Jordan algebras, geometry of Hermitian symmetric spaces and non-commutative Hardy spaces

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These notes were written following lectures I had the pleasure of giving on this subject at Keio University, during November and December 2004.

The first part is about new applications of Jordan algebras to the geometry of Hermitian symmetric spaces and to causal semi-simple symmetric spaces of Cayley type.

The second part will present new contributions for studying (non commutative) Hardy spaces of holomorphic functions on Lie semigroups which is a part of the so called *Gelfand-Gindikin program*.

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Contents

Let G/K be a Riemannian symmetric space of non compact type. Suppose that G/K is an irreducible Hermitian symmetric space, *i.e.* the center $\mathfrak{z}(\mathfrak{k})$ of $\mathfrak{k}$ (the Lie algebra of K) is non trivial.

There are two essentially equivalent ways to treat Hermitian symmetric spaces.

The first one uses the theory of semi-simple Lie groups; it was in this way that the basic facts of the theory were established by Élie Cartan in the 1930's and by Harish-Chandra in the 1950's.

The second way avoids semi-simple Lie groups theory and uses Jordan algebra and triple systems; it is essentially due to Koecher and his school in the 1980's.

In our presentation we will mainly use the Jordan theory to point out some development in the geometry and analysis on Hermitian symmetric spaces.

Let θ be the Cartan involution of the Hermitian symmetric space G/K. There exists a non trivial involution τ of the group G which commutes with the given Cartan involution. Let H be the $\tau-$fixed points of G. Then the symmetric space G/H belongs to an important class of non Riemannian symmetric spaces, namely, the class of causal symmetric spaces introduced by Ólafsson and Ørsted in the 1990's.

Suppose G/K of tube type, holomorphically equivalent, via the Cayley transform, to the tube $V + i\Omega$, where V is a Euclidean Jordan algebra and Ω its symmetric cone. We prove (see section 4) that in this case G/H is a causal symmetric space of Cayley type. In section 5 we give a conformal compactification of such spaces. We also investigate the semigroup associated with the order on G/H, see section 6. We prove that it is related to the semigroup S_Ω of compressions of Ω. Each element of S_Ω is a contraction for the Riemannian metric as well as for the Hilbert metric of Ω.

Let S be the Shilov boundary of G/K. In section 6, we study the causal structure of S. This boundary has many geometric invariants, in particular the transversality index (see section 7) and the triple Malsov index (see section 8). The universal covering of S is needed to study other geometric invariants introduced by Souriau and Arnold-Leray in the Lagrangian case. In section 9 we give an explicit construction of this universal covering. We also generalize the Souriau index (see section

10) and the Arnold-Leray index (see section 11). Finally, in section 12, we use the Souriau index to generalize of the Poincaré rotation number.

A further interesting development is the so-called Gelfand-Gindikin program. In 1977 Gelfand and Gindikin proposed a new approach to study the Plancherel formula for semi-simple Lie groups. The idea is to consider functions on G as boundary values of holomorphic functions of a domain (Lie semi-groups) in the complexification of G and to study the action of G on these holomorphic functions. When G is the group of holomorphic diffeomorphisms of a Hermitian symmetric space of tube type, the Gelfand-Gindikin program has been developed by Olshanskiĭ in several papers. This study is related to harmonic analysis of bounded symmetric domains and the decompositions of the Hardy spaces of Lie semi-groups, which involves the holomorphic discrete series of representations of G.

We present in section 13 the theory of Olshanskiĭ. In sections 16, 17 and 18 we develop this program for the groups $Sp(r,\mathbb{R})$, $SO^*(2\ell)$ and $U(p,q)$. We give a new approach of studying Hardy spaces on Lie semi-groups. More precisely, we introduce a new Cayley transform, which allows us to compare the classical Hardy space and the Hardy space on the Lie semi-group.

Part 1

Cayley type symmetric spaces, transversality and the Maslov index

1. Causal symmetric spaces

In this section we will recall the notion of causal symmetric spaces introduced by Ólafsson and Ørsted and give some examples. For non-compactly causal symmetric spaces, we will introduce the corresponding Olshanskiĭ semigroup.

1.1. Causal structures. Let V be a vector space. A subset C of V is called a *causal cone* if it is non-zero closed convex and proper (i.e. $C \cap -C = \{0\}$) cone.
Let $\mathcal{M}$ be a $n-$dimensional manifold. A *causal structure* on $\mathcal{M}$ is an assignment, $x \mapsto C_x$, to each point x of $\mathcal{M}$ a causal cone C_x in the tangent space $T_x(\mathcal{M})$ of $\mathcal{M}$ at x such that C_x depends smoothly on x. A $\mathcal{C}^1$ curve $\gamma : [\alpha, \beta] \to \mathcal{M}$ is said to *causal curve* (resp; *anti-causal curve*) if $\dot\gamma(t) \in C_{\gamma(t)}$ (resp. $\dot\gamma(t) \in -C_{\gamma(t)}$) for all t. If there is no non-trivial closed causal curves, the causal structure of $\mathcal{M}$ is said to *global*. We can then define a partial order $\preccurlyeq$ on $\mathcal{M}$:

$x \preccurlyeq y$ if there exists a causal curve $\gamma : [\alpha, \beta] \to \mathcal{M},\ \gamma(\alpha) = x,\ \gamma(\beta) = y.$

If $\mathcal{M} = G/H$ is a homogeneous space, where G is a Lie group and H is a closed subgroup of G, then the causal structure of $\mathcal{M}$ is said to be $G-$*invariant* if, for every $g \in G$,

$$C_{g\cdot x} = Dg(x)(C_x),$$

where $Dg(x)$ is the derivative of g at x. Let $x_o = eH$ be the base point. An invariant causal structure on $\mathcal{M} = G/H$ is determined by a causal cone $C_o \subset T_{x_o}(\mathcal{M})$ which is invariant under the action of H.

1.2. Causal symmetric spaces. Suppose that $\mathcal{M} = G/H$ is a symmetric space : there exists an involution σ of G such that $(G^\sigma)^\circ \subset H \subset G^\sigma$, where $(G^\sigma)^\circ$ is the connected component of G^σ, the subgroup of fixed points of σ in G.
Let $\mathfrak{g}$ be the Lie algebra of G. Put $\mathfrak{h} = \mathfrak{g}(+1, \sigma)$ and $\mathfrak{q} = \mathfrak{g}(-1, \sigma)$ the eigenspaces of σ. Then $\mathfrak{h}$ is the Lie algebra of H and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. The tangent space of $\mathcal{M}$ at the point x_o can be identified with $\mathfrak{q}$. With this identification, the derivative $Dh(x_o)$, $h \in H$ corresponds to $\mathrm{Ad}(h)$. Therefore an invariant causal structure on $\mathcal{M}$ is determined by a causal cone $C \subset \mathfrak{q}$ which is $\mathrm{Ad}(H)-$invariant.

Suppose G semi-simple and has a finite center and suppose that the pair $(\mathfrak{g}, \mathfrak{h})$ is irreducible (i.e. there is no non-trivial ideal in $\mathfrak{g}$, invariant by σ). In this case there exists a Cartan involution θ of G such that $\sigma\theta = \theta\sigma$. Let $K = G^\theta$, then K is a maximal compact subgroup of G.

Let $\mathfrak{k} = \mathfrak{g}(+1, \theta)$ and $\mathfrak{p} = \mathfrak{g}(-1, \theta)$. Then $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is the Cartan decomposition of $\mathfrak{g}$. Moreover we have

$$\mathfrak{g} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{q} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p}) \oplus (\mathfrak{q} \cap \mathfrak{p}).$$

Let $\mathrm{Cone}_H(\mathfrak{q})$ be the set of $\mathrm{Ad}(H)-$invariant causal cones in $\mathfrak{q}$.

Definition 1.1. *Let $\mathcal{M} = G/H$ be an irreducible non-Riemannian semi-simple symmetric space. Then we call $\mathcal{M} = G/H$*

(CC) *compactly causal symmetric space, if there exists $C \in \mathit{Cone}_H(\mathfrak{q})$ such that $C^\circ \cap \mathfrak{k} \neq \emptyset$.*
(NCC) *a non-compactly causal symmetric space, if there exists $C \in \mathit{Cone}_H(\mathfrak{q})$ such that $C^\circ \cap \mathfrak{p} \neq \emptyset$.*
(CT) *symmetric space of Cayley type, if both (CC) and (NCC) hold.*
(CAU) *causal symmetric space if either (CC) or (NCC) holds.*

Let

$$\mathfrak{q}^{H\cap K} = \{X \in \mathfrak{q} \ , \ \forall k \in H \cap K \ : \ \mathrm{Ad}(k)X = X\}.$$

Then, there exists an $\mathrm{Ad}(H)-$invariant causal cone in $\mathfrak{q}$ if and only if $\mathfrak{q}^{H\cap K} \neq \{0\}$.

Proposition 1.2. *Let $\mathcal{M} = G/H$ be an irreducible symmetric space.*

(1) *$\mathcal{M}$ is compactly causal if and only if $\mathfrak{q}^{H\cap K} \cap \mathfrak{k} \neq \{0\}$.*
(2) *$\mathcal{M}$ is non-compactly causal if and only if $\mathfrak{q}^{H\cap K} \cap \mathfrak{p} \neq \{0\}$.*

Examples 1.3. We give some examples of causal symmetric spaces.

(CC) (a) The group case. Let G_1 be a semi-simple Lie group. Let $G = G_1 \times G_1$ and define $\sigma(a, b) = (b, a)$. Then $H = \{(a, a) \ ; \ a \in G_1\} \simeq G_1$. The symmetric space $\mathcal{M} = G/H \simeq G_1$ has an invariant causal structure if and only if the Lie algebra $\mathfrak{g}_1$ of G_1 is of Hermitian type. In this case $\mathcal{M}$ is a compactly causal symmetric space.

(b) The hyperboloid $Q^n_+ = \{v \in \mathbb{R}^{n+1} \ ; \ v_1^2 + v_2^2 - v_3^2 - \ldots - v_{n+1}^2 = 1\}$. We have

$$Q^{1,n}_+ \simeq SO_o(2, n)/SO_o(1, n)$$

and it is a symmetric space where the involution σ is the conjugation by the matrix

$$J_{1,n} = \begin{pmatrix} 1 & 0 \\ 0 & -I_n \end{pmatrix}.$$

In this case $\mathfrak{q}$ is the set of all matrices

$$q_+(v) = \begin{pmatrix} 0 & -v^t \\ v & 0 \end{pmatrix} \; ; \; v \in \mathbb{R}^{n+1}$$

in which $SO_0(1,n)$ acts as usual. Let

$$C_+ = \{q_+(v) \in \mathfrak{q} \; ; \; v_1^2 - v_2^2 - \ldots - 1 \geq 0, v_1 \geq 0\}.$$

Then C_+ is a causal cone in $\mathfrak{q}$ invariant under the group $SO_0(1,n)$ and $\mathrm{Cone}_{SO_0(1,n)}(\mathfrak{q}) = \{C_+, -C_+\}$. In particular $SO_o(2,n)/SO_o(1,n)$ carries a compactly causal structure.

(NCC) (a) The Ol'shanskiĭ symmetric spaces. Let $\mathfrak{g}_1$ be a semisimple Hermitian Lie algebra. Let $\mathfrak{g} = (\mathfrak{g}_1)_{\mathbb{C}}$ and let σ be the complex conjugation. let $G_{\mathbb{C}}$ be a complex analytic group with Lie algebra $\mathfrak{g}$. Let G_1 be the analytic subgroup of $G_{\mathbb{C}}$ corresponding to $\mathfrak{g}_1$. Then $G_1 = (G_{\mathbb{C}}^{\sigma})^{\circ}$ and $\mathcal{M} = G_{\mathbb{C}}/G_1$ is a symmetric space. In this case $\mathfrak{q} = i\mathfrak{g}_1$ and $\mathcal{M}$ is non-compactly causal symmetric space.

(b) The hyperboloid $Q_-^n = \{v \in \mathbb{R}^{n+1} \; ; \; v_1^2 - v_2^2 - \ldots - v_{n+1}^2 = -1\}$. We have

$$Q_-^n \simeq SO_0(1,n)/SO_0(1,n-1)$$

and it is a symmetric space. In this case $\mathfrak{q}$ is the set

$$q_-(w) = \begin{pmatrix} 0 & w \\ w^t & 0 \end{pmatrix} \; ; \; w \in \mathbb{R}^{n+1}$$

in which the group $SO_0(1,n)$ acts. Let

$$C_- = \{q_-(w) \in \mathfrak{q} \; ; \; w_1^2 - w_2^2 - \ldots - 1 \geq 0, w_1 \geq 0\}.$$

Then C_- is a causal cone in $\mathfrak{q}$, invariant under the group $SO_0(1,n)$. Furthermore, $\mathrm{Cone}_{SO_0(1,n)}(\mathfrak{q}) = \{C_-, -C_-\}$ and $SO_0(1,n+1)/SO_0(1,n)$ is a non-compactly causal symmetric space.

(TC) The hyperboloid of one sheet in $\mathbb{R}^3$, $\mathrm{SO}_o(2,1)/\mathrm{SO}_o(1,1)$ is a symmetric space of Cayley type. One can realize it as the off-diagonal subset of $S^1 \times S^1$, where S^1 is the unit circle.

$$\mathcal{M} \simeq \{(u,v) \in S^1 \times S^1 \; ; \; u \neq v\}.$$

We will see, in section **??**, that this happens for any symmetric space of Cayley type.

If $\mathcal{M} = G/H$ is noncompactly causal symmetric space, then there exists a causal cone $C \in \mathrm{Cone}_H(\mathfrak{q})$ such that $C^\circ \cap \mathfrak{p} \neq \emptyset$ and $C \cap \mathfrak{k} = \{0\}$. The causal structure is then global and we can define and order $\preccurlyeq$ on $\mathcal{M}$. The set

$$S_{\preccurlyeq} = \{g \in G \ ; \ x_o \preccurlyeq g \cdot x_o\}$$

is a closed semigroup called the *Ol'shanskiĭ semigroup*. One has the *the Olshanskiĭ decomposition*

$$S_{\preccurlyeq} = H \exp(C).$$

2. Jordan algebras and symmetric cones

In this section we recall the notion of Euclidean Jordan algebras and fix notations. Our presentation is mainly based on [**F–K**].

