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Abstract: The steady state system of isothermal Navier-Stokes equations is considered in two dimensional domain including an obstacle. The shape optimisation problem of drag minimisation with respect to the admissible shape of the obstacle is defined. The generalized solutions for the Navier-Stokes equations are introduced. The existence of an optimal shape is proved in the class of admissible domains. In general the solution to the problem under consideration is not unique.

Key-words: Compressible fluids, Navier-Stokes equations, generalized solutions, shape optimization, drag minimization

Mathematics Subject Classification (2000): 35D05, 35Q30, 49Q10, 49Q20, 76N25

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Sur l'optimisation de forme pour les équations de Navier-Stokes en compressible isotherme

Résumé : On considère le système des équations de Navier-Stokes dans le cas d'un fluide compressible et d'un domaine plan contenant un obstacle. On définit le problème d'optimisation géométrique consistant à minimiser la traînée dans un ensemble de formes d'obstacle admissibles. On introduit les solutions généralisées des équations de Navier-Stokes. On prouve l'existence d'une forme optimale dans la classe de domaines admissibles. En général la solution du problème considéré n'est pas unique.

Mots-clés : Fluide compressible, équations de Navier-Stokes, solutions généralisées, optimisation de forme, minimisation de traînée

1 Introduction

1.1 Problem formulation

Suppose that compressible Newtonian fluid occupies the bounded region $\Omega \subset \mathbb{R}^2$. We will assume that $\Omega = B \setminus S$, where B is a sufficiently large hold all containing inside a compact obstacle S. We could take, e.g., for B a ball of radius R, $B = \{x||x| < R\}$. We do not impose restrictions on the topology of the flow region. The cases of S with a finite number of connected components or $S = \emptyset$ are taken into consideration.

The fluid density $\rho : \Omega \mapsto \mathbb{R}^+$ and the velocity field $\mathbf{u} : \Omega \mapsto \mathbb{R}^2$ are governed by the Navier-Stokes equations

$$\begin{aligned}&-\nu\Delta\mathbf{u} - \xi\nabla\mathrm{div}\,\mathbf{u} + \rho\mathbf{u}\nabla\mathbf{u} + \nabla\rho = \rho\mathbf{f}\ ,\\ &\mathrm{div}\ (\rho\mathbf{u}) = 0\ ,\end{aligned}$$

where ν, ξ are positive viscous coefficients and $\mathbf{f} : \Omega \mapsto \mathbb{R}^2$ is a given vector field. If the viscous stress tensor is defined by the equality

$$\Sigma = \nu(\nabla\mathbf{u} + \nabla\mathbf{u}^\top) + (\xi - \nu)\mathrm{div}\,\mathbf{u}\,\mathbf{I}\ ,$$

then the governing equations can be written in the equivalent divergence form

$$\mathrm{div}\,(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\rho - \rho\mathbf{f} = \mathrm{div}\,\Sigma \quad \text{in} \quad \Omega\ , \tag{1.1a}$$

$$\mathrm{div}\,(\rho\mathbf{u}) = 0 \quad \text{in} \quad \Omega\ . \tag{1.1b}$$

Equations (1.1) should be supplemented with the boundary conditions. In view of possible applications e.g., to the shape optimisation problem of a wing it is supposed that the velocity field satisfies the non-homogeneous boundary conditions

$$\mathbf{u} = 0 \quad \text{on} \quad \partial S\ , \qquad \mathbf{u} = \mathbf{U}^\infty \quad \text{on} \quad \Gamma\ , \tag{1.2a}$$

and the density distribution is prescribed on the entrance set

$$\rho = \rho^\infty \quad \text{on} \quad \Gamma^+ = \{x \in \partial B :\ \mathbf{U}^\infty \cdot \mathbf{n}(x) < 0\}\ . \tag{1.2b}$$

Here $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$. It is assumed that $\mathbf{U}^\infty \in \mathbb{R}^2$ is a given vector, and $\rho^\infty \in L_\infty(\Gamma^+)$ is a given non-negative function.

Boundary condition (1.2a) can be written in the form of the equality $\mathbf{u} = \mathbf{u}^\infty$ on $\partial\Omega$, where $\mathbf{u}^\infty(x)$ is a smooth function defined for any $x \in \mathbb{R}^2$, which vanishes in the vicinity of S and coincides with $\mathbf{U}^\infty$ in an open neighbourhood of ∂B.

For $\mathbf{u}^\infty = 0$ problem (1.1)-(1.2) becomes the classical boundary value problem with no slip condition on the boundary of the flow region

$$\mathbf{u} = 0 \quad \text{on} \quad \partial\Omega\ . \tag{1.3a}$$

In this particular case there are no boundary conditions for the density and the total mass $\mathcal{M}$ of the gas must be prescribed

$$\int_\Omega \rho dx = \mathcal{M} \ . \tag{1.3b}$$

Other physical quantities which characterise the flow include the kinetic energy $\mathcal{E}$, the volume rate of energy dissipation $\mathcal{D}$ and the drag J, and are defined respectively by

$$\mathcal{E} = \frac{1}{2}\int_\Omega \rho|\mathbf{u}|^2 dx \ , \quad \mathcal{D} = \int_\Omega (\nu|\nabla \mathbf{u}|^2 + \xi|\operatorname{div}\mathbf{u}|^2)dx \ , \quad J = -\mathbf{U}^\infty \cdot \int_{\partial S} (\mathbf{\Sigma} - \rho \mathbf{I})\cdot \mathbf{n} dS \ . \tag{1.4}$$

The drag J accounts for the reaction of the surrounding fluid on the obstacle S. For our purposes, the formula for the drag can be written in the equivalent form, see Appendix,

$$J(\rho, \mathbf{u}, \Omega) = \int_\Omega (\mathbf{\Sigma} - \rho\mathbf{u}\otimes\mathbf{u} - \rho\mathbf{I}) : \nabla\mathbf{u}^\infty dx + \int_\Omega (\mathbf{U}^\infty - \mathbf{u}^\infty)\cdot \mathbf{f}\rho dx \ . \tag{1.5}$$

We will consider the physically reasonable solutions to problems (1.1)-(1.2) and (1.1)-(1.3) for which the density is non-negative and the quantities (1.4) are bounded from above.

On the other hand, the peculiarity of problem (1.1) is that the equations do not allow us to control any L_r norm of the density ρ even for $r = 1$. Moreover, we can not eliminate the possibility of concentration of finite mass of gas in very small domains. The simplest way to bypass this difficulty is to suppose that the mass of gas is a Borel measure μ_ρ in Ω. This means that the mass contained in any measurable set E is simply $\mu_\rho(E)$. In the paper the standard notation is used for the function spaces. The space $H^{1,p}(\Omega)$ is the Sobolev space of functions integrable along with the first order generalized derivatives in $L_p(\Omega)$ equipped with its natural norm. For $p = 2$ we use the notation $H^{1,2}(\Omega)$ rather than $H^1(\Omega)$, and for real $m > 0$ we denote the Sobolev space of order m by $H^{m,2}(\Omega)$.

Definition 1.1. For given $\mathbf{U}^\infty \in \mathbb{R}^2$ and $\mathbf{f} \in C(\Omega)^2$ a generalized solution to problem (1.1)-(1.2) is the pair $(\mu_\rho, \mathbf{u})$, where μ_ρ is a non-negative Borel measure in Ω and $\mathbf{u} - \mathbf{u}^\infty \in H^{1,2}_0(\Omega)$, which satisfies the following conditions:

(a) The measure μ_ρ does not charge null capacity sets i.e., $\mu_\rho(E) = 0$ for any Borel set with $\operatorname{cap} E \equiv \inf\{\int_\Omega |\nabla\varphi|^2 dx : \varphi \in C^\infty_0(\Omega), \ \varphi \geq 1 \text{ on } E\} = 0$ and

$$\int_\Omega d\mu_\rho(x) = \mu_\rho(\Omega) = \mathcal{M} < \infty \ . \tag{1.6}$$

It implies, in particular, that for any continuous function $f : \mathbb{R} \mapsto \mathbb{R}$ the composed function $f(\mathbf{u})$, more precisely its quasicontinuous representative, is measurable with respect to μ_ρ.

(b) The scalar function $|\mathbf{u}|^2$ is integrable with respect to measure μ_ρ i.e.,

$$\mathcal{E} = \frac{1}{2}\int_\Omega |\mathbf{u}|^2 d\mu_\rho(x) < \infty \ .$$

This means that the kinetic energy $\mathcal{E}$ of the flow is finite. It follows from this condition that the functions u_i and $u_i u_j$, where u_i, $i = 1, 2$, are the components of the velocity field $\mathbf{u} = (u_1, u_2)$, are integrable with respect to μ_ρ.

(c) The energy dissipation satisfies the inequality

$$\mathcal{D} \le \int\limits_\Omega \Big(\mathbf{\Sigma} : \nabla \mathbf{u}^\infty + \frac{\xi}{2}|\mathrm{div}\,\mathbf{u}|^2 + \frac{1}{2\xi}\Big)dx - \int\limits_\Omega (\mathbf{u}\otimes\mathbf{u} + \mathbf{I}) : \nabla\mathbf{u}^\infty d\mu_\rho + \int\limits_\Omega \mathbf{f}\cdot(\mathbf{u}-\mathbf{u}^\infty)d\mu_\rho - \int\limits_{\Gamma+} \rho_\infty \log(1+\rho_\infty)\mathbf{U}^\infty \cdot \mathbf{n} ds \ . \quad (1.7)$$

(d) The integral identities

$$\int\limits_\Omega (\mathbf{u}\otimes\mathbf{u}+\mathbf{I}) : \nabla\varphi d\mu_\rho + \int\limits_\Omega \mathbf{f}\cdot\varphi d\mu_\rho = \int\limits_\Omega \mathbf{\Sigma} : \nabla\varphi dx \ , \quad (1.8a)$$

$$\int\limits_\Omega \mathbf{u}\cdot\nabla\psi d\mu_\rho + \int\limits_{\Gamma+} \psi\rho^\infty \mathbf{U}^\infty \cdot \mathbf{n} d\Gamma = 0 \quad (1.8b)$$

hold for all vector fields $\varphi \in C_0^1(\Omega)^2$ and all functions $\psi \in C^1(\Omega)$ vanishing on $\partial B \setminus \Gamma^+$. Here, $C_0^k(\Omega) \subset C^k(\Omega)$ stands for the linear subspace of compactly supported functions.

In the same way we can define generalized solutions to problem (1.1),(1.3).

Definition 1.2. For given $\mathcal{M}$ and $\mathbf{f} \in C(\Omega)^2$ a generalized solution to problem (1.1),(1.3) is a pair $(\mu_\rho, \mathbf{u})$, where μ_ρ is a Borel measure in Ω and $\mathbf{u} \in H_0^{1,2}(\Omega)$. The generalized solution satisfies conditions (a)-(b) of Definition 1.1 and the bound on the rate of dissipation of energy

$$\mathcal{D} \le \int\limits_\Omega \Big(\frac{\xi}{2}|\mathrm{div}\,\mathbf{u}|^2 + \frac{1}{2\xi}\Big)dx + \int\limits_\Omega \mathbf{f}\cdot\mathbf{u} d\mu_\rho \ . \quad (1.9)$$

Furthermore, the integral identities

$$\int\limits_\Omega (\mathbf{u}\otimes\mathbf{u}+\mathbf{I}) : \nabla\varphi d\mu_\rho + \int\limits_\Omega \mathbf{f}\cdot\varphi d\mu_\rho = \int\limits_\Omega \mathbf{\Sigma} : \nabla\varphi dx \ , \quad (1.10a)$$

$$\int\limits_\Omega \mathbf{u}\cdot\nabla\psi d\mu_\rho = 0 \quad (1.10b)$$

hold for all vector fields $\varphi \in C_0^1(\Omega)^2$ and all functions $\psi \in C_0^1(\Omega)$.

□

Conditions (a)-(b) in Definition 1.1 imply that for generalized solutions the drag functional can be defined as follows

$$J(\rho, \mathbf{u}, \Omega) = \int\limits_\Omega \mathbf{\Sigma} : \nabla\mathbf{u}^\infty dx - \int\limits_\Omega (\mathbf{u}\otimes\mathbf{u}+\mathbf{I}) : \nabla\mathbf{u}^\infty d\mu_\rho + \int\limits_\Omega (\mathbf{U}^\infty - \mathbf{u}^\infty)\cdot\mathbf{f} d\mu_\rho \ . \quad (1.11)$$

1.2 Preliminaries and main results

The cost functional for shape optimisation problems is the drag $J(\Omega, \mathbf{u}, \mu_\rho)$ defined by formula (1.11). In applications, the drag is usually minimised within the class of admissible shapes. To our best knowledge there are no results on the shape optimisation problem in the framework of generalized solutions, the simpler case of evolution equations is considered in [1]. The drag depends on the solution $(\mu_\rho, \mathbf{u})$ to problem (1.1)-(1.2), however such a solution is not in general unique. Furthermore, the drag depends on an admissible shape of the obstacle S. The dependence of the drag on the admissible shapes is twofold, first, it depends directly on Ω since the integrals in (1.11) are defined over Ω, and it depends on the generalized solutions defined in Ω. The restrictions on the shapes of admissible obstacles S are defined in such a way that the set of admissible shapes and of the associated generalized solutions is compact. The precise conditions for admissible shapes are established below. In the present paper we do not provide the necessary optimality conditions for the problem of drag minimisation, we prove only the compactness of the set of solutions over the set of admissible shapes. We establish as well the relation between the drag defined by (1.11) compared to the particular case of incompressible flow in absence of volume forces under assumption of sufficiently small data. In order to formulate the main results we introduce some notations which will be used throughout the paper.

First, we introduce the auxiliary functions σ, ω, h_m and list the required properties (1.12)-(1.18) of the functions, which are referred to as Condition $\mathfrak{D}$.

Condition $\mathfrak{D}$

Let there be given positive and strictly decreasing functions $a, b : (0,\infty) \mapsto [1,\infty)$. We denote

$$\sigma(s) = s\int_s^\infty \frac{d\tau}{\tau^2 a(\tau)}\ , \quad \omega(s) = \int_0^s b(\tau)\sigma'(\tau)d\tau\ , \quad h_m(s) = s^{2(m-1)}b(s)\ . \tag{1.12}$$

We suppose that the following conditions are satisfied.

$$s^\varepsilon(a(s) + b(s) + \sigma(s)^{-1}) \searrow 0 \quad \text{as} \quad s \searrow 0 \quad \text{for any} \quad \varepsilon > 0\ , \tag{1.13}$$

$$\int_0^{R_\Omega} \frac{ds}{sb(s)} < \infty\ , \quad \text{where} \quad R_\Omega = \text{diam}\ \Omega, \tag{1.14}$$

$$|b'(s)| \le cb(s)/s, \quad b(ks)/b(s) \le c(k) \quad \text{for any} \quad s > 0, k > 0\ , \tag{1.15}$$

$$\frac{1}{s}\int_0^s h_m(\tau)d\tau \le c(m)h_m(s) \quad \text{for} \quad 0 < s \le R_\Omega \quad \text{and} \quad 1/2 < m \le 1\ , \tag{1.16}$$

$$\sigma(s)b(s) \searrow 0, \quad \omega(s) \searrow 0 \quad \text{as} \quad s \searrow 0. \tag{1.17}$$

Furthermore, it is assumed that there exists $\delta > 0$ such that $\omega(s)$ increases for $s < \delta$, $\omega : (0,\delta] \mapsto (0,\omega_0]$ is an homeomorphism, the function is onto i.e., $\omega((0,\delta]) = (0,\omega_0]$, and there exists a positive exponent $0 < \kappa < 1/2$ such that

$$b(s) \le \omega(s)^{-\kappa}, \quad s \in (0,\delta) . \tag{1.18}$$

For example, the functions

$$a(s) = \ln(e + s^{-1})^{\alpha}, \quad b(s) = \ln(e + s^{-1})^{\beta}, \quad \alpha > 1 + \beta(1+\kappa^{-1}), \quad \beta > 1,$$

meet all requirements of Condition $\mathfrak{D}$. In this case $\sigma(s) \sim |\ln s|^{-\alpha}$, $\omega(s) \sim |\ln s|^{1+\beta-\alpha}$ as $s \searrow 0$.

Finally, $d : \Omega \mapsto \mathbb{R}^+$ is the distance function $d(x) = \inf_{y\in\partial\Omega} |y - x|$ and we set $\Omega(r) = \{x \in \Omega : \ d(x) \le r\}$.

We are now in position to introduce the set of admissible shapes.

Definition 1.3. For every positive T and C_Ω denote by $\mathfrak{S}(T, C_\Omega)$ the class of domains $\Omega = B \setminus S$ satisfying the following conditions.

(α) The domain B is C^2 and there exists a compact set $B_0 \Subset B$ such that $S \subset B_0$.

(β) The so-called both side cone condition holds which means that for every $x \in \partial\Omega$ the set $\partial\Omega \bigcap B(x,T)$ is a graph of a Lipschitz function, and the Lipschitz constant does not exceed C_Ω.

(γ) The distance function $d(x)$ belongs to the space $H^{2,\infty}_{loc}(\Omega)$ and satisfies the inequalities

$$\frac{C_\Omega}{d(x)}\mathbf{I} \ge D^2 d(x) \ge -\frac{C_\Omega}{d(x)a(d(x))}\mathbf{I} \ \text{ a.e. in } \ \Omega \ , \tag{1.19}$$

where the symmetric matrix $D^2 d(x)$ stands for the Hessian of d. □

The following lemma with the proof relegated to Appendix shows that the family $\mathfrak{S}(T, C_\Omega)$ supplemented with the Hausdorff metric is a compact set.

Lemma 1.4. *(i) For positive constants T, C_Ω the family of obstacles S such that $\Omega = B \setminus S \in \mathfrak{S}(T, C_\Omega)$ is compact with respect to the Hausdorff metric.*

(ii) If $S \Subset B$ is either a convex set having an interior point or a piecewise C^2-smooth curvilinear polygon with the interior angles strictly between 0 and π, then $\Omega = B \setminus S$ belongs to the class $\mathfrak{S}(T, C_\Omega)$ with some constants T, C_Ω. Such a class includes e.g., the typical admissible shapes of wings in applied gas dynamics.

Definition 1.5. In the sequel we denote by c a generic constant which depend on the quantities $\|\mathbf{u}^\infty\|_{C^1(\Omega)}$, $\|\mathbf{f}\|_{L^\infty(\Omega)}$, T, C_Ω and R_Ω. We denote by c_α constants depending on the same quantities and, in addition, on the parameter α i.e.,

$$c = c(\|\mathbf{u}^\infty\|_{C^1(\Omega)}, \|\mathbf{f}\|_{L^\infty(\Omega)}, T, C_T, \operatorname{diam}\Omega)$$

and

$$c_\alpha = c(\alpha, \|\mathbf{u}^\infty\|_{C^1(\Omega)}, \|\mathbf{f}\|_{L^\infty(\Omega)}, T, C_T, \operatorname{diam}\Omega) \ .$$

□

We associate with the generalized solution $(\mu_\rho, \mathbf{u})$ the measure $d\mu_e = (2+|\mathbf{u}|^2)d\mu_\rho$, this means that for any bounded Borel function $g : \Omega \mapsto \mathbb{R}$

$$\int_\Omega g(x)d\mu_e = \int_\Omega g(x)(2+|\mathbf{u}(x)|^2)d\mu_\rho \ . \tag{1.20}$$

The boundedness of $\mu_e(\Omega)$ is equivalent to the boundedness of the total mass and of the kinetic energy of the gas.

