The Number of Cylindrical Shells

Olivier Devillers

published in Discrete and Computational Geometry 20, 2003, p.453–458.

On the number of cylindrical shells

Olivier Devillers*

Abstract

Given a set P of n points in three dimensions, a cylindrical shell (or zone cylinder) is formed by two circular cylinders with the same axis such that all points of P are between the two cylinders. We prove that the number of cylindrical shells enclosing P passing through combinatorially different subsets of P has size $\Omega(n^3)$ and $O(n^4)$ (the previously known bound was $O(n^5)$). As a consequence, the minimum enclosing shell can be found in $O(n^4)$ time.

1 Introduction

Consider a set P of n points in three dimensions. A *cylindrical shell* is formed by the space between two coaxial circular cylinders in three dimensions. The shell is said to enclose P if the set P is between the two cylinders. The difference between the radii of the two cylinders is the *width* of the shell.

Finding the minimum-width shell enclosing P is an important metrology problem. To control the quality of cylindrical objects (e.g. mechanical pieces, pistons, etc) points are probed on the boundary of the cylinder and the width of the minimum-width enclosing shell is a recognized measure of the quality of the piece [1]. Computing the minimum-width enclosing shell has already received attention in the literature. In the special case of small width, Devillers and Preparata compute a provably good approximation of the minimum-width enclosing shell by reducing the problem to linear programming [5]. Agarwal, Aronov and Sharir compute a solution which approximates the optimum up to a constant factor in quadratic time [2]. Har-Peled and Varadarajan propose to compute an $(1+\epsilon)$-approximation in time linear in n and exponential in $\frac{1}{\epsilon}$ [6].

Few exact algorithms have been developed for computing the minimum-width enclosing shell. In general position, six points define a constant number of shells having these six points on their boundary (less than 150) [4], thus a naive algorithm will have $O(n^7)$ complexity. Agarwal, Aronov and Sharir [2]

*INRIA, BP93, 06902 Sophia-Antipolis, France, Olivier.Devillers@sophia.inria.fr

have proposed a reduction to convex hull in 10 dimensions and deduced an algorithm of complexity $O(n^5)$.

In this paper, we prove new upper and lower bounds for the number of combinatorially different enclosing cylindrical shells for a set of n points in three-dimensional space. Using the new upper bound, the complexity for the exact algorithm of Agarwal, Aronov and Sharir computing the smallest-width enclosing shell can be reduced to $O(n^4)$.

2 Lower bound example

Theorem 1 *Given n points in 3 dimensions, the number of combinatorially different enclosing shells is $\Omega(n^3)$.*

Proof: We construct below a set of $3n$ points with more than n^3 combinatorially different enclosing shells. We consider three sets of n points:

$$\begin{aligned} p_i &= (\frac{i}{n}, 0, 0),\ 1 \leq i \leq n \\ q_i &= (\frac{i}{n}, 0, \zeta),\ 1 \leq i \leq n \\ r_i &= (x_i, \frac{i}{n}, 0),\ 1 \leq i \leq n \end{aligned}$$

such that the r_i belong to the circle $\mathcal{C}$ in plane $z = 0$ with center $c = (3, \frac{1}{2}, 0)$ and radius 6, and ζ is some positive parameter to be chosen later (see Figure 1).

Then for any triple (i, j, k) we construct a shell such that the internal cylinder is tangent to line $y = z = 0$ at p_i and to line $y = \zeta - z = 0$ at q_j and such that the external cylinder contains and is tangent to circle $\mathcal{C}$ at r_k.

To guarantee the tangencies at points p_i and q_j, it is enough to have a cylinder with axis parallel to $p_i q_j$ that passer through a point of the line l_i through p_i and parallel to the y axis. We choose that point to be the intersection point γ_{ik} of the line through c and r_k and the line l_i. Then the cylinder with axis α_{ijk}, passing through p_i, also passes through q_j and is tangent to the plane Oxz, so that points $p_l, l \neq i$, and $q_l, l \neq j$, are outside the cylinder.