Let V be a *Euclidean Jordan algebra* with identity element e and of dimension n. This means that V is a $n-$dimensional Euclidean vector space equipped with a bilinear product such that

$$xy = yx,$$
$$x^2(xy) = x(x^2y),$$
$$(xy|z) = (x|yz). \tag{1}$$

For $x \in V$, denote by $L(x)$ the linear operator defined by $y \mapsto L(x)y = xy$, and introduce the *quadratic representation* $P(\cdot)$ and the "square" operator $\Box$, defined by

$$P(x) = 2L(x)^2 - L(x^2), \quad x\Box y = L(xy) + [L(x), L(y)],$$

where the brackets denote the commutator. For simplicity we assume V to be simple. In other words, there is non non-trivial ideal in V. An element x is said to be *invertible* if there exists an element $y \in \mathbb{R}[x]$ such that $xy = e$. Since $\mathbb{R}[x]$ is associative, y is unique. It is called the *inverse* of x and is denoted by $y = x^{-1}$. Let $V^\times$ the set of invertible elements of V. The connected component of the unit e in $V^\times$ is the set Ω of squares,

$$\Omega = \{x^2 \ ; \ x \in V^\times\}.$$

The set Ω is open, convex, proper, generating, symmetric, homogeneous cone. This is the *symmetric cone* associated with the Jordan algebra V. Let $G(\Omega)$ be the subgroup of linear transformations of V which preserve Ω. Then $G(\Omega)$ is a reductive group, which acts transitively on Ω. The same properties hold for its neutral component, which we denote by G_0. The stabilizer $K_0 = (G_0)_e$ of the unit e is a maximal compact subgroup of G_0 and it is the neutral component of $Aut(V)$, the *automorphism group* of the Jordan algebra V. Moreover $K_0 = G_0 \cap O(V)$, where $O(V)$ is the orthogonal group of the inner product on V. The space $\Omega \simeq G_0/K_0$ is a Riemannian symmetric space.

Let c be an *idempotent* element in $V : c^2 = c$. Then the only possible eigenvalues of $L(c)$ are 1, $\frac{1}{2}$, 0 and V is the direct sum of the corresponding eigenspaces $V(c,1)$, $V(c,\frac{1}{2})$ and $V(c,0)$. The decomposition

$$V = V(c,1) \oplus V(c,\frac{1}{2}) \oplus V(c,0),$$

is called the *Peirce decomposition* of V with respect to the idempotent c. It is an orthogonal decomposition with respect to any scalar product satisfying (1).

Two idempotents c and d are said to be *orthogonal* if $(c|d) = 0$, which is equivalent to $cd = 0$. An idempotent is said to be *primitive* if it is not the sum of two non-zero idempotents. An idempotent c is primitive if and only if $\dim V(c,1) = 1$. We say that $(c_j)_{1\le j\le m}$ is a *Jordan frame* if each c_j is a primitive idempotent and

$$c_i c_j = 0, \quad i \neq j$$

$$c_1 + c_2 + \ldots + c_m = e.$$

All the Jordan frames have the same number of elements which we denote by r. The integer r is the *rank* of the Jordan algebra V.
The group K acts transitively on the set of primitive idempotents, and also on the set of Jordan frames. Therefore if we fix a Jordan frame $(c_j)_{j=1}^r$, then every element $x \in V$ can be written in the form

$$x = k(\sum_{j=1}^{r} \lambda_j c_j)$$

where $k \in K$ and $\lambda_1, \ldots, \lambda_r$ real numbers. The scalars $(\lambda_j)_{1\le j\le r}$ are unique and called the *spectral values* of x. We define the *determinant* and the *trace* of the Jordan algebra by

$$\mathbf{det}(x) = \prod_{j=1}^{r} \lambda_j, \quad \mathbf{tr}(x) = \sum_{j=1}^{r} \lambda_j.$$

The trace is a linear form of V and the determinant is a homogeneous polynomial on V of degree r. One can show that

$$V^\times = \{x \in V \ , \ \mathbf{det}(x) \neq 0\}.$$

From now we assume that the scalar product of V is given by

$$(x|y) = \mathbf{tr}(xy). \tag{2}$$

EXAMPLE 2.1. *The vector space $V = Sym(r,\mathbb{R})$ of $r \times r$ real symmetric matrices is a Euclidean Jordan algebra with the product $x \circ y = \frac{1}{2}(xy + yx)$ and the scalar product $(x|y) = \mathbf{Tr}(xy)$. The quadratic representation is given by $P(x)y = xyx$. In this case the determinant and the trace are the usual matrix determinant and trace. The corresponding symmetric cone is $\Omega = Sym^{++}(r,\mathbb{R})$ the set of definite positive symmetric matrices. An idempotent is an orthogonal projection $c = \begin{pmatrix} I_p & 0 \\ 0 & 0 \end{pmatrix}$ with $r = p + q$. Then*

$$V(c,1) = \{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \ ; \ a \ : \ p \times p \ \textit{symmetric matrix}\},$$

$$V(c, \frac{1}{2}) = \{\left(\begin{smallmatrix} 0 & d \\ d^t & 0 \end{smallmatrix}\right) \ ; \ d \ : \ p \times q \ \mathit{matrix}\},$$
$$V(c, 0) = \{\left(\begin{smallmatrix} 0 & 0 \\ 0 & b \end{smallmatrix}\right) \ ; \ b \ : \ q \times p \ \mathit{symmetric\ matrix}\}.$$

3. Hermitian symmetric spaces of tube type

In this section we characterize irreducible Hermitian symmetric spaces of tube type using the Jordan theory and give the classification of such spaces.

By complexification, we get a complex Jordan algebra $V_{\mathbb{C}}$. We denote the complex conjugation with respect to V by η. We also use the notation

$$\eta(z) = \bar{z}. \tag{3}$$

We extend the inner product (2) of V to the Hermitian inner product of $V_{\mathbb{C}}$ defined by

$$(z|w) = \mathbf{tr}(z\bar{w}). \tag{4}$$

Let

$$S = \{\sigma \in V_{\mathbb{C}}\ ,\ \sigma^{-1} = \bar{\sigma}\}.$$

This is a connected compact sub-manifold of $V_{\mathbb{C}}$.

Proposition 3.1 ([**F–K**]). *For $\sigma \in V_{\mathbb{C}}$ the following properties are equivalent :*

(i) $\sigma \in S$,
(ii) $\sigma = \exp(iu)$, *where* $u \in V$,
(iii) There exists a Jordan frame $(c_j)_{1\le j\le r}$ *of* V *and complex numbers* $(\xi_j)_{1\le j\le r}$ *of modulus 1, such that* $\sigma = \sum_{j=1}^{r} \xi_j c_j$.

Denote by $\mathbb{L} = Str(V_{\mathbb{C}})$ be the *structure group* of $V_{\mathbb{C}}$, i.e. the set of $g \in GL(V_{\mathbb{C}})$ such that

$$P(gz) = gP(z)g'$$

or equivalently

$$g(z\Box w)g^{-1} = (gz)\Box(g'^{-1}w).$$

Consider the group

$$L(S) = \{g \in GL(V_{\mathbb{C}})\ ,\ g(S) = S\}.$$

Then $L(S) = \mathbb{L} \cap U(V_{\mathbb{C}})$, where $U(V_{\mathbb{C}})$ is the unitary group of the Hermitian inner product (4) on $V_{\mathbb{C}}$. It acts transitively on S. Moreover, the stabilizer of e in $L(S)$ coincides with $Aut(V)$ (we extend the automorphisms of V as complex linear automorphisms of $V_{\mathbb{C}}$). The involution (3) preserves S and e is its unique isolated fixed point. The set of fixed points of the corresponding involution of $L(S)$, $g \mapsto \eta\circ g\circ\eta$, is $L(S)_e = Aut(V)$. Hence, with the metric induced by the Hermitian product (4), S is a Riemannian symmetric space of compact type isomorphic to $L(S)/Aut(V)$.

Let U be the identity component of $L(S)$, and U_e the stabilizer of e in U. Then we have $Aut(V)^\circ \subset U \subset Aut(V)$, and

$$S \simeq U/U_e.$$

Let $(c_j)_{1\leq j\leq r}$ be a Jordan frame of V. Then every element $z \in V_{\mathbb{C}}$ can be written as in the form

$$z = u(\sum_{j=1}^{r} \lambda_j c_j),$$

where $u \in U$ and $0 \leq \lambda_1, \ldots, \lambda_r$. The *spectral norm* of z is then defined by

$$|z| = \sup_{1\leq j\leq r} \lambda_j.$$

It turns out to be a norm on $V_{\mathbb{C}}$, invariant under the group U.
Introduce the domain D in $V_{\mathbb{C}}$ as th open unit ball for the spectral norm

$$D = \{z \in V_{\mathbb{C}}\ ,\ |z| < 1\}.$$

Recall that the *Shilov boundary* of D is the smallest closed set in $\overline{D}$ where the principle of the maximum holds.

THEOREM 3.2 ([**F–K**]). *D is a bounded symmetric domain and S is its Shilov boundary.*

There is a realization of the domain D as a tube domain through the *Cayley transform.* Let T_Ω be the tube over the symmetric cone Ω,

$$T_\Omega = V + i\Omega = \{z = x + iy \in V_{\mathbb{C}}\ ,\ y \in \Omega\}.$$

The Cayley transform c and its inverse p are given (in their domains of definition) by

$$\begin{aligned} p(z) &= (z - ie)(z + ie)^{-1} \\ c(w) &= i(e + w)(e - w)^{-1}. \end{aligned}$$

PROPOSITION 3.3 ([**F–K**]). *The map p induces a biholomorphic isomorphism from T_Ω onto D, and*

$$p(V) = \{\sigma \in S\ ,\ \mathbf{det}(e - \sigma) \neq 0\}.$$

Both domains T_Ω and D are biholomorphically equivalent, and V can be thought of as the Shilov boundary of T_Ω and its image under the transformation p is an open dense in S.

Let $G = G(D)$ be the neutral component of the group of biholomorphic diffeomorphisms of D. It is a semi-simple Lie group and the stabilizer of $0 \in D$ in G is a maximal compact subgroup of G which coincides with U.

To describe the group G, we use the Cayley transform. Let $G^c := G(T_\Omega)$ be the neutral component of the group of biholomorphic diffeomorphisms of T_Ω. Then

$$c^{-1} \circ G \circ c = G^c.$$

We already know some subgroups of G^c. In fact , an element of G_0 acts on T_Ω and we can identify G_0 with a subgroup of G^c.

For $v \in V$, the translation

$$t_v : z \mapsto z + v$$

is a holomorphic automorphism of T_Ω and the group of all real translations t_v is an Abelian subgroup N^+ of G^c isomorphic to the vector space V.
The *inversion*

$$j : z \mapsto z^{-1}$$

belongs to G^c. We set $N^- = j \circ N^+ \circ j$. It is the subgroup of G^c of the maps

$$\widetilde{t_v} = j \circ t_v \circ j : z \mapsto (z^{-1} - v)^{-1}, \quad v \in V,$$

and it is an Abelian subgroup of G^c isomorphic to V.

Theorem 3.4 ([**F–K**]). *The subgroups G_0 and N^+, together with the inversion j, generate G^c.*

The semi-direct product $P^+ = G_0N^+$ is a maximal parabolic subgroup of G^c. The homogeneous space G^c/P^+ is then a (real) compact manifold which contains V as an open dense subset,

$$V \to G^c/P^+ \,:\, v \mapsto g_v P^+, \tag{5}$$

where $g_v(z) = j(z) + v$. The manifold G^c/P^+ is the *conformal compactification* of the Jordan algebra V, and it is isomorphic to the Shilov boundary S of D.

Example 3.5. *If V is the Jordan algebra $Sym(r, \mathbb{R})$, then D is the Siegel domain*

$$D = \{z \in Sym(r, \mathbb{C}),\ I_r - zz^* >> 0\}.$$

It is holomorphically isomorphic to the upper half domain

$$T_\Omega = \{z = x + iy \in Sym(r, \mathbb{C}),\ y \in Sym^{++}(r, \mathbb{R})\}.$$

In this case, $G = Sp(r, \mathbb{R})/\{\pm Id\}$, where $Sp(r, \mathbb{R})$ is the symplectic group.

Here we give the classification of tube domains, their Shilov boundaries and the corresponding Euclidean Jordan algebras.

V	$V_{\mathbb{C}}$	$D \simeq G/U$	$S \simeq U/U_e$
$Sym(m, \mathbb{R})$	$Sym(m, \mathbb{C})$	$Sp(2m, \mathbb{R})/U(m)$	$U(m)/O(m)$
$Herm(m, \mathbb{C})$	$Mat(m, \mathbb{C})$	$SU(m, m)/S(U(m) \times U(m))$	$U(m)$
$Herm(m, \mathbb{H})$	$Skew(2m, \mathbb{C})$	$SO^*(4m)/U(2m)$	$U(2m)/SU(m, \mathbb{H})$
$\mathbb{R} \times \mathbb{R}^{q-1}$	$\mathbb{C} \times \mathbb{C}^{q-1}$	$SO_0(2, q)/SO(2) \times SO(q)$	$(U(1) \times S^{q-1})/\mathbb{Z}_2$
$Herm(3, \mathbb{O})$	$Mat(3, \mathbb{O})$	$E_{7(-25)}/U(1)E_6$	$U(1)E_6/F_4$

TABLE 1. Tube domains and their Shilov boundaries

V	n	r	d
$Sym(m, \mathbb{R})$	$\frac{1}{2}m(m+1)$	m	1
$Herm(m, \mathbb{C})$	m^2	m	2
$Herm(m, \mathbb{H})$	$m(2m-1)$	m	4
$\mathbb{R} \times \mathbb{R}^{q-1}$	q	2	$q-2$
$Herm(3, \mathbb{O})$	27	3	8

TABLE 2. The dimension, rank and the Peirce invariant

4. Cayley type symmetric spaces

In this section we characterize the causal symmetric spaces of Cayley type; we prove in particular that if G/K is a Hermitian symmetric space of tube type, then G/H is a causal symmetric space of Cayley type.

