{ik} \in P_{ik}$ are chosen according to algoritm 4 which constructs an exact solution of (2.6), (2.8).

Algorithm 4 : Construct a solution of (2.6), (2.8).

0. Choose $\xi^{i0} \in \tau_i$, $i = 1,\ldots,N$. Set $t^0 = 0$, $k = 0$.
   Set $\omega_h(\xi^{i0},0) = \nabla\Lambda u_0(\xi^{i0})$.

1. For i = 0 to N do
   . compute the solution of (2.6), (2.7)
   . with procedure $F(\xi^{ik}, t^k, \Delta t, u_h, 1)$ compute all the nodes of $\{x_m^{ik}, t_m^{ik}\}_1^{m_{ik}}$
   of the characteristics solution of

$$(2.9)\qquad \frac{dX}{d\tau} = u_h(X,\tau)$$

for all $\tau \in]t^k, t^k + \Delta t]$ with $X(t^k) = \xi^{ik}$.

Set $\xi^{j(i),k+1} = x_{m_{ik}}^{ik}$, $t^{k+1} = t^k + \Delta t$,

$$\omega_h(\xi^{j(i),k+1}, t^{k+1}) = \omega_h(\xi^{ik}, t^k)$$

where j(i) is such that $\xi^{j(i),k+1} \in P_{j(i),k+1}$

2. For all $i' \leq N$ such that $j(i) \neq i'$, $\forall i$ do
   with procedure $F(\xi^{i'0}, t^{k+1}, t^{k+1}, -u_h, -1)$ compute all the nodes $\{x_m^{ik}, t_m^k\}_1^{m_{ik}}$ of the characteristic solution of (2.9) for all

$\tau \in]0, t^{k+1}[$ with $X(t^{k+1}) = \xi^{i'0}$, and set $\xi^{i',k+1} = \xi^{i',0}$

$$\omega_h(\xi^{i',k+1}, t^{k+1}) = \omega_h(\xi^{\ell,o})$$

where ℓ is such that $X(o) \in P_{\ell,o}$

Remark 1 : Algorithm 4 proceed exactly like algorithm 2 to integrate (2.8). The fact that u_h depends upon ω_h is not a problem because ω_h can be compute on $[k\Delta t,(k+1)\Delta t]$ if u_h is known on $[0,k\Delta t]$.

Remark 2 : As before one could choose a θ and compute the characteristics backward. On a time interval θ only.

2.2. Error bounds

The equation (2.8) is a transport equation therefore the sharpest a priori bound for the solution u_h, ω_h is given by the L^∞ norm of ω_h. Namely we have

$$(2.10) \qquad |\omega_h(.,t)|_{L^\infty(\Omega_h)} \leq |\omega_0(.)|_{L^\infty(\Omega)} .$$

The situation is similar when one tries to prove the regularity of the exact solution of the Euler equation in two dimensions. From the relation (2.10) one deduces that ω is bounded uniformly in $L^\infty(\Omega)$.
Therefore u solution of the elliptic system

$$\nabla \Lambda u = \omega \ , \quad \nabla . u = 0 \ , \quad u.\nu_{|\partial\Omega} = 0$$

is uniformly bounded in $L^\infty(\mathbb{R}^+ ; W^{1,p}(\Omega))$.

for $1 \leq p < \infty$ but not for $p = \infty$, and it is not possible to prove directly that the higher order derivatives of u remain bounded.
This fact is related to the nature of the green function of the Laplace equation : the solution of the equation

$$- \Delta\varphi = \omega \ , \quad \varphi_{|\partial\Omega} = 0 \ ,$$

u given by the formula :

$$u = \nabla\wedge\Psi = \left(\frac{\partial\Psi}{\partial x_2}, -\frac{\partial\Psi}{\partial x_1}\right)$$

is not lipschitzian, but satisfies the following a priori estimate.

$$(2.11) \qquad |u(x) - u(y)| \leq C|x-y| \ \text{Log} \ \frac{D}{|x-y|} \ |\omega|_\infty$$

where D denotes the diameter of the open set Ω. Now let x(t) and y(t) be the trajectories of two particlesof the fluid and denote by $\varphi(t) = |x(t) - y(t)|$ the Euclidian distance between x(t) and y(t). Since x(t) and y(t) are the solutions of the differential equations :

$$(2.12) \qquad \dot{x}(t) = u(x(t),t), \ \dot{y}(t) = u(y(t),t) \ ,$$

one deduces from (2. 11) the a priori estimates

$$(2.13) \qquad |\dot{\rho}(t)| = |u(x(t),t) - u(y(t),t)|$$

$$\leq C \rho \ \text{Log} \ \frac{D}{\rho} \ |\omega|_\infty \ .$$

By a comparison argument one finally obtains :

$$(2.14) \qquad \left(\frac{\Psi(0)}{D}\right)^{e^{C|\omega|_\infty t}} \leq \frac{\Psi(t)}{D} \leq \left(\frac{\Psi(0)}{D}\right)^{e^{-C|\omega|_\infty t}}$$

The first inequality of (2.14) is the corner stone of the proof of the regularity of ω (in $C^{0,\alpha}(\Omega)$). Then one obtains easily that u is bounded in $C^{1,\alpha}$ (see Wolibner [14], Scheaffer [13] and Kato [11]) for details), but the bounds involve constants, like those in (2.14), rapidly growing with t.