We consider now the cylinder with the same axis α_{ijk} passing through r_k. First we remark that, in the horizontal plane, the circle of center γ_{ik} passing through r_k is tangent externally to $\mathcal{C}$ since its radius is bigger than the radius of $\mathcal{C}$. Since the cylinder is not vertical, its intersection with the horizontal plane is not a circle but an ellipse, although, by choosing ζ large enough it is possible to make this ellipse close enough to a circle and to preserve the property that r_k is on the cylinder and the $r_l, l \neq k$ are inside the cylinder. ■

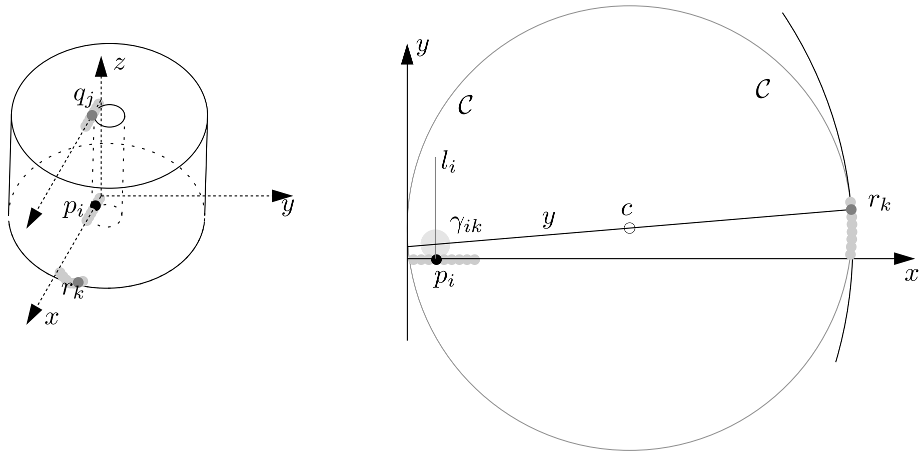

Figure 1: $\Omega(n^3)$ lower bound example.

3 Projection theorem

We will prove in this section that the convex hull of certain special configurations of points in d dimensions cannot reach the worst-case complexity of $O(n^{\lfloor \frac{d}{2} \rfloor})$. These special configurations are point sets in a d-dimensional space of size $2n$. They are constructed by taking n points in a hyperplane and adding to them their parallel projection on another hyperplane. In such configurations, the convex hull in d dimensions is the extrusion of the convex hull in one of the hyperplanes and has complexity $O(n^{\lfloor \frac{d-1}{2} \rfloor})$. More precisely, we have:

Theorem 2 *Let π, π' be two hyperplanes in $\mathbb{R}^d$ and $\vec{v}$ be a vector of $\mathbb{R}^d$ which is not parallel to either π or π'. Let $\mathcal{U} \subset \pi$ be a finite set of n points all lying on the same side of π' and $\mathcal{U}' \subset \pi'$ be the projection of $\mathcal{U}$ on π' along direction $\vec{v}$. Let $\mathcal{V} = \mathcal{U} \cup \mathcal{U}'$. The convex hull $CH(\mathcal{V})$ of $\mathcal{V}$ has the same asymptotic complexity as $CH(\mathcal{U})$, that is $O(n^{\lfloor \frac{d-1}{2} \rfloor})$.*

Proof: The space is divided in four quadrants by the hyperplanes π and π' (see Figure 2). Three of these quadrants are obviously separated from $\mathcal{V}$ by π or π' and thus cannot intersect $CH(\mathcal{V})$. Consider a face F of $CH(\mathcal{V})$ in the fourth quadrant; we first claim that $\vec{v}$ is parallel to F. We denote by H a hyperplane supporting F and by n_H the normal to H. Let x be a vertex of F in π and x' its projection on π'. Similarly, let y' a vertex of F in π' and y its projection on π. Since H is a supporting hyperplane we must have x' and y on the same side of H, that is, $\text{sign}(\vec{xx'} \cdot n_H) = \text{sign}(\vec{y'y} \cdot n_H)$, but since $\vec{xx'}$ and $\vec{y'y}$ are collinear to $\vec{v}$ and in opposite directions, we get that $\vec{v}$ is parallel to H.

In fact we have that the faces of $CH(\mathcal{V})$ are either faces of $CH(\mathcal{U})$, faces of $CH(\mathcal{U}')$ or faces linking a face of $CH(\mathcal{U})$ to its projection in $CH(\mathcal{U}')$. Thus the total number of faces of $CH(\mathcal{V})$ is three times the number of faces of $CH(\mathcal{U})$. ■