Let $\mathfrak{g}$ be the Lie algebra of $G = G(D)$ and $\mathfrak{g}^c$ be the Lie algebra of $G^c = G(T_\Omega)$. Let $\mathfrak{g}_0$ be the Lie G_0. The Lie algebra of K_0 is set of all derivations $Der(V)$ of V. Let $\mathfrak{p}_0 = \{L(v)\ ,\ v \in V\}$. Then the Cartan decomposition is given by $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. The Lie algebra of U is $\mathfrak{u} = \mathfrak{k}_0 \oplus i\mathfrak{p}_0$. Let $G_{\mathbb{C}}$ be the Lie group generated by $Str(V_{\mathbb{C}})$, the complex translation of $V_{\mathbb{C}}$ and j. Then G and G^c are two real forms of $G_{\mathbb{C}}$. The Lie algebra $\mathfrak{g}_{\mathbb{C}}$ of $G_{\mathbb{C}}$ is the set of vector fields X on $V_{\mathbb{C}}$ of the form

$$X(z) = u + Tz - P(z)v,$$

with $u, v \in V_{\mathbb{C}}$ and $T \in \mathfrak{str}(V_{\mathbb{C}})$, where $\mathfrak{str}(V_{\mathbb{C}})$ is the Lie algebra of the structure group of $V_{\mathbb{C}}$ and coincides with $(\mathfrak{g}_0)_{\mathbb{C}}$. If

$$\begin{aligned} X_1(z) &= u_1 + T_1 z - P(z)v_1, \\ X_1(z) &= u_2 + T_2 z - P(z)v_2, \end{aligned}$$

then the bracket $[X_1, X_2]$ is given by

$$[X_1, X_2](z) = u + Tz - P(z)v,$$

with

$$\begin{aligned} u &= T_1 u_2 - T_2 u_1, \\ T &= [T_1, T_2] + 2(u_1 \Box v_2) - 2(u_2 \Box v_1), \\ v &= -T_1^* v_2 + T_2^* v_1. \end{aligned}$$

A vector field X in $\mathfrak{g}_{\mathbb{C}}$

$$X(z) = u + Tz - P(z)v,$$

can be identified with $(u, T, v) \in V_{\mathbb{C}} \times \mathfrak{str}(V_{\mathbb{C}}) \times V$, then

$$\mathfrak{g}_{\mathbb{C}} \simeq V_{\mathbb{C}} \times \mathfrak{str}(V_{\mathbb{C}}) \times V_{\mathbb{C}}.$$

With this identification we have,

$$\begin{aligned} \mathfrak{g}_0 &= \{(u, T, u)\ ,\ u \in V, T \in \mathfrak{k}_0\} \simeq V \times \mathfrak{k}_0 \times V, \\ \mathfrak{g}^c &= \{(u, T, v)\ ,\ u,\ v \in V, T \in \mathfrak{g}_0\} \simeq V \times \mathfrak{g}_0 \times V, \\ \mathfrak{g} &= \{(w, T, \bar{w})\ ,\ w \in V_{\mathbb{C}}, T \in \mathfrak{u}\}, \\ \mathfrak{u} &= \{(u, T, -u)\ ,\ u \in V, T \in \mathfrak{k}_0\}. \end{aligned}$$

Define $X_0 = (0, I, 0)$, then $\mathrm{ad}(X_0)$ has eigenvalues $1, 0, -1$ with the eigenspaces

$$\begin{aligned} \mathfrak{g}_1^c &= \{(u,0,0)\ ,\ u \in V\} \simeq V, \\ \mathfrak{g}_0^c &= \{(0,T,0)\ ,\ T \in \mathfrak{g}_0\} \simeq \mathfrak{g}_0, \\ \mathfrak{g}_{-1}^c &= \{(0,0,v)\ ,\ v \in V\} \simeq V. \end{aligned}$$

The decomposition

$$\mathfrak{g}^c = \mathfrak{g}_1^c + \mathfrak{g}_0^c + \mathfrak{g}_{-1}^c$$

is called the *Kantor-Koecher-Tits* decomposition of $\mathfrak{g}^c$.

Consider the involutions σ^c and θ^c of G^c given by

$$\begin{aligned} \sigma^c(g) &= (-j) \circ g \circ (-j) \\ \theta^c(g) &= j \circ g \circ j. \end{aligned}$$

We keep the same notation for the corresponding involutions of the Lie algebra $\mathfrak{g}^c$.

Proposition 4.1 ([**Ko**], [**Ko**$_1$], [**Ko**$_2$]). *θ^c is the Cartan involution of G^c. It commutes with the involution σ^c. If $X = (u, T, v) \in \mathfrak{g}^c$, then*

$$\begin{aligned} \sigma^c(X) &= (v, -T^*, u), \\ \theta^c(X) &= (-u, -T^*, -v). \end{aligned}$$

Similarly, consider the involution σ and θ of G given by

$$\begin{aligned} \sigma(g) &= \nu \circ g \circ \nu \\ \theta^c(g) &= (-\nu) \circ g \circ (-\nu). \end{aligned}$$

where $\nu(z) = \bar{z}$, and keep the same notation for the corresponding involutions of the Lie algebra $\mathfrak{g}$.

Proposition 4.2 ([**Ko**], [**Ko**$_1$], [**Ko**$_2$]). *θ is the Cartan involution of G. It commutes with the involution σ. If $X = (w, T, \bar{w}) \in \mathfrak{g}$, then*

$$\begin{aligned} \sigma(X) &= (\bar{w}, \bar{T}, w), \\ \theta(X) &= (-w, -T, -\bar{w}). \end{aligned}$$

Let τ be the element of G^c given by

$$\tau(z) = (e + z)(e - z)^{-1}.$$

Then its inverse is

$$\tau^{-1}(z) = (z - e)(z + e)^{-1}.$$

Notice that $c(z) = i\tau(z)$ and $c^{-1}(z) = \tau^{-1}(iz)$. For any element $g \in G_0$,

$$c^{-1} \circ g \circ c = \tau \circ (g^*)^{-1} \circ \tau^{-1}.$$

Since G_0 is a reductive group, we have

$$c^{-1} \circ G_0 \circ c = \tau \circ G_0 \circ \tau^{-1}.$$

Let $H = G^\sigma$ and $H^c = (G^c)^{\sigma^c}$. Then we have

THEOREM 4.3 ([**Ko**], [**Ko**$_1$], [**Ko**$_2$]). (1) *$H = c^{-1} \circ G_0 \circ c$ and $H^c = \tau \circ G_0 \circ \tau^{-1}$.*

(2) *The subgroups H and H^c coincides and $H = H^c = G \cap G^c$.*

(3) *The symmetric space $\mathcal{M} = G/H \simeq G^c/H^c$ is a Cayley type symmetric space and any symmetric space of Cayley type is given in this way.*

More precisely, we have

$$\begin{aligned}
\mathfrak{h}^c &= \{(u, T, u)\ ;\ u \in V, T \in \mathfrak{k}_0\},\\
\mathfrak{q}^c &= \{(u, L(v), -u)\ ;\ u, v \in V\},\\
\mathfrak{k}^c &= \{(u, T, -u)\ ;\ u \in V, T \in \mathfrak{k}_0\},\\
\mathfrak{p}^c &= \{(u, L(v), u)\ ;\ u, v \in V\}.
\end{aligned}$$

Let C_1, respectively C_2, be the cone in $\mathfrak{q}^c$ given by

$$C_1 = \{(u, 2L(v), -u)\ ;\ (u + v) \in -\overline{\Omega}, (u - v) \in \overline{\Omega}\},$$

respectively

$$C_2 = \{(u, 2L(v), -u)\ ;\ (u + v) \in \overline{\Omega}, (u - v) \in \overline{\Omega}\}.$$

C_1 and C_2 are two $\mathrm{Ad}(H^c)-$invariant, regular cones isomorphic to $\overline{\Omega} \times \overline{\Omega}$. Moreover

$$C_1 \cap \mathfrak{p}^c \neq \emptyset,\ C_1 \cap \mathfrak{k}^c = \{0\}$$

and

$$C_2 \cap \mathfrak{k}^c \neq \emptyset,\ C_1 \cap \mathfrak{p}^c = \{0\}.$$

Thus C_1 (respectively C_2) defines a non-compactly (resp. compactly) causal structure on G^c/H^c.

5. The 2-transitivity property on $S^2_\top$ and Cayley type symmetric spaces

In this section we will give a causal compactification of cuasal symmetric spaces of Cayley type.

Definition 5.1. *Let $\mathcal{M}$ be a causal $G-$manifold. A causal compactification of $\mathcal{M}$ is a pair $(\mathcal{N}, \Phi)$ such that*

(1) *$\mathcal{N}$ is a compact causal $G-$manifold.*
(2) *The map $\phi : \mathcal{M} \to \mathcal{N}$ is causal.*
(3) *The map Φ is $G-$equivariant, i.e., $\Phi(g\cdot x) = g\cdot\Phi(x)$, for every $g \in G$ and every $x \in \mathcal{M}$.*
(4) *$\Phi(\mathcal{M})$ is open and dense in $\mathcal{N}$.*

Two points $z, w \in V_{\mathbb{C}}$ are called *transversal*, and we write $z\top w$, if and only if $\mathbf{det}(z-w) \neq 0$. This condition is equivalent to $\mathrm{Det}\, P(z-w) \neq 0$. We denote the set of transversal elements in S^2 by

$$S^2_\top = \{(\sigma,\tau) \in S^2 \ ,\ \sigma\top\tau\} = \{(\sigma,\tau) \in S^2 \ ,\ \mathbf{det}(\sigma-\tau) \neq 0\},$$

Theorem 5.2 ([**Ko**], [**Ko**$_1$], [**Ko**$_2$]). *The group G acts transitively on $S^2_\top$. The stabilizer of the element $(e,-e) \in S^2_\top$ in G is the group $H = c^{-1}\circ G_0 \circ c$.*

Hence, the Cayley symmetric space is $G-$equivariant to $S^2_\top$,

$$G/H \simeq S^2_\top.$$

Since $S^2_\top$ is open dense in S^2, and since S^2 is a compact causal $G-$manifold (because we already prove that S is a causal $G-$manifold), the manifold S^2 is a causal compactification of G/H.
The "non bounded" realization of $\mathcal{M} = G^c/H^c$ is such that

$$\mathcal{M} \cap (V\times V) = \{(x,y) \in V\times V \ ;\ \mathbf{det}(x-y) \neq 0\}.$$

Examples 5.3. (1) *If $\mathcal{M} = SU(n,n)/GL(n,\mathbb{C})\mathbb{R}^+$. Then*

$$D = SU(n,n)/S(U(n)\times U(n)) = \{z \in Mat(n,\mathbb{C}) \ ;\ I_n - z^*z \gg 0\},$$

its Shilov boundary is $S = U(n)$ and

$$\mathcal{M} \simeq \{(z,w) \in U(n)\times U(n) \ ;\ \mathrm{Det}(z-w) \neq 0\}.$$

(2) *If $\mathcal{M} = Sp(n,\mathbb{R})/GL(n,\mathbb{R})\mathbb{R}^+$. Then*

$$D = Sp(n,\mathbb{R})/U(n) = \{z \in Sym(n,\mathbb{C}) \ ;\ I_n - z^*z \gg 0\},$$

its Shilov boundary is the Lagrange Grassmann manifold $S = U(n)/O(n) = \{z \in U(n) \ ;\ z^t = z\}$ and

$$\mathcal{M} \simeq \{(z,w) \in U(n)\times U(n) \ ;\ z^t = z, w^t = w, \mathrm{Det}(z-w) \neq 0\}.$$

6. The Lie semigroup associated with the Cayley type symmetric space

In this section we inverstigate the semigroup S_Ω of compressions of Ω. We prove in particular a triple decomposition and that S_Ω is the real part of the holomorphic semigroup of compression of the tube domain. We give a new characterization of the Riemannian metric of Ω and prove that S_Ω is a semigroup of contractions of this metric. The Hilbert metric on Ω is also studied

6.1. The compression semigroup of the Hermitian domain D. Let $C^c_{\max}$ be the maximal cone in $\mathfrak{g}^c$. It is the closed convex cone given by

$$C^c_{\max} = \{X \in \mathfrak{g}^c \ ; \ X(v) \in \overline{\Omega}, \forall v \in V\}.$$

The cone $C_{\max} = \mathrm{Ad}(c)(C^c_{\max})$ is the maximal cone in $\mathfrak{g}$. Let

$$\Gamma(C_{\max}) = G\exp(iC_{\max}).$$

The following theorem is due to Olshanskiĭ

Theorem 6.1 ([**O**$_2$]). *The set $\Gamma(C_{max})$ is a Lie semigroup (associated with C_{max}) and it is the semigroup of compressions of the Hermitian domain D,*

$$\Gamma(C_{max}) = \{g \in G_{\mathbb{C}} \ ; \ g(D) \subset D\}.$$

Moreover

$$\Gamma(C_{max})^\circ = \{g \in G_{\mathbb{C}} \ ; \ g(\overline{D}) \subset D\}.$$

The convex cone C_1 is the maximal cone $c^c_{\max}$ in $\mathfrak{q}^c$. It defines a non-compactly causal structure on $\mathcal{M}$. Therefore $\mathcal{M}$ is and ordering symmetric space Let Γ the semigroup associated with the order of $\mathcal{M}$,

$$S_{\succeq} = \{g \in G^c \ ; \ g(e,-e) \succeq (e,-e)\}.$$

Then we have

Theorem 6.2 ([**Ko**], [**Ko**$_1$], [**Ko**$_2$]). *The semigroup $S_{\succeq}$ satisfies*

(1) $S_{\succeq} = \exp(c^c_{max})H$.
(2) $\Gamma(C_{max}) \cap G^c = S_{\succeq}^{-1}$

6.2. The compression semigroup of the symmetric cone Ω. Recall that when P is the parabolic subgroup $P = G_0N^+$, the compact symmetric space $\mathcal{X} = G^c/P$ is conformal compactification of V, and the imbedding of V into $\mathcal{X}$ given by (5) is an open dense embedding. A Lie semigroup that is naturally related to the action of G^c on $\mathcal{X}$ occurs as the semigroup of compressions of Ω in G^c :

$$S_\Omega = \{\gamma \in G^c \ ; \ \gamma\Omega \subset \Omega\}. \tag{6}$$

Since the closure $\widetilde{\Omega}$ of Ω in $\mathcal{X}$ is compact with Ω as interior, the compression semigroup S_Ω is a closed semigroup of G^c. Moreover S_Ω contains G_0, and its interior is

$$S_\Omega^\circ = \{\gamma \in G^c \ ; \ \gamma\widetilde{\Omega} \subset \Omega\}.$$

Now let

$$\begin{aligned} S_\Omega^+ &= \{\gamma_v^+ : z \mapsto z + v \ ; \ v \in \overline{\Omega}\}, \\ S_\Omega^- &= \{\gamma_v^- : z \mapsto (z^{-1} + v)^{-1} \ ; \ v \in \overline{\Omega}\}. \end{aligned}$$

Then it is easy to see that $S_\Omega^\pm$ and G_0 are closed sub-semigroups in S_Ω. Hence $S_\Omega^+ G_0 S_\Omega^- \subset S_\Omega$.