The first theorem shows that the set of solutions to problem (1.1) with the uniformly bounded cost function is compact.

Theorem 1.6. *Fix* $\mathbf{f} \in C(\mathbb{R}^2)$*. Let the sequence of domains* $\Omega_n = B \setminus S_n$ *belong to the class* $\mathfrak{S}(T, C_\Omega)$ *with some positive* T, C_Ω *and let* $(\mu_{\rho,n}, \mathbf{u}_n)$ *be generalized solutions to problem* (1.1)-(1.2) *in* Ω_n *such that*

$$\sup_n \mathcal{M}_n < \infty, \quad \sup_n J(\mu_{\rho,n}, \mathbf{u}_n, \Omega_n) < \infty \ .$$

Suppose that $\mu_{\rho,n}$ *and* $\mathbf{u}_n$ *denote the measures and functions extended by* 0 *over the obstacles* $S_n \Subset B$*, respectively. Then there exists a subsequence, still denoted by* $(\Omega_n, \mu_{\rho,n}, \mathbf{u}_n)$*, a domain* $\Omega = B \setminus S \in \mathfrak{S}(T, C_\Omega)$*, measures* μ_ρ*,* μ_e*, and a velocity field* $\mathbf{u} \in H^{1,2}(B)$*, such that the subsequence of domains* Ω_n *converges in Hausdorff metric to the domain* $\Omega = B \setminus S$*,*

$$\mu_{\rho,n} \to \mu_\rho, \mu_{e,n} \to \mu_e \quad \text{*-weakly in } C_0^*(B), \quad \mathbf{u}_n \to \mathbf{u} \quad \text{weakly in } H^{1,2}(B) \ .$$

Moreover $\mu_e(S) = 0$ *and*

$$\mathcal{M}_n \to \mathcal{M} = \mu_\rho(\Omega), \quad \mu_{e,n}(\Omega_n) \to \mu_e(\Omega) \ .$$

According to our definition, the pair $(\mu_\rho, \mathbf{u})$ *is a generalized solution to problem* (1.1)-(1.2) *in* Ω *and*

$$-\infty < J(\mu_\rho, \mathbf{u}, \Omega) = \lim_{n\to\infty} J(\mu_{\rho,n}, \mathbf{u}_n, \Omega_n) \ .$$

For problem (1.1),(1.3) the cost function is equal to zero and the value of total mass is prescribed. Thus Theorem 1.6 implies the following result on the compactness of the set of solutions to the boundary value problem with no slip condition.

Theorem 1.7. *Fix* $\mathbf{f} \in C(\mathbb{R}^2)^2$ *and* $\mathcal{M} \in \mathbb{R}^+$*. Let the sequence of domains* $\Omega_n = B \setminus S_n$ *belong to the class* $\mathfrak{S}(T, C_\Omega)$ *with some positive* T, C_Ω *and* $(\mu_{\rho,n}, \mathbf{u}_n)$ *are generalized solutions to problem* (1.1),(1.3) *in* Ω_n*. Suppose that* $\mu_{\rho,n}$ *and* $\mathbf{u}_n$ *denote the measures and functions extended by* 0 *over the obstacles* $S_n \Subset B$*. Then there exists a subsequence still denoted by* $(\Omega_n, \mu_{\rho,n}, \mathbf{u}_n)$*, a domain* $\Omega = B \setminus S \in \mathfrak{S}(T, C_\Omega)$*, measures* μ_ρ*,* μ_e*, and a velocity field* $\mathbf{u} \in H_0^{1,2}(B)$*, such that the subsequence of domains* Ω_n *converges in Hausdorff metric to the domain* $\Omega = B \setminus S$*,*

$$\mu_{\rho,n} \to \mu_\rho, \mu_{e,n} \to \mu_e \quad \text{*-weakly in } C_0^*(B), \quad \mathbf{u}_n \to \mathbf{u} \quad \text{weakly in } H_0^{1,2}(B) \ .$$

Moreover $\mu_e(S) = 0$ *and*

$$\mathcal{M}_n \to \mathcal{M} = \mu_\rho(\Omega), \quad \mu_{e,n}(\Omega_n) \to \mu_e(\Omega) \ .$$

The pair $(\mu_\rho, \mathbf{u})$ *is a generalized solution to problem* (1.1),(1.3) *in* Ω.

The proofs of these results are given in Section 5.

At the present time the question on the existence of solutions to non-homogeneous boundary problems for the steady compressible Navier-Stokes equations is still open. Moreover, for isothermal flows, for which the pressure is equal to const ρ, there are no results on the non-local solvability even for the classical no slip conditions. Our approach allows to prove the following theorem which is the last main result of the article.

Theorem 1.8. *For any* $\mathbf{f} \in C(\mathbb{R}^2)^2$, $\mathcal{M} \in \mathbb{R}^+$ *and* $\partial\Omega$ *of a class* $C^{2+\alpha}$, $0 < \alpha < 1$, *problem* (1.1)-(1.3) *has at least one generalized solution.*

1.3 Stationary compressible Navier-Stokes equations

The general theory of boundary value problems for stationary compressible Navier-Stokes equations is proposed in [2], where the existence of generalized solutions is proved in the case of the pressure $p = \rho^\gamma$, $\gamma > 1$, and for the velocity field which vanishes at the boundary. The question on the solvability boundary value problems for isothermal flows with $p = \rho$, which are described by equations (1.1), is still open. On the other hand, there exists well developed local theory of the existence, uniqueness and stability of solutions for the isothermal Navier-Stokes equations, see review [3] and paper [4] for further references. In particular, in [5] it is shown that if for a given positive $\mathcal{M}$ the external forces admit the representation $\rho\nabla F + g$ with the small norm $\|g\|_{H^{1,2}(\Omega)}$ and the boundary of the flow region is sufficiently smooth, then there exists a strong solution to the isothermal Navier-Stokes equations which is close to

$$\mathbf{u} = 0, \quad \rho(x) = \mathcal{M} \exp F(x) \Big\{ \int\limits_\Omega \exp F(x) dx \Big\}^{-1} . \tag{1.21}$$

In order to understand the peculiarity of problem (1.1)-(1.3) it is useful to compare Theorem 1.8 with the existence results, [2], for the equations

$$\alpha\rho\mathbf{u} + \operatorname{div}(\rho\mathbf{u} \otimes \mathbf{u}) + \nabla\rho - \rho\mathbf{f} = \operatorname{div}\Sigma \ \text{ in } \ \Omega \ , \tag{1.22}$$

$$\alpha\rho + \operatorname{div}(\rho\mathbf{u}) = 0 \ \text{ in } \ \Omega \ , \tag{1.23}$$

containing additional terms with positive parameter α and supplemented with boundary conditions (1.3). In this case the first energy estimate gives the bounds depending on α for the rate of energy dissipation $\mathcal{D}$, the kinetic energy $\mathcal{E}$ and $\|\rho \ln(1+\rho)\|_{L_1(\Omega)}$. The energy estimates along with the Trudinger inequality imply the bound for L_1-norm of $\rho\mathbf{u} \otimes \mathbf{u}$ in which L_1 can not be replaced by L_r with some $r > 1$. This leads to so-called concentration problem in the proof of compactness of solutions to equations (1.22)-(1.23). In [2] this difficulty was overcome by using sharp results on the structure of concentrations [6]. In our case the energy inequality does not give the bounds for the density distribution and the kinetic energy. Therefore, the main task is to obtain the estimates of $\rho\mathbf{u}\otimes\mathbf{u}$ using only the bound for $\|\rho\|_{L_1(\Omega)}$ given by (1.3b). This problem is closely connected with the question on the non-negativity of the density. It is easily seen that in the stationary case the governing equations themselves do not guarantee the non-negativity of the density of the gas, see [3],[7] for a discussion. We show, in Lemma 1.10, that the solutions obtained via regularisation method satisfy this condition. In particular, the regularisation used in [2] also leads to physically acceptable solutions.

Unfortunately, for non-homogeneous boundary conditions the governing equation does not allow us to control the total mass of the gas. Nevertheless, for optimal control problem with the drag as the cost function we can estimate the kinetic energy $\mathcal{E}$ and the rate of the energy dissipation $\mathcal{D}$ in terms of $\mathcal{M}$.

1.4 Estimates of measures

The proof of the main Theorem 1.6 is based on the following result on the estimates of measure-valued tensor fields. Let us consider the symmetric matrix $\mathbf{\Lambda} = (\mu_{i,j})_{1\leq i,j\leq 2}$ which elements $\mu_{i,j}$ are Borel measures in Ω and denote $\mu = \mathrm{tr}\mathbf{\Lambda} = \mu_{11} + \mu_{22}$. We assume that the quadratic form associated with $\mathbf{\Lambda}$ is non-negative and the inequalities

$$0 \leq \int_\Omega \xi\otimes\xi : d\mathbf{\Lambda}(x) \leq \int_\Omega |\xi(x)|^2 d\mu(x) \tag{1.24}$$

hold for every bounded Borel function $\xi : \Omega \mapsto \mathbb{R}^2$.

Remark. Condition (1.24) implies that for $\mathbf{G} \in C(\Omega)^4$,

$$\int_\Omega \mathbf{G}(x) : d\mathbf{\Lambda}(x) \leq c \int_\Omega |\mathbf{G}(x)| d\mu(x) \tag{1.25}$$

with some absolute constant c and

$$\int_\Omega \mathbf{G}(x) : d\mathbf{\Lambda}(x) \geq 0 \quad \text{for} \quad \mathbf{G} \geq 0\ . \tag{1.26}$$

Suppose that $\mathbf{\Lambda}$ satisfies the following differential equation

$$\mathrm{div}\,\mathbf{\Lambda} = \mathrm{div}\,\mathbf{G} + \varrho \quad \text{in} \quad \Omega\ , \tag{1.27}$$

where $\mathbf{G} : \Omega \mapsto \mathbb{R}^4$ is symmetric integrable matrix and ϱ is a vector Borel measure in Ω. Equation (1.27) holds in $\mathcal{D}'(\Omega)$ i.e., in the sense of distributions. This means that

$$\int_\Omega \nabla\varphi(x) : d\mathbf{\Lambda}(x) = \int_\Omega \nabla\varphi(x) : \mathbf{G}(x)dx - \int_\Omega \varphi(x) d\varrho(x) \tag{1.28}$$

for all $\varphi \in C_0^1(\Omega)$.

Theorem 1.9. *Assume that relations* (1.24) *and* (1.27) *are verified with* $\mathbf{G} \in L_q(\Omega)^4$, $1 < q \leq 2$, *and the variation* $|\varrho|$ *of the vector measure* ϱ *satisfies the inequality* $|\varrho| \leq c\mu$. *Then there exist positive constants* $c_{m,q}$, c_q *such that:*

(i) The inequality

$$|\int_K F(x)d\mu(x)| \leq c_{m,q}\Big(h_m(R)\mu(\Omega) + \|G\|_{L_q(\Omega)}\Big)\|F\|_{H_0^{m,2}(\Omega)} \tag{1.29}$$

holds for any function $F \in H_0^{m,2}(\Omega)$, $q^{-1} < m \leq 1$, *and every compact set* K *with* $R = dist(K, \partial\Omega) > 0$. *Here the function* h_m *is defined by* (1.12).

(ii) If $F \in H_0^{1,2}(\Omega)$ *and* $b(d)^{1/2}\nabla F \in L_2(\Omega)$, *then*

$$|\int_\Omega F(x)d\mu(x)| \leq c_q\Big(\mu(\Omega) + \|G\|_{L_q(\Omega)}\Big)\|b(d)^{1/2}\nabla F\|_{L_2(\Omega)} \ . \tag{1.30}$$

(iii) The measure μ *near the boundary of* Ω *satisfies the inequality*

$$\int_{\Omega(t)} \nabla d(x) \otimes \nabla d(x) : d\mathbf{\Lambda}(x) \leq c_q\Big(\sigma(t)\mu(\Omega) + t^{1-1/q}\|G\|_{L_q(\Omega)}\Big), \quad t \leq T \ , \tag{1.31}$$

in which σ *is given by* (1.12).

The subsequent two sections are devoted to the proof of Theorem 1.9. The solvability of problem (1.1),(1.3) is established by using the regularisation method. We introduce the regularised equations in the form

$$\operatorname{div}(\rho\mathbf{u} \otimes \mathbf{u} - \varepsilon\nabla\rho \otimes \mathbf{u}) + \nabla\rho - \rho\mathbf{f} = \operatorname{div}\Sigma \ \text{ in } \ \Omega \ , \quad \mathbf{u} \in H_0^{1,2}(\Omega) \ , \tag{1.32a}$$

$$-\varepsilon\Delta\rho + \operatorname{div}(\rho\mathbf{u}) = 0 \ \text{ in } \ \Omega \ , \tag{1.32b}$$

$$\nabla\rho \cdot \mathbf{n} = 0 \ \text{ on } \ \partial\Omega \ , \tag{1.32c}$$

where $\varepsilon > 0$ is a small parameter and $\mathbf{n}$ is the outward normal to $\partial\Omega$. The existence of solution to problem (1.32) follows from the energy estimates and from an application of the fixed point theory. The main difficulty is the proving of the positivity of the density ρ. This fact results from the following lemma which proof is given in Section 6

Lemma 1.10. *Let* Ω *be a bounded region in* $\mathbb{R}^n$ *with a boundary* $\partial\Omega \in C^{2+\alpha}$, $0 < \alpha < 1$ *and a vector field* $\mathbf{u} \in C^{1+\alpha}(\Omega)$ *vanishing at* $\partial\Omega$. *Then for any positive* $\mathcal{M}$ *there exist a unique positive solution to problem* (1.32b)-(1.32c) *which satisfies*

$$\int_\Omega \rho(x)dx = \mathcal{M} \ .$$

The compactness of the set of solutions to regularised equations follows from Theorem 1.7

2 Estimates of mass density

In this section we prove the estimates for the density of the measure μ. We denote a closed ball of the centre y and of the radius R by $B(y,R)=\{x\ :\ |x|\le R\}$. For the sake of simplicity the same symbol $B(y,R)$ is used for an open ball when integrating over $B(y,R)$.

Lemma 2.1. *Under the assumptions of Theorem 1.9, whenever a closed ball $B(y,R)\subset\Omega$ and $0<r\le R$,*

$$\frac{1}{r}\int\limits_{B(y,r)} d\mu \le \frac{c}{R}\int\limits_{B(y,R)} d\mu(x) + c\int\limits_{B(y,r)} K_{r,R}(|x-y|)|G|dx\ , \tag{2.1}$$

where the kernel $K_{r,R}$ is given by

$$K_{r,R}(s)=\frac{1}{r}-\frac{1}{R}\ \ \textit{for}\ \ 0\le s\le r,\quad K_{r,R}(s)=\frac{2}{s}-\frac{1}{R}\ \ \textit{for}\ \ r\le s\le R\ . \tag{2.2}$$

Proof. For an arbitrary positive s,ε such that $s+\varepsilon<R$ and define the function $\zeta_\varepsilon:\mathbb{R}^+\mapsto[0,1]$ by the equalities

$$\zeta_\varepsilon(\tau)=\begin{cases}1 & \text{for}\ \ 0\le\tau\le s\ ,\\ 1+(s-\tau)/\varepsilon & \text{for}\ \ s\le\tau\le s+\varepsilon\ ,\\ 0 & \text{otherwise.}\end{cases} \tag{2.3}$$

Substituting into the integral identity (1.28) the vector field $\varphi(x)=\zeta_\varepsilon(|x-y|)(x-y)$ we obtain the equality

$$\int\limits_{\Omega}\zeta_\varepsilon(|x-y|)d\mu(x)-\frac{1}{\varepsilon}\int\limits_{s<|x-y|\le s+\varepsilon}|x-y|\nu(x)\otimes\nu(x):d\mathbf{\Lambda}(x)=$$
$$\int\limits_{\Omega}\zeta_\varepsilon(|x-y|)\ \mathrm{tr}\mathbf{G}dx-\frac{1}{\varepsilon}\int\limits_{s\le|x-y|\le s+\varepsilon}|x-y|(\nu\otimes\nu):\mathbf{G}dx-$$
$$\int\limits_{\Omega}\zeta_\varepsilon(|x-y|)(x-y)d\varrho(x)\ ,$$

where $\nu(x)=|x-y|^{-1}(x-y)$. We conclude from the latter equality and (1.24) that

$$\int\limits_{\Omega}\zeta_\varepsilon(|x-y|)d\mu(x)-\frac{s+\varepsilon}{\varepsilon}\int\limits_{s\le|x-y|\le s+\varepsilon}d\mu(x)\le$$
$$\int\limits_{\Omega}\zeta_\varepsilon(|x-y|)|\mathbf{G}|dx+\frac{s+\varepsilon}{\varepsilon}\int\limits_{s\le|x-y|\le s+\varepsilon}|\mathbf{G}|dx+c(s+\varepsilon)\int\limits_{\Omega}\zeta_\varepsilon(|x-y|)d\mu(x)\ . \tag{2.4}$$

The function $M : [0, R] \mapsto \mathbb{R}^+$ defined by

$$M(s) = \int\limits_{|x-y|\le s} d\mu(x) \ , \tag{2.5}$$

is monotone, and therefore it is differentiable at almost every point of the interval $[0, R]$. Moreover, for all continuous non-negative functions $f : [0, s] \mapsto \mathbb{R}$,

$$\int\limits_{B(y,s)} f(|x-y|)d\mu = \int\limits_{[0,s]} f(s)dM(s) \ge \int\limits_{[0,s]} f(s)M'(s)ds \ . \tag{2.6}$$

The equality in (2.6) obviously follows from the definition of $M(s)$ and the inequality in (2.6) is a consequence of the Lebesgue theorem. Next, we introduce the function

$$G(s) = \int\limits_{B(y,s)} |\mathbf{G}|dx = \int\limits_0^s \Big(\int\limits_{\partial B(y,\tau)} |\mathbf{G}|dH_1 \Big) d\tau \quad \text{for any} \quad s \le R \ .$$

Hence, the non-negative function

$$g(s) = \int\limits_{\partial B(y,s)} |\mathbf{G}|dH_1$$

is integrable over $[0, R]$. Moreover, $G \in H^{1,1}(0, R)$, $G'(s) = g(s)$ a.e. on $[0, R]$ and

$$\int\limits_r^R f(s)g(s)ds = \int\limits_{B(y,R)\setminus B(y,r)} f(|x-y|)|\mathbf{G}(x)|dx \tag{2.7}$$

for any continuous function $f : [0, R] \mapsto \mathbb{R}$. Passing to the limit in (2.4) with $\varepsilon \searrow 0$ we obtain

$$M(s) - sM'(s) - csM(s) \le cG(s) + csG'(s) \quad \text{for a.e.} \quad s \in [0, R] \ .$$

Multiplying both sides of the above inequality by $s^{-2}\exp(cs)$, taking into account that $s \le \text{diam } \Omega$, we can rewrite the inequality in the form

$$-\frac{d}{ds}\left(\frac{e^{cs}}{s}M(s)\right) \le c\left(\frac{1}{s^2}G(s) + \frac{g(s)}{s}\right) \ . \tag{2.8}$$

Since $B(y, s)$ is a closed ball we note that

$$\frac{e^{cr}}{r}M(r) - \frac{e^{cR}}{R}M(R) = -\int\limits_{(r,R]} \left(\frac{e^{cs}}{s}\right)' M(s)ds - \int\limits_{(r,R]} \frac{e^{cs}}{s}dM(s) \le$$

$$-\int\limits_{(r,R]} \left(\left(\frac{e^{cs}}{s}\right)' M(s) + \frac{e^{cs}}{s}M'(s)\right) ds = -\int\limits_r^R \left(\frac{e^{cs}}{s}M(s)\right)' ds \ .$$

>From this relation and (2.8) we obtain

$$\frac{1}{r}M(r) \le \frac{c}{R}M(R) + c\int\limits_r^R \Big(\frac{1}{s^2}G(s) + \frac{g(s)}{s}\Big)ds\ . \tag{2.9}$$

Integrating the integral in the right-hand side of (2.9) by parts and using (2.7) we arrive at

$$\begin{aligned}
&\int\limits_r^R \left(\frac{G(s)}{s^2} + \frac{g(s)}{s}\right) ds = \frac{1}{r}G(r) - \frac{1}{R}G(R) + 2\int\limits_r^R \frac{g(s)}{s} ds = \\
&\frac{1}{r}\int\limits_{B(y,r)} |\mathbf{G}|dx - \frac{1}{R}\int\limits_{B(y,R)} |\mathbf{G}|dx + 2\int\limits_{B(y,R)\setminus B(y,r)} \frac{|\mathbf{G}|dx}{|x-y|} = \\
&\int\limits_{B(y,R)} K_{r,R}(|x-y)|G(x)|dx\ .
\end{aligned}$$

Substituting the last expression in the right-hand side of (2.9) results in (2.1) and the proof of Lemma 2.1 is completed. □

The next lemma gives the similar estimates near the boundary of Ω.

Lemma 2.2. *Under the assumptions of Theorem 1.9,*

$$\int\limits_{\Omega(t)} \nu(x)\otimes\nu(x) : d\mathbf{\Lambda} \le c\sigma(t)\mu(\Omega) + ct^{1-1/q}\|\mathbf{G}\|_{L_q(\Omega)}, \quad 0 \le r \le \operatorname{diam}\Omega\ , \tag{2.10}$$

where $\nu(x) = \nabla d(x)$ *is the unit normal vector to the level curve* $\{d(\cdot) = d(x)\}$ *and the function* σ *is given by* (1.12)*.*

Proof. The proof is based on the same arguments as in the proof of Lemma 2.1, however the justification of the appropriate differential inequalities is obtained in a more complicated way because of the complexity of condition (1.19). It suffices to prove inequality (2.10) for $t \le T$, where T is the constant from Definition 1.3 of the class $\mathfrak{S}$. Suppose that d is extended by 0 outside Ω and set $\mathfrak{d}_\lambda = k_\lambda * d$ and $\mathfrak{a}_\lambda = k_\lambda * (da(d))^{-1}$ where the mollifier $k_\lambda(x) = \lambda^{-2}k(x/\lambda)$ and $k \in C^\infty(\mathbb{R}^2)$ is a non-negative function satisfying

$$\operatorname{supp} k \subset B(0,1), \quad \int_{\mathbb{R}^2} k(x)dx = 1\ .$$

Choose an arbitrary $\delta > 0$ and continuous, non-negative $C^1(\mathbb{R})$ function f which vanishes in the open set $(-\infty,\delta)\cup(T-\delta,\infty)$. Introduce also the family of smooth potential vector fields defined by

$$\varphi_\lambda(x) = f(\mathfrak{d}_\lambda(x))\nabla\mathfrak{d}_\lambda(x), \quad 0 < \lambda < \delta/2\ .$$

Obviously supp $\varphi_\lambda \subset \Omega(T-\delta/2)\setminus\Omega(\delta/2)$ and

$$\nabla\varphi_\lambda = f'(\mathfrak{d}_\lambda)\nabla\mathfrak{d}_\lambda\otimes\nabla\mathfrak{d}_\lambda + f(\mathfrak{d}_\lambda)D^2\mathfrak{d}_\lambda\ .$$

From this and Definition 1.3 we obtain the inequality

$$\nabla\varphi_\lambda \geq f'(\mathfrak{d}_\lambda)\nabla\mathfrak{d}_\lambda\otimes\nabla\mathfrak{d}_\lambda - C_\Omega f(\mathfrak{d}_\lambda)$$

which along with (1.26) and the identity $\mathbf{I}:d\mathbf{\Lambda} = d\mu$ implies

$$\int_\Omega \nabla\varphi_\lambda : d\mathbf{\Lambda} \geq \int_\Omega f'(\mathfrak{d}_\lambda)\nabla\mathfrak{d}_\lambda\otimes\nabla\mathfrak{d}_\lambda : d\mathbf{\Lambda} - C_\Omega\int_\Omega f(\mathfrak{d}_\lambda)\mathfrak{a}_\lambda d\mu\ . \tag{2.11}$$

Next, by Definition 1.3, the functions ∇d, d^{-1}, $a(d)^{-1}$ are continuous on the set $K_\delta = \mathbf{cl}\ \Omega(T)\setminus\Omega(\delta/2)$. Hence $\nabla\mathfrak{d}_\lambda$, $\mathfrak{a}_\lambda$ and $f(\mathfrak{d}_\lambda)$ converge with $\lambda\searrow 0$ uniformly on K_δ to ∇d, $(da(d))^{-1}$ and $f(d)$, respectively. Passing to the limit in (2.11) with $\lambda\searrow 0$ we obtain

$$\liminf_{\lambda\searrow 0}\int_\Omega \nabla\varphi_\lambda : d\mathbf{\Lambda} \geq \int_\Omega f'(d)\nabla d\otimes\nabla d : d\mathbf{\Lambda} - c\int_\Omega f(d)d^{-1}a(d)^{-1}d\mu\ . \tag{2.12}$$

Note that, by Definition 1.3, $D^2d(x)$ is bounded on the set cl $\Omega(T)\setminus\Omega(\delta/2)$ and hence $D^2\mathfrak{d}_\lambda$ converges to D^2d in any space L_r on this set, which along with (1.19) gives

$$\lim_{\lambda\searrow 0}\int_\Omega \nabla\varphi_\lambda : \mathbf{G}dx = \int_\Omega f'(d)\nabla d\otimes\nabla d : \mathbf{G}dx + \int_\Omega f(d)D^2d : \mathbf{G}dx \leq$$
$$\int_\Omega f'(d)\nabla d\otimes\nabla d : \mathbf{G}dx + c\int_\Omega f(d)d^{-1}|\mathbf{G}|dx\ . \tag{2.13}$$

On the other hand,

$$\lim_{\lambda\searrow 0}\int_\Omega \varphi_\lambda d\varrho dx = \int_\Omega f(d)d\varrho \leq c\int_\Omega f(d)d\mu\ . \tag{2.14}$$

Substituting φ_λ into integral identity (1.28), passing to the limits as $\lambda\to 0$ and using inequalities (2.12)-(2.14) we obtain

$$\int_\Omega f'(d)\nabla d\otimes\nabla d : d\mathbf{\Lambda} \leq c\int_\Omega f(d)d^{-1}a(d)^{-1}d\mu + \int_\Omega f'(d)\nabla d\otimes\nabla d : \mathbf{G}dx +$$
$$c\int_\Omega f(d)d^{-1}|\mathbf{G}|dx + c\int_\Omega f(d)d\mu\ . \tag{2.15}$$

Now fix positive s, ε such that $s+\varepsilon < T$ and set

$$f_\delta(t) = \frac{\eta_\delta(t-\delta)}{\delta} \int\limits_{t-\delta}^{t} \zeta_\varepsilon(\tau)d\tau \quad \text{for} \quad t \in (0,\infty) \ ,$$

where ζ_ε are defined by (2.3) and $\eta_\delta(t) = 0$ for $t < 0$,

$$\eta_\delta(t) = t^2/(2\delta) \quad \text{for} \quad 0 < t \le \delta \quad \text{and} \quad \eta_\delta(t) = (2t-\delta)/2 \quad \text{otherwise} \quad .$$

For any $\delta > 0$ functions f_δ are continuously differentiable. Moreover, they are uniformly continuous and

$$\lim_{\delta \searrow 0} f_\delta(t) = t\zeta_\varepsilon(t), \quad \limsup_{\delta \searrow 0} t^{-1} f_\delta(t) \le \zeta_\varepsilon(t) \quad \text{for} \quad t > 0$$

$$\lim_{\delta \searrow 0} f_\delta'(t) = \begin{cases} 1 & \text{for} \quad 0 \le t \le s, \\ 1 + (s-t)/\varepsilon - t/\varepsilon & \text{for} \quad s < t \le s+\varepsilon, \\ 0 & \text{otherwise.} \end{cases} \quad .$$

Substituting $f = f_\delta$ into (2.15), passing to the limit as $\delta \searrow 0$ and applying the Lebesgue theorem we obtain

$$\int\limits_{\Omega(s)} \nabla d \otimes \nabla d : d\mathbf{\Lambda} + \int\limits_{\{s<d(x)\le s+\varepsilon\}} \left(\frac{s+\varepsilon-d}{\varepsilon} - \frac{d}{\varepsilon}\right) \nabla d \otimes \nabla d : d\mathbf{\Lambda} \le$$
$$\int\limits_{\Omega(s+\varepsilon)} a(d)^{-1} d\mu + c \int\limits_{\Omega(s)} |\mathbf{G}| dx +$$
$$\int\limits_{\{s<d(x)\le s+\varepsilon\}} \left(\frac{s+\varepsilon-d}{\varepsilon} - \frac{d}{\varepsilon}\right) |\mathbf{G}| dx + c \int\limits_{\Omega(s+\varepsilon)} d(x) d\mu \ .$$

Introduce the measure μ_0 defined by $d\mu_0 = \nabla d \otimes \nabla d : d\mathbf{\Lambda}$. Since $a(t)$ decreases, it follows from the above inequality that

$$\int\limits_{\Omega(s)} d\mu_0 - \frac{s+\varepsilon}{\varepsilon} \int\limits_{\{s<d(x)\le s+\varepsilon\}} d\mu_0 \le ca(s+\varepsilon)^{-1} \int\limits_{\Omega(s+\varepsilon)} d\mu +$$
$$c \int\limits_{\Omega(s+\varepsilon)} |\mathbf{G}| dx + \frac{s+\varepsilon}{\varepsilon} \int\limits_{\{s<d(x)\le s+\varepsilon\}} |\mathbf{G}| dx + c(s+\varepsilon) \int\limits_{\Omega} d\mu \ . \quad (2.16)$$

Since $\nabla d(x)$ is the unit normal vector to the C^1 level surface of the distance function and $|\mathbf{G}|$ is integrable in Ω, so an application of the Fubini theorem implies

$$G_\Omega(s) \equiv \int\limits_{\Omega(s)} |\mathbf{G}| dx = \int\limits_0^s \Big(\int\limits_{\{d(x)=\tau\}} |\mathbf{G}| dH_1 \Big) d\tau \quad \text{for any} \quad s \le T \ .$$

Hence the non-negative function

$$g_\Omega(s) = \int\limits_{\{d(x)=s\}} |\mathbf{G}|dH_1$$

is integrable over the interval $[0,T]$. Moreover, $G_\Omega \in H^{1,1}(0,T)$ and $G'_\Omega(s) = g_\Omega(s)$ a.e. in $[0,T]$, and

$$\int\limits_r^T f(s)g_\Omega(s)ds = \int\limits_{\Omega(T)\setminus\Omega(r)} f(d(x))|\mathbf{G}(x)|dx \tag{2.17}$$

for any continuous function $f : (0,T] \mapsto \mathbb{R}$. On the other hand, the function

$$N(s) = \int\limits_{\Omega(s)} d\mu_0, \quad s \in [0,T] ,$$

is monotone and differentiable at almost every point of $[0,T]$. Hence we can pass to the limit with $\varepsilon \to 0$ in the both sides of inequality (2.16) to obtain for a.e. $s \in [0,T]$

$$N(s) - sN'(s) \le ca(s)^{-1}\mu(\Omega) + G_\Omega(s) + sg_\Omega(s) .$$

Using the same arguments as in the proof of Lemma 2.1, in view of the equality (1.12) for σ, we arrive at

$$\frac{1}{t}N(t) \le \frac{1}{T}N(T) + c\frac{\sigma(t)}{t}\mu(\Omega) + c\int\limits_t^T \Big(\frac{1}{\tau^2}G_\Omega(\tau) + \frac{1}{\tau}g_\Omega(\tau)\Big)\,d\tau . \tag{2.18}$$

Integrating the right-hand side by parts and using (2.17) we get

$$\int\limits_t^T \Big(\frac{1}{\tau^2}G_\Omega(\tau) + \frac{1}{\tau}g_\Omega(\tau)\Big)\,d\tau = \frac{1}{t}G_\Omega(t) - \frac{1}{T}G_\Omega(T) + 2\int\limits_t^T \frac{g_\Omega(\tau)}{\tau}d\tau =$$

$$\frac{1}{t}\int\limits_{\Omega(t)} |\mathbf{G}|dx - \frac{1}{T}\int\limits_{\Omega(T)} |\mathbf{G}|dx + 2\int\limits_{\Omega(T)\setminus\Omega(r)} |\mathbf{G}|\frac{dx}{d(x)} = \int\limits_{\Omega(T)} K_{t,T}(d(x))|\mathbf{G}(x)|dx .$$

Since $N(T) \le \mu(\Omega)$ and $t \le c\sigma(t)$, from the above equality and (2.18) we obtain the inequality

$$N(t) \le c\sigma(t)\mu(\Omega) + ct\int\limits_{\Omega(T)} K_{t,T}(d(x))|\mathbf{G}(x)|dx \le$$

$$c\sigma(t)\mu(\Omega) + t\|\mathbf{G}\|_{L_q(\Omega)}\Big(\int\limits_{\Omega(T)} K_{t,T}(d(x))^p dx\Big)^{1/p} , \tag{2.19}$$

where $p = q/(q-1)$. Obviously

$$\int\limits_{\Omega(T)} K_{t,T}(d(x))^p dx \le \frac{1}{t^p} \int\limits_{\Omega(t)} dx + c \int\limits_{\Omega(T)\setminus\Omega(t)} \frac{dx}{d(x)^p} \le \frac{c}{t^{p-1}} + c\int\limits_t^T \frac{d\tau}{\tau^p} \le \frac{c}{t^{p-1}} \ .$$

Substituting the above inequality into the right-hand side of (2.19), taking into account that $t^{1-(p-1)/p} = t^{1-1/q}$, (2.10) follows, which completes the proof. □

In Lemma 2.3 below the estimates for the convolution of the kernel h_m with the measure μ are established. The convolution $h_m * \mu(x)$ over the set E is defined by the equality

$$\int\limits_E h_m(|x-y|)d\mu(x) = \lim_{N\to\infty} \int\limits_E h_{m,N}(|x-y|)d\mu(x) \ ,$$

for all $y \in \Omega$ and all Borel sets $E \subset \Omega$, where $h_{m,N}(s)$ is the cut off of the function $h_m(s)$ at the level N. Let K be an arbitrary compact subset of Ω and $\operatorname{dist}(K, \partial\Omega) > 2R > 0$.

Lemma 2.3. *Under the assumptions of Theorem 1.9 the inequality*

$$\int\limits_K h_m(|x-y|)d\mu(x) \le c_m h_m(R)\mu(K \bigcup B(y,R)) + c_{m,q}\|\mathbf{G}\|_{L_q(B(y,R))} \ . \tag{2.20}$$

holds for all $y \in \Omega$ such that dist$(y, K) \le R$.

Proof. It is clear that

$$\int\limits_K h_m(|x-y|)d\mu(x) \le \int\limits_{K\setminus B(y,R)} h_m(|x-y|)d\mu(x) + \lim_{N\to\infty} \int\limits_{B(y,R)} h_{m,N}(|x-y|)d\mu(x) \ . \tag{2.21}$$

Let us estimate the last integral in the right-hand side. Equality (2.6) implies

$$\int\limits_{B(y,R)} h_{m,N}(|x-y|)d\mu(x) = \int\limits_{[0,R]} h_{m,N}(s)dM(s) =$$
$$NM(r_N) + \int\limits_{(r_N,R]} h_{m,N}(s)dM(s) \ ,$$

where $h(r_N) = N$, $r_N \searrow 0$ as $N \to \infty$. Recall that the function $M(s) = \mu(B(y,s))$ is continuous from the right since by definition $B(y,s)$ is a closed ball. Integrating the second integral in the right-hand side by parts, taking into account the equality $h_{m,N}(r) = h_m(r)$ for $r \in [r_N, R]$, since h_m decreases on the interval $(0, \operatorname{diam}\Omega)$, we obtain

$$\int\limits_{(r_N,R]} h_{m,N}(s)dM(s) = h_m(R)M(R) - h_m(r_N)M(r_N) - \int\limits_{(r_N,R]} h'_m(s)M(s)ds \ ,$$

which gives

$$\int\limits_{B(y,R)} h_{m,N}(|x-y|)d\mu(x) = h_m(R)M(R) - \int\limits_{(r_N,R]} h'_m(s)M(s)ds\ . \tag{2.22}$$

>From Lemma 2.1 we have the inequality

$$M(s) = \int\limits_{B(y,s)} d\mu \leq \frac{cs}{R}M(R) + cs\int\limits_{B(y,s)} K_{s,R}(|x-y|)|\mathbf{G}|dx\ , \tag{2.23}$$

where the kernel $K_{s,R}$ is defined in (2.2). Substituting (2.23) into the right-hand side (2.22), and noting that $h'(s) < 0$, we obtain

$$\int\limits_{B(y,R)} h_{m,N}(|x-y|)d\mu(x) \leq \Big(h_m(R) - \frac{c}{R}\int\limits_{r_N}^{R} sh'_m(s)ds\Big)M(R) - \tag{2.24}$$

$$\int\limits_{0}^{R}\Big(\int\limits_{B(y,R)} K_{s,R}(|y-x|)|\mathbf{G}(x)|dx\Big)sh'_m(s)ds\ .$$

The function $s \to sh'_m(s)$ satisfies the inequality $s|h'_m(s)| \leq c_m h_m(s)$ by (1.12), (1.15), thus in view of (1.13) $sh'_m(s)$ is integrable over the interval $(0, \operatorname{diam}\Omega)$, so (1.16) yields

$$h_m(R) - \frac{c}{R}\int\limits_{r_N}^{R} sh'_m(s)ds \leq c\Big(h_m(R) - \frac{1}{R}\int\limits_{0}^{R} sh'_m(s)ds\Big) = \tag{2.25}$$

$$\frac{c}{R}\int\limits_{0}^{R} h_m(s)ds \leq c_m h_m(R)\ .$$

>From (2.24) and (2.25) we obtain

$$\lim_{N\nearrow\infty}\int\limits_{B(y,R)} h_{m,N}(|x-y|)d\mu(x) \leq \tag{2.26}$$

$$c_m h_m(R)M(R) - \int\limits_{0}^{R}\Big(\int\limits_{B(y,R)} K_{s,R}(|y-x|)|\mathbf{G}(x)|dx\Big)sh'_m(s)ds\ .$$

Definition (2.2) of the kernel $K_{s,R}(\tau)$, the inequalities $h'_m(s) < 0$ and (1.16) imply that

$$-\int_0^R K_{s,R}(\tau) s h'_m(s) ds = -\int_0^\tau \Big(\frac{2}{\tau} - \frac{1}{R}\Big) s h'_m(s) ds - \int_\tau^R \Big(\frac{1}{s} - \frac{1}{R}\Big) s h'_m(s) ds \le$$
$$-\frac{2}{\tau}\int_0^\tau s h'_m(s) ds - 2\int_\tau^R h'_m(s) ds \le \frac{2}{\tau}\int_0^\tau h_m(s) ds \le c_m h_m(\tau)\ .$$

Hence

$$\int_0^R \Big(\int_{B(y,R)} K_{s,R}(|y-x|)|\mathbf{G}(x)|\Big) s h'_m(s) ds \le c \int_{B(y,R)} h_m(|y-x|)|\mathbf{G}(x)| dx\ .$$

Substituting this estimate into the right side of (2.26) we obtain the inequality

$$\lim_{N\to\infty} \int_{B(y,R)} h_{m,N}(|x-y|) d\mu(x) \le \tag{2.27}$$
$$c_m h_m(R) M(R) + c \int_{B(y,R)} h_m(|y-x|)|\mathbf{G}(x)| dx \le$$
$$c_m h_m(R) M(R) + \|\mathbf{G}\|_{L_q(\Omega)} \Big(\int_{B(0,R)} |h_m(|z|)|^p dz\Big)^{1/p}\ .$$

Recall that $p = q/(q-1)$ and, by our assumptions $(m-1)p > -1$, which along with (1.13) yields

$$\int_{B(0,R)} |h_m(|z|)|^p dz \le 2\pi \int_0^R s^{2(m-1)p+1} b(s)^p ds \le c_{m,q}\ .$$

Substituting this inequality into the right-hand side (2.27) gives

$$\int_K h_m(|x-y|) d\mu(x) \le$$
$$\int_{K\setminus B(y,R)} h_m(|x-y|) d\mu(x) + c_m h_m(R)\mu(B(y,R)) + c_{m,q}\|\mathbf{G}\|_{L_q(B(y,R))}\ .$$

Noting that $h_m(|x-y|) \le c h_m(R)$ on the set $K \setminus B(y,R)$ and $\mu(K\setminus B(y,R)) + \mu(B(y,R)) = \mu(K \bigcup B(y,R))$ we obtain the needed inequality (2.20) which completes the proof. □

3 Proof of Theorem 1.9

In order to prove assertion (i) of Theorem 1.9 we recall that any function $F \in H_0^{1,m}(\Omega)$, $1/2 < m \le 1$, extended by zero outside of Ω can be represented by the Riesz potential

$$F(x) = c(m) \int\limits_\Omega |x-y|^{m-2} f(y)dy$$

with the inequalities

$$C^{-1}\|F\|_{H_0^{1,m}(\Omega)} \le \|f\|_{L_2(\Omega)} \le C\|F\|_{H_0^{1,m}(\Omega)} ,$$

where the constant C depends only on diamΩ. Hence it is sufficient to prove that under the assumptions of the theorem, the inequality

$$\int\limits_K F(x)d\mu(x) \le c_{m,q}\Big(h_m(R)\mu(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}\Big) \tag{3.1}$$

holds for every compact set $K \Subset \Omega$ such that dist$(K, \Omega^c) \ge 2R$ and all functions F represented by Riesz potentials

$$F(x) = \int\limits_\Omega |x-y|^{m-2} f(y)dy, \quad f \ge 0, \quad \|f\|_{L_2(\Omega)} = 1 .$$

It is clear that

$$|x-y|^{m-2}f(y) \le \frac{1}{2|x-y|^2 b(|x-y|)} + \frac{b(|x-y|)}{2|x-y|^{2(1-m)}}|f(y)|^2 \equiv \frac{1}{2|x-y|^2 b(|x-y|)} + \frac{1}{2}h_m(|x-y|)|f(y)|^2 .$$

Observe that, by inequality (1.14),

$$\int\limits_\Omega \frac{dy}{|x-y|^2 b(|x-y|)} \le \int\limits_{B(y,R_\Omega)} \frac{dy}{|x-y|^2 b(|x-y|)} \le c\int\limits_0^{R_\Omega} \frac{ds}{sb(s)} \le c$$

whence

$$F(x) \le c + \int\limits_\Omega h_m(|x-y|) f^2(y)dy . \tag{3.2}$$

>From (3.2) and inequality (2.20) we obtain

$$\int_K F(x)d\mu(x) \le c_m\mu(\Omega) + \int_\Omega f^2(y)\left\{\int_K h_m(|x-y|)d\mu(x)\right\}dy \le$$
$$c_m\mu(\Omega) + c_{m,q}\Big(h_m(R)\mu(K\bigcup B(y,R)) + \|\mathbf{G}\|_{L_q(\Omega)}\Big)\int_\Omega f^2(y)dy \le$$
$$c_{m,q}\Big(h_m(R)\mu(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}\Big)$$

and assertion (i) follows.

It is well known, we refer to [8] for a proof, that the elements $F \in H_0^{1,2}(\Omega)$ admit the representation

$$F(x) = (2\pi)^{-1}\int_\Omega |x-y|^{-2}(x-y)\nabla F(y)dy$$

thus

$$|F(x)| \le (2\pi)^{-1}\int_\Omega |x-y|^{-1}|\nabla F(y)|dy\ .$$

Therefore, in order to prove the assertion (ii) it suffices to show that for any function F with the representation

$$F(x) = \int_\Omega |x-y|^{-1}f(y)dy, \quad f = b(d)^{-1/2}f^* \ge 0, \quad \|f^*\|_{L_2(\Omega)} = 1, \tag{3.3}$$

the following inequality is valid

$$\int_\Omega F(x)d\mu(x) \le c_{m,q}(\mu(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)})\ . \tag{3.4}$$

First, we introduce the notation. For any integer j we denote $D_j = \{x \in \Omega :\ 2^{-j} < d(x) \le 2^{-j+1}\}$. Obviously

$$\Omega = \bigcup_{j=n}^{\infty} D_j, \qquad \text{for an integer} \ \ n\ .$$

Inequality (3.2) along with the identity $h_1 = b$ yields

$$\int_\Omega F(x)d\mu(x) \le c\mu(\Omega) + \int_\Omega f^2(y)\Big(\int_\Omega b(|x-y|)d\mu(x)\Big)dy\ . \tag{3.5}$$

The second integral in the right-hand side of (3.5) can be represented by the series

$$\int_\Omega f^2(y)\Big(\int_\Omega b(|x-y|)d\mu(x)\Big)dy = \sum_{i,j=n}^{\infty} I_{i,j}\ , \tag{3.6}$$

$$I_{i,j} = \int_{D_j}\Big(\int_{D_i} b(|x-y|)d\mu(x)\Big)f^2(y)dy\ .$$

Now our task is to obtain the estimates for the terms $I_{i,j}$. Let j be a fixed integer and choose $y \in D_j$. Set $K = D_{j-1}\cup D_j \cup D_{j+1}$ and $R = 2^{-j-2}$. Since $\mathrm{dist}(K, \partial\Omega) \geq 2R = 2^{-j-1}$, we have $B(y,R) \subset K$. Applying inequality (2.20) from Lemma 2.3 with $m = 1$ and noting that $h_1 = b$ we obtain

$$\int_{D_{j-1}\cup D_j\cup D_{j+1}} b(|x-y|)d\mu(x) \leq cb(2^{-j-2})\mu(\Omega) + c\|\mathbf{G}\|_{L_q(\Omega)}\ . \tag{3.7}$$

Using the property of $b(\cdot)$, namely the second inequality in (1.15) yields $b(2^{\pm 1}s) \leq cb(s)$, which combined with (3.6) leads to

$$I_{j-1,j} + I_{j,j} + I_{j+1,j} \leq c\left(b(2^{-j})\mu(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}\right)\int_{D_j} f^2(y)dy\ . \tag{3.8}$$

On the other hand , $|x-y| \geq 2^{-j-2}$ for all $y \in D_j$ and $x \in D_i$ with $|i-j| \geq 2$. Since b decreases, we get

$$I_{i,j} \leq cb(2^{-j})\mu(D_i)\int_{D_j} f^2(y)dy \quad \text{for} \quad |i-j| \geq 2\ .$$

Combining the latter estimate with (3.8) results in the inequality

$$\int_{D_j} f^2(y)\Big(\int_\Omega b(|x-y|)d\mu(x)\Big)dy = \sum_{i=n}^{\infty} I_{i,j} \leq \tag{3.9}$$

$$c\Big(b(2^{-j})\mu(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}\Big)\int_{D_j} f^2(y)dy\ .$$

Since $b(s)$ decreases, we have

$$b(2^{-j})b(d(y))^{-1} \leq b(2^{-j})b(2^{-j+1})^{-1} \leq cb(2^{-j+1})b(2^{-j+1})^{-1} = c$$

for all $y \in D_j$, hence

$$b(2^{-j})\int_{D_j} f^2(y)dy \leq c\int_{D_j} f^{*2}(y)dy\ .$$

Using the above estimate and (3.9), in view of the inequality $f \le f^*$, we obtain

$$\int_\Omega f^2(y)\Big(\int_\Omega b(|x-y|)d\mu(x)\Big)dy = \sum_{j=n}^{\infty}\int_{D_j} f^2(y)\Big(\int_\Omega b(|x-y|)d\mu(x)\Big)dy \le$$

$$c\Big(\mu(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}\Big)\sum_{j=n}^{\infty}\int_{D_j} f^{*2}(y) = c\Big(\mu(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}\Big)\ ,$$

which implies (3.4) and the assertion (*ii*) follows. To complete the proof we note that the assertion (*iii*) of Theorem 1.9 obviously follows from Lemma 2.2. □

4 A priori estimates and compactness of solutions to momentum equation for ideal isothermal flow

In this section the compactness of the set of generalized solutions to the momentum equation

$$\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) + \nabla\rho = \operatorname{div}\mathbf{G} + \rho\mathbf{f}\ , \tag{4.1}$$

is shown and the following result which plays the key role in the proof of Theorems 1.6-1.8 is established.

Theorem 4.1. *Let $\Omega \in \mathfrak{S}(T, C_\Omega)$, $\mathbf{f} \in C(\Omega)^2$, $\mathbf{G} \in L_q(\Omega)^4$, $1 < q \le 2$ and $\mathbf{u} - \mathbf{u}^\infty \in H_0^{1,2}(\Omega)^2$. Suppose that the tensor field $\mathbf{u}\otimes\mathbf{u}$ is integrable with respect to a non-negative Borel measure μ_ρ in Ω and satisfies the integral identity*

$$\int_\Omega \nabla\varphi : (\mathbf{u}\otimes\mathbf{u} + \mathbf{I})d\mu_\rho = \int_\Omega \nabla\varphi : \mathbf{G}dx - \int_\Omega \varphi\cdot\mathbf{f}d\mu_\rho \tag{4.2}$$

for all $\varphi \in C_0^1(\Omega)^2$. Then there exists a constant c depending only on C_Ω, T, $\|\mathbf{f}\|_{C(\Omega)}$, $\|\mathbf{u}^\infty\|_{C^1(\Omega)}$ and q such that :

(α) The inequality

$$\int_{\Omega(t)} d\mu_e(x) \le c\omega(t)^{1/3}(1+\mathcal{D})^{1/3}(\mu_e(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}) \tag{4.3}$$

holds for any $t \in (0, T_0)$, where T_0 is a positive constant which depends only on the function ω given by (1.12). We recall that μ_e is the Borel measure defined by the equality $d\mu_e = (|\mathbf{u}|^2 + 2)d\mu_\rho$.

(β)

$$\mathcal{E} + 2\mathcal{M} = \mu_e(\Omega) \le c\|\mathbf{G}\|_{L_q(\Omega)} + c\mathcal{M}(1+\mathcal{D})^{1+2\kappa}\ , \tag{4.4}$$

where $\mathcal{E}$, $\mathcal{M}$, $\mathcal{D}$ are given by (1.4) *and κ is exponent from* (1.18). *Furthermore, for any compact set $K \Subset \Omega$ and all $F \in H_0^{m,2}(\Omega)$ with $m > q^{-1}$,*

$$\int_K |F| d\mu_e \le c_m(K)\Big(\|\mathbf{G}\|_{L_q(\Omega)} + c\mathcal{M}(1+\mathcal{D})^{1+2\kappa}\Big)\|F\|_{H_0^{m,2}(\Omega)} \ . \tag{4.5}$$

Proof. We start with the observation that the tensor measure $\mathbf{\Lambda} = (\mu_{i,j})$ with $d\mu_{i,j} = (u_i u_j + 2\delta_{i,j})d\mu_\rho$ satisfies condition (1.24) and

$$\mu \equiv \mu_{1,1} + \mu_{2,2} = \mu_e \ .$$

On the other hand, the vector measure $d\varrho = \mathbf{f} d\mu_\rho$ obviously satisfies the inequality $|\varrho| \le c\mu_\rho \le c\mu_e$. Hence the measure μ_e meets all requirements of Theorem 1.9. Applying inequality (1.29) from Theorem 1.9 we can conclude that

$$\int_K F(x) d\mu_e(x) \le c_{m,q}\Big(h_m(R)\mu_e(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}\Big)\|F\|_{H_0^{m,2}(\Omega)} \tag{4.6}$$

for all $F \in H_0^{m,2}(\Omega)$ with $1/q < m \le 1$ and for all compact sets $K \Subset \Omega$ such that dist $(K, \partial\Omega) \ge 2R$. Inequality (1.30) shows that

$$\int_\Omega F(x) d\mu_e(x) \le c\Big(\mu(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}\Big)\|\sqrt{b(d)}\nabla F\|_{L_2(\Omega)} \tag{4.7}$$

for all $F \in H_0^{1,2}(\Omega)$ such that $\sqrt{b(d)}\nabla F \in L_2(\Omega)$. Finally, noting that

$$d\mu_0 = \nabla d(x) \otimes \nabla d(x) : d\mathbf{\Lambda} = [(\nabla d(x)\cdot \mathbf{u})^2 + 2] d\mu_\rho \ge 2 d\mu_\rho$$

and applying inequality (1.31) we get

$$\int_{\Omega(t)} d\mu_\rho \le c\sigma(t)\Big(\mu_e(\Omega) + \|G\|_{L_q(\Omega)}\Big), \quad t \le T \ . \tag{4.8}$$

The remaining part of the proof is divided into two steps. First, we prove inequality (4.3). Since $|\mathbf{u}|^2 \le \eta|\mathbf{u}|^3 + 4\eta^{-2}/27$ for every positive η, it is easy to see that the inequality

$$|\mathbf{u}|^2 + 2 \le \eta(|\mathbf{u}|^2 + 2)(|\mathbf{u} - \mathbf{u}^\infty| + |\mathbf{u}^\infty|) + c\eta^{-2}, \quad c = 58/27 \tag{4.9}$$

holds for all $\eta \in (0,1)$. Since $b \ge 1$ we can replace η by $\eta b(d)^{-1/2}$ in (4.9). Integrating the resulting inequality over $\Omega(t)$ with respect to the measure μ_ρ and taking into account $\eta b(d)^{-1/2} \le \eta$ leads to

$$(1 - \eta\|\mathbf{u}^\infty\|_{C(\Omega)}) \int_{\Omega(t)} d\mu_e \le \eta \int_{\Omega(t)} F(x) d\mu_e(x) + c\eta^{-2} \int_{\Omega(t)} b(d) d\mu_\rho \ , \tag{4.10}$$

where $F = b(d)^{-1/2}|\mathbf{u}-\mathbf{u}^\infty|$. Since $|\nabla d| = 1$, we have

$$b(d)^{1/2}|\nabla F| \le \frac{1}{2}b(d)^{-1}|b'(d)||\mathbf{u}-\mathbf{u}^\infty| + |\nabla\mathbf{u} - \nabla\mathbf{u}^\infty|\ .$$

We combine the above inequality with the inequality $|b'(d)|/b(d) \le cd^{-1}$ following from (1.15) to obtain

$$\int\limits_{\Omega(t)} b(d)|\nabla F|^2 \le c\int\limits_{\Omega} |\frac{\mathbf{u}-\mathbf{u}^\infty}{d}|^2 dx + \int\limits_{\Omega} |\nabla\mathbf{u} - \nabla\mathbf{u}^\infty|^2 dx\ .$$

By our assumptions the admissible domain Ω uniformly satisfies the cone condition and $\mathbf{u}-\mathbf{u}^\infty \in H_0^{1,2}(\Omega)$ which implies, [9],

$$\int\limits_{\Omega} |\frac{\mathbf{u}-\mathbf{u}^\infty}{d}|^2 dx \le c\int\limits_{\Omega} |\nabla\mathbf{u} - \nabla\mathbf{u}^\infty|^2 dx \le c(1+\mathcal{D})\ .$$

From the latter inequality we can conclude that

$$\|\sqrt{b(d)}\nabla F\|_{L_2(\Omega)} \le c(1+\mathcal{D})^{1/2}\ .$$

Now, inequality (4.7) yields

$$\int\limits_{\Omega} F(x)d\mu_e(x) \le c(\mu(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)})(1+\mathcal{D})^{1/2}\ . \tag{4.11}$$

On the other hand, we have

$$\int\limits_{\Omega(t)} b(d)d\mu_\rho = \int\limits_0^t b(s)dN_\rho(s), \quad \text{where} \quad N_\rho(s) = \int\limits_{\Omega(s)} d\mu_\rho\ .$$

Note that, by relation (1.17), $b(s)\sigma(s) \searrow 0$ which along with (4.8), gives $b(s)N_\rho(s) \searrow 0$ with $s \searrow 0$. Integrating the integral in the right-hand side by parts we obtain

$$\int\limits_{\Omega(t)} b(d)d\mu_\rho = b(t)N_\rho(t) - \int\limits_0^t b'(s)N_\rho(s)ds\ .$$

Recalling that $b'(s) \le 0$ and noting that, by estimate (4.8),

$$N_\rho(t) \le c\sigma(t)\Big(\mu_e(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}\Big)$$

we arrive at the inequality

$$\int\limits_{\Omega(t)} b(d)d\mu_\rho \le c\left(b(t)\sigma(t) - \int\limits_0^t b'(s)\sigma(s)ds\right)\Big(\mu_e(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}\Big) = \tag{4.12}$$

$$c\omega(t)\Big(\mu_e(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}\Big) .$$

Substituting inequalities (4.11)-(4.12) into the right side of (4.10) we obtain that for any $\eta \in (0,1]$

$$(1-\eta\|\mathbf{u}^\infty\|_{C(\Omega)})\int\limits_{\Omega(t)} d\mu_e \le \tag{4.13}$$

$$c\Big(\eta^{-2}\omega(t) + \eta(1+\mathcal{D})^{1/2}\Big)\Big(\mu_e(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}\Big) .$$

By relation (1.17) we can choose the positive number T_0 such that $\omega(t) \le 1$ for any $t \in [0, T_0]$. Fix an arbitrary $\zeta \in (0,1)$ and set $\eta = \zeta\omega(t)^{1/3}(1+\mathcal{D})^{-1/6}$. Note that $\eta \le \zeta < 1$ for $t \in [0, T_0]$. Substituting η in (4.13) gives

$$(1-\zeta\|\mathbf{u}^\infty\|_{C(\Omega)})\int\limits_{\Omega(t)} d\mu_e \le c\omega(t)^{1/3}(1+\mathcal{D})^{1/3}(\zeta^{-2}+\zeta)\Big(\mu_e(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}\Big) .$$

Choosing $\zeta = 1/(2\|\mathbf{u}^\infty\|_{C(\Omega)})$ we obtain the required inequality (4.3).

In the last step of the proof inequality (4.4) is obtained. Choose an arbitrary $t \in (0, T_0)$ and set $K = \mathrm{cl}\ (\Omega \setminus \Omega(t))$. Since dist $(K, \partial\Omega) = 2R = t$ and $h_1(t/2) = b(t/2) \le cb(t)$, inequality (4.6) implies

$$\int\limits_K F(x)d\mu_e(x) \le c_q\Big(b(t)\mu_e(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}\Big)\|F\|_{H_0^{1,2}(\Omega)} \tag{4.14}$$

for all $F \in H_0^{1,2}(\Omega)$. On the other hand, integrating both sides of (4.9) over K with respect to the measure μ_ρ gives

$$(1-\eta\|\mathbf{u}^\infty\|_{C(\Omega)})\int\limits_K d\mu_e \le \eta\int\limits_K F d\mu_e + c\eta^{-2}\mathcal{M} , \tag{4.15}$$

where $F = |\mathbf{u} - \mathbf{u}^\infty| \in H_0^{1,2}(\Omega)$ and $\eta \in (0,1]$ is an arbitrary number. Noting that $\|F\|_{H_0^{1,2}(\Omega)} \le c(1+\mathcal{D})^{1/2}$ and applying inequality (4.14) we obtain

$$(1-\eta\|\mathbf{u}^\infty\|_{C(\Omega)})\int\limits_K d\mu_e \le c_0\eta\Big(b(t)\mu_e(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}\Big)(1+\mathcal{D})^{1/2} + c_0\eta^{-2}\mathcal{M} , \tag{4.16}$$

where $c_0 \geq 1$ depends only on C_Ω, T, $\|\mathbf{u}^\infty\|_{C^1(\Omega)}$, $\|\mathbf{f}\|_{C(\Omega)}$ and q. Denote

$$\eta = \min\left\{\frac{1}{2\|\mathbf{u}^\infty\|_{C(\Omega)}}, \frac{1}{4c_0 b(t)(1+\mathcal{D})^{1/2}}\right\} .$$

Substituting η in (4.16) and noting that $b \geq 1$ we arrive at

$$\int_K d\mu_e \leq \frac{1}{2}\Big(\mu_e(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}\Big) + c\mathcal{M}(1+\mathcal{D})b(t)^2 .$$

Combining this inequality with (4.3) and noting that $\mu_e(\Omega) \leq \mu_e(K) + \mu_e(\Omega(t))$ we get the estimate

$$\int_\Omega d\mu_e \leq \Big(\frac{1}{2} + c_1\omega(t)^{1/3}(1+\mathcal{D})^{1/3}\Big)\Big(\mu_e(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}\Big) + c\mathcal{M}(1+\mathcal{D})b(t)^2 , \qquad (4.17)$$

where the constant $c_1 \geq 1$ depends only on the constants C_Ω, T, $\|\mathbf{u}^\infty\|_{C^1(\Omega)}$, $\|\mathbf{f}\|_{C(\Omega)}$ and q. Set

$$k = \min\{\omega_0, 64^{-1}(1+\mathcal{D})^{-1}c_1^{-3}, 1\} ,$$

where ω_0 is the constant from condition $\mathfrak{D}$. Recall that by this condition the function ω transforms homeomorphically the interval $(0, \delta]$ onto the interval $(0, \omega_0]$. Hence, there exists t^* such that $\omega(t^*) = k$. Since $k \leq 1$, we have $t^* \leq T_0$. For such a choice of t^* we have $c_1\omega(t^*)^{1/3}(1+\mathcal{D})^{1/3} \leq 1/4$ which combined with (4.17) leads to

$$\int_\Omega d\mu_e \leq \frac{3}{4}(\mu_e(\Omega) + \|\mathbf{G}\|_{L_q(\Omega)}) + c\mathcal{M}(1+\mathcal{D})b(t^*)^2 . \qquad (4.18)$$

Noting that by (1.18)

$$b(t^*)^2 \leq \omega(t^*)^{-2\kappa} \leq k^{-2\kappa} \leq c(1+\mathcal{D})^{2\kappa} .$$

>From this and (4.18) we finally obtain

$$\int_\Omega d\mu_e \leq 4\|\mathbf{G}\|_{L_q(\Omega)} + c\mathcal{M}(1+\mathcal{D})^{1+2\kappa} ,$$

which implies (4.4) and the proof of Theorem 4.1 is completed. □

The following result establishes the required compactness properties of the solution set to equation (4.1). Let us fix $\mathbf{f} \in C(B)^2$ and the sequence $\mathbf{G}_n \in L_q(\Omega_n)^4$ which converges weakly in $L_q(B)^4$ to the limit $\mathbf{G}$.

In Theorem 4.2 below we assume that there is given a sequence of domains $\Omega_n = B \setminus S_n$, $n \geq 1$, and a domain Ω, in the class $\mathfrak{S}(T, C_\Omega)$, with some positive T, C_Ω, such that the sequence Ω_n converges to Ω in the Hausdorff metric.

Theorem 4.2. *Suppose that, under the above assumptions on the data, the sequence of pairs* $(\mu_{\rho,n}, \mathbf{u}_n)$, $n \geq 1$, *where* $\mathbf{u}_n - \mathbf{u}^\infty \in H_0^{1,2}(\Omega)$ *and* $\mu_{\rho,n}$ *are non-negative Borel measures in* Ω_n, *satisfy the integral identity*

$$\int_\Omega \nabla\varphi : \Big(\mathbf{u}_n \otimes \mathbf{u}_n + \mathbf{I}\big) d\mu_{\rho,n} = \int_\Omega \nabla\varphi : \mathbf{G}_n dx - \int_\Omega \varphi \cdot \mathbf{f} d\mu_{\rho,n}, \quad \text{for all} \quad \varphi \in C_0^1(\Omega_n)^2 \ . \tag{4.19}$$

Assume that for the sequence the total mass and the energy dissipation are uniformly bounded

$$\sup_n \mathcal{M}_n + \sup_n \mathcal{D}_n \leq C < \infty \ . \tag{4.20}$$

Furthermore, let $\mu_{\rho,n}$ *and* $\mathbf{u}_n$ *denote the measures and functions extended by zero over the obstacles* $S_n \Subset B$. *Suppose that the functions* $u_{i,n}u_{j,n}$ *are integrable with respect to the measures* $\mu_{\rho,n}$ *and the measures* $\mu_{e,n}$ *are defined by* $d\mu_{e,n} = (|\mathbf{u}_n|^2 + 2)d\mu_{\rho,n}$. *Then there exists a subsequence still denoted by* $(\Omega_n, \mu_{\rho,n}, \mathbf{u}_n)$, *the Borel measure* μ_ρ *in* Ω *and the velocity field* $\mathbf{u} \in H^{1,2}(\Omega)$, *such that* $\mu_{\rho,n} \to \mu_\rho$ **-weakly on every open set* $K \Subset \Omega$ *and* $\mathbf{u}_n \to \mathbf{u}$ *weakly in* $H^{1,2}(B)$. *Moreover* $\mathbf{u} - \mathbf{u}^\infty \in H_0^{1,2}(\Omega)$ *and*

$$\int_{\Omega_n} \psi u_{i,n} u_{j,n} d\mu_{\rho,n} \to \int_\Omega \psi u_i u_j d\mu_\rho \quad \textit{with} \quad n \to \infty \ , \tag{4.21}$$

$$\int_{\Omega_n} \psi u_{i,n} d\mu_{\rho,n} \to \int_\Omega \psi u_i d\mu_\rho, \quad \int_{\Omega_n} \psi d\mu_{\rho,n} \to \int_\Omega \psi d\mu_\rho, \qquad \textit{as} \quad n \to \infty \ , \tag{4.22}$$

for all $\psi \in C(B)$ *and* $i, j = 1, 2$. *In particular,*

$$\mathcal{M}_n \to \mathcal{M} = \mu_\rho(\Omega), \quad \mu_{e,n}(\Omega_n) \to \mu_e(\Omega) \ , \tag{4.23}$$

where the measure μ_e *is defined by* $d\mu_e = (|\mathbf{u}|^2 + 2)d\mu_\rho$. *The pair* $(\mu_\rho, \mathbf{u})$ *is a generalized solution of equation* (4.1) *in* Ω *and satisfies integral identity* (4.2).

Proof. We start with the observation that, since the sequence Ω_n satisfies uniformly the cone condition, the limit of the sequence Ω is an open set with one connected component. Hence the sequence of spaces $H_0^{1,2}(\Omega_n)$ converges to the space $H_0^{1,2}(\Omega)$ in the sense of Mosco, [10]. Therefore, there exists a subsequence of the velocity fields, still denoted by $\mathbf{u}_n$, which converges weakly in $H^{1,2}(B)^2$ to the vector field $\mathbf{u} \in H^{1,2}(B)^2$ with $\mathbf{u} - \mathbf{u}^\infty \in H_0^{1,2}(\Omega)^2$.

Next, note that Ω_n, $\mathbf{G}_n$ and the pairs $(\mu_{\rho,n}, \mathbf{u}_n)$ meet all requirements of Theorem 4.1. Thus, inequalities (4.4) along with (4.20) imply that the total variations $\mu_{e,n}(\Omega_n)$ of the measures $\mu_{e,n}$ are uniformly bounded from above. Applying inequality (4.3) with Ω, μ_e, $\mathcal{D}$ and $\mathbf{G}$ replaced by Ω_n,$\mu_{e,n}$, $\mathcal{D}_n$ and $\mathbf{G}_n$, respectively, we conclude that there exists a sequence $t_l \searrow 0$ with $l \nearrow \infty$ such that

$$\mu_{e,n}(\Omega_n(2t_l)) \leq 1/l \quad \text{for all} \quad n \geq 1 \ . \tag{4.24}$$

Denote $K_l = \Omega \setminus \Omega(t_l)$ and observe that dist $(K_l, \partial\Omega) = t_l$. Since Ω_n converge to Ω in the Hausdorff metric, then for any l there is N_l such that

$$\Omega_n \setminus \Omega_n(2t_l) \subset K_l \subset \Omega_n \setminus \Omega_n(t_l/2) \quad \text{for} \quad n \geq N_l \ . \tag{4.25}$$

Since the total variations of the measures $\mu_{\rho,n}$ and $\mu_{e,n}$ are uniformly bounded for $n = 1, 2, \ldots$, we can assume, after passing to a subsequence if necessary, that the measures $\mu_{\rho,n}$ and $\mu_{e,n}$ converge weakly with $n \nearrow \infty$ on each K_l to non-negative measures μ_ρ, μ^*, respectively, defined on the Borel subsets of Ω. Moreover, the measures $\mu_{ij,n}$ defined by $d\mu_{ij,n} = u_{i,n}u_{j,n}d\mu_{\rho,n}$, $i, j = 1, 2$, converge weakly on each K_l to some measures μ^*_{ij}. Obviously

$$2\mu_\rho \leq \mu^*, \quad \mu^*_{11} + \mu^*_{22} + 2\mu_\rho = \mu^*.$$

Inclusion (4.25) along with inequality (4.5) from Theorem 4.1 imply that for $l \geq 1$ and $q^{-1} < m \leq 1$, the inequality

$$\int\limits_{K_l} |F(x)| d\mu_{e,n}(x) \leq c(l, m, C) \|F\|_{H^{m,2}_0(B)} \quad \text{for} \quad m > q^{-1} \ . \tag{4.26}$$

holds for all $n \geq N_l$ and $F \in C^1_0(\Omega)$ with supp $F \Subset \Omega_n$. After passing to the limit with $n \nearrow \infty$ we obtain

$$\int\limits_{K_l} |F(x)| d\mu_\rho(x) \leq \int\limits_{K_l} |F(x)| d\mu^*(x) \leq c(l, m, C) \|F\|_{H^{m,2}_0(B)} \quad \text{for} \quad m > q^{-1} \ . \tag{4.27}$$

and for all smooth compactly supported in K_l functions F. Hence the quasicontinuous representative of a function $F \in H^{m,2}_0(K_l)$ with $m > q^{-1}$ is integrable with respect to the measures μ_ρ, μ^* over every set K_l.

Let us prove that $\mu^* = \mu_e$ and $d\mu^*_{ij} = u_i u_j d\mu_\rho$. Note that in view of (4.20) the sequence $\mathbf{u}_n$ is bounded in $H^{1,2}(B)$ thus, it is bounded in all spaces $L_r(B)$ for $r < \infty$. Therefore, the sequences $u_{i,n}u_{j,n}$ are bounded in the Sobolev spaces $H^{1,p}(B)$ for all $p < 2$. By the Sobolev embedding theorem, the sequences are relatively compact sets in $H^{m,2}(B)$ for all $m < 1$.

Fix an arbitrary $l \geq 1$, $m \in (q^{-1}, 1]$ and let the test function $\chi \in C^1_0(K_{l+1})$ be such that

$$0 \leq \chi \leq 1 \quad \text{and} \quad \chi(x) = 1 \quad \text{in} \quad K_l \ . \tag{4.28}$$

Obviously the functions $\chi u_i u_j$ belong to $H^{m,2}(K_{l+1})$. Hence the functions $u_i u_j$ are integrable with respect to the measures μ^*, μ_ρ over K_l. Choose an arbitrary test function $\xi \in C^1_0(K_l)$ and introduce the set

$$\mathcal{C}_{i,j} = \text{cl}\ \{\xi u_{i,n} u_{j,n}\}_{n\geq 1} \subset H^{m,2}_0(K_l) \ .$$

The set $\mathcal{C}_{i,j}$ is relatively compact and has the unique limit $\xi u_i u_j$. Therefore, for any positive ε there exists a finite collection of functions $\psi^k_{ij} \in H^{m,2}_0(K_l)$, $1 \leq k \leq N(\varepsilon)$, such that

$$\inf_{n\geq 1} \|\psi^k_{ij} - \xi u_{i,n} u_{j,n}\|_{H^{m,2}_0(K_l)} < \varepsilon \ .$$

Without loss of the generality we can suppose that ψ^k_{ij} belong to $C^1_0(K_l)$. In particular, there exists a sequence $\psi^{k(n)}_{ij}$ and an element $\psi^{k(\infty)}_{ij}$ such that

$$\|\psi^{k(n)}_{ij} - \xi u_{i,n} u_{j,n}\|_{H^{m,2}_0(K_l)} < \varepsilon, \quad \|\psi^{k(\infty)}_{ij} - \xi u_i u_j\|_{H^{m,2}_0(K_l)} < \varepsilon \ . \tag{4.29}$$

We can assume that the sequence $\psi^{k(n)}_{ij}$ converges in $C^1(K_l)$ to $\psi^{k(\infty)}_{ij}$, after passing to a subsequence if necessary. In fact, the functions $\psi^{k(n)}_{ij}$ coincide with the functions $\psi^{k(\infty)}_{ij}$ for sufficiently large n. Thus, we get

$$\lim_{n\to\infty} \int\limits_{K_l} \psi^{k(n)}_{ij} d\mu_{\rho,n} = \int\limits_{K_l} \psi^{k(\infty)}_{ij} d\mu_\rho \ . \tag{4.30}$$

On the other hand, under our assumptions,

$$\lim_{n\to\infty} \int\limits_{K_l} \xi u_{i,n} u_{j,n} d\mu_{\rho,n} = \int\limits_{K_l} \xi d\mu^*_{i,j} \ . \tag{4.31}$$

Inequality $\mu_{\rho,n} \le \mu_{e,n}$ along with (4.26) implies

$$|\int\limits_{K_l} \psi^{k(n)}_{ij} d\mu_{\rho,n} - \int\limits_{K_l} \xi u_{i,n} u_{j,n} d\mu_{\rho,n}| \le c(l,m,C) \|\psi^{k(n)}_{ij} - \xi u_{i,n} u_{j,n}\|_{H^{m,2}_0(K_l)} \le c\varepsilon \ . \tag{4.32}$$

By inequality (4.27) we have

$$|\int\limits_{K_l} \psi^{k(\infty)}_{ij} d\mu_\rho - \int\limits_{K_l} \xi u_i u_j d\mu_\rho| \le c(l,m,C) \|\psi^{k(\infty)}_{ij} - \xi u_i u_j\|_{H^{m,2}_0(B)} \le c\varepsilon \ . \tag{4.33}$$

Combining (4.30) and (4.31) with (4.32) and (4.33) we obtain

$$|\int\limits_{K_l} \xi u_i u_j d\mu_\rho - \int\limits_{K_l} \xi d\mu^*_{i,j}| \le c\varepsilon \ .$$

Since ε and l are arbitrary numbers and ξ is an arbitrary element of $C^\infty_0(K_l)$ we can conclude that $d\mu^*_{j,j} = u_i u_j d\mu_\rho$ and $\mu^* = \mu_e$. The repeating of the same arguments implies that the measures $u_i d\mu_{\rho,n}$ converge to $u_i d\mu_\rho$ weakly on every set K_l.

In order to prove (4.21) note that if $\chi \in C^1_0(K_{l+1})$ satisfies (4.28), then

$$\begin{aligned}\Delta_n = & \int\limits_{\Omega_n} \psi u_{i,n} u_{j,n} d\mu_{\rho,n} - \int\limits_{\Omega} \psi u_i u_j d\mu_\rho = \\ & \int\limits_{K_{l+1}} \chi\psi u_{i,n} u_{j,n} d\mu_{\rho,n} - \int\limits_{K_{l+1}} \chi\psi u_i u_j d\mu_\rho + \\ & \int\limits_{\Omega_n\setminus K_l} (1-\chi)\psi u_{i,n} u_{j,n} d\mu_{\rho,n} - \int\limits_{\Omega\setminus K_l} (1-\chi)\psi u_i u_j d\mu_\rho\end{aligned}$$

for all $n > N_{l+1}$. Noting that $\mu_{i,j,n}$ converge weakly in K_{l+1} to $u_iu_jd\mu_\rho$ and passing to the limit with $n \to \infty$ we obtain

$$\limsup_{n\to\infty} |\Delta_n| \leq \limsup_{n\to\infty} |\int\limits_{\Omega_n\setminus K_l} (1-\chi)\psi u_{i,n}u_{j,n}d\mu_{\rho,n}| + |\int\limits_{\Omega\setminus K_l} (1-\chi)\psi u_iu_jd\mu_\rho|.$$

Inclusion (4.24) along with inequality (4.25) imply

$$|\int\limits_{\Omega_n\setminus K_l} (1-\chi)\psi u_{i,n}u_{j,n}d\mu_{\rho,n}| \leq \|\psi\|_{C(B)} \int\limits_{\Omega_n\setminus K_l} d\mu_{e,n} \leq \|\psi\|_{C(B)}\frac{1}{l}.$$

On the other hand, we have

$$|\int\limits_{\Omega_n\setminus K_l} (1-\chi)\psi u_iu_jd\mu_\rho| \leq \|\psi\|_{C(B)} \int\limits_{\Omega\setminus K_l} d\mu_e.$$

Thus we get

$$\limsup_{n\to\infty} |\Delta_n| \leq \|\psi\|_{C(B)}\frac{1}{l} + \|\psi\|_{C(B)}\mu_e(\Omega\setminus K_l). \tag{4.34}$$

Recall that the sequence $K_l \subset K_{l+1}$ and $\cup_l K_l = \Omega$ which implies $\lim\limits_{l\to\infty} \mu_e(\Omega\setminus K_l) = 0$. Passing to the limits in (4.34) with $l \to \infty$ leads to the equality $\lim\limits_{n\to\infty} \Delta_n = 0$ which implies (4.21). The same arguments give (4.22). From (4.21), and (4.19) we can conclude that $(\mathbf{u}, \mu_\rho)$ is a generalized solution to (4.1), which completes the proof. □

5 Proofs of Theorem 1.6 and Theorem 1.7

Proof of Theorems 1.6. First, we show that under the assumptions of the theorem, the sequence $\mathcal{D}_n$ is bounded from above. Note that $\mu_{\rho,n}$ and $\mathbf{u}_n$ are generalized solutions to problem (1.1)-(1.2) and hence satisfy inequality (1.7). Next, observe that formula (1.11) for the cost functional J implies

$$\int\limits_{\Omega_n} (\mathbf{u}_n\otimes\mathbf{u}_n + \mathbf{I}) : \nabla\mathbf{u}^\infty d\mu_{\rho_n} =$$
$$- J(\mu_{\rho,n}, \mathbf{u}_n, \Omega_n) + \int\limits_{\Omega_n} \mathbf{\Sigma}_n : \nabla\mathbf{u}^\infty dx + \int\limits_{\Omega_n} (\mathbf{U}^\infty - \mathbf{u}^\infty)\cdot\mathbf{f}d\mu_{\rho,n}\ .$$

Substituting this expressions into (1.7) with $\mathbf{u}$, $\mathbf{\Sigma}$, μ_ρ, and Ω replaced by $\mathbf{u}_n$, $\mathbf{\Sigma}_n$, $\mu_{\rho,n}$, and Ω_n, respectively, leads to

$$\mathcal{D}_n \le \int\limits_{\Omega_n} \Big(\frac{\xi}{2}|\mathrm{div}\,\mathbf{u}_n|^2 + \frac{1}{2\xi}\Big)dx +$$

$$\int\limits_{\Omega_n} \mathbf{f}\cdot(\mathbf{u}_n - \mathbf{U}^\infty)d\mu_{\rho,n} - \int\limits_{\Gamma_n^+} \rho_\infty \log(1+\rho_\infty)\mathbf{U}^\infty\cdot\mathbf{n}ds + J(\mu_{\rho,n},\mathbf{u}_n,\Omega_n)\ .$$

We have also

$$\mathcal{D}_n - \int\limits_{\Omega_n} \frac{\xi}{2}|\mathrm{div}\,\mathbf{u}_n|^2dx = \int\limits_{\Omega_n}(\nu|\nabla\mathbf{u}_n|^2 + \frac{\xi}{2}|\mathrm{div}\,\mathbf{u}_n|^2)dx \ge \frac{1}{2}\mathcal{D}_n\ .$$

Thus, by combining the above inequalities we can conclude that

$$\mathcal{D}_n \le 2J(\mu_{\rho,n},\mathbf{u}_n,\Omega_n) + 2\int\limits_{\Omega_n}\mathbf{f}\cdot(\mathbf{u}_n-\mathbf{U}^\infty)d\mu_{\rho,n} + c\ . \tag{5.1}$$

The integral identity (1.8a) from the definition of generalized solutions implies that $\mu_{\rho,n}$, $\mathbf{u}_n$ satisfy integral identity (4.19) with

$$\mathbf{G}_n = \mathbf{\Sigma}_n \ \text{ in } \ \Omega_n \ \text{ and } \ \mathbf{G}_n = 0 \ \text{ on } \ S_n\ . \tag{5.2}$$

Obviously $\|\mathbf{\Sigma}_n\|_{L_2(\Omega)} \le c(\mathcal{D}_n^{1/2}+1)$. Now, estimate (4.4) from Theorem 4.1 with $q=2$ implies

$$\int\limits_{\Omega_n} d\mu_{e,n} \le c(\mathcal{M}_n(1+\mathcal{D}_n)^{1+2\kappa} + \|\mathbf{G}_n\|_{L_q(\Omega_n)}) \le c(\mathcal{M}_n(1+\mathcal{D}_n)^{1+2\kappa} + \mathcal{D}_n^{1/2}+1)\ . \tag{5.3}$$

On the other hand,

$$\int\limits_{\Omega_n}\mathbf{f}\cdot(\mathbf{u}_n-\mathbf{U}^\infty)d\mu_{\rho,n} \le c\int\limits_{\Omega_n}|\mathbf{u}_n|d\mu_{\rho,n} + c\mathcal{M}_n\ . \tag{5.4}$$

Noting that

$$|\mathbf{u}_n| \le (1+\mathcal{D}_n)^{1/2+\kappa} + |\mathbf{u}_n|^2(1+\mathcal{D}_n)^{-1/2-\kappa}$$

we find

$$\int\limits_{\Omega_n}|\mathbf{u}_n|d\mu_{\rho,n} \le (1+\mathcal{D}_n)^{-1/2-\kappa}\int\limits_{\Omega_n}|\mathbf{u}_n|^2d\mu_{\rho,n} + (1+\mathcal{D}_n)^{1/2+\kappa}\int\limits_{\Omega_n}d\mu_{\rho,n}\ .$$

Substituting this inequality into (5.4) leads to

$$\int\limits_{\Omega_n} \mathbf{f}\cdot(\mathbf{u}_n-\mathbf{U}^\infty)d\mu_{\rho,n} \le c(1+\mathcal{D}_n)^{-1/2-\kappa}\mu_{e,n}(\Omega_n)+c\mathcal{M}_n(1+\mathcal{D}_n)^{1/2+\kappa}\ ,$$

which along with (5.3) yields

$$\int\limits_{\Omega_n} \mathbf{f}\cdot(\mathbf{u}_n-\mathbf{U}^\infty)d\mu_{\rho,n} \le c(1+\mathcal{D}_n)^{1/2+\kappa}(1+\mathcal{M}_n)\ .$$

Finally, substituting this inequality into (5.1) we obtain

$$\mathcal{D}_n \le 2J(\mu_{\rho,n},\mathbf{u}_n,\Omega_n)+c(1+\mathcal{D}_n)^{1/2+\kappa}(1+\mathcal{M}_n)\ . \tag{5.5}$$

By the hypotheses, $\kappa < 1/2$ and $\sup\limits_n \mathcal{M}_n + \sup\limits_n J(\mu_{\rho,n},\mathbf{u}_n,\Omega_n) < \infty$, which along with (5.5) implies the boundedness of the sequence $\mathcal{D}_n$. Moreover, it follows from (5.5) that the sequence $J(\mu_{\rho,n},\mathbf{u}_n,\Omega_n)$ is bounded from below.

By Lemma 1.4, we can choose a subsequence, still denoted by $\Omega_n \in \mathfrak{S}(T,C_\Omega)$, such that the sequence of obstacles S_n converges in the Hausdorff metric to a compact subset $S \Subset B$ and the limit $\Omega = B\setminus S \in \mathfrak{S}(T,C_\Omega)$. In addition, it follows from the convergence of Ω_n to Ω that the sequence of Sobolev spaces $H^{1,2}_0(\Omega_n)$ converges to $H^{1,2}_0(\Omega)$ in the sense of Mosco. Therefore, after passing to a subsequence we can assume that the sequence of functions $\mathbf{u}_n$, extended by $\mathbf{u}^\infty$ outside of Ω, converges weakly in $H^{1,2}(B)^2$ to the function $\mathbf{u}$ such that $\mathbf{u}-\mathbf{u}^\infty \in H^{1,2}_0(\Omega)^2$. In particular, the sequence of tensors $\mathbf{G}_n$, defined by (5.2), converges weakly in $L_2(B)^4$ to the limit

$$\mathbf{G}=\mathbf{\Sigma}\ \text{ in }\ \Omega\ \text{ and }\ \mathbf{G}=0\ \text{ on }\ S\ .$$

Hence for the sequence $(\mu_{\rho,n},\mathbf{u}_n)$ all requirements of Theorem 4.2 are satisfied with $\mathbf{G}_n=\mathbf{\Sigma}_n$ and $q=2$. By the hypotheses, the pair $(\mathbf{u}_n,\mu_{\rho,n})$ satisfies the integral identities

$$\int\limits_{\Omega_n}(\mathbf{u}_n\otimes\mathbf{u}_n+\mathbf{I}):\nabla\varphi d\mu_{\rho,n}+\int\limits_{\Omega_n}\mathbf{f}\cdot\varphi d\mu_{\rho,n}=\int\limits_{\Omega_n}\mathbf{\Sigma}_n:\nabla\varphi dx,$$

$$\int\limits_{\Omega_n}\mathbf{u}_n\cdot\nabla\psi d\mu_{\rho,n}+\int\limits_{\Gamma+}\psi\rho^\infty\mathbf{U}^\infty\cdot\mathbf{n}d\Gamma=0\ .$$

Passing to the limit with $n\to\infty$ and using relations (4.21)-(4.22) from Theorem 4.2 we can conclude that $\mathbf{u}$ and μ_ρ satisfy the integral identities (1.8).

Furthermore, the pair $(\mathbf{u}_n,\mu_{\rho,n})$ satisfies the energy inequality

$$\mathcal{D}_n-\int\limits_{\Omega_n}\frac{\xi}{2}|\operatorname{div}\mathbf{u}_n|^2dx \le \int\limits_{\Omega_n}\Big(\mathbf{\Sigma}_n:\nabla\mathbf{u}^\infty+\frac{1}{2\xi}\Big)dx-\int\limits_{\Omega_n}(\mathbf{u}_n\otimes\mathbf{u}_n+\mathbf{I}):\nabla\mathbf{u}^\infty d\mu_{\rho,n}+$$
$$\int\limits_{\Omega_n}\mathbf{f}\cdot(\mathbf{u}_n-\mathbf{u}^\infty)d\mu_{\rho,n}-\int\limits_{\Gamma+}\rho_\infty\log(1+\rho_\infty)\mathbf{U}^\infty\cdot\mathbf{n}ds\ . \tag{5.6}$$

Since $\nabla \mathbf{u}_n$ converges weakly in $L_2(B)$ to $\nabla \mathbf{u}$ and vanishes on S_n

$$\mathcal{D} - \int\limits_{\Omega} \frac{\xi}{2} |\mathrm{div}\, \mathbf{u}|^2 dx = \int\limits_{B} (\nu |\nabla \mathbf{u}|^2 + \frac{\xi}{2} |\mathrm{div}\, \mathbf{u}|^2) dx \le$$

$$\liminf_{n\to\infty} \int\limits_{B} (\nu |\nabla \mathbf{u}_n|^2 + \frac{\xi}{2} |\mathrm{div}\, \mathbf{u}_n|^2) dx = \liminf_{n\to\infty} \left(\mathcal{D}_n - \int\limits_{\Omega_n} \frac{\xi}{2} |\mathrm{div}\, \mathbf{u}_n|^2 dx \right) .$$

On the other hand, relations (4.21) and (4.22) yields

$$\int\limits_{\Omega_n} (\mathbf{u}_n \otimes \mathbf{u}_n + \mathbf{I}) : \nabla \mathbf{u}^\infty d\mu_{\rho,n} \to \int\limits_{\Omega} (\mathbf{u} \otimes \mathbf{u} + \mathbf{I}) : \nabla \mathbf{u}^\infty d\mu_\rho, \tag{5.7}$$

$$\int\limits_{\Omega_n} \mathbf{f} \cdot (\mathbf{u}_n - \mathbf{u}^\infty) d\mu_{\rho,n} \to \int\limits_{\Omega} \mathbf{f} \cdot (\mathbf{u} - \mathbf{u}^\infty) d\mu_\rho \quad \text{as} \quad n \to \infty . \tag{5.8}$$

Passing to the limits in (5.6) we can conclude that the pair $(\mathbf{u}, \mu_\rho)$ satisfies inequality (1.7) and hence it is a generalized solution to problem (1.1)-(1.2).

It remains to note that formula (1.11) along with the relations (4.21)-(4.22) and (5.7)-(5.8) implies

$$\lim_{n\to\infty} J(\mu_{\rho,n}, \mathbf{u}_n, \Omega_n) = J(\mu_\rho, \mathbf{u}, \Omega)$$

which completes the proof. □

Proof of Theorem 1.7 It suffices to note that Theorem 1.7 is a particular case of Theorem 1.6 for the choice $\mathbf{U}^\infty = 0$. □

6 Proof of Theorem 1.8

6.1 Positivity of density. Proof of Lemma 1.10

Denote by $\mathbb{K} \subset L_2(\Omega)$ the cone of all functions which are non-negative a.e. in Ω. Obviously $L_2(\Omega)$ is a linear hull of $\mathbb{K}$. Recall that the linear operator $A : L_2(\Omega) \mapsto L_2(\Omega)$ is positive if and only if $A(\mathbb{K}) \subset \mathbb{K}$. Let us consider boundary value problem (1.32b)-(1.32c). Choose a positive number $k > \sup\limits_{\Omega} |\mathrm{div}\, \mathbf{u}(x)|$ and introduce the linear operator

$$L\rho \equiv -\varepsilon \Delta \rho + \mathrm{div}\, (\rho \mathbf{u}) + k\rho \equiv -L_0 \rho , \tag{6.1}$$

where

$$L_0 \rho \equiv \varepsilon \Delta \rho - \nabla \rho \cdot \mathbf{u} + c\rho, \quad c = -\mathrm{div}\, \mathbf{u} - k < 0 .$$

It follows from the general theory of boundary value problems for the second order elliptic differential equations [8] that the Neumann problem

$$L\rho = f \in L_2(\Omega), \quad \nabla \rho \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega \tag{6.2}$$

admits a unique solution $\rho \in H^{2,2}(\Omega)$ for any $f \in L_2(\Omega)$. Note that by the embedding theorem $\rho \in H^{2,2}(\Omega)$ belongs to any space $C^\alpha(\Omega)$ with the index $\alpha \in [0,1)$. By the maximum principle for strong solutions ρ is non-negative if f is non-negative. Hence the bounded positive operator is defined by

$$A \equiv L^{-1} : L_2(\Omega) \mapsto H^{2,2}(\Omega) \ . \tag{6.3}$$

Note that a solution to problem (6.2) satisfies the relation $L_0\rho \leq 0$. Since dim $\Omega = 2$ we can apply the strong maximum principle for strong solutions ([8], Theorem 9.6) to obtain that ρ does not attain the non-positive minimum in Ω. Hence a solution to problem (6.2) is strictly positive inside Ω for all $f \in \mathbb{K} \setminus \{0\}$. Moreover, the solution is strictly positive in $\mathbf{cl}\,\Omega = \overline{\Omega}$. In order to prove the last statement we choose a compact $K \Subset \Omega$ such that $f > 0$ a.e. on K and we denote $f_K = \chi_K f$, where χ_K is the characteristic function of K. Since $f_K \leq f$, we have $Af_K \leq Af$ and it suffices to prove that Af_K is strictly positive in cl Ω. Note that Af_K is strictly positive in Ω and hence in $\Omega \setminus K$. Suppose, on the contrary to our claim, that $Af_K(x_0) = 0$ at some point $x_0 \in \partial\Omega$. Observe that the function $v = -Af_K$ is negative and satisfies the equation $L_0 v = 0$ in the open set $\Omega \setminus K$. Furthermore, the second order derivatives of v are continuous in an open neighbourhood of the boundary of Ω and takes the zero value for the maximum at x_0. By Lemma 3.4 in [8] we have $\mathbf{n} \cdot \nabla v(x_0) > 0$ which is in contradiction with the boundary condition $\mathbf{n} \cdot \nabla v = 0$ at $\partial\Omega$.

Set $\rho_0 = A\mathbf{1}$, where $\mathbf{1} = \chi_\Omega$ is the characteristic function of Ω. Since Af is continuous and strictly positive, for every $f \in \mathbb{K} \setminus \{0\}$ there exist positive constants α, β depending on f such that

$$\alpha\rho_0 \leq Af \leq \beta\rho_0 \ .$$

Hence $A : L_2(\Omega) \mapsto L_2(\Omega)$ is a compact ρ_0-positive operator, [11]. Classical results from theory of positive operators, see Theorems 2.8, 2.10 and 2.13 from [11] for example, imply that A has a positive simple eigenvalue λ_0 such that the corresponding eigenfunction is strictly positive and $\lambda_0 > |\lambda|$ for any eigenvalue $\lambda \neq \lambda_0$.

We observe that main problem (1.32b)-(1.32c) is equivalent to the operator equation

$$k^{-1}\rho = A\rho, \quad \rho \in L_2(\Omega) \ . \tag{6.4}$$

Hence the proof of Lemma 6.1 is completed if we prove that $1/k$ is the maximal eigenvalue of the operator A. First, we note that the operator equation $k^{-1}\rho - A\rho = \mathbf{1}$ is equivalent to the boundary value problem

$$-\varepsilon\Delta\rho + \operatorname{div}(\rho\mathbf{u}) = k \ \text{ in } \ \Omega, \quad \nabla\rho \cdot \mathbf{n} = 0 \ \text{ on } \ \partial\Omega \ . \tag{6.5}$$

If a solution exists, then it belongs to $C^2(\Omega)$. Integrating both the sides of this equation over Ω we obtain $k = 0$. Hence this problem has no solution. Therefore, by the Fredholm alternative, k^{-1} is an eigenvalue of the operator A.

Let us prove that k^{-1} is the maximal eigenvalue. If this assertion is false, then there exists the positive eigenvalue $\lambda_0 > k^{-1}$. By the definition of A, the eigenfunction $\rho_0 \in H^{2,2}(\Omega)$ satisfying the equation $\lambda_0\rho_0 - A\rho_0 = 0$ is a solution to the boundary value problem

$$-\varepsilon\Delta\rho_0 + \operatorname{div}(\rho_0\mathbf{u}) = -\nu\rho_0 \ \text{ in } \ \Omega, \quad \nabla\rho_0 \cdot \mathbf{n} = 0 \ \text{ on } \ \partial\Omega \ , \tag{6.6}$$

where $\nu = k - \lambda_0^{-1} > 0$. Let us consider the parabolic boundary value problem

$$\frac{\partial v}{\partial t} - \varepsilon\Delta v + \operatorname{div}(v\mathbf{u}) = 0 \text{ in } \Omega \times (0,\infty), \quad \nabla v \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \times (0,\infty) ,$$

$$v(x,0) = v_0(x) \text{ in } \Omega .$$

For any $v_0 \in C(\Omega)$ this problem has the unique smooth solution, which is positive if v_0 is non-negative. Introduce the operator $V(t) : v_0(\cdot) \mapsto v(\cdot,t)$. Obviously $V(t)$ preserves the order and the charge i.e.,

$$V(t)v_0' \geq V(t)v_0'' \quad \text{for any } v_0' \geq v_0'' \text{ and } \int_\Omega V(t)v_0(x)dx = \int_\Omega v_0 dx .$$

Since for every $u, v \in C(\Omega)$ the function $\max\{u,v\} \in C(\Omega)$ we can apply the Crandall-Tartar Theorem [12] which implies that $V(t)$ is a non-expansive operator in the metric of $L_1(\Omega)$. In particular, we have $\|V(t)v_0\|_{L_1(\Omega)} = \|v_0\|_{L_1(\Omega)}$. On the other hand, equations (6.6) imply that for $v_0 = \rho_0$ the solution of the parabolic problem is given by $V(t)v_0 = e^{\nu t}v_0$ and $\|V(t)v_0\|_{L_1(\Omega)} \nearrow \infty$ as $t \nearrow \infty$. Hence $\nu = 0$ which gives $\lambda_0 = k^{-1}$ and the lemma follows.

□

6.2 Proof of Theorem 1.8

We start with consideration of the regularised problem depending on the parameters $\varepsilon, s \in (0,1)$,

$$\nu\Delta\mathbf{u} + \xi\nabla\operatorname{div}\mathbf{u} = \operatorname{div}[(\rho\mathbf{u} - \varepsilon\nabla\rho)\otimes\mathbf{u}] + \nabla\rho - (1+s\rho)^{-1}\rho\mathbf{f} , \text{ in } \Omega \tag{6.7a}$$

$$\varepsilon\Delta\rho = \operatorname{div}(\rho\mathbf{u}) , \text{ in } \Omega \tag{6.7b}$$

$$\mathbf{u} = 0, \quad \nabla\rho\cdot\mathbf{n} = 0 , \text{ on } \partial\Omega \tag{6.7c}$$

$$\int_\Omega \rho(x)dx = \mathcal{M} . \tag{6.7d}$$

Introduce the function $v : \Omega \mapsto \mathbb{R}^+$ defined by

$$v = \sqrt{\varepsilon(1+\rho)}, \quad \varepsilon\rho = v^2 - \varepsilon . \tag{6.8}$$

Our first task is the proving of the existence of a classical solution to this problem. The result is given by

Lemma 6.1. *Let, under the assumptions of Theorem 1.8,* $\mathbf{f} \in C^\alpha(\Omega)^2$, $\alpha \in (0,1)$ *and* $\mathcal{M} > 0$. *Then problem* (6.7) *has a solution* $(\rho,\mathbf{u}) \in C^{2+\alpha}(\Omega)$ *which satisfies the inequality*

$$\|v\|^2_{H^{1,2}(\Omega)} + \|\mathbf{u}\|^2_{H^{1,2}_0(\Omega)} \leq c\|\mathbf{f}\|_{C(\Omega)} \int_\Omega |\mathbf{u}|\rho dx + \varepsilon c(1+\mathcal{M}) . \tag{6.9}$$

The constant c depends on the constants T, C_Ω from Definition 1.3 and does not depend on $\mathbf{f}$*,* $\mathcal{M}$ *and* s, ε*.*

Proof. Let us consider the family of boundary value problems depending on the parameter $t \in [0,1]$:

$$\nu\Delta\mathbf{u} + \xi\nabla\mathrm{div}\,\mathbf{u} = t\mathrm{div}\,[(t\rho\mathbf{u} - \varepsilon\nabla\rho)\otimes\mathbf{u}] + t\nabla\rho - t(1+s\rho)^{-1}\rho\mathbf{f}\ , \quad \text{on} \quad \Omega \tag{6.10a}$$

$$\varepsilon\Delta\rho = t\mathrm{div}\,(\rho\mathbf{u})\ , \quad \text{on} \quad \Omega \tag{6.10b}$$

$$\mathbf{u} = 0, \quad \nabla\rho\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega\ , \tag{6.10c}$$

$$\int_\Omega \rho(x)dx = \mathcal{M}\ . \tag{6.10d}$$

First we prove that a solution to problem (6.10) satisfies inequality (6.9). Suppose that $(\mathbf{u},\rho) \in C^{2+\alpha}(\Omega)$ satisfies (6.10). Multiplying both the sides of (6.10a) by $\mathbf{u}$, integrating over Ω and noting that $\mathrm{div}\,(t\rho\mathbf{u} - \varepsilon\nabla\rho) = 0$ we obtain

$$t^{-1}\int_\Omega (\nu|\nabla\mathbf{u}|^2 + \xi|\mathrm{div}\,\mathbf{u}|^2)dx = \int_\Omega \frac{\rho}{1+s\rho}\mathbf{f}\cdot\mathbf{u}dx + \int_\Omega \rho\mathrm{div}\,\mathbf{u}dx\ . \tag{6.11}$$

It follows from (6.10b) that the identity

$$-\varepsilon\Delta a(\rho) + t(\rho a'(\rho) - a(\rho))\mathrm{div}\,\mathbf{u} + \varepsilon a''(\rho)|\nabla\rho|^2 = 0 \tag{6.12}$$

holds for any smooth function a. Choosing $a(\rho) = \rho\ln(1+\rho)$ gives

$$(\rho a'(\rho) - a(\rho))\mathrm{div}\,\mathbf{u} = \rho\mathrm{div}\,\mathbf{u} - \frac{\rho}{1+\rho}\mathrm{div}\,\mathbf{u}\ .$$

Integrating this equality over Ω and using identity (6.12) leads to

$$\int_\Omega \rho\mathrm{div}\,\mathbf{u}dx = \int_\Omega \frac{\rho}{1+\rho}\mathrm{div}\,\mathbf{u}dx - t^{-1}\varepsilon\int_\Omega a''(\rho)|\nabla\rho|^2dx \le$$
$$\int_\Omega |\mathrm{div}\,\mathbf{u}|dx - t^{-1}\varepsilon\int_\Omega a''(\rho)|\nabla\rho|^2dx\ .$$

Substituting this inequality into (6.11) we arrive at

$$t^{-1}\int_\Omega (\nu|\nabla\mathbf{u}|^2 + \xi|\mathrm{div}\,\mathbf{u}|^2)dx + t^{-1}\varepsilon\int_\Omega a''(\rho)|\nabla\rho|^2dx \le \tag{6.13}$$
$$\int_\Omega \frac{\rho}{1+s\rho}\mathbf{f}\cdot\mathbf{u}dx + \int_\Omega |\mathrm{div}\,\mathbf{u}|dx\ .$$

It is easy to see that

$$\varepsilon a''(\rho)|\nabla\rho|^2 = \varepsilon\left(\frac{1}{1+\rho}+\frac{1}{(1+\rho)^2}\right)|\nabla\rho|^2 \geq 2|\nabla v|^2 \ ,$$

which along with (6.13) gives

$$t^{-1}\int_\Omega(\nu|\nabla\mathbf{u}|^2+\xi|\operatorname{div}\mathbf{u}|^2)dx + t^{-1}\int_\Omega|\nabla v|^2dx \leq \int_\Omega\frac{\rho}{1+s\rho}\mathbf{f}\cdot\mathbf{u}dx+\int_\Omega|\operatorname{div}\mathbf{u}|dx \ . \tag{6.14}$$

By the definition of v we have $\|v\|^2_{L_2(\Omega)} = \varepsilon(\mathcal{M}+\text{meas}\ \Omega)$ thus

$$\|v\|^2_{H^{1,2}(\Omega)} = \|\nabla v\|^2_{L_2(\Omega)} + \|v\|^2_{L_2(\Omega)} \leq \|\nabla v\|^2_{L_2(\Omega)} + \varepsilon\mathcal{M} + c \ .$$

On the other hand, $|\operatorname{div}\mathbf{u}| \leq \xi|\operatorname{div}\mathbf{u}|^2+\xi^{-1}$. Substituting these inequalities into (6.14) we get

$$\nu\int_\Omega|\nabla\mathbf{u}|^2dx + \|v\|^2_{H^{1,2}(\Omega)} \leq t\int_\Omega\frac{\rho}{1+s\rho}|\mathbf{f}||\mathbf{u}|dx + \varepsilon\mathcal{M} + c \ .$$

Furthermore, by the Poincaré inequality $\|\mathbf{u}\|_{H^{1,2}(\Omega)} \leq c\|\nabla\mathbf{u}\|_{L_2(\Omega)}$ therefore

$$\|\mathbf{u}\|^2_{H^{1,2}(\Omega)} + \|v\|^2_{H^{1,2}(\Omega)} \leq ct\|\mathbf{f}\|_{C(\Omega)}\int_\Omega\frac{\rho}{1+s\rho}|\mathbf{u}|dx + c(\varepsilon\mathcal{M}+1) \ , \tag{6.15}$$

where c depends only on Ω which yields inequality (6.9).

Our next ask is to obtain the strong estimates depending on s and ε. Since

$$\int_\Omega(1+s\rho)^{-1}\rho|\mathbf{u}|dx \leq c(s)\|\mathbf{u}\|_{H^{1,2}(\Omega)} \ ,$$

inequality (6.15) implies the estimate

$$\|\mathbf{u}\|^2_{H^{1,2}(\Omega)} + \|v\|^2_{H^{1,2}(\Omega)} \leq C_0(\Omega,\mathcal{M},\|\mathbf{f}\|_{C(\Omega)},s,\varepsilon) \ , \tag{6.16}$$

where the constant C_0 is independent of $\mathbf{u}$, v and t. Recall that the embedding $H^{1,2}(\Omega) \hookrightarrow L_m(\Omega)$ is bounded for all $m<\infty$. Hence

$$\|\mathbf{u}\|_{L_m(\Omega)} + \|v\|_{L_m(\Omega)} \leq C_0(\Omega,\mathcal{M},\|\mathbf{f}\|_{C(\Omega)},s,\varepsilon). \tag{6.17}$$

Noting that $|\rho\mathbf{u}| \leq \varepsilon^{-1}v^2|\mathbf{u}|$ we can conclude that

$$\|\rho|\mathbf{u}|\|_{L_m(\Omega)} \leq C_0(\Omega,\mathcal{M},\|\mathbf{f}\|_{C(\Omega)},s,\varepsilon)$$

for each $m < \infty$. From this and a priori estimates for solutions to boundary value problem (6.10b)-(6.10d) we obtain

$$\|\rho\|_{C(\Omega)} \leq c(\Omega)\|\rho\|_{H^{1,m}(\Omega)} \leq C_0(\Omega, \mathcal{M}, \|\mathbf{f}\|_{C(\Omega)}, s, \varepsilon) \quad \text{for all} \quad m \in (2, \infty). \tag{6.18}$$

In order to obtain the same estimate for the velocity field note that (6.17)-(6.18) imply the inequalities

$$\|(t\rho\mathbf{u} - \varepsilon\nabla\rho) \otimes \mathbf{u}\|_{L_m(\Omega)} + \|\nabla\rho\|_{L_m(\Omega)} \leq C_0(\Omega, \mathcal{M}, \|\mathbf{f}\|_{C(\Omega)}, s, \varepsilon) \quad \text{for all} \quad m < \infty.$$

Standard estimates for solutions to elliptic problem (6.10a),(6.10c) now imply

$$\|\mathbf{u}\|_{C(\Omega)} \leq c(\Omega)\|\mathbf{u}\|_{H^{1,m}(\Omega)} \leq C_0(\Omega, \mathcal{M}, \|\mathbf{f}\|_{C(\Omega)}, s, \varepsilon) \quad \text{for all} \quad m \in (2, \infty). \tag{6.19}$$

Equations (6.10) can be rewritten formally as the boundary value problem

$$\nu\Delta\mathbf{u} + \xi\nabla\operatorname{div}\mathbf{u} = tF_0 \quad \text{in} \quad \Omega\ , \quad \mathbf{u} = 0 \quad \text{on} \quad \partial\Omega\ , \tag{6.20a}$$

$$\varepsilon\Delta\rho = tF_1 \quad \text{in} \quad \Omega\ , \quad \nabla\rho\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\Omega\ , \tag{6.20b}$$

$$\int\limits_\Omega \rho(x)dx = \mathcal{M}\ , \tag{6.20c}$$

where

$$F_0 = (t\rho\mathbf{u} - \varepsilon\nabla\rho)\nabla\mathbf{u} + \nabla\rho - (1 + s\rho)^{-1}\rho\mathbf{f},$$
$$F_1 = \nabla\rho\cdot\mathbf{u} + \rho\operatorname{div}\mathbf{u}$$

We recall that, by construction, the mean value of F_1 over Ω is equal to zero. It follows from (6.18)-(6.19) that the right-hand sides of equations (6.20a)-(6.20b) admit the bounds

$$\|F_i\|_{L_m(\Omega)} \leq C_0(\Omega, \mathcal{M}, \|\mathbf{f}\|_{C(\Omega)}, s, \varepsilon) \quad \text{for all} \quad m < \infty\ ,$$

which along with the classical a priori estimates for the solutions to elliptic boundary value problems yields the estimates

$$\|\mathbf{u}\|_{H^{2,m}(\Omega)} + \|\rho\|_{H^{2,m}(\Omega)} \leq C_0(\Omega, \mathcal{M}, \|\mathbf{f}\|_{C(\Omega)}, s, \varepsilon) \quad \text{for all} \quad m < \infty$$

Now choose a number $m > 2$ so that $\alpha < 1 - 2/m$, where $\alpha \in (0, 1)$ is the index from the conditions of Theorem 1.8. Since the embedding operator $H^{2,m}(\Omega) \hookrightarrow C^{1+\alpha}(\Omega)$ is bounded, we finally obtain the a priori estimate for solutions to problem (6.10)

$$\|\mathbf{u}\|_{C^{1+\alpha}(\Omega)} + \|\rho\|_{C^{1+\alpha}(\Omega)} \leq C_2(\alpha, \Omega, \mathcal{M}, \|\mathbf{f}\|_{C(\Omega)}, s, \varepsilon)\ . \tag{6.21}$$

Let us consider the family of the mappings $(\mathbf{u}, \rho) = \Psi(t, \mathbf{u}^*, \rho^*)$ defined as a solution to the boundary value problem

$$\nu\Delta\mathbf{u} + \xi\nabla\text{div}\,\mathbf{u} = t\text{div}\,[(\rho\mathbf{u}^* - \varepsilon\nabla\rho^*)\otimes\mathbf{u}^*] + t\nabla\rho^* - t(1+s\rho^*)^{-1}\rho^*\mathbf{f} \ \text{ in } \ \Omega\,,$$

$$\mathbf{u} = 0 \ \text{ on } \ \partial\Omega\,, \tag{6.22a}$$

$$\varepsilon\Delta\rho = t\text{div}\,(\rho\mathbf{u}^*) \ \text{ in } \ \Omega\,, \tag{6.22b}$$

$$\nabla\rho\cdot\mathbf{n} = 0 \quad \text{ on } \ \partial\Omega\,, \quad \int\limits_\Omega \rho(x)dx = \mathcal{M}\,. \tag{6.22c}$$

Here the existence of the positive solution to problem (6.22b)-(6.22c) is guaranteed by Lemma 1.10. Denote by $\mathcal{G}$ the subset of $C^{1+\alpha}(\Omega)^3$ defined by the inequalities (6.21). Obviously the mapping $\Psi : [0,1]\times\mathcal{G} \mapsto C^{2+\alpha}(\Omega)^3$ is continuous. Therefore the operator $\Psi(t,\cdot) : \mathcal{G} \mapsto C^{1+\alpha}(\Omega)^3$ is continuous and compact. It is easy to see that $\Psi(0,\cdot) \equiv 0$. On the other hand, inequality (6.21) and Lemma 1.10 imply that $\Psi(t,\cdot)$ has no fixed point at the boundary of $\mathcal{G}$ for any $t \in [0,1]$. Applying the Leray-Schauder Theorem we conclude that the equation $(\mathbf{u},\rho) = \Psi(1,\mathbf{u},\rho)$ has a solution $(\mathbf{u},\rho) \in \mathcal{G}\cap C^{2+\alpha}(\Omega)^3$. It remains to note that by (6.15) this solution satisfies inequality (6.9) and the lemma follows. □

The next lemma gives the bounds for solutions to problem (6.7). Introduce the tensor

$$\mathbf{G} = \varepsilon\nabla\rho\otimes\mathbf{u} + \mathbf{\Sigma}, \quad \text{where} \ \ \mathbf{\Sigma} = \nu(\nabla\mathbf{u} + \nabla\mathbf{u}^\top) + (\xi-\nu)\text{div}\,\mathbf{u}\mathbf{I}\,. \tag{6.23}$$

Lemma 6.2. *Under the assumptions of Lemma 6.1,*

$$\|v\|^2_{H^{1,2}(\Omega)} + \|\mathbf{u}\|^2_{H^{1,2}_0(\Omega)} \le C(\Omega, \|\mathbf{f}\|_{C(\Omega)}, \mathcal{M})\,, \tag{6.24}$$

$$\|\mathbf{G}-\mathbf{\Sigma}\|_{L_{3/2}(\Omega)} \le \varepsilon^{1/6}C(\Omega, \|\mathbf{f}\|_{C(\Omega)}, \mathcal{M})\,, \tag{6.25}$$

where $C(\Omega, \|\mathbf{f}\|_{C(\Omega)}, \mathcal{M})$ does not depend on s and ε.

Proof. Choose an arbitrary λ from the interval $\max\{2^{-1}, 2\kappa\} < \lambda < 1$. Noting that

$$\int\limits_\Omega |\mathbf{u}|\rho dx \le (1+\mathcal{D})^\lambda\mathcal{M} + (1+\mathcal{D})^{-\lambda}\int\limits_\Omega |\mathbf{u}|^2\rho dx$$

we can rewrite inequality (6.9) in the form

$$\|v\|^2_{H^{1,2}(\Omega)} + \|\mathbf{u}\|^2_{H^{1,2}_0(\Omega)} \le c\|\mathbf{f}\|_{C(\Omega)}(1+\mathcal{D})^{-\lambda}\int\limits_\Omega |\mathbf{u}|^2\rho dx +$$

$$(1+\|\mathbf{f}\|_{C(\Omega)})c(1+\mathcal{M})(1+\mathcal{D})^\lambda\,. \tag{6.26}$$

Since $\mathbf{u}$ vanishes at the boundary Ω, we have

$$c^{-1}\|\mathbf{u}\|^2_{H^{1,2}_0(\Omega)} \le \mathcal{D} \le c\|\mathbf{u}\|^2_{H^{1,2}_0(\Omega)}\,,$$

which along with (6.26) implies the inequality

$$\|v\|^2_{H^{1,2}(\Omega)} + \|\mathbf{u}\|^2_{H^{1,2}_0(\Omega)} \leq c_0(1+\|\mathbf{u}\|^2_{H^{1,2}(\Omega)})^{-\lambda} \int\limits_\Omega |\mathbf{u}|^2\rho dx + c_1(1+\|\mathbf{u}\|^2_{H^{1,2}(\Omega)})^{\lambda} \ . \quad (6.27)$$

Here, and below, the constants c_i, $i = 0,..,5$, depend on $\|\mathbf{f}\|_{C(\Omega)}$, Ω and $\mathcal{M}$ and are independent of s and ε. Furthermore, the functions $\mathbf{u}$, ρ satisfy the equation

$$\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u}) = \operatorname{div}(\mathbf{G}) + (1+s\rho)^{-1}\rho\mathbf{f} \ ,$$

which meets all requirements of Theorem 4.1. Noting that in this case $d\mu_\rho = \rho dx$, $d\mu_e = (|\mathbf{u}|^2+2)\rho dx$ and applying the inequality (4.4) from this theorem we get the estimate

$$\int\limits_\Omega |\mathbf{u}|^2\rho dx \leq c\Big(\|\mathbf{G}\|_{L_{3/2}(\Omega)} + \mathcal{M}(1+\|\mathbf{u}\|^2_{H^{1,2}(\Omega)})^{1+2\kappa}\Big) \ ,$$

where c depends on Ω and $\|\mathbf{f}\|_{C(\Omega)}$ only. Substituting the inequality into the right-hand side of (6.27) we have

$$\|v\|^2_{H^{1,2}(\Omega)} + \|\mathbf{u}\|^2_{H^{1,2}_0(\Omega)} \leq c_2(1+\|\mathbf{u}\|^2_{H^{1,2}(\Omega)})^{-\lambda}\|\mathbf{G}\|_{L_{3/2}(\Omega)} + c_3(1+\|\mathbf{u}\|^2_{H^{1,2}(\Omega)})^{\beta} \ ,$$

where $\beta = \max\{1+2\kappa-\lambda, \lambda\} \in (0,1)$. Next note that

$$\|\mathbf{G}\|_{L_{3/2}(\Omega)} \leq \|\mathbf{G}-\mathbf{\Sigma}\|_{L_{3/2}(\Omega)} + \|\mathbf{\Sigma}\|_{L_{3/2}(\Omega)} \leq \|\mathbf{G}-\mathbf{\Sigma}\|_{L_{3/2}(\Omega)} + c\|\mathbf{u}\|_{H^{1,2}(\Omega)} \ ,$$

which along with the previous inequality yields

$$\|v\|^2_{H^{1,2}(\Omega)} + \|\mathbf{u}\|^2_{H^{1,2}_0(\Omega)} \leq c_2(1+\|\mathbf{u}\|^2_{H^{1,2}(\Omega)})^{-\lambda}\|\mathbf{G}-\mathbf{\Sigma}\|_{L_{3/2}(\Omega)} + c_4(1+\|\mathbf{u}\|^2_{H^{1,2}(\Omega)})^{\beta} \ . \quad (6.28)$$

It remains to estimate $\|\mathbf{G}-\mathbf{\Sigma}\|_{L_{3/2}(\Omega)}$. It is easy to see that

$$\mathbf{G}-\mathbf{\Sigma} = \varepsilon\nabla\rho\otimes\mathbf{u} = 2v\nabla v\otimes\mathbf{u} \ .$$

Applying the Hölder inequality combined with $\|\mathbf{u}\|_{L_6(\Omega)} \leq c(\Omega)\|\mathbf{u}\|_{H^{1,2}(\Omega)}$, we obtain

$$\|\mathbf{G}-\mathbf{\Sigma}\|_{L_{3/2}(\Omega)} \leq c(\Omega)\|\nabla v\|_{L_2(\Omega)}\|v\|_{L_6(\Omega)}\|\mathbf{u}\|_{L_6(\Omega)} \leq$$
$$c(\Omega)\|\mathbf{u}\|_{H^{1,2}(\Omega)}\|v\|_{H^{1,2}(\Omega)}\|v\|_{L_6(\Omega)} \ .$$

The interpolation inequality and the identity $v^2 = \varepsilon(1+\rho)$ imply

$$\|v\|_{L_6(\Omega)} \leq c(\Omega)\|v\|^{2/3}_{H^{1,2}(\Omega)}\|v\|^{1/3}_{L_2(\Omega)} \leq c(\Omega)\|v\|^{2/3}_{H^{1,2}(\Omega)}\varepsilon^{1/6}(1+\mathcal{M})^{1/6} \ .$$

Substituting this estimate into the right-hand side of the previous inequality we obtain

$$\|\mathbf{G}-\mathbf{\Sigma}\|_{L_{3/2}(\Omega)} \leq c_5\varepsilon^{1/6}\|v\|^{5/3}_{H^{1,2}(\Omega)}\|\mathbf{u}\|_{H^{1,2}(\Omega)} \ . \quad (6.29)$$

By the choice of λ we have

$$(1 + \|\mathbf{u}\|^2_{H^{1,2}(\Omega)})^{-\lambda}\|\mathbf{u}\|_{H^{1,2}(\Omega)} \le 1$$

so by substitution of (6.29) into the right-hand side of (6.28) we get the inequality

$$\|v\|^2_{H^{1,2}(\Omega)} + \|\mathbf{u}\|^2_{H^{1,2}_0(\Omega)} \le c_2 c_5 \varepsilon^{1/6}\|v\|^{5/3}_{H^{1,2}(\Omega)} + c_4(1 + \|\mathbf{u}\|^2_{H^{1,2}(\Omega)})^{\beta} \quad , \tag{6.30}$$

which implies the required estimate (6.24). It remains to note that estimate (6.25) is a consequence of (6.24) and (6.28) and the proof is completed. □

Now, we are in position to complete the proof of Theorem 1.8. Choose the sequence of the functions $\mathbf{f}_s \in C^{1+\alpha}(\Omega)^2$ which converges in $C(\Omega)^2$ to $\mathbf{f}$ as $s \searrow 0$ and consider problem (6.7) with $\mathbf{f}$ replaced by $\mathbf{f}_s$. By Lemma 6.1 this problem admits a solution $(\mathbf{u}_{\varepsilon s}, \rho_{\varepsilon s}) \in C^{2+\alpha}(\Omega)^3$ satisfying inequalities (6.25). Multiplying both the sides of equations (6.7a),(6.7b) by arbitrary functions $\varphi \in C^\infty_0(\Omega)^2$, $\psi \in C^\infty_0(\Omega)$, respectively, and integrating the result over Ω we get the integral identities

$$\int_\Omega \Big((\mathbf{u}_{\varepsilon s} \otimes \mathbf{u}_{\varepsilon s} : \nabla\varphi + \operatorname{div}\varphi\Big)\rho_{\varepsilon s} dx = \int_\Omega \mathbf{G}_{\varepsilon s} : \nabla\varphi dx + \int_\Omega (1 + \rho_{\varepsilon s})^{-1}\mathbf{f}_s \cdot \varphi dx \tag{6.31}$$

$$\int_\Omega \mathbf{u}_{\varepsilon s} \cdot \nabla\psi \rho_{\varepsilon s} dx + \varepsilon \int_\Omega \rho_{\varepsilon s}\Delta\psi dx = 0 \ . \tag{6.32}$$

The identity $\nabla\rho = 2\varepsilon^{-1}v\nabla v$ along with the estimate (6.24) imply that for fixed $\varepsilon > 0$ the sequence $(\mathbf{u}_{\varepsilon s}, \rho_{\varepsilon s})$, $0 < s < 1$, is bounded in $H^{1,2}_0(\Omega)^2 \times H^{1,m}(\Omega)$ for any $m < 2$. Therefore after passing to a subsequence we can assume that there are $(\mathbf{u}_\varepsilon, \rho_\varepsilon) \in H^{1,2}_0(\Omega)^2 \times H^{1,m}(\Omega)$ such that

$$\rho_{\varepsilon s} \to \rho_\varepsilon, \quad \mathbf{u}_{\varepsilon s} \to \mathbf{u}_\varepsilon \text{ in any } L_r(\Omega), \quad L_r(\Omega)^2, \text{ respectively}$$
$$\nabla\rho_{\varepsilon s} \rightharpoonup \nabla\rho_\varepsilon \text{ weakly in } L_m(\Omega)^2, \quad \nabla\mathbf{u}_{\varepsilon s} \rightharpoonup \nabla\mathbf{u}_\varepsilon \text{ weakly in } L_2(\Omega)^4$$

with $s \searrow 0$. Hence, passing to the limits in (6.31),(6.32), we obtain

$$\int_\Omega \Big((\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla\varphi + \operatorname{div}\varphi\Big)\rho_\varepsilon dx = \int_\Omega \mathbf{G}_\varepsilon : \nabla\varphi dx + \int_\Omega \mathbf{f} \cdot \varphi dx \tag{6.33}$$

$$\int_\Omega \mathbf{u}_\varepsilon \cdot \nabla\psi \rho_\varepsilon dx + \varepsilon \int_\Omega \rho_\varepsilon \Delta\psi dx = 0 \ . \tag{6.34}$$

Lemma 6.2 implies that the sequence $\mathbf{u}_\varepsilon$ is bounded in $H^{1,2}_0(\Omega)^2$. Introduce the Borel measures $\mu_{\rho\varepsilon}$ defined by $d\mu_{\rho\varepsilon} = \rho_\varepsilon dx$. By construction the total variation of $\mu_{\rho\varepsilon}$ equals to $\mathcal{M}$. Hence, after extraction of a subsequence if necessary, we can assume that there are $\mathbf{u} \in H^{1,2}_0(\Omega)^2$ and the Borel measure μ_ρ in Ω such that

$$\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} \text{ weakly in } H^{1,2}_0(\Omega)^2 \text{ and } \mu_{\rho\varepsilon} \rightharpoonup \mu_\rho \text{ *-weakly in } C^*_0(\Omega) \ .$$

Recall that

$$\mathbf{G}_\varepsilon = \varepsilon\nabla\rho_\varepsilon \otimes \mathbf{u}_\varepsilon + \mathbf{\Sigma}_\varepsilon, \quad \mathbf{\Sigma}_\varepsilon = \nu(\nabla\mathbf{u}_\varepsilon + \nabla\mathbf{u}_\varepsilon^\top) + (\xi - \nu)\mathrm{div}\,\mathbf{u}_\varepsilon\mathbf{I}\ .$$

It follows from Lemma 2.27 that the sequence $\mathbf{G}_\varepsilon$ is bounded in $L_{3/2}(\Omega)^4$ and converges to $\mathbf{\Sigma}$.

Obviously the sequences $(\mathbf{u}_\varepsilon, \mu_{\rho\varepsilon})$ and $\mathbf{G}_\varepsilon$ satisfy all conditions of Theorem 4.2. Applying this theorem and passing to the limit in (6.33)-(6.34) we conclude that the pair which consists of the vector field $\mathbf{u}$ and the measure μ_ρ is a generalized solution to problem (1.1)-(1.3) and the theorem follows. □

A Appendix

Proof of equality (1.5)**.** Suppose that a smooth vector field $\mathbf{u}$ and a function ρ satisfies equations (1.1)-(1.2). Since $\mathbf{u}$ vanishes at the boundary of the obstacle, we have

$$J(\rho, \mathbf{u}, \Omega) = -\mathbf{U}^\infty \cdot \int\limits_{\partial S} (\mathbf{\Sigma} - \rho\mathbf{u}\otimes\mathbf{u} - \rho\mathbf{I})\cdot\mathbf{n}dS\ .$$

Using the Green formula we get the equality

$$J(\rho, \mathbf{u}, \Omega) = \mathbf{U}^\infty \cdot \int\limits_{\partial B} (\mathbf{\Sigma} - \rho\mathbf{u}\otimes\mathbf{u} - \rho\mathbf{I})\cdot\mathbf{n}dS - \mathbf{U}^\infty \cdot \int\limits_{\Omega} \mathrm{div}\,(\mathbf{\Sigma} - \rho\mathbf{u}\otimes\mathbf{u} - \rho\mathbf{I})dx\ . \quad \text{(A.1)}$$

Since $\mathbf{u} = \mathbf{u}^\infty = \mathbf{U}^\infty$ on ∂B we have,

$$\mathbf{U}^\infty \cdot \int\limits_{\partial B} (\mathbf{\Sigma} - \rho\mathbf{u}\otimes\mathbf{u} - \rho\mathbf{I})\cdot\mathbf{n}dS = \int\limits_{\partial B} (\mathbf{\Sigma} - \rho\mathbf{u}\otimes\mathbf{u} - \rho\mathbf{I}) : \mathbf{u}^\infty\otimes\mathbf{n}dS\ .$$

Applying the Green formula to the left side of this equality, taking into account that $\mathbf{u}^\infty = 0$ on the boundary of S we obtain

$$\mathbf{U}^\infty \cdot \int\limits_{\partial B} (\mathbf{\Sigma} - \rho\mathbf{u}\otimes\mathbf{u} - \rho\mathbf{I})\cdot\mathbf{n}dS =$$
$$\int\limits_{\Omega} \mathrm{div}\,(\mathbf{\Sigma} - \rho\mathbf{u}\otimes\mathbf{u} - \rho\mathbf{I})\cdot\mathbf{u}^\infty dx + \int\limits_{\Omega} (\mathbf{\Sigma} - \rho\mathbf{u}\otimes\mathbf{u} - \rho\mathbf{I}) : \nabla\mathbf{u}^\infty dx\ .$$

Substituting this equality into the right-hand side of (A.1) leads to

$$J(\rho, \mathbf{u}, \Omega) = \int\limits_{\Omega} \mathrm{div}\,(\mathbf{\Sigma} - \rho\mathbf{u}\otimes\mathbf{u} - \rho\mathbf{I})\cdot(\mathbf{u}^\infty - \mathbf{U}^\infty)dx + \int\limits_{\Omega} (\mathbf{\Sigma} - \rho\mathbf{u}\otimes\mathbf{u} - \rho\mathbf{I}) : \nabla\mathbf{u}^\infty dx\ .$$

It remains to note that

$$\operatorname{div}(\mathbf{\Sigma} - \rho\mathbf{u}\otimes\mathbf{u} - \rho\mathbf{I}) = -\rho\mathbf{f}$$

and (1.5) follows.

Proof of Lemma 1.4 We start with the proof of assertion (i). Choose an arbitrary sequence $\Omega_n \in \mathfrak{S}(T, C_\Omega)$. Then there exists a subsequence, still denoted by Ω_n, such that the sequence of obstacles S_n converges in the Hausdorff metric to some compact $S \Subset B$. Since the sequence Ω_n uniformly satisfies the cone condition from Definition 1.3, the limit $\Omega = B \setminus S$ satisfies the cone condition with the same constants T, C_Ω. Let us prove that $\Omega \subset \mathfrak{S}(T, C_\Omega)$. It is sufficient to show that the distance function satisfies inequality (1.19). Choose an arbitrary disk $B(a, r) \subset \Omega$ such that

$$0 < \text{ dist } (\partial\Omega, B(a,r)) < T \ .$$

It is clear that for all sufficiently large n the disk $B(a,r)$ is included strictly inside of the sets $\Omega_n(T)$ and hence the second order derivatives of the distance functions $d_n(x)$ are uniformly bounded on the disk. Since the sequence $d_n \to d$ converges uniformly on $B(a,r)$, the sequence of the second order derivatives $D^2 d_n$ converges to $D^2 d$ weakly in any space $L_p(B(a,r))$ with $p \in [1, \infty)$, which provides the lower bound in (1.19). Since the determinant $|D^2 d_n|$ is a convex function of the matrix $D^2 d_n$ we obtain

$$d(x)|D^2 d(x)| \le w - \lim_{n\to\infty} d_n(x)|D^2 d_n(x)| \le C_\Omega$$

which implies upper bound in (1.19). Therefore, the limit $\Omega \in \mathfrak{S}(T, C_\Omega)$ and the proof of assertion (i) is completed.

Let us prove that if $S \subset B_0$ is a convex set with the nonempty interior, then $\Omega = B \setminus S$ belongs to some class $\mathfrak{S}(T, C_\Omega)$. Here B_0 is the compact set from Definition 1.17. Note that for any convex set S such that $B(x_0, r) \subset S \subset B_0$ there exist a sequence of convex C^2 domains S_n, $B(x_0, r) \subset S_n \subset B_0$ which converges to S in the Hausdorff metric. Therefore, it is sufficient to prove that for any C^2 convex set S, $B(x_0, r) \subset S \subset B_0$, the domain $B \setminus S$ belongs to a class $\mathfrak{S}(T, C_\Omega)$ with the constants T, C_Ω depending only on B_0 and r.

Note that there exist $T > 0$ and $C_\Omega > 0$, depending only on r and the diameter of S, such that S satisfies the cone condition β from Definition 1.3. Without loss of the generality we can assume that $T < 2^{-1}\text{dist }(S, \partial B)$. Next, for any point $x \in \Omega$ with dist $(x, S) \le T$ there exists the unique $y \in \partial S$ such that $|x - y| = d(x)$. Moreover, $x = y + d(x)\nu(y)$ where $\nu(y)$ is the unit normal vector to ∂S at the point y directed into Ω. It is known, [8], that

$$\text{tr } D^2 d(x) = \Delta d(x) = k(y)(1 + d(x)k(y))^{-1} \ . \tag{A.2}$$

Here $k(y)$ is the curvature of the boundary ∂S at the point y with the sign chosen in such a way that the curvature k is positive if S is a ball. Since S is convex, we have $k(y) \ge 0$ and $D^2 d(x) \ge 0$. From this property and (A.2) we can conclude that

$$\frac{1}{d}\mathbf{I} \ge \text{tr } D^2 d(x)\mathbf{I} \ge D^2(x) \ge 0 \ .$$

Hence for dist $(x, S) \leq T$ the function $d(x)$ satisfies inequalities (1.19) with $C_\Omega = 1$, which completes the proof.

Let us consider the case of the boundary ∂S in the form of a C^2 curvilinear polygon, and with the values of interior angles in the interval $(0, \pi)$. Without loss of the generality we can assume that the boundary is given by

$$\partial S = \{y \in B : \quad y = Y(s),\ s \in [0, L],\ Y(0) = Y(L)\}$$

with the vertexes

$$Y_k = Y(s_k), \quad s_0 = 0 < s_1 < ... < s_n < s_{n+1} = L \ . \tag{A.3}$$

We identify Y_0 with Y_{n+1}. By the hypothesis, the function $Y(s)$ belongs to the class $C^2(s_k, s_{k+1})$ for $k \in [0, n]$, and $|Y'(s)| = 1$ for $s \neq s_k$. For all points $y = Y(s) \neq Y_k$ we denote by $\nu(y) = Y'(s)^\perp$ the unit normal vector directed into Ω. Set $\nu_k^\pm = \lim_{s \to s_k \pm 0} \nu(Y(s))$. Now, we choose an arbitrary positive T_0 such that

$$T_0 \leq 2^{-1} \text{dist } (S, \partial B) \quad \text{and} \quad T_0 < \inf_{\partial S \setminus \cup \{Y_k\}} R(y) \ ,$$

where $R(y) = |\nu(Y(s))'|^{-1}$ is the inverse of the absolute value of the curvature of ∂S at the point $y = Y(s)$.

For each integer $k \in [0, n]$ and $T < T_0$ we introduce the mapping $X_k : (s, t) \mapsto x$ which is defined by

$$\begin{aligned} &x = Y(s) + t\nu(Y(s)) \quad \text{for} \quad (s, t) \in (s_k, s_{k+1}) \times (0, T], \\ &x = Y_k + t\nu_k^+ \quad \text{for} \quad s = s_k \quad \text{and} \quad t \in (0, T] \\ &x = Y_{k+1} + t\nu_{k+1}^- \quad \text{for} \quad s = s_{k+1} \quad \text{and} \quad t \in (0, T] \ . \end{aligned} \tag{A.4}$$

Since the arc $\overbrace{Y_k, Y_{k+1}}$ belongs to the class C^2, this mapping transforms diffeomorphically the rectangle $[s_k, s_{k+1}] \times (0, T]$ onto curvilinear rectangular domain $\Omega_k(T)$. It is easy to see that if $(s, t) = X_k^{-1}(x)$, then $t = \text{dist } (x, \overbrace{Y_k, Y_{k+1}})$. Moreover, since the interior angles of the polygon S are smaller than π, there is a positive $T_1 < T_0$ such that $\Omega_i(T_1) \cap \Omega_j(T_1) \setminus S = \emptyset$ for $i \neq j$ and $t = \text{dist } (x, S) = d(x)$ and for all $x \in \Omega_k(T_1)$. Therefore, $d \in C^2(\Omega_k(T_1))$.

By construction, the set $O_S(T_1) = \{x \in \Omega : \quad \text{dist } (x, \partial S) \leq T_1\}$ has the representation

$$O_S(T_1) = \bigcup_{k=0}^{n} \big(\Omega_k(T_1) \cup \sigma_k(T_1)\big) \ ,$$

where $\sigma_k(T_1)$ is the sector bounded by the segments $x = Y_k + t\nu_k^\pm$, $t \in (0, T_1]$ and the arc $|x - Y_k| = T_1$. It is obvious that $d(x) = |x - Y_k|$ in $\sigma_k(T_1)$. Hence the function $d \in C^1(O_s(T_1))$ has the bounded second order derivatives in $\Omega_k(T_1)$ and satisfies the inequalities

$$\frac{1}{d(x)} \mathbf{I} \geq D^2 d(x) \geq 0 \quad \text{in} \quad \sigma_k(T_1) \ . \tag{A.5}$$

Therefore, we can conclude that d satisfies inequalities (1.19) and the proof is completed. □

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