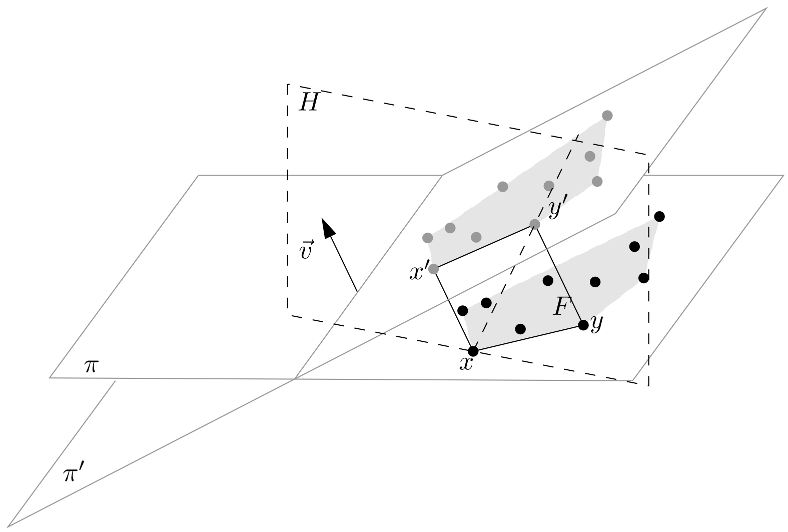

Figure 2: For the proof of Theorem 2.

Another proof of this result can be obtained using a projective transformation of the space θ sending the plane generated by $\pi \cap \pi'$ and $\vec{v}$ to infinity. Such a transformation does not change the combinatorial structure of a convex hull while at the same time making the two hyperplanes parallel and leaving $\mathcal{U}'$ a projection of $\mathcal{U}$.

Unfortunately, Theorem 2 does not generalize (as stated) to the case where π and π' are k-flats with $k \leq d-2$. For example if $\mathcal{U} = \{p_i = (i, i^2, 0, 0); 1 \leq i \leq n\}$ and $\mathcal{U}' = \{p'_j = (0, 0, j, j^2); 1 \leq j \leq n\}$ then the $O(n^2)$ simplices $(p_i, p_{i+1}, p'_j, p'_{j+1})$ are convex hull faces.

4 Minimum-width cylindrical shell algorithm

Given a set $\mathcal{S}$ of n three-dimensional points Agarwal, Aronov and Sharir [2, page 511] exhibit a transformation mapping a 3D point $p = (x, y, z)$ in two half-spaces $P^\star$ and $P^\dagger$ in 10 dimensions :

$$\begin{aligned} P^\star : \varphi_9 \quad &\leq \quad (x^2+y^2) + 2x\varphi_1 + 2y\varphi_2 + 2z\varphi_3 - 2xy\varphi_4 - 2xz\varphi_5 - 2yz\varphi_6 \\ &\qquad + (y^2+z^2)\varphi_7 + (x^2+z^2)\varphi_8 \\ P^\dagger : \varphi_{10} \quad &\geq \quad (x^2+y^2) + 2x\varphi_1 + 2y\varphi_2 + 2z\varphi_3 - 2xy\varphi_4 - 2xz\varphi_5 - 2yz\varphi_6 \\ &\qquad + (y^2+z^2)\varphi_7 + (x^2+z^2)\varphi_8 \end{aligned}$$

where $\varphi_1, \ldots \varphi_{10}$ are the coordinates in 10 dimensions. They prove that the complexity of the intersection of the $2n$ half-spaces $P^\star$ and $P^\dagger$ is a bound on the number of combinatorially distinct enclosing shells. Moreover, computing this intersection can be done in time proportional to its worst-case size [3], thus enabling us to enumerate all possible shells and thereby find the minimum-width one. Since the complexity of the intersection of $2n$ half-spaces in 10 dimensions is $O(n^5)$, the same complexity is obtained for the minimum-width enclosing shell algorithm.

The problem of intersecting half-spaces can be transformed to a convex hull problem through the well-known duality between points and hyperplanes, provided that a point inside the intersection of half-spaces is known. Here we can remark that for some α large enough, the point $(0,0,0,0,0,0,0,0,0,\alpha)$ is certainly strictly inside the half-spaces $P^\star$ and $P^{\dagger}$[1]. By a translation by that vector, the hyperplane

$$0 = \psi_0 + \sum_{i=1}^{10} \psi_i \varphi_i$$

is transformed to the hyperplane

$$0 \quad = \quad (\psi_0 + \psi_{10}\alpha) + \sum_{i=1}^{10} \psi_i \phi_i$$

where ϕ_i are the coordinates in the new frame. This equation can be normalized by dividing it by ψ_8 and by duality we get the point

$$(\frac{\psi_0 + \psi_{10}\alpha}{\psi_8}, \frac{\psi_1}{\psi_8}, \frac{\psi_2}{\psi_8}, \frac{\psi_3}{\psi_8}, \frac{\psi_4}{\psi_8}, \frac{\psi_5}{\psi_8}, \frac{\psi_6}{\psi_8}, \frac{\psi_7}{\psi_8}, \frac{\psi_9}{\psi_8}, \frac{\psi_{10}}{\psi_8}).$$