Theorem 6.3 ([**Ko**], [**Ko$_2$**]). (1) *The compression semigroup S_Ω is equal to the semigroup $S_{\succeq}$*

(2) *The sub-semigroups S_Ω^+ and S_Ω^-, together with the subgroup G_0, generate S_Ω. More precisely, one has the following decomposition*

$$S_\Omega = S_\Omega^+ G_0 S_\Omega^- = N^+ G_0 N^- \cap S_\Omega. \tag{7}$$

If $\gamma = \gamma_u^+ g \gamma_v^- \in S_\Omega$, then we write

$$n^+(\gamma) := u, \ A(\gamma) := g \text{ and}, \ n^-(\gamma) := v. \tag{8}$$

6.3. The semigroup of contractions. The family of bilinear forms g_x given by,

$$\mathrm{g}_x(u, v) = (P(x)^{-1}u|v), \quad x \in \Omega, u, v \in V,$$

defines a $G(\Omega)-$invariant Riemannian metric on Ω, see [**F–K**, Theorem III.5.3]. Therefore, Ω is a Riemannian symmetric space isomorphic to $G(\Omega)_\circ/K(\Omega)_\circ$.

Theorem 6.4 ([**Ko$_3$**]). *Let $x, y \in \Omega$. Then there exists a unique curve of shortest length joining x and y. The length of this curve is given by*

$$\delta(x, y) = \Big(\sum_{k=1}^r \log^2 \lambda_k(x, y)\Big)^{1/2},$$

where $\lambda_1(x, y), \ldots, \lambda_r(x, y)$ are the spectral values of of $P(y)^{-1/2}x$.

$\delta(x, y)$ is the Riemannian distance of x and y, and the scalars

$$\mu_k(x, y) := \log^2(\lambda_k(x, y)$$

are by definition the *angles* or the *compounds distance*.
Using the notations (8), we set

$$S_1 = \{\gamma \in S_\Omega \mid n^+(\gamma) \in \Omega\},$$

and

$$S_2 = \{\gamma \in S_\Omega \mid n^-(\gamma) \in \Omega\}.$$

THEOREM 6.5 ([**Ko**$_3$]). *Let $k \in \{1, \dots, r\}$. The following holds:*

(1) *For any $\gamma \in S$ and for any $x, y \in \Omega$: $\mu_k(\gamma \cdot x, \gamma \cdot y) \leq \mu_k(x, y)$.*
(2) *For any $\gamma \in S_1 \cup S_2$ and for any $x, y \in \Omega$: $\mu_k(\gamma \cdot x, \gamma \cdot y) < \mu_k(x, y)$.*
(3) *For any $\gamma \in S_1 \cap S_2$, there exists $\kappa(\gamma)$, $0 < \kappa(\gamma) < 1$, such that for any $x, y \in \Omega$: $\mu_k(\gamma \cdot x, \gamma \cdot y) \leq \kappa(\gamma)\mu_k(x, y)$.*

As an easy consequence, Theorem 6.5 implies that the elements of the semigroup S_Ω are contractions of the distance δ. More precisely we have

COROLLAIRY 6.6 ([**Ko**$_3$]). *The following holds:*

(1) *For any $\gamma \in S_\Omega$, and $x, y \in \Omega$: $\delta(\gamma \cdot x, \gamma \cdot y) \leq \delta(x, y)$.*
(2) *For any $\gamma \in S_1 \cup S_2$ and $x, y \in \Omega$: $\delta(\gamma \cdot x, \gamma \cdot y) < \delta(x, y)$.*
(3) *For any $\gamma \in S_1 \cap S_2$, there exists $\kappa(\gamma)$, $0 < \kappa(\gamma) < 1$, such that, for all $x, y \in \Omega$: $\delta(\gamma \cdot x, \gamma \cdot y) \leq \kappa(\gamma)\, \delta(x, y)$.*

6.4. Hilbert's projective metric. Let E be a real Banach space and C be a closed convex pointed cone, where pointed means $C \cap -C = \{0\}$. The relation $\preccurlyeq$ is defined on E by saying that $x \preccurlyeq y$ if and only if $y - x \in C$.
For $x \in E$ and $y \in \mathrm{int}(C)$ we let

$$M(x, y) := \inf\{\lambda \mid x \preccurlyeq \lambda y\},$$

and

$$m(x, y) := \sup\{\mu \mid \mu y \preccurlyeq x\}.$$

Hilbert's projective metric is defined on $\mathrm{int}(C)$ by

$$d(x, y) = \log \frac{M(x, y)}{m(x, y)}. \tag{9}$$

In the case of $\mathbb{R}^n_+$, Hilbert's projective metric is $d(x, y) = \log \frac{\max \frac{x_i}{y_i}}{\min \frac{x_i}{y_i}}$ where $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are two vectors of $\mathbb{R}^n_+$.

The Hilbert projective metric may be applied to variety of problems involving positive matrices and positive integral operators. For example one can use it to solve some Volterra equations. It is also particularly useful in proving the existence of the fixed point for positive operators defined in a Banach space. In this way, it has been shown by Bushell [**Bu**] that Hilbert projective metric may be applied to prove that, if T

a real nonsingular $r \times r$ matrix, then there exists a unique real positive definite symmetric $r \times r$ matrix A such that

$$T'AT = A^2. \tag{10}$$

Notice that if T is neither symmetric nor orthogonal the existence and the uniqueness of A is not an elementary problem, even if $r = 2$.

We will formulate the Hilbert projective metric on symmetric cones in a way most convenient for our purpose using Jordan algebra theory and extend Bushell's Theorem to this class of convex cones.

If we consider the cone Ω_{Sym} of real symmetric positive definite $r \times r$ matrices, then one can easily express the Hilbert projective metric (9) in terms of eigenvalues of elements of Ω_{Sym}. Indeed, if A and B are in Ω_{Sym}, then

$$M(A,B) := \inf\{\lambda \mid \lambda B - A \preccurlyeq 0\} = \max_{\|x\|=1} \frac{(Ax|x)}{(Bx|x)},$$

and

$$m(A,B) := \sup\{\lambda \mid \lambda B - A \preccurlyeq 0\} = \min_{\|x\|=1} \frac{(Ax|x)}{(Bx|x)},$$

which are respectively the greatest and the least eigenvalue of $B^{-1}A$. Observe that eigenvalues of the matrix $B^{-1}A$ are the same of the matrix $B^{-\frac{1}{2}}AB^{-\frac{1}{2}} = P(B^{-\frac{1}{2}})A$.

More generally, for symmetric cones, Hilbert's projective metric can be also formulated in terms of extremal eigenvalues : let x and y be in Ω and let $\lambda_M(x,y) > 0$ and $\lambda_m(x,y) > 0$ denote the greatest and the least eigenvalue of the element $P(y^{-\frac{1}{2}})x \in \Omega$. Then one can prove that

$$\lambda_M(x,y) = \max_{c \in \mathcal{J}(V)} \frac{(x|c)}{(y|c)},$$

and

$$\lambda_m(x,y) = \min_{c \in \mathcal{J}(V)} \frac{(x|c)}{(y|c)},$$

see [**Ko**$_3$, Thoerem 4.2]. Consequently, we have :

Proposition 6.7 ([**Ko**$_4$]). *If x, $y \in \Omega$, then the Hilbert metric of x and y is given by*

$$d(x,y) = \log \frac{\lambda_M(x,y)}{\lambda_m(x,y)} = \log\big[\lambda_M(x,y)\lambda_M(y,x)\big]. \tag{11}$$

Furthermore, one can prove (see [**Ko**$_4$]) that (Ω, d) is a pseudo-metric space. In other words, for any x, y, $z \in \Omega$, the following holds,

(a) $d(x,y) \geq 0$
(b) $d(x,y) = d(y,x)$

(c) $d(x,z) \leq d(x,y) + d(y,z)$
(d) $d(x,y) = 0 \Leftrightarrow \exists \lambda > 0 : x = \lambda y$.

Now the characterization (11) of the Hilbert metric allows us to prove the completeness :

PROPOSITION 6.8 ([**Ko**$_4$]). *$(\Omega \cap S(V), d)$ is a complete metric space.*

As application, we prove a generalization of the Bushell theorem :

THEOREM 6.9 ([**Ko**$_4$]). *Let $g \in G(\Omega)$ and $p \in \mathbb{R}$ such that $|p| > 1$. Then there exists a unique element a in Ω such that $g(a) = a^p$.*

7. The 2-transitivity property on S^2 and the transversality index

We introduce here, the transversality index, a new invariant on the Shilov boundary which characterize the action of G on $S \times S$. In the particular case of symmetric matrices this invariant has been studied by Hua.

Fix a Jordan frame $(c_j)_{1\leq j\leq r}$ and for $k = 0, 1 \dots, r$ let

$$\epsilon_0 = -e,\ \epsilon_k = \sum_{j=1}^{k} c_j - \sum_{j=k+1}^{r} c_j,\ \epsilon_r = e.$$

Proposition 7.1. *There are exactly $r+1$ orbits in $V^\times$ under the action of G_0. The elements ϵ_k, $0 \leq k \leq r$ are the set of representatives of all the orbits.*

Using this proposition and the Cayley transform one can prove the following

Theorem 7.2. *There are exactly $r+1$ orbits in $S \times S$ under the action of G represented by the family (e, ϵ_k), $0 \leq k \leq r$.*

Let $(\sigma, \tau) \in S \times S$. The *transversality index* of the pair (σ, τ) is defined by

(12) $$\mu(\sigma, \tau) = k$$

where k is the unique integer, $0 \leq k \leq r$ such that (σ, τ) is conjugate under G to the pair (e, ϵ_k).

The transversality index can also be understood as follows : Recall that the rank $\mathrm{rank}(x)$ of an element $x \in V$ is by the number of its non-zero spectral values with their multiplicities counted. This is an invariant under the action of G_0. Observe that x and y have the same rank if and only if $P(x)$ and $P(y)$ have the same rank.

Proposition 7.3 ([**C–K**]). *Let $\sigma, \tau \in S$, then there exists $u \in U$ such that $u(\sigma)$ and $u(\tau)$ are transversal to e. In addition the integer*

$$\mathrm{rank}[c(u(\sigma)) - c(u(\tau))]$$

does not depends on the element $u \in U$ and

$$\mu(\sigma, \tau) = r - \mathrm{rank}[c(u(\sigma)) - c(u(\tau))].$$

Notice that

$$\mu(\sigma,\tau) = k \iff \text{rank } P(\sigma - \tau) = k + \frac{k(k-1)}{2}d.$$

In particular,

$$\mu(\sigma,\tau) = 0 \iff \sigma \top \tau.$$

8. The 3-transitivity property on $S^3_\top$ and the triple Maslov index

In this section we recall another invariant which characterize the action of G on $(S \times S \times S)^\top$. This invariant was introduced by Clerc and Ørsted and provides a generalization of the triple Malsov index.

Let $S^3_\top$ be set of pairwise transversal elements in S^3,

$$S^3_\top = \{(\sigma_1, \sigma_2, \sigma_3) \in S^3\ ,\ \sigma_i \top \sigma_j,\ 1 \leq i \neq j \leq j\}.\}.$$

Choose a Jordan frame $(c_j)_{1 \leq j \leq r}$, $\sum_{j=1}^r c_j = e$, then we have

THEOREM 8.1 ([**C–Ø**$_1$]). *There are exactly $r+1$ orbits in $S^3_\top$ under the action of G, represented by the family $(e, -e, -i\epsilon_k)$, $0 \leq k \leq r$.*

Let $(\sigma_1, \sigma_2, \sigma_3) \in S^3\top$. The *triple Maslov index* $\imath(\sigma_1, \sigma_2, \sigma_3)$ of the triplet $(\sigma_1, \sigma_2, \sigma_3)$ is defined by

$$\imath(\sigma_1, \sigma_2, \sigma_3) = k - (r - k) = 2k - r \tag{13}$$

where k is the unique integer $0 \leq k \leq r$ such that $(\sigma_1, \sigma_2, \sigma_3)$ is conjugate under G to the triplet $(e, -e, -i\epsilon_k)$. One can prove that the triple Maslov index $\imath$ satisfies :

PROPOSITION 8.2 ([**C–Ø**$_1$]). *The triple Maslov index is*

(1) *an integer valued function : for all $(\sigma_1, \sigma_2, \sigma_3) \in S^3\top$,*

$$-r \leq \imath(\sigma_1, \sigma_2, \sigma_3) \leq r$$

(2) *invariant under the action of G : for all $(\sigma_1, \sigma_2, \sigma_3) \in S^3\top$ and all $g \in G$,*

$$\imath(g(\sigma_1), g(\sigma_2), g(\sigma_3)) = \imath(\sigma_1, \sigma_2, \sigma_3)$$

(3) *skew symmetric : for all $(\sigma_1, \sigma_2, \sigma_3) \in S^3\top$ and all permutation π of $\{1, 2, 3\}$,*

$$\imath(\sigma_{\pi(1)}, \sigma_{\pi(2)}, \sigma_{\pi(3)}) = \imath(\sigma_1, \sigma_2, \sigma_3)$$

(4) *a cocycle : for all $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in S$ such that $\sigma_i \top \sigma_j$, $1 \leq i \neq j \leq 4$,*

$$\imath(\sigma_1, \sigma_2, \sigma_3) = \imath(\sigma_1, \sigma_2, \sigma_4) + \imath(\sigma_2, \sigma_3, \sigma_4) + \imath(\sigma_3, \sigma_1, \sigma_4).$$

There is another construction of the triple Maslov index as the integral of the Kaehler form of the Hermitian domain D. Let $z_1, z_2, z_3 \in D$. Form the oriented geodesic triangle $\Delta(z_1, z_2, z_3)$, and consider any surface Σ in D which has this triangle as boundary. Let ω be the Kaehler form of the domain D. Then the real number

$$\varphi(z_1, z_2, z_3) = \int_\Sigma \omega$$

is not depending on Σ, since the Kaehler form is closed, and is called the *symplectic area* of the triangle $\Delta(z_1, z_2, z_3)$. As the Kaehler form is invariant under G, this gives and invariant for the oriented triples in D. Now for $(\sigma_1, \sigma_2, \sigma_3) \in S^3_\top$, then the limit

$$\lim_{z_j \to \sigma_j} \varphi(z_1, z_2, z_3)$$

exists and is equals to the triple Maslov index $\imath(\sigma_1, \sigma_2, \sigma_3)$, see [**C–Ø**$_2$]. This definition of the triple Maslov index extends for general $(\sigma_1, \sigma_2, \sigma_3) \in S^3$, without the transversality condition. This requires a notion of *radial convergence*, see [**C**$_2$] for more details.

9. The universal covering of the Shilov boundary

In this section we introduce the causal structure of the Shilov boundary S and give an explicit construction of the universal covering of S.

9.1. The causal structure of the Shilov boundary. Usually, one demands that the cones in the causal structures be closed. For our purpose we will suppose them open.
The Jordan algebra V has a natural causal structure modelled after the symmetric cone Ω. It is simply given by the symmetric cone Ω viewed as a causal cone in the tangent space $T_x(V) = V$ at any point $x \in V$. It is clear that this causal structure is G^c-invariant .
Consider $-e$ as the base point of S. The tangent space $T_{-e}(S)$ can be identified to iV. Moreover $p(0) = -e$ and the Cayley transform p is well defined in a neighbourhood of 0. Its derivative at 0 is given by

$$Dp(0) = -2i\mathrm{Id}_{V_{\mathbb{C}}}.$$

This allows us to transfer the causal structure form V to S. We define the causal cone C_{-e} to be

$$C_{-e} = Dc(-e)(\Omega) = -i\Omega.$$

This is an invariant cone by the stabilizer of $-e$ in G. We can then define a $G-$invariant causal structure on S as follow : Let $\sigma \in S$, then there exists $g \in G$ such that $g(-e) = \sigma$, then we define the causal cone C_σ to be

$$C_\sigma = Dg(-e)(C_{-e}) = Dg(-e)(-i\Omega). \tag{14}$$

The family $(C_\sigma)_{\sigma \in S}$ is the unique $G-$invariant causal structure of the Shilov boundary S modelled after Ω.

9.2. The construction of the universal covering. The Shilov boundary $S \simeq U/U_e$ is not a semi-simple symmetric space. To construct the universal covering of S, we prefer to deal with its semi-simple part.
Consider the set

$$S_1 = \{\sigma \in S \mid \mathbf{det}(\sigma) = 1\},$$

then S_1 is a connected sub-manifold of S. There exists a character χ of the structure group $Str(V_{\mathbb{C}})$ such that

$$\mathbf{det}(gz) = \chi(g)\,\mathbf{det}(z)$$

for all $g \in Str(V_{\mathbb{C}})$ and all $z \in V_{\mathbb{C}}$. Let

$$U_1 = \{u \in U \mid \chi(u) = 1\}.$$

Then U_1 is a compact semi-simple group and acts transitively on S_1 and

$$S_1 \simeq U_1^\circ/(U_1^\circ \cap Aut(J)),$$

where U_1° is the neutral component of the group U_1. Thus we have

PROPOSITION 9.1 ([**C–K**]). *S_1 a semi-simple Riemannian symmetric space of compact type.*

The Lie algebra of U_1 is $\mathfrak{u}_1 = \mathfrak{k} \oplus i\mathfrak{p}_1$ where

$$\mathfrak{p}_1 = \{L(v) \mid v \in V, \mathbf{tr}(v) = 0\}.$$

Let

$$\mathfrak{a}_1 = \{L(a) \mid a = \sum_{j=1}^{r} a_j c_j,\ a_j \in \mathbb{R},\ \mathbf{tr}(a) = 0\}.$$

Then $\mathfrak{a}_1$ is Cartan subspace of $\mathfrak{p}_1$. We consider now the fundamental lattice Λ_0 of $\mathfrak{a}_1$,

$$\Lambda_0 = \text{lattice generated by } \{2\pi \frac{A_{j,k}}{(A_{j,k}|A_{j,k})} \mid 1 \leq j \neq k \leq r\},$$

where $A_{j,k}$ is the covector of $\frac{1}{2}(a_j - a_k)$. The unitary lattice Λ of $\mathfrak{a}_1$ is given by

$$\Lambda = \{H \in \mathfrak{a}_1 \mid \exp(iH)e = e\}.$$

One can prove (see [**C–K**]) that $\Lambda_0 = \Lambda$. According [**Lo**, Theorem 3.6] (see also [**He**$_2$, Ch. VII. Theorem 8.4 et Theorem 9.1]) we have

THEOREM 9.2 ([**C–K**]). *The symmetric space S_1 is simply connected.*

Following a classical method, we will realize the universal covering of S. let

$$\widetilde{S} = \{(\sigma, \theta) \in S \times \mathbb{R} \mid \mathbf{det}(\sigma) = e^{ir\theta}\},$$

with the topology induced by the topology of $S \times \mathbb{R}$.

THEOREM 9.3 ([**C–K**]). *$\widetilde{S}$ is the universal covering of S.*

In fact we prove that map

$$S_1 \times \mathbb{R} \longrightarrow \widetilde{S} \quad (\sigma, \theta) \longmapsto (e^{i\theta}\sigma, \theta)$$

is a homeomorphism and bijective.

Now we will describe a covering of the conformal group G and give and explicit action of it on the universal covering of the Shilov

boundary.
For $g \in G$ and $z \in D$ we define

$$j(g,z) = \chi(Dg(z)).$$

This is an element of the structure group $Str(V_{\mathbb{C}})$. it is easy to see that $j(g,z) \neq 0$. Since D is simply connected, we can find a determination φ_g of the argument of $j(g,\cdot)$, that is

$$\forall z \in \mathcal{D}, \quad e^{i\varphi_g(z)} = \frac{j(g,z)}{|j(g,z)|}.$$

Two such determinations differs by $2\pi k$.
Consider the following group

$$\Gamma = \big\{ \big(g, \varphi_g\big) \mid g \in G \big\}.$$

The multiplicative law being given by

$$(g, \varphi(g,\cdot))(h, \psi(h,\cdot)) = \big(gh, \varphi(g, h(\cdot)) + \psi(h,\cdot)\big).$$

Observe that Γ can be identified with the closed sub-set of $G \times \mathbb{R}$ given by

$$\big\{ (g,\theta) \in G \times \mathbb{R} \ \mid \ e^{i\theta} = j(g,0) \big\}.$$

Thus Γ becomes a topological group.

Proposition 9.4 ([**C–K**]). *For $(g,\varphi_g) \in \Gamma$ and $(\sigma,\theta) \in \widetilde{S}$, set*

$$(g,\varphi_g) \cdot (\sigma,\theta) = \big(g(\sigma), \theta + \frac{1}{r}\varphi(g,\sigma)\big).$$

Then this defines a continuous action of Γ on $\widetilde{S}$.

To prove this proposition we need the following formula (see [**C–K**, Lemme 3.6])

$$\mathbf{det}(g(\sigma)) = \frac{j(g,\sigma)}{|j(g,\sigma)|}\mathbf{det}(\sigma), \quad for g \in G, \ \sigma \in S. \tag{15}$$

This requires the causal structure of S.

10. The Souriau index

In this section we construct a primitive m of the Maslov cocycle. That is an integer valued function

$$m : \widetilde{S} \times \widetilde{S} \to \mathbb{Z}$$

which is skew symmetric and satisfies the following cohomology property

$$\imath(\sigma_1, \sigma_2, \sigma_3) = m(\widetilde{\sigma}_1, \widetilde{\sigma}_2) + m(\widetilde{\sigma}_2, \widetilde{\sigma}_3) + m(\widetilde{\sigma}_3, \widetilde{\sigma}_1).$$

Let σ belongs to the set $S_\top(-e)$ of elements in S transversal to $-e$. We can define the logarithm of σ by

$$\log \sigma = \int_{-\infty}^{0} \big((se - \sigma)^{-1} - (s-1)^{-1} e\big) ds \in V_{\mathbb{C}}. \tag{16}$$

The function log has the following standard properties :

PROPOSITION 10.1 ([**C–K**]). *For any $\sigma \in S_\top(-e)$,*

(*i*) $\exp(\log \sigma) = \sigma$.
(*ii*) $e^{\mathbf{tr}(\log \sigma)} = \mathbf{det}(\sigma)$.
(*iii*) $\log \sigma^{-1} = -\log \sigma$.
(*iv*) $\log(k\sigma) = k \log \sigma$, *for any $k \in Aut(J)$.*

Let $\widetilde{\sigma} = (\sigma, \theta)$ and $\widetilde{\tau} = (\tau, \phi)$ are two elements of $\widetilde{S}$. We say that $\widetilde{\sigma}$ and $\widetilde{\tau}$ are transversal if the projections are transversal, $\sigma \top \tau$. Then there exists $u \in U$ such that $u^{-1}(\tau) = -e$ and $u^{-1}(\sigma) \top -e$. We can apply (16) to $u^{-1}(\sigma)$ and define the *Souriau index* of the pair $(\widetilde{\sigma}, \widetilde{\tau})$ to be

$$m(\widetilde{\sigma}, \widetilde{\tau}) = \frac{1}{\pi}\Big[\frac{1}{i}\,\mathbf{tr}(\log u^{-1}(\sigma)) - r(\theta - \phi)\Big]. \tag{17}$$

THEOREM 10.2 ([**C–K**]). *The Souriau index is $\mathbb{Z}-$valued continuous function on $S^2_\top$ and is invariant under the action of the covering Γ.*

We also prove the following essential cohomological property

THEOREM 10.3 ([**C–K**]). *If $\widetilde{\sigma}_1, \widetilde{\sigma}_2, \widetilde{\sigma}_3 \in \widetilde{S}$ have pairwise transverse projections $\sigma_1, \sigma_2, \sigma_3$, then*

$$m(\widetilde{\sigma}_1, \widetilde{\sigma}_2) + m(\widetilde{\sigma}_2, \widetilde{\sigma}_3) + m(\widetilde{\sigma}_3, \widetilde{\sigma}_1)$$

is an integer and coincides with the triple Maslov index of the triplet $(\sigma_1, \sigma_2, \sigma_3)$,

$$\imath(\sigma_1, \sigma_2, \sigma_3) = m(\widetilde{\sigma}_1, \widetilde{\sigma}_2) + m(\widetilde{\sigma}_2, \widetilde{\sigma}_3) + m(\widetilde{\sigma}_3, \widetilde{\sigma}_1)$$

Following an idea of [$\mathbf{deG}_1$], we extend the definition of the Souriau index to $S\times S$ and prove Theorem 10.3 without the transversality condition.

Now, fix a Jordan frame $(c_j)_{1\leq j\leq r}$, then we have

Theorem 10.4 ([**C–K**]). *Fix*

$$\widetilde{\sigma}_1=(\sum_{j=1}^{\ell}e^{i\theta_j}c_j+\sum_{j=\ell+1}^{r}e^{i\theta_j}c_j,\theta),\ \ \widetilde{\sigma}_2=(\sum_{j=1}^{\ell}e^{i\theta_j}c_j+\sum_{j=\ell+1}^{r}e^{i\varphi_j}c_j,\varphi)\in\widetilde{S}$$

such that $\theta_j-\varphi_j\notin 2\pi\mathbb{Z}$ *for* $\ell+1\leq j\leq r$. *Then*

$$m(\widetilde{\sigma}_1,\widetilde{\sigma}_2)=\frac{1}{\pi}\Big[\sum_{j=\ell+1}^{r}\{\theta_j-\varphi_j+\pi\}-r(\theta-\varphi)\Big]. \tag{18}$$

In particular, if $\widetilde{\sigma}_1=(-e,-\pi)$ and $\widetilde{\sigma}_2=(-\sum_{j=1}^{\ell}c_j+\sum_{j=\ell+1}^{r}e^{i\varphi_j}c_j,\varphi)$, where

(*i*) $-\pi<\varphi_j<\pi$, $\forall j$, $\ell+1\leq j\leq r$, et
(*ii*) $r\varphi=-\ell\pi+\sum_{j=\ell+1}^{r}\varphi_j+2k\pi$, with $k\in\mathbb{Z}$,

then

$$m(\widetilde{\sigma}_1,\widetilde{\sigma}_2)=2k+r-\ell=2k+r-\mu(\sigma_1,\sigma_2), \tag{19}$$

where $\mu(\sigma_1,\sigma_2)$ is the transversality index of the pair (σ_1,σ_2).

11. The Arnold-Leray index

In this section we generalize the notion of Maslov cycles, prove their stratification and their causal orientation. We use these geometric properties to construct another primitive to the triple Maslov index.

11.1. The Maslov cycles. Given $\sigma_0 \in S$ we consider the following sub-manifold of point of S which are not transversal to σ_0,

$$\Sigma(\sigma_0) = S \setminus S_\top(\sigma_0) = \{\sigma \in S \mid \mathbf{det}(\sigma - \sigma_0) = 0\}.$$

This is the *Maslov cycle* attached to σ_0.

THEOREM 11.1 ([**C–K**]). *The Maslov cycle $\Sigma(\sigma_0)$ is a stratified sub-manifold of S of codimension 1 and its singularity is of codimension ≥ 3 in S.*

In fact the stratums are the sets

$$S_k(\sigma_0) = \{\sigma \in S \ ; \ \mu(\sigma, \sigma_0) = k\}, \ 0 \leq k \leq r.$$

We prove that $S_k(\sigma_0)$ is a sub-manifold of S of codimension $k + \frac{k(k-1)}{2}d$, and

$$\Sigma(\sigma_0) = \bigsqcup_{1 \leq k \leq r} S_k(\sigma_0) = \overline{S_1(\sigma_0)}.$$

The singularity of the Maslov cycle $\Sigma(\sigma_0)$ is $\bigsqcup_{2 \leq k \leq r} S_k(\sigma_0) = \overline{S_2(\sigma_0)}$, and the set of regular points is $S_1(\sigma_0)$.

Now we wish to prove that the Maslov cycle is in addition oriented. For this purpose we use one more time the causal structure of the Shilov boundary S and prove the following fundamental fact :

PROPOSITION 11.2 ([**C–K**]). *Let $\sigma \in S_1(\sigma_0)$ be a regular point of $\Sigma(\sigma_0)$. Set $H_{\sigma_0}(\sigma) = T_\sigma(\Sigma(\sigma_0))$. Then the tangent vectors of all causal curves starting from σ are all contained in the same half space of $T_\sigma(S)$ limited by $H_{\sigma_0}(\sigma)$.*

Hence $H_{\sigma_0}(\sigma)$ has two sides : $+$ side and $-$ side. The $+$ side is the one which contains all mentioned tangent vectors. We will denote it by $H^+_{\sigma_0}(\sigma)$. The family

$$(H^+_{\sigma_0}(\sigma))_{\sigma \in S_1(\sigma_0)}$$

is called the *canonical transverse orientation* of the Maslov cycle $\Sigma(\sigma_0)$. We now claim that this orientation is compatible with the action of the group G : let $g \in G$, then the transverse orientation of the Maslov cycle $\Sigma(g\sigma_0)$ is given by the family

$$Dg(\sigma)\left[H^+_{\sigma_0}(\sigma)\right] = H^+_{g\sigma_0}(g\sigma), \ \ \sigma \in S_1(\sigma_0),$$

since the causal structure of S is $G-$invariant.

11.2. The Arnold-Leray index. We now wish to use the topological properties of Maslov cycles to construct a homotopy invariant. Let $\sigma_0 \in S$ and set $\Sigma_0 = \Sigma(\sigma_0)$. A *proper path* (relatively to Σ_0) in S is a smooth path $\gamma : [0,1] \to S$ such that $\gamma(0) \notin \Sigma_0$, $\gamma(1) \notin \Sigma_0$ and intersects $S_1(\sigma_0)$ transversally in a finite number of crossings, say in $t_1, t_2, \ldots, t_k$.

We define the *Arnold number* $\nu_A(\gamma)$ of the proper path γ to be the number of intersections of γ with $S_1(\sigma_0)$, each counted with sign $\pm$ according to whether the crossing is in the positive or negative direction. Or in the same thing,

$$\nu_A(\gamma) = \epsilon_1 + \ldots + \epsilon_k, \quad \text{with} \quad \epsilon_j = \begin{cases} +1 & \text{if } \dot{\gamma}(t_j) \in H^+_{\sigma_0}(\gamma(t_j)) \\ -1 & \text{if } \dot{\gamma}(t_j) \in H^-_{\sigma_0}(\gamma(t_j)) \end{cases}$$

It is easy to show that Arnold number satisfies the following properties (see [**C–K**]) :

- Every homotopy class of a given path contains a proper path with the same endpoints.
- Two homotopic proper paths with the same endpoints have the same Arnold number.

This allows us to define the Arnold number for any path γ to be the Arnold number of a proper path γ_{proper} homotopic to γ with the same endpoints,

$$\nu_A(\gamma) = \nu_A(\gamma_{\text{proper}}).$$

We will now define an index of a pair of point of the universal covering $\widetilde{S}$ by using the invariance by homotopy of the Arnold number.
Let $\widetilde{\sigma}_0, \widetilde{\tau}_0 \in \widetilde{S}$, and $\sigma_0, \tau_0 \in S$ their projections. Let $\sigma(t)$, $0 \le t \le 1$ be a causal curve such that $\sigma(0) = \sigma_0$. Let $\tau(t)$, $0 \le t \le 1$ be an anti-causal curve such that $\tau(0) = \tau_0$. Let $\widetilde{\sigma}(t)$ be the lift of $\sigma(t)$ with origin $\widetilde{\sigma}_0$, and $\widetilde{\tau}(t)$ be the lift of $\tau(t)$, with origin $\widetilde{\tau}_0$.

Then we claim, see [**C–K**], that there exists $\epsilon > 0$ such that for all t, $0 < t < \epsilon$ the points $\sigma(t)$ and $\tau(t)$ are outside of Maslov cycle $\Sigma(\sigma_0)$ attached with σ_0. Fix a such t and let $\gamma_t(s)$, $0 \le s \le 1$ be a proper path (relatively to $\Sigma(\sigma_0)$) with origin $\sigma(t)$ and end $\tau(t)$ such that its lift is of origin $\widetilde{\sigma}(t)$ and end $\widetilde{\tau}(t)$.

We define the *Arnold index* $\nu(\widetilde{\sigma}_0, \widetilde{\tau}_0)$ of the pair $(\widetilde{\sigma}_0, \widetilde{\tau}_0)$ to be the Arnold number of the path γ_t,

$$\nu(\widetilde{\sigma}_0, \widetilde{\tau}_0) = \nu_A(\gamma_t).$$

Clearly, for fixed t, this index does not depend on the choice of the path γ_t, since any other path having the same properties is homotopic to γ_t and thus has the same Arnold number. Moreover we prove that $\nu(\widetilde{\sigma}_0, \widetilde{\tau}_0)$ does not depend on the parameter t, and on which of the (causal or anti-causal) curves $\sigma(t)$ and $\tau(t)$ we use.

The construction of the Arnold index uses only the invariant concepts by causal transformations (causal curves, Maslov cycles, transversality), and thus the Arnold index is invariant under the action of the group Γ.

We will now calculate the index of Arnold “ in coordinates ”, as we did for the index of Souriau. One fixes for that a Jordan frame $(c_j)_{1\leq j\leq r}$ of V. Thanks to the results concerning the orbits of the action of G in $S\times S$ (see section 9.1), the following proposition covers the general case.

Proposition 11.3 ([**C–K**]). *Let*

$$\widetilde{\sigma}_0 = \widetilde{-e} = (-e, -\pi) \ \ \textit{and} \ \ \widetilde{\tau}_0 = \Big(-\sum_{j=1}^{\ell} c_j + \sum_{j=\ell+1}^{r} e^{i\varphi_j} c_j, \ \varphi\Big) \in \widetilde{S}.$$

One notes $\sigma_0 = -e$ and τ_0 their corresponding projections on S. Suppose

(i) $-\pi < \varphi_j < \pi$, $\forall j$, $\ell + 1 \leq j \leq r$;
(ii) $r\varphi = -\ell\pi + \sum_{j=\ell+1}^{r} \varphi_j + 2k\pi$, *with* $k \in \mathbb{Z}$.

Then

$$\nu(\widetilde{\sigma}_0, \widetilde{\tau}_0) = -\ell + k = k - \mu(\sigma_0, \tau_0). \tag{20}$$

Corollairy 11.4 ([**C–K**]).

$$\nu(\widetilde{\sigma}, \widetilde{\tau}) = \frac{1}{2}\big[m(\widetilde{\sigma}, \widetilde{\tau}) - \mu(\sigma, \tau) - r\big]. \tag{21}$$

A consequence of this corollary is that the right-hand side of the formula (21) is an integer. This allows us to introduce the *index of inertia* and the *Arnold-Leray* index.

Let $(\sigma_1, \sigma_2, \sigma_3) \in S^3$. We define the inertia index of the triplet $(\sigma_1, \sigma_2, \sigma_3)$ to be

$$\jmath(\sigma_1, \sigma_2, \sigma_3) = \frac{1}{2}\big(\iota(\sigma_1, \sigma_2, \sigma_3) + \mu(\sigma_1, \sigma_2) - \mu(\sigma_1, \sigma_3) + \mu(\sigma_2, \sigma_3) + r\big).$$

Let $\widetilde{\sigma}_1, \widetilde{\sigma}_2 \in \widetilde{S}$, and σ_1, σ_2 the corresponding projections. We define the Arnold-Leray index to be

$$n(\widetilde{\sigma}_1, \widetilde{\sigma}_2) = \nu(\widetilde{\sigma}_1, \widetilde{\sigma}_2) + \mu(\sigma_1, \sigma_2) + r.$$

One finishes this section by announcing this theorem

THEOREM 11.5 ([**C–K**]). *The index of inertia satisfies the following :*

(i) *$\jmath$ is $\mathbb{Z}-$valued function.*
(ii) *$\jmath$ is a 2-cocycle*[1]

$$\jmath(\sigma_1, \sigma_2, \sigma_3) - \jmath(\sigma_1, \sigma_2, \sigma_4) + \jmath(\sigma_1, \sigma_3, \sigma_4) - \jmath(\sigma_2, \sigma_3, \sigma_4) = 0$$

fro all $\sigma_1, \sigma_2, \sigma_3 \in S$.

(iii) *The Arnold-Leray index is a primitive of the index of inertia, i.e.*

$$\jmath(\sigma_1, \sigma_2, \sigma_3) = n(\widetilde{\sigma}_1, \widetilde{\sigma}_2) - n(\widetilde{\sigma}_1, \widetilde{\sigma}_3) + n(\widetilde{\sigma}_2, \widetilde{\sigma}_3)$$

for all $\widetilde{\sigma}_1, \widetilde{\sigma}_2, \widetilde{\sigma}_3 \in \widetilde{S}$, with the corresponding projections $\sigma_1, \sigma_2, \sigma_3$.

[1]It will be observed that the index of inertia $\jmath$ does not satisfy the skew-symmetric property that has the triple Maslov index $\imath$.

12. The Poincaré rotation number of the conformal group

In This section we use the Souriau index to generalize the notion of Poincaré rotation number

A group G is *uniformly perfect* , if there exists an integer k such that, every element $g \in G$ is a product of k commutators at the maximum. A such group has the following property :

Let G be a uniformly perfect group and let

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \Gamma \xrightarrow{\pi} G \longrightarrow 1$$

be a central extension of G. Let $T = \iota(1)$. Then there exists at the maximum one map $\Phi : \Gamma \to \mathbb{R}$ such that

(1) $\Phi(\gamma T) = \Phi(\gamma) + 1$, $\forall \gamma \in \Gamma$
(2) $\Phi(\gamma_1\gamma_2) - \Phi(\gamma_1) - \Phi(\gamma_2)$ *is bounded on* $\Gamma \times \Gamma$
(3) $\Phi(\gamma^n) = n\Phi(\gamma)$, $\forall \gamma \in \Gamma$, $\forall n \in \mathbb{Z}$.

A map Φ satisfying (2) is called a *quasi-morphism.* A map Φ satisfying (2) and (3) is called a *homogeneous quasi-morphism.*

If Φ exists, then the function

$$c : G \times G \to \mathbb{R}, \quad c(g_1, g_2) = \Phi(\gamma_1\gamma_2) - \Phi(\gamma_1) - \Phi(\gamma_2)$$

(where γ_i is the lift of g_i) is well defined and is a 2-cocycle, *i.e.*

$$c(g_1, g_2) + c(g_1g_2, g_3) = c(g_1, g_2g_3) + c(g_2, g_3).$$

Let us consider the example $G = Homeo^+(S^1)$, the group of all homeomorphisms of the circle preserving the orientation, where S^1 is the oriented unite circle. This group is uniformly perfect and we consider the central extension

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\iota} \widetilde{Homeo^+}(S^1) \xrightarrow{\pi} Homeo^+(S^1) \longrightarrow 1$$

where $\widetilde{Homeo^+}(S^1)$ is the universal covering of $Homeo^+(S^1)$. We will exhibit a function satisfying (1), (2) and (3).
Let us introduce the cyclic order. Let $p, q, r \in S^1$, the *cyclic order* of p, q, r is defined by

$$\operatorname{ord}(p, q, r) = \begin{cases} 0 & \text{if 2 points coincide} \\ 1 & \text{if } q \in)p, r(\\ -1 & \text{if } q \in)r, p(. \end{cases}$$

Let

$$\Phi_{\mathrm{ord}} : \widetilde{Homeo^+}(S^1) \to \mathbb{Z},\ \Phi_{\mathrm{ord}}(\widetilde{f}) = 2E(\widetilde{f}(0))$$

where

$$E(x) = \begin{cases} x & \text{if } x \in \mathbb{Z}, \\ [x] + \frac{1}{2} & \text{if } x \notin \mathbb{Z}. \end{cases}$$

Then Φ_{ord} is (non homogeneous) quasi-morphism. Indeed,

$$\Phi_{\mathrm{ord}}(\widetilde{f} \circ \widetilde{g}) - \Phi_{\mathrm{ord}}(\widetilde{f}) - \Phi_{\mathrm{ord}}(\widetilde{g}) = \mathrm{ord}(1, f(1), f \circ g(1))$$

which is bounded. Moreover, the following limit exists

$$\lim_{n \to +\infty} \frac{1}{n} \Phi_{\mathrm{ord}}(\widetilde{f}^n) = 2 \lim_{n \to +\infty} \frac{E(\widetilde{f}^n(0))}{n} = 2\tau(\widetilde{f}).$$

The function $\tau : \widetilde{Homeo^+}(S^1) \to \mathbb{R}$ is the so-called *Poincaré translation number* . This function satisfies (1), (2) and (3). Passing to the quotient, we get a function $\rho : Homeo^+(S^1) \to \mathbb{R}/\mathbb{Z} \simeq S^1$, which is the so-called *Poincaré rotation number* .

We return to the general case of Hermitian symmetric spaces of tube type $\mathcal{D} = G/K$. We will use the triple Maslov index and the Souriau index to generalize the notion of the Poincaré rotation number. We prove first the following

Proposition 12.1 ([**C–K**]). *Let $G = KAN$ be the Cartan decomposition of G, then*

(*i*) *Every element of N is a commutator.*
(*ii*) *Every element of A is a product of r commutators at the maximum.*

We also need the following lemma

Lemma 12.2 ([**C–K**]). *Consider the following central extension of G*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\ \imath\ } \Gamma \xrightarrow{\ p\ } G \longrightarrow 1.$$

Then there exists at the maximum one map $\Phi : \Gamma \to \mathbb{R}$ such that

(0) Φ *is continuous*
(1) $\Phi(\gamma T) = \Phi(\gamma) + 1$, $\forall \gamma \in \Gamma$
(2) $\Phi(\gamma_1\gamma_2) - \Phi(\gamma_1) - \Phi(\gamma_2)$ *is bounded on* $\Gamma \times \Gamma$
(3) $\Phi(\gamma^n) = n\Phi(\gamma)$, $\forall \gamma \in \Gamma$, $\forall n \in \mathbb{Z}$.

We begin now to construct the Poincaré rotation number on G. Let $\widetilde{o}$ be a base point of $\widetilde{S}$, the universal covering of the Shilov boundary. Then the function

$$\mathsf{c} : \Gamma \to \mathbb{Z} \; : \; \mathsf{c}(\gamma) = m(\gamma \cdot \widetilde{o}, \widetilde{o})$$

where m is the Souriau index, is a (non homogeneous) quasi-morphism. Indeed,

$$\begin{aligned} \mathsf{c}(\gamma_1\gamma_2) - \mathsf{c}(\gamma_1) - \mathsf{c}(\gamma_2) \quad &= m(\gamma_1\gamma_2 \cdot \widetilde{o}, \widetilde{o}) + m(\widetilde{o}, \gamma_1 \cdot \widetilde{o}) + m(\gamma_1 \cdot \widetilde{o}, \gamma_1\gamma_2 \cdot \widetilde{o}) \\ &= \imath(o, g_1 \cdot o, g_1 g_2 \cdot o), \end{aligned}$$

which is bounded by the rank r, where γ_j is the lift of g_j, $j = 1, 2, 3$. Hence, for $\gamma \in \Gamma$, the sequence $\mathsf{c}_k = \mathsf{c}(\gamma^k)$ satisfies

$$|\mathsf{c}_{k+\ell} - \mathsf{c}_k - \mathsf{c}_\ell| \leq r.$$

Thus, the following limit exists

$$\lim_{k \to +\infty} \frac{1}{k} \mathsf{c}(\gamma^k) := \tau(\gamma).$$

THEOREM 12.3 ([**C–K**]). *The function $-\frac{1}{2}\tau$ is a continuous homogeneous quasi-morphism of Γ and it is independent of the choice of $\widetilde{o}$.*

The function τ is the *generalized Poincaré translation number* . Passing to the quotient, the function

$$\rho(g) = -\frac{1}{2}\tau(\gamma) \mod (\mathbb{Z})$$

where γ is the lift of g, is the *generalized Poincaré rotation number* of G. Finally, we prove the following

PROPOSITION 12.4 ([**C–K**]). *The function ρ satisfies ;*

(1) *ρ is invariant by conjugaison.*
(2) *If $g \in G$ fixes a point in S, then $\rho(g) = 0$.*
(3) *If $u \in U$, then $e^{2i\pi\rho(u)} = \chi(u)$*

Part 2

Non-commutative Hardy spaces

13. Hardy spaces on Lie semi-groups

In this section we recall the theory of Hardy spaces on Lie semi-groups due to Olshanskiĭ

Let $\mathfrak{g}$ be a simple Lie algebra over the reals $\mathbb{R}$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$. Let $\mathfrak{t} \subset \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{k}$. We shall suppose that $\mathfrak{k}$ has a non–zero center $\mathfrak{z}$; then $\mathfrak{z}$ is one dimensional and $\mathfrak{t}$ is also a Cartan subalgebra of $\mathfrak{g}$.

Let $G_{\mathbb{C}}$ be the simply connected complex Lie group corresponding to $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} + i\mathfrak{g}$, and let G, K and T be the connected subgroups in $G_{\mathbb{C}}$ corresponding to $\mathfrak{g}$, $\mathfrak{k}$ and $\mathfrak{t}$ respectively. By the Kostant–Paneitz–Vinberg Theorem [**V**], there are non–trivial regular cones C in $i\mathfrak{g}$ which are $\mathrm{Ad}(G)-$invariant, where regular means, convex, closed, pointed ($C \cap -C = \{0\}$) and generating ($C - C = i\mathfrak{g}$). Let $\mathrm{Cone}(i\mathfrak{g})$ be the set of all regular $\mathrm{Ad}(G)-$invariant cones in $i\mathfrak{g}$.

For such a cone C in $\mathrm{Cone}(i\mathfrak{g})$, Ol'shanskiĭ associates a semigroup $\Gamma(C) := G\exp(C)$ in $G_{\mathbb{C}}$, and for this semigroup he associates a "non-commutative" Hardy space $H^2(\Gamma(C))$ which is the set of holomorphic functions f on the complex manifold $\Gamma(C)^{\circ} = G\exp(C^{\circ})$, the interior of $\Gamma(C)$, such that

$$\sup_{\gamma\in\Gamma(C)^{\circ}} \int_G |f(g\gamma)|^2\, dg < \infty.$$

For any $\gamma \in \Gamma(C)^{\circ}$ the linear functional $f \longmapsto f(\gamma)$ is continuous on $H^2(\Gamma(C))$. Therefore by the Riesz representation theorem, there exists a vector $K_\gamma \in H^2(\Gamma(C))$ such that $(f, K_\gamma) = f(\gamma)$. The reproducing kernel K which is called the *Cauchy–Szegö kernel* is defined by

$$K(\gamma_1, \gamma_2) = K_{\gamma_2}(\gamma_1).$$

It is Hermitian, holomorphic in γ_1 and anti–holomorphic in γ_2.

Let $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ be the set of roots of $\mathfrak{g}_{\mathbb{C}}$ relative to $\mathfrak{t}_{\mathbb{C}}$. Let $\Delta^+ \subset \Delta$ be the set of positive roots relative to some order (namely the one where the center of $\mathfrak{k}$ comes first), $\Delta^+_{\mathfrak{k}}$ and $\Delta^+_{\mathfrak{p}}$ the set of positive compact and non-compact roots , respectively. Put $\mathfrak{t}_{\mathbb{R}} := i\mathfrak{t} \subset \mathfrak{t}_{\mathbb{C}}$. We identify $\mathfrak{t}_{\mathbb{R}}$ with its own dual via the Cartan–Killing form. Then we can consider $\Delta \subset \mathfrak{t}_{\mathbb{R}}$. Let $\mathcal{P} \subset \mathfrak{t}^*_{\mathbb{R}} \simeq \mathfrak{t}_{\mathbb{R}}$ be the set of weights relative to T and let $\mathcal{R}$ be the set of all highest weights relative to $\Delta^+_{\mathfrak{k}}$,

$$\mathcal{R} = \{\lambda \in \mathcal{P} \mid (\forall \alpha \in \Delta^+_{\mathfrak{k}})\ \langle \lambda, \alpha \rangle \geq 0\}.$$

Let ρ be the half sum of all positive roots. Then by Harish–Chandra ([**HC**$_1$],[**HC**$_2$],[**HC**$_3$]) the holomorphic discrete series representations for the group G are those irreducible unitary representations of G that are square–integrable with a highest weight λ belonging to

$$\mathcal{R}' = \{\lambda \in \mathcal{R} \mid (\forall \beta \in \Delta_{\mathfrak{p}}^+)\ \langle \lambda + \rho, \beta \rangle < 0\}.$$

We will say that $\lambda \in \mathcal{R}$ satisfies the *Harish–Chandra condition* if

$$\langle \lambda + \rho, \beta \rangle < 0, \quad \forall \beta \in \Delta_{\mathfrak{p}}^+.$$

By Vinberg [**V**], there exists in $\mathrm{Cone}(i\mathfrak{g})$ a unique (up to multiplication by -1) maximal cone $C_{\max}$, such that

$$C_{\max} \cap \mathfrak{t}_{\mathbb{R}} = c_{\max} := \{X \in \mathfrak{t}_{\mathbb{R}} \mid (\forall \alpha \in \Delta_{\mathfrak{p}}^+)\ \langle X, \alpha \rangle \geq 0\},$$

and a unique minimal cone $C_{\min} = C_{\max}^*$, such that $C_{\min} \cap \mathfrak{t}_{\mathbb{R}} = c_{\min}$ is the convex cone spanned by all α in $\Delta_{\mathfrak{p}}^+$. A unitary representation π

of G in a Hilbert space $\mathcal{H}$ is said to be $C-$*dissipative* if for all $X \in C$ and all $\xi \in \mathcal{H}^\infty$, the space of $\mathcal{C}^\infty$ vectors in $\mathcal{H}$,

$$(\pi(X)\xi|\xi) \leq 0.$$

We can now state the Theorem B of Ol'shanskiĭ [$\mathbf{O}_2$] on the non–commutative Hardy spaces

THEOREM 13.1 ([$\mathbf{O}_2$]). *The Hardy space $H^2(\Gamma(C))$ is a non–trivial Hilbert space for any $C \in \mathrm{Cone}(i\mathfrak{g})$.*
The representation of G in $H^2(\Gamma(C))$ can be decomposed into a direct sum of irreducible unitary representations of G. The components of this decomposition are precisely all the holomorphic discrete series representations of G which are $C-$dissipative.

The group $G \times G$ acts on $H^2(\Gamma(C))$ via left and right regular representations. Therefore

$$H^2(\Gamma(C)) = \bigoplus_{\lambda \in (C^* \cap \mathfrak{t}_{\mathbb{R}}) \cap \mathcal{R}'} \pi_\lambda \otimes \pi_\lambda^*, \tag{22}$$

where π_λ is the *contraction representation* of $\Gamma(C)$ corresponding to a unitary highest weight representation of G with highest weight λ. Moreover, the corresponding function of the Cauchy–Szegö kernel K of $H^2(\Gamma(C))$ can be written on $\Gamma(C)^\circ$ as follows

$$K(\gamma) := K(\gamma, e) = \sum_{\lambda \in (C^* \cap \mathfrak{t}_{\mathbb{R}}) \cap \mathcal{R}'} d_\lambda \, \mathbf{tr}(\pi_\lambda(\gamma)), \tag{23}$$

where d_λ denotes the *formal dimension* of the representation π_λ. The series for K converges uniformly on compact subsets in $\Gamma(C)^\circ$.

Remark 13.2. *Whenever* C *is the minimal cone,* C_{min}*, the decompositions (22) and (23) are over all the holomorphic discrete series, i.e. over* $\lambda \in \mathcal{R}'$.
One of the most important problems in this areas is to give an explicit formula for the function $K(\gamma)$.

14. The contraction semigroup

In this section we restrict our self to $G = Sp(r,\mathbb{R})$, $G = SO^(2\ell)$ and $G = U(p,q)$. We prove that in this case the Lie semigroup is a semigroup of contractions. We also prove that the image of this semigroup under a new Cayley transform is the tube domain (modulo some singular points).

From now on we assume that G is one of the classical groups $U(p,q)$, $Sp(r,\mathbb{R})$ or $SO^*(2l)$. Let σ be an involution in $\mathfrak{g}_\mathbb{C}$ such that

$$\mathfrak{g} = \{X \in \mathfrak{g}_\mathbb{C} \mid \sigma(X) = -X\}.$$

Then

$$\sigma(X) = JX^*J,$$

where X^* is the adjoint matrix and

$$\begin{array}{lll} \text{for} & \mathfrak{g} = \mathfrak{u}(p,q), & J = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix} \\ \text{for} & \mathfrak{g} = \mathfrak{sp}(r,\mathbb{R}), & J = \begin{pmatrix} -I_r & 0 \\ 0 & I_r \end{pmatrix} \\ \text{for} & \mathfrak{g} = \mathfrak{o}^*(2l), & J = \begin{pmatrix} -I_l & 0 \\ 0 & I_l \end{pmatrix}. \end{array}$$

REMARK 14.1. *$U(p,q)$ is not a simple Hermitian Lie groups. Since*

$$U(p,q) \simeq \big(U(1) \times SU(p,q)\big)/\mathbb{Z}_{p+q},$$

the holomorphic discrete series representations of $U(p,q)$ are the holomorphic discrete series representations of the circle times the Hermitian group $SU(p,q)$ which are trivial on $(\xi, \xi^{-1}I_n)$, for $\xi^n = 1$ ($n = p+q$). Therefore one can easily generalize the results of section 1 to the reductive group $U(p,q)$.

Let C be the regular cone in $i\mathfrak{g}$ defined by

$$C := \{X \in i\mathfrak{g} \mid JX \leq 0\},$$

and let $\Gamma(C) := G\exp(C)$ be the corresponding Ol'shanskiĭ semigroup. An element γ of $G_\mathbb{C}$ is said to be a *$J-$contraction* (resp. a *strict $J-$contraction*) if $J - \gamma^*J\gamma \geq 0$ (resp. $J - \gamma^*J\gamma \gg 0$).

PROPOSITION 14.2 ([**K–Ø**$_3$]). *The semigroup $\Gamma(C)$ is the $J-$contractions semigroup,*

$$\Gamma(C) = \{\gamma \in G_\mathbb{C} \mid J - \gamma^*J\gamma \geq 0\},$$

and $\Gamma(C)^\circ$ is the semigroup of strict $J-$contractions,

$$\Gamma(C)^\circ = \{\gamma \in G_\mathbb{C} \mid J - \gamma^*J\gamma \gg 0\}.$$

Now, let V be one of the *Jordan algebras* $Herm(n, \mathbb{C})$, $Sym(2r, \mathbb{R})$ or $Herm(l, \mathbb{H})$ and let Ω be the corresponding *symmetric cone*. Then $\Omega = V^+$ is the set of positive definite matrices in V. The tube domain $T_\Omega := V + i\Omega$ is a Hermitian symmetric space isomorphic to $G^\flat/K^\flat$, where $G^\flat$ is $SU(n,n)$, $Sp(2r, \mathbb{R})$ or $SO^*(4l)$ respectively and $K^\flat$ the corresponding maximal compact subgroup, i.e. $S\big(U(n) \times U(n)\big)$, $U(2r)$ or $U(2l)$ respectively.

Let $\mathtt{C}$ be the *Cayley transform* defined by

$$\mathtt{C}(Z) := (Z - iJ)(Z + iJ)^{-1}$$

whenever the matrix $(Z + iJ)$ is invertible.

PROPOSITION 14.3 ([**K–Ø**$_3$]). *The Cayley transform* $\mathtt{C}$ *is a biholomorphic bijection from an open subset of the tube domain* T_Ω *onto the complex manifold* $\Gamma(C)^\circ$. *More precisely, if* Σ *denotes the hypersurface* $\Sigma = \{Z \in T_\Omega \mid \mathbf{det}(Z + iJ) = 0\}$, *then*

$$\mathtt{C}(T_\Omega \setminus \Sigma) = \Gamma(C)^\circ. \tag{2.3}$$

Here “**det**” denotes the determinant of the Jordan algebra $V_\mathbb{C}$ (see [**F–K**]).

15. The holomorphic discrete series

In this section we recall the holomorphic discrete series representations of $G^\flat$ and explain our strategy to compare the Hardy space of the Lie semigroup and the Hardy space of the tube domain.

Let N and R be the dimension and the rank of the Jordan algebra V. For a complex manifold $\mathcal{M}$ we denote by $\mathcal{O}(\mathcal{M})$ the space of holomorphic functions on $\mathcal{M}$.

The group $G^\flat$ acts on T_Ω via

$$g \cdot Z = (AZ+B)(CZ+D)^{-1}, \quad g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right),$$

and the scalar–valued holomorphic discrete series representations of $G^\flat$ are

$$\big(U_\lambda(g)f\big)(Z) = \mathbf{det}(CZ+D)^{-\lambda} f(g^{-1}\cdot Z), \quad g^{-1} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$$

for $\lambda \geq 2\frac{N}{R}$, which all are unitary and irreducible in the Hilbert spaces

$$\mathcal{H}_\lambda(T_\Omega) := \{f \in \mathcal{O}(T_\Omega) \mid \int_{T_\Omega} |f(X+iY)|^2 \,\mathbf{det}(Y)^{\lambda - 2\frac{N}{R}} dX dY < \infty\}.$$

Moreover the reproducing kernel of $\mathcal{H}_\lambda(T_\Omega)$ is given by

$$K_\lambda^{T_\Omega}(Z,W) = \mathbf{det}\Big(\frac{Z-W^*}{2i}\Big)^{-\lambda}.$$

The *classical Hardy space* $H^2(T_\Omega)$ on T_Ω is defined as the space of holomorphic functions f on T_Ω such that

$$\sup_{Y\in\Omega} \int_V |f(X+iY)|^2 \, dX < \infty.$$

Proposition 15.1. *The Hardy space $H^2(T_\Omega)$ may be thought of as the space $\langle_\lambda(T_\Omega)$ for $\lambda = \frac{N}{R}$, and the Cauchy–Szegö kernel of T_Ω is given by*

$$K(Z,W) = \mathbf{det}\Big(\frac{Z-W^*}{2i}\Big)^{-N/R}.$$

We list here the groups G and the corresponding group $G^\flat$, Jordan algebra V, its rank R, its dimension N, and the determinant $\mathbf{det}$:

G	$G^\flat$	V	N	R	$\mathbf{det}$
$Sp(r,\mathbb{R})$	$Sp(2r,\mathbb{R})$	$Sym(2r,\mathbb{R})$	$r(2r+1)$	$2r$	Det
$SO^*(2l)$	$SO^*(4l)$	$Herm(l,\mathbb{H})$	$l(2l-1)$	l	$\mathrm{Det}^{1/2}$
$U(p,q)$	$SU(n,n)$	$Herm(n,\mathbb{C})$	n^2	n	Det

A crucial point is to compare holomorphic functions on the tube domain with their pull-backs on the semigroup via the Cayley transform, and vice versa. In particular, it will be important to know the rate of growth of the functions near the singularity Σ above. Assuming $\gamma = \mathtt{C}(Z)$ we have that

$$Z + iJ = 2(I - \gamma)^{-1} iJ$$

so that to approach the singularity in the Z variable, means that $\mathbf{det}(I - \gamma)$ tends to infinity in the γ variable. Clearly this condition is invariant under conjugation with G, so we may reduce the question of the growth near the singularity to a question on the compact Cartan subspace. Suppose the holomorphic functions f and F are related by

$$f(Z) = \mathbf{det}(I - \gamma)^p F(\gamma)$$

so that F is holomorphic on $\Gamma(C)^\circ$ and f therefore holomorphic on $T_\Omega \setminus \Sigma$. Then for f to admit a holomorphic continuation to all of T_Ω it is necessary and sufficient that it stays bounded as the determinant factor tends to infinity, i.e. that F satisfies a decay condition related to p. This is what we shall make precise in the following.

16. The case of $G = Sp(r, \mathbb{R})$

In the section we will give an explicit construction of double covering of the Lie semigroup and compare the two Hardy spaces. In the case $G = Sp(r, \mathbb{R})$, the classical Hardy space is isomorphic to the odd part of the Olshanskiĭ Hardy space

We assume that $G = Sp(r, \mathbb{R})$. Then the Hardy parameter is $\frac{N}{R} = r + \frac{1}{2} \in \mathbb{Z} + \frac{1}{2}$ and **det** coincides with the usual matrix determinant Det. This suggests that the operator $\mathtt{C}_{\frac{N}{R}} = \mathtt{C}_{r+\frac{1}{2}}$,

$$f = \mathtt{C}_{r+\frac{1}{2}}(F) \ : \ f(Z) = \mathrm{Det}(Z + iJ)^{-(r+\frac{1}{2})} F(\gamma) \tag{24}$$

may be an intertwining operator between $H^2(T_\Omega)$ and the odd part of the Hardy space $H^2(\Gamma(C)_2)$ on the double covering $\Gamma(C)_2$ of $\Gamma(C)$.

16.1. The explicit construction of the two sheeted covering semigroup. For the open subset $\Gamma(C)_2{}^\circ$ we have a new and explicit construction: Let $J^\flat = \left(\begin{smallmatrix} J & 0 \\ 0 & J \end{smallmatrix}\right)$ and let $G^\flat := Sp(2r, \mathbb{R})$ be the group of all matrices in $Sp(2r, \mathbb{C})$ satisfying

$$g^* J^\flat g = J^\flat.$$

We imbed G in a natural way in $G^\flat$ as follows :

$$g \longmapsto \begin{pmatrix} g & 0 \\ 0 & I_{2r} \end{pmatrix}.$$

We also view the Cayley transform $\mathtt{C}$ as the element of $G^\flat_{\mathbb{C}}$ given by the matrix

$$\mathtt{C} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_{2r} & -iJ \\ I_{2r} & iJ \end{pmatrix}.$$

Our precise definition of G_2 is to be the set of all pairs $(g, \omega(g^{\mathtt{C}}, \cdot))$ with $g \in Sp(r, \mathbb{R})$, $g^{\mathtt{C}} = \mathtt{C}^{-1}\, g\, \mathtt{C} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, $\omega(g^{\mathtt{C}}, \cdot)^2 = \mathrm{Det}(C\ \cdot + D)^{-1}$ and $Z \longrightarrow \omega(g^{\mathtt{C}}, Z)$ is holomorphic on T_Ω. Note that this is analogous to the definition of the double cover of $SU(1,1)$, where we take all pairs $(g, \sqrt{cz + d})$ with $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU(1,1)$ and $\sqrt{cz + d}$ a holomorphic choice of square root of the non-zero function $cz + d$ on the unit disc. Indeed, it sometimes is convenient to think in terms of such multivalued functions when doing practical calculations, but of course, the precise definition is behind this. We also recall the more informal definition of G_2 as follows:

Take again $Z \in T_\Omega$ and $g \in Sp(r, \mathbb{R})$ such that $\mathtt{C}^{-1}\, g\, \mathtt{C} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$. A determination on T_Ω of the square root $\mathrm{Det}(CZ+D)^{-\frac{1}{2}}$ is completely determined by its value on $Z = iI$. For each $g \in Sp(n,\mathbb{R})$ we choose a determination of $\mathrm{Det}(CZ+D)^{-\frac{1}{2}}$. This is a global determination. We consider here Z as a variable, since the group (and indeed all contractions) acts on the tube domain, and we consider the function

$$Z \longmapsto \omega(g^{\mathtt{C}}, Z) := \mathrm{Det}(CZ+D)^{-\frac{1}{2}}$$

from T_Ω into $\mathbb{C}\setminus\{0\}$, where $g^{\mathtt{C}} = \mathtt{C}^{-1}\, g\, \mathtt{C}$. We read ω as "a holomorphic choice of square root of the determinant". It follows that ω may be viewed as a cocycle for G_2, and it gives a choice of square root at the product of two elements as follows:

$$\omega(g_1^{\mathtt{C}} g_2^{\mathtt{C}}, Z) = \omega(g_1^{\mathtt{C}}, g_2^{\mathtt{C}} \cdot Z)\omega(g_2^{\mathtt{C}}, Z).$$

This equation is to be understood as an equation for the two-valued function ω; it does not hold for any single-valued function. More generally, assuming the determinant to be non-zero, we let

$$\omega_2(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), Z) := \mathrm{Det}(CZ+D)^{-1}$$

and correspondingly $\omega(\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right), Z)$ a choice of one of the two square roots of this, either (as here) global and holomorphic, or (as below) local, i.e. at the fixed point Z. Then we may consider our double covering group to be

$$G_2 := \{\widetilde{g} := (g, \omega(g^{\mathtt{C}}, \cdot)) \mid g \in Sp(r,\mathbb{R})\},$$

endowed with the group law

$$\text{(25)} \qquad \big(g_1, \omega(g_1^{\mathtt{C}}, Z)\big)\big(g_2, \omega(g_2^{\mathtt{C}}, Z)\big) = \big(g_1 g_2, \omega(g_1^{\mathtt{C}}, g_2^{\mathtt{C}} \cdot Z)\omega(g_2^{\mathtt{C}}, Z)\big)\,.$$

G_2 is a two–sheeted covering group of G, since we are considering both choices of square root. G_2 is called the *metaplectic group.*

Now we wish to give another version of the double covering construction. Here N will be the open semigroup, realized as a subset of the tube domain as in Proposition 14.3. For $Z \in T_\Omega \setminus \Sigma$ and for a choice of a local determination of $\mathrm{Det}(Z+iJ)^{-\frac{1}{2}}$ we note that up to a constant

$$\mathrm{Det}(Z+iJ)^{-\frac{1}{2}} = \omega(\mathtt{C}, Z).$$

This is again an identity between two-valued functions. Hence at each fixed point Z we make a choice between the two possible values of the square root, so here the notation does not consider Z as a variable. Note that we may extend our cocycle to the complexified group in the natural way. Therefore, the complex manifold

$$\Gamma(C)_2^\circ := \{\widetilde{\gamma} = (\gamma, \omega(\mathtt{C}, Z))\,; \gamma \in \Gamma(C)^\circ,\ \gamma = \mathtt{C}(Z),\ Z \in T_\Omega \setminus \Sigma\},$$

is a two–sheeted covering of the semigroup $\Gamma(C)^\circ$. As before, we consider both choices of square root here, and corresponding to the modern point of view, the more precise definition of $\Gamma(C)_2^\circ$ is the set of $\widetilde{\gamma} = (\gamma, w) \in \Gamma(C)^\circ \times \mathbb{C}$ such that

$$\gamma = \mathtt{C}(Z),\ Z \in T_\Omega \setminus \Sigma \text{ and } w^2 = \mathrm{Det}(Z + iJ)^{-1}.$$

In particular, w is just a complex number.

Lemma 16.1 ([**K–Ø**$_3$]). *The group G_2 acts on the right on the manifold $\Gamma(C)_2^\circ$*

Indeed, letting Z' satisfy $g^{-1}\gamma = \mathtt{C} \cdot Z'$, which implies that $Z = g^{\mathtt{C}} \cdot Z'$, then

$$\big(\gamma, \omega(\mathtt{C}, Z)\big) \cdot \big(g, \omega(g^{\mathtt{C}}, \cdot)\big) = \big(g^{-1}\gamma, \omega(\mathtt{C}, Z')\big).$$

To show that $\Gamma(C)_2^\circ$ is a semigroup we consider the following manifold

$$\Gamma(C)_2^{\circ\prime} := \{\big(\gamma, \omega(\gamma^{\mathtt{C}}, \cdot)\big) \mid \gamma \in \Gamma(C)^\circ\}.$$

It is clear that $\Gamma(C)_2^{\circ\prime}$ is a double covering of $\Gamma(C)^\circ$ and has a semigroup structure with respect to the law (25).

Consider the map φ from $\Gamma(C)_2^{\circ\prime}$ to $\Gamma(C)_2^\circ$ defined by

$$\big(\gamma, \omega(\gamma^{\mathtt{C}}, Z)\big) \longmapsto \big(\gamma, \omega(\mathtt{C}, Z)\big), \quad \text{where } \gamma = \mathtt{C}(Z) \in \Gamma(C)^\circ.$$

Lemma 16.2 ([**K–Ø**$_3$]). *φ is a homeomorphism from $\Gamma(C)_2^{\circ\prime}$ onto $\Gamma(C)_2^\circ$.*

Remark 16.3. *The semigroup $\Gamma(C)_2^\circ$ is isomorphic to the interior of the metaplectic semigroup or the Howe oscillator semigroup. We call it the open metaplectic semigroup.*

16.2. The Hardy space on $\Gamma(C)_2$. The Hardy space $H^2(\Gamma(C)_2)$ on the *metaplectic semigroup* $\Gamma(C)_2$ is the space of holomorphic functions $F \in \mathcal{O}(\Gamma(C)_2^\circ)$ such that

$$\sup_{\widetilde{\gamma} \in \Gamma(C)_2^\circ} \int_{G_2} |F(\widetilde{\gamma}\widetilde{g})|^2 \, d\widetilde{g} < \infty.$$

The compact maximal subgroup K of $G = Sp(r, \mathbb{R})$ is isomorphic to $U(r)$ and the maximal split Abelian subalgebra

$$\mathfrak{t}_{\mathbb{R}} = \Big\{ \left(\begin{smallmatrix} X & 0 \\ 0 & -X \end{smallmatrix}\right) \in \mathcal{M}(r \times r, \mathbb{R}) \mid X = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_r \end{pmatrix} \Big\},$$

can be identified with $\mathbb{R}^r$. Let $\epsilon_1, \ldots, \epsilon_r$ be the canonical basis of $\mathfrak{t}_\mathbb{R}^* = \mathfrak{t}_\mathbb{R} = \mathbb{R}^r$. Then the root system $\Delta = \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{t}_\mathbb{C})$ is of type C_r :

$$\begin{aligned}
\Delta &= \{\pm(\epsilon_i \pm \epsilon_j) \ (1 \le i < j \le r) \ , \ \pm 2\epsilon_i \ (1 \le i \le r)\} \\
\Delta^+ &= \{\epsilon_i \pm \epsilon_j \ (1 \le i < j \le r) \ , \ 2\epsilon_i \ (1 \le i \le r)\} \\
\Delta_\mathfrak{k}^+ &= \{\epsilon_i - \epsilon_j \ (1 \le i < j \le r)\} \\
\Delta_\mathfrak{p}^+ &= \{\epsilon_i + \epsilon_j \ (1 \le i \le j \le r)\} \\
\rho &= r\epsilon_1 + (r-1)\epsilon_2 + \ldots + \epsilon_r \\
&\simeq (r, r-1, \ldots, 1).
\end{aligned}$$

Furthermore $\mathcal{P}$ is the lattice $\mathbb{Z}^r$, the set of highest weights relative to $\Delta_\mathfrak{k}^+$ is given by

$$\mathcal{R} = \{\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \mid \lambda_1 \ge \ldots \ge \lambda_r\},$$

and $\lambda \in \mathcal{R}$ satisfies the Harish–Chandra condition if

$$-r > \lambda_1 \ge \ldots \ge \lambda_r,$$

which gives the set $\mathcal{R}'$.

Let $K_2 \subset G_2$, resp. $T_2 \subset K_2$ be the corresponding covering of K and T. Then the corresponding $\mathcal{P}_2$, $\mathcal{R}_2$ and $\mathcal{R}'_2$ are given by

$$\begin{aligned}
\mathcal{P}_2 &= \mathbb{Z}^r \cup (\mathbb{Z}^r + \frac{1}{2}) = \mathcal{P} \cup (\mathcal{P} + \frac{1}{2}) = \mathcal{P}_{2,\mathrm{even}} \cup \mathcal{P}_{2,\mathrm{odd}}, \\
\mathcal{R}_2 &= \{\lambda \in \mathcal{P}_2 \mid (\forall \alpha \in \Delta_\mathfrak{k}^+) \ \langle \lambda, \alpha \rangle \ge 0\} \\
&= \{\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \cup (\mathbb{Z}^r + \frac{1}{2}) \mid \lambda_1 \ge \ldots \ge \lambda_r\}, \\
&= \mathcal{R}_{2,\mathrm{even}} \cup \mathcal{R}_{2,\mathrm{odd}} \\
\mathcal{R}'_2 &= \{\lambda \in \mathcal{R}_2 \mid \langle \lambda + \rho, \beta \rangle < 0, \ \ \forall \beta \in \Delta_\mathfrak{p}^+\}, \\
&= \{\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \cup (\mathbb{Z}^r + \frac{1}{2}) \mid -r > \lambda_1 \ge \ldots \ge \lambda_r\}, \\
&= \mathcal{R}'_{2,\mathrm{even}} \cup \mathcal{R}'_{2,\mathrm{odd}},
\end{aligned}$$

where $\frac{1}{2}$ stands for the tuple $(\frac{1}{2}, \ldots, \frac{1}{2})$. The holomorphic discrete series representations for the metaplectic group G_2 are those irreducible unitary representations π_λ of G_2 that are square–integrable with a highest weight $\lambda \in \mathcal{R}'_2 = \mathcal{R}'_{2,\mathrm{even}} \cup \mathcal{R}'_{2,\mathrm{odd}}$. Therefore

$$H^2(\Gamma(C)_2) = \bigoplus_{\lambda \in (C^* \cap \mathfrak{t}_\mathbb{R}) \cap \mathcal{R}'_2} \pi_\lambda \otimes \pi_\lambda^*. \tag{5.4}$$

The cone C is the minimal one in $i\mathfrak{g}$, so the above summation is over $\mathcal