Nevertheless for smooth data we have :

$$u \in W^{1,\infty}(Q)$$

To get an error estimate on $|\omega-\omega_h|_\infty$ we shall need the following lemmas :

Lemma 1 (TARTAR)

If $o < \alpha < 1$ and if

$$(2.15)\qquad \begin{aligned} &\left|\frac{d}{dt}\,\delta(\tau)\right| \le c_4\, h^{1-\alpha} + c_5\quad \delta(o) + c_6\,\delta(\tau) \qquad \forall \tau \in]o,t[\\ &\delta(t) \le c_7(h+\Delta t) \end{aligned}$$

then

$$(2.16)\qquad \delta(\tau) \le c_8(h^{1-\alpha} + \Delta t) \qquad \forall \tau \in]o,t[$$

Proof :

Note first of all that if (2.15) was an equality then

$$(2.17)\qquad \delta(\tau) = \left[\frac{c_4}{c_6}\,h^{1-\alpha}\left[1 + \frac{c_5}{c_6}(e^{-\varepsilon c_6 t} - 1)\right] + \frac{c_5}{c_6}\,(h+\Delta t)c_7 e^{-\varepsilon c_6 t}\right](e^{\varepsilon c_6(\tau-t)} - 1) + (h+\Delta t)\,c_7\, e^{\varepsilon c_6(\tau-t)}$$

with $\varepsilon = \pm 1$; therefore the result is reasonable. Now to proove it one has to use the Bellman Gronwall lemma on time intervals of length θ such that

$$(2.18)\qquad \theta \le \frac{1}{c_6}\log(1+1/2c_5)\ .$$

Indeed on $]o,\theta[$ (2.15) implies

$$(2.19)\qquad \delta(o)\,[1-c_5(e^{c_6\theta}-1)/\,c_6] \le \delta(\theta)\,e^{c_6\theta} + (e^{c_6\theta}-1)\,c_4 h^{1-\alpha}/\,c_6$$

and (2.18) yields then

$$(2.20)\qquad \delta(o) \le 2c_6 e^{c_6\theta}\,\delta(\theta) + 2(e^{c_6\theta}-1)\,c_4 h^{1-\alpha}$$

In turn (2.15) and (2.20) imply

$$(2.21)\quad \left|\frac{d}{d\tau}\,\delta(\tau)\right| \le c_4 h^{1-\alpha}\left[1+2\left(e^{c_6\theta}-1\right)c_5\right] + 2c_6c_5\, e^{c_6\theta}\,\delta(\theta) + c_6\,\delta(\tau)\ .$$

So by repeating the same argument on $\tau \in]\theta,2\theta[$ and so onone finds eventually

$$(2.22)\quad \left|\frac{d}{d\tau}\,\delta(\tau)\right| \le c_9\, h^{1-\alpha} + c_{10}(h+\Delta t) + c_6\,\delta(\tau)$$

which yields (2.16).

Lemma 2

Assume that Ω is a bounded convex open set, let T_h denote a regular triangulation of Ω. We will assume that $\Omega_h = \bigcup k \;\; k \in T_h$ is contained in Ω and that every interval belonging to $\partial\Omega_h$ has it vertices on $\partial\Omega$.
As usual we will assume that the diameter of k is of the order of h and that there exist a constant C independant of h and k_h that one has, for every triangle $k_h \in T_h$:

$$(2.23)\quad (\text{diameter of } k_h) \le C\ (\text{diameter of the circle inscribed in } k_h).$$

For every function $\omega \in L^\infty(\Omega)$, we introduce the functions ψ and ψ_h by the relations :

$$(2.24)\quad \psi \in H_0^1(\Omega),\ \int_\Omega \nabla\psi\,\nabla\theta = \int_\Omega \omega\,\theta\,dx \quad \forall\theta_h \in V_h\ .$$

$$(2.25)\quad \psi_h \in V_h,\ \int_{\Omega_h} \nabla\psi_h\,\nabla\theta_h = \int_{\Omega_h} \omega\,\theta_h\,dx \quad \forall\theta_h \in V_h\ .$$

Then we have for any p, $1 \le p < \infty$:

$$(2.26) \qquad |\psi - \psi_h|_{W^{1,\infty}(\Omega)} \leq C\, h^{1-1/p} |\omega|_{L^\infty(\Omega)}$$

In (2.26) C denotes a constant independant of h and ω.

Proof : The proof follows with slight modifications the line of the proof given by Nitsche [12] which is also described in the book of Ciarlet [4]. First we notice that any $\theta_h \in V_h$ can be exented by zero in $\Omega - \Omega_h$ and defines a function still denoted θ_h which belongs to $H_0^1(\Omega)$. Therefore V_h is a closed subspace of $H_0^1(\Omega)$. Conversely let $\Pi_h\psi$ be the P_1 Lagrange interpolate of any function $\psi \in H_0^1(\Omega) \cap H^2(\Omega)$, ($\psi$ being continuous this expression make sense). $\Pi_h\psi$ vanishes on $\partial\Omega_h$ therefore $\Pi_h\psi \in V_h$ and we have : (cf. Ciarlet [4] (3.1. 39) chap. 3, § 3.1., p. 123)

$$(2.27) \qquad |\psi - \Pi_h\psi|_{W^{k,\infty}(\Omega_h)} \leq C\, h^{2-k-2/p} |\psi|_{W^{2,P}(\Omega_h)}$$

Using (2.24) and the fact that ω is uniformly bounded in $L^\infty(\Omega)$ we deduce from (2.27) that we have for k = 0,1

$$(2.28) \qquad |\psi - \Pi_h\psi|_{W^{k,\infty}(\Omega_h)} \leq C\, h^{2-k-2/p} |\omega|_{L^\infty(\Omega)}$$

(As usual (2.25) is valid only for $p < +\infty$).

Next on the space $H_0^1(\Omega_h) \cap W^{1,\infty}(\Omega_h)$ and on the space V_h we will use the weighted norms :

$$(N1) \quad v \rightarrow |v|_{0,\infty,\Omega} + h|\log h|\, |v|_{1,\infty,\Omega}$$

and

$$(N2) \quad v \rightarrow |\log h|^{-\frac{1}{2}} |v|_{0,\infty,\Omega} + h|v|_{1,\infty,\Omega}$$

From (2.25) we deduce that ψ_h is the projection of ψ : $P_h\psi$ (for the scalar product $\int_\Omega \nabla u\, \nabla v\, dx$) on V_h. Therefore we have (Ciarlet [4] (3.3.6.1.), page 163).

$$(2.29)\qquad |\log h|^{-\frac{1}{2}}\,|P_h\psi|_{0,\infty,\Omega_h} + h|P_h\psi|_{1,\infty,\Omega_h}$$

$$\leq C(|\Psi|_{0,\infty,\Omega_h} + h|\mathrm{Log}\ h|\ |\Psi|_{1,\infty,\Omega_h})$$

Next we have

$$(2.30)\qquad |\log h|^{-\frac{1}{2}}|\psi-\psi_h|_{0,\infty,\Omega_h} + h|\psi-\psi_h|_{1,\infty,\Omega_h}$$

$$= |\log h|^{-\frac{1}{2}}\,|(I-P_h)\psi|_{0,\infty,\Omega_h} + h|\psi-\psi_h|_{1,\infty,\Omega_h}$$

$$= |\log h|^{-\frac{1}{2}}\,|(I-P_h)(\psi-v_h)|_{0,\infty,\Omega_h} + h|\psi-\psi_h|_{1,\infty,\Omega_h}$$

$$\leq C\,(|\psi-v_h|_{0,\infty,\Omega_h} + h|\log h|\ |\psi-v_h|_{1,\infty,\Omega_h})$$

where C denotes a constant independant of h for h small enough, and v_h any element of V_h. Now taking for v_h the interpolate $\Pi_h\psi$ and using (2.27) we obtain :

$$(2.31)\qquad |\psi-\psi_h|_{0,\infty,\Omega_h} + h|\log h|^{\frac{1}{2}}\,|\psi-\psi_h|_{1,\infty,\Omega_h} \leq$$

$$(|\log h|\ h^{2-2/p} + h|\log h|^{3/2}\ h^{1-2p})|\psi|_{W^{2,p}(\Omega_h)}$$

and now we can use the fact that Ω is smooth and $\omega = \Delta\psi$ is bounded in $L^\infty(\Omega)$ to obtain an estimate of the following type.

$$(2.32)\qquad |\psi|_{2,p,\Omega} \leq C\,|\omega|_{L^\infty(\Omega)}\ .$$

These estimates remain valid for domain Ω with corner of angler Θ_i, provided one has

$$(2.33)\qquad \max\left(\frac{\Pi}{\Theta_i}\right) < p\ .$$

With the relation (2.21) and the Lemma 2.1 we can give an error estimate :

Theorem 3 :

Assume that Ω is a smooth regular convex open set of $\mathbb{R}^2$ and that the initial data $u_0(.)$ is a smooth function. ($u_0(.)$ belongs to $C^2(\Omega)$ for instance) then the solution of (P_h) : (2.6), (2.7) and (2.8) converges to the solution of (2.1), (2.2) (or equivalently to the solution of (2.3), (2.4), (2.5)). More precisely, denoting by Δt the time step and by h the parameter of the triangulation, we have :

$\forall \varepsilon \in]0,1[$ there exists h^o, Δt^o such that :

$$(2.34) \quad |\omega_h(x,t) - \omega(x,t)| \leq C(t)\,(h^{1-\varepsilon} + \Delta t) \qquad \forall h^o,\ \Delta t \leq \Delta t^o. \quad \blacksquare$$

Remark :

In (2.34) $C(t)$ denotes a constant depending of u_0 and t, but not of h, Δt, furthermore $C(t)$ is rapidly growing towards $C(T)$ when t increases. As we mentioned in the begining of the paragraph $C(T)$ may itself grow very fast with T. Nethertheless theorem 3 shows that the method of order $h^{1-\varepsilon}+\Delta t$

Proof of theorem 3 ;

By construction :

$$(2.35) \quad |\omega_h(x,t)-\omega(x,t)| \leq |\omega_{oh}(X_h^{\xi^{ij},t^j}(0))-\omega_0(X^{x,t}(0))|$$
$$\leq |\omega_{oh}-\omega_o|_\infty + |\nabla\omega_o|_\infty \,|\, X_h^{\xi^{ij},t^j}(0) - X^{x,t}(0)|$$

where the second inequality was obtained by adding and substracting $\omega_0(X_h^{\xi^{ij},t^j}(0))$ in the first one.

Therefore

$$(2.36) \quad |\omega_h(.,t)-\omega(.,t)|_\infty \leq C_{11}h + |\nabla\omega_0|_\infty \sup_{x,t} |X_h^{\xi^{ij}(x),t^j}(0)-X^{x,t}(0)|$$

Now by construction of X_h and X we have

$$(2.37) \quad \left| \frac{d}{d\tau} (X_h^{\xi^{ij},t^j}(\tau)-X^{x,t}(\tau))\right| = |u_h(X_h^{\xi^{ij},t^j}(\tau),\tau) - u(X^{x,t}(\tau),\tau)|,$$
$$\leq |u_h(.,\tau)-u(.,\tau)|_\infty + |\nabla u|_\infty |X_h^{\xi^{ij},t^j}(\tau)-X^{x,t}(\tau)|$$

(again to get the last inequality one adds and substract $u(X_h^{\xi^{ij},t^j}(\tau),\tau)$).

Let $\tilde{\psi}$ be the solution of

$$(2.38) \qquad -\Delta\tilde{\psi} = \omega_h \text{ in } \Omega\ ,\ \tilde{\psi}|_\Gamma = 0.$$

Then according to lemma 2 :

$$(2.39)\ |u_h(.,\tau)-u(.,\tau)|_\infty \le |\nabla^\perp\psi_h(.,t)-\nabla^\perp\tilde{\psi}(.,t)|_\infty + |\nabla^\perp\tilde{\psi}(.,t)-\nabla^\perp\psi(.,t)|_\infty$$

$$\le C_{12}\, h^{1-\varepsilon}\ |\omega_h(.,t)|_\infty + C_{13}|\omega_h(.,t)-\omega(.,t)|_\infty\ ;$$

the last term comes from the continuity of $-\Delta$ from $L^\infty(\Omega)$ into $W^{1,\infty}(\Omega)$. Thus (2.35), (2.37) and (2.39) yield

$$(2.40)\ \left|\frac{d}{d\tau}(X_h^{\xi^{ij},t^j}(\tau)-X_h^{x,t}(\tau))\right| \le C_{12}|\omega_h(.,t)|_\infty\, h^{1-\varepsilon} + C_{13}(|\omega_{oh}-\omega_o|_\infty + |\nabla\omega_o|_\infty$$

$$\sup_{x,t}\ |X_h^{\xi^{ij},t^j}(x),t^j(t)(o) - X^{x,t}(o)|) + |\nabla u|_\infty\ |X_h^{\xi^{ij},t^j}(\tau)-X^{x,t}(\tau)|$$

If we take $x = x^*, t = t^*$ where the * denotes the values for which the sup in (2.40) is attained then Lemma 2 gives

$$|X_h^{\xi^{ij}(x^*),t^j(t^*)}(o) - X^{x^*,t^*}(o)| \le C_{14}\ (h^{1-\varepsilon} + \Delta t)\ ;$$

then the Bellman Gronwall lemma yields the result. ∎

2. Numerical experiments

Test 1 : we have solved problem (2.3) (2.5) with

$$\Omega =]0,1[^2$$

$$\omega_o = \begin{cases} 10 & \text{in } [.22, .33] \times [.11, .22] \\ 1 & \text{elsewhere.} \end{cases}$$

and we have triangulated Ω with 3 families of parallel lines $h = 1/9$, $\Delta t = 1$. At later times u can only be equal to 10 or 1. Therefore the initial spot, $u = 10$, is expected to turn with the flow and the area of $\{x : u(x,t) = 10\}$ remains constant.

Since our scheme is conservative in a statistical sens only the area mentionned above is not constant, but it fluctivates around the exact value. The spot can be seen to turn with the flow.

Test 2 : Similar problem in a ring.
These problems are particularly difficult for dissipative numerical methods.

Test 3 : This test simulates the dispersion of a pollutant (cheminey smoke) by a wind. The pollutant comes out of Γ_1 a part of the boundary Γ with initial velocity 27 times the initial velocity of the wind and a vorticity of 10, while the wind has no initial vorticity ; Ω contains a hump (hill) to illustrate the feasibility of the finite element method.

Therefore we have solved

$$\frac{\partial \omega}{\partial t} + (\nabla \Lambda \Psi) \nabla \omega = 0$$

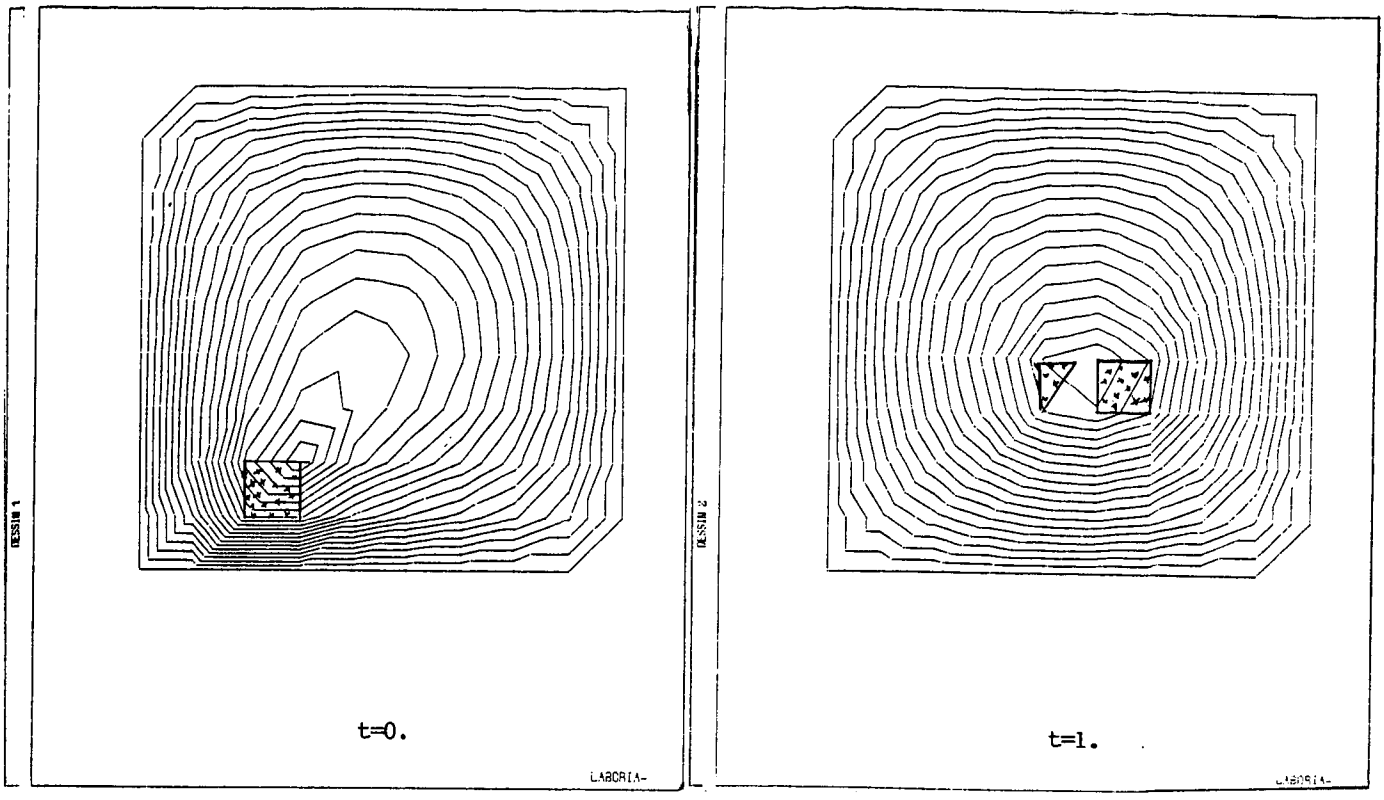
$$\omega(t = 0) = 0$$

$$\omega|_{\Gamma_1} = 10 \ , \ \omega|_{\Gamma_2} = 0.$$

$$-\Delta \Psi = \omega$$

$$\frac{\partial \Psi}{\partial y}\Big|_{\Gamma_2 \cup \Gamma_3} = 1 \ , \ -\frac{\partial \Psi}{\partial x}\Big|_{\Gamma_1} = 2.7 \ , \ \Psi|_{\Gamma_4} = \text{constant}.$$

The triangulation shows 225 nodes and $\Delta t = .2$.
Computing times are 30" on an IBM 370. 168 for tests 1 and 2 and 1 mn for test 3. The plottings of the vorticity (shaded area on the figures) are done by hand because it is piecewise constant on the triangulation.



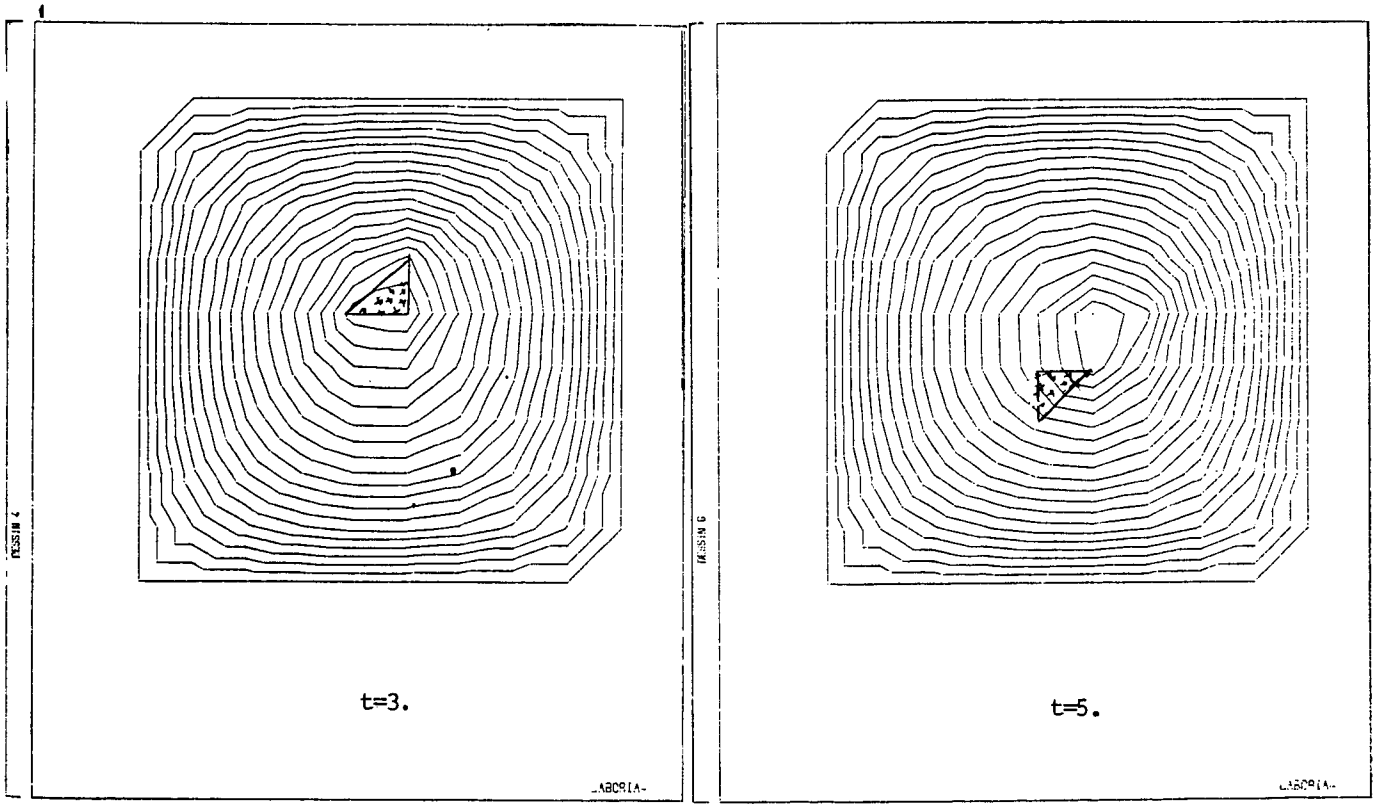


Figure 1

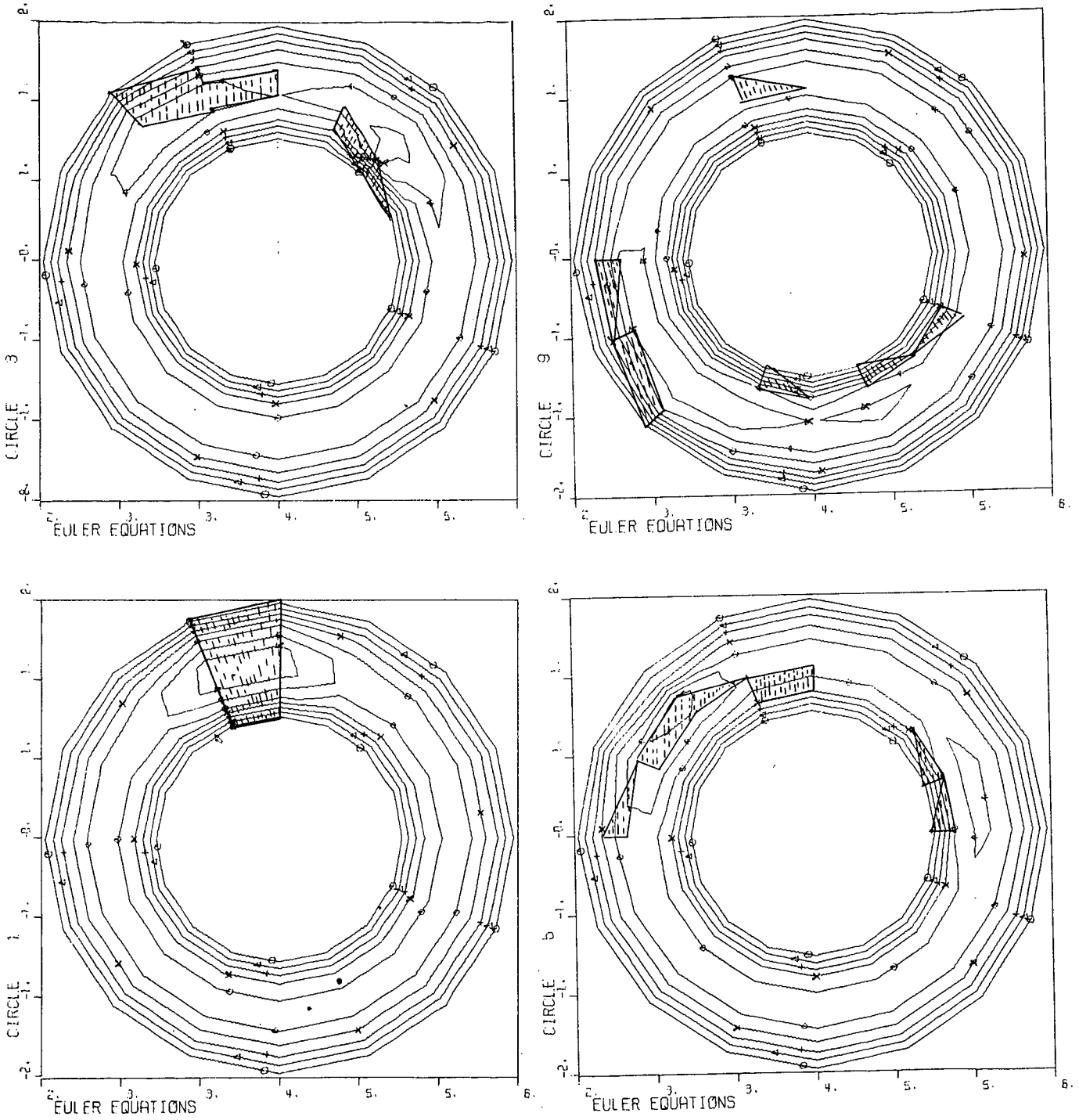

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Figure 2

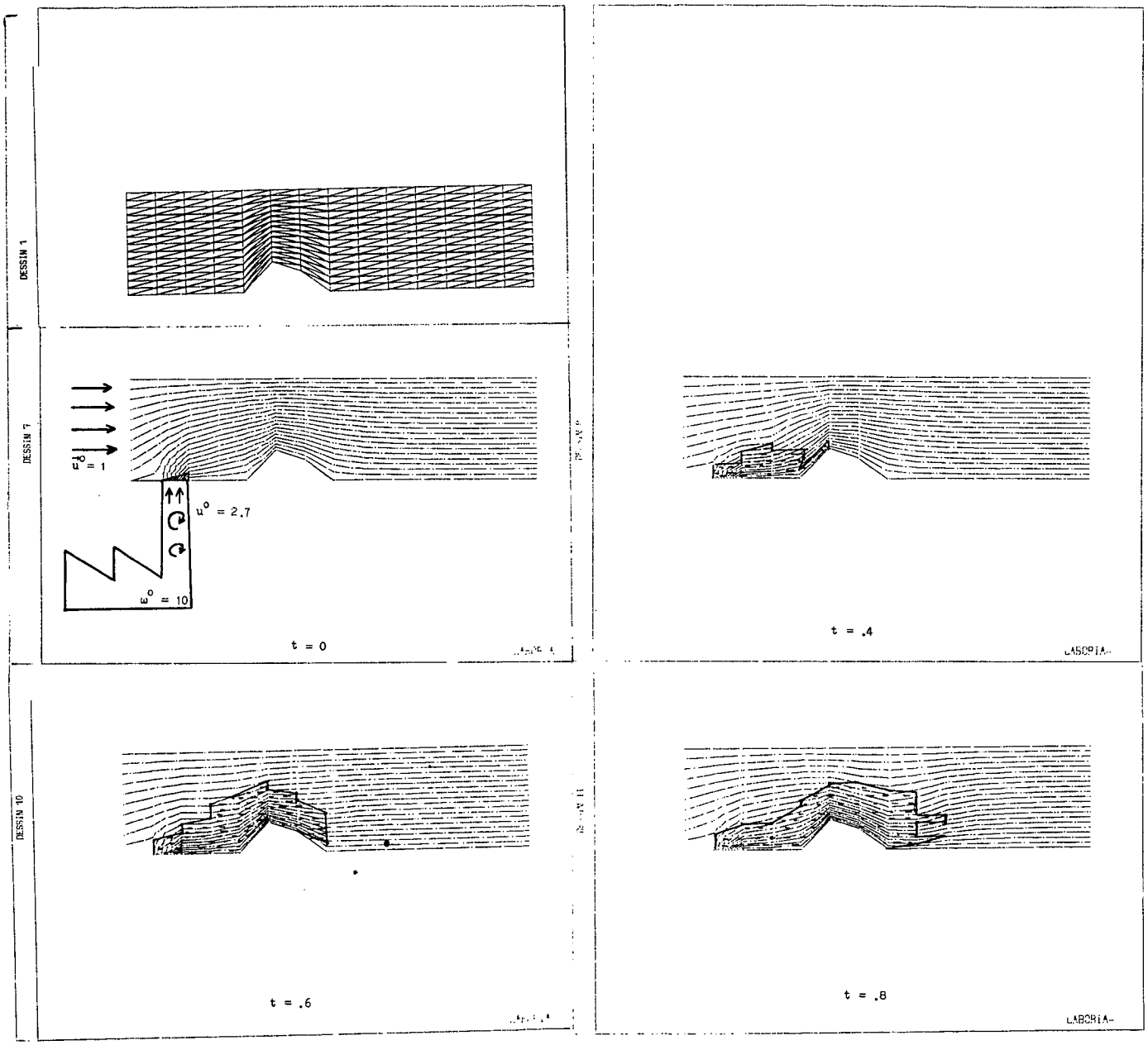
DESSIN 1
DESSIN 7
DESSIN 10
$u^\delta = 1$
$u^\circ = 2.7$
$\omega^\circ = 10$
t = 0
t = .4
t = .6
t = .8

Figure 3

References

[1] Arnold V. Méthodes mathématiques en Mécanique classique.

[2] Baker G. The clouds and cells technique applied to the roll up of vortex sheets, J. of Comp. phys. 75-95 (1979).

[3] Bardos C. Existence et Unicité de la solution de l'équation d'Euler en dimension deux. J. of Math. Anal. and Appl. 40, 769-790 (1972).

[4] Ciarlet Ph. The finite element method for elliptic problems. Studied in Math. and its applications, North Holland (1978).

[5] Chouh A.J. Numerical study of slightly viscous flow, J. of Fluid Mech. 57, 784-796 (1973).

[6] Christiansen J.P. Numerical solution of hydrodynamics by the method of point vortices J. of Comp. Phys. 13, 863-879 (1973).

[7] Covet B., Buneman O. and Leonard A. Three dimensional simulation of the free shear Layer using the vortex in cell method, Proceeding 2^d symposium on Turbulent shear flows, London (1979).

[8] Grisward P. Behaviour of the solution of an elliptic boundary value problem in a polygonal of polyhedral domain. Numerical solutions of Differential Equation, 3 (1976), (Ed. B. A. Press, 267-274.

[9] Hald M.O. Convergence of vortex method for the Euler equation. Siam J. Numer. Anal. vol. 16, n° 5, October 1979, 726-755.

[10] Hald M.O. and Manceri del Prete V. Convergence of vortex method for the Euler equation. Math. Comp. 27 (1973) 719-728.

[11] Kato T. On the classical solution of the two dimensional non stationary Euler equation, Arch. Rat. Mech. Anal. 24 (1967) 302-324.

[12] Nitsche J. L_∞ convergence of finite element approximation "Mathematical Aspects of Finite Element method", Rome (1975), Springer.

[13] Schaeffer A.C. Existence Theorem for the flow of an incompressible fluid in two-dimension, Trans. Am. Math. Soc. 42, 497 (1967).

[14] Wolibner W. Un théorème sur l'existence du mouvement plan d'un fluide parfait homogène et incompressible pendant un temps infiniment long, Math. Zeit. 37 (1935) 727-738.

[15] Zerner, M. Equations d'évolution quasi-linéaire du premier ordre : le cas lipschitzien (à paraître).