Through this process, a three-dimensional point $p = (x, y, z)$ gives two half-spaces $P^\star$ and $P^\dagger$ in 10 dimensions and then by duality two points $p^\star$ and $p^\dagger$ in 10 dimensions:

$$\begin{aligned}
p^\star \quad &= \quad \Big(\frac{x^2+y^2}{x^2+z^2}, \frac{2x}{x^2+z^2}, \frac{2y}{x^2+z^2}, \frac{2z}{x^2+z^2}, \\
&\qquad \frac{-2xy}{x^2+z^2}, \frac{-2xz}{x^2+z^2}, \frac{-2yz}{x^2+z^2}, \frac{y^2+z^2}{x^2+z^2}, \frac{1}{x^2+z^2}, 0\Big) \\
p^\dagger \quad &= \quad \Big(\frac{x^2+y^2}{x^2+z^2} + \frac{\alpha}{x^2+z^2}, \frac{2x}{x^2+z^2}, \frac{2y}{x^2+z^2}, \frac{2z}{x^2+z^2}, \\
&\qquad \frac{-2xy}{x^2+z^2}, \frac{-2xz}{x^2+z^2}, \frac{-2yz}{x^2+z^2}, \frac{y^2+z^2}{x^2+z^2}, 0, \frac{1}{x^2+z^2}\Big) \\
&= \quad p^\star + \frac{1}{x^2+z^2}\,(\alpha, 0, 0, 0, 0, 0, 0, 0, -1, 1)\,.
\end{aligned}$$

We remark that $p^\star$ belongs to the half-hyperplane $\pi^+ : \phi_{10} = 0, \phi_9 \geq 0$ and that $p^\dagger$ is the projection of $p^\star$ on $\pi' : \phi_9 = 0$ along direction $\vec{v} =$

[1] This the case if the origin is chosen outside $\mathcal{S}$ and if α is larger than the square of the distance between the origin and its farthest neighbor in $\mathcal{S}$.

$(\alpha, 0, 0, 0, 0, 0, 0, 0, -1, 1)$. The hypotheses for Theorem 2 are satisfied and we conclude that the complexity of the convex hull of the set of $2n$ points formed by the $p^\star$ and $p^\dagger$ ($p \in \mathcal{S}$) can be reduced to $O(n^4)$. The convex hull can be actually computed in that time by computing the hull of points $p^\star$ [3] and reconstructing the facets linking π and π' from that hull. We can thus state the following theorem, improving the previous result by a factor of n :

Theorem 3 *Given a set $\mathcal{S}$ of n points in $\mathbb{R}^3$, a minimum-width cylindrical shell containing $\mathcal{S}$ can be computed in $O(n^4)$ time.*

5 Open problems

Our lower bound of $\Omega(n^3)$ applies to combinatorially different cylindrical shells enclosing a set of n points without consideration of width. We may ask, if the bound applies also to the number of local or global minima of the width function on the set of enclosing shells.

Another obvious problem is to reduce the gap between $\Omega(n^3)$ and $O(n^4)$.

Acknowledgments

Thanks to Menelaos Karavelas and Hazel Everett for their help in preparing this note.

References

[1] ISO/DIS 12180. Geometrical product specifications (gps) – cylindricity, 2000.

[2] Pankaj K. Agarwal, Boris Aronov, and Micha Sharir. Exact and approximation algorithms for minimum-width cylindrical shells. In *Proc. 11th ACM-SIAM Sympos. Discrete Algorithms*, pages 510–517, 2000.

[3] Bernard Chazelle. An optimal convex hull algorithm in any fixed dimension. *Discrete Comput. Geom.*, 10:377–409, 1993.

[4] Olivier Devillers, Bernard Mourrain, Franco P. Preparata, and Philippe Trebuchet. On circular cylinders by four or five points in space. Rapport de recherche 4195, INRIA, 2001.

[5] Olivier Devillers and Franco P. Preparata. Evaluating the cylindricity of a nominally cylindrical point set. In *Proc. 11th ACM-SIAM Sympos. Discrete Algorithms*, pages 518–527, 2000.

[6] Sariel Har-Peled and Kasturi R. Varadarajan. Approximate shape fitting via linearization. In *Proc. 42th Annu. IEEE Sympos. Found. Comput. Sci.*, 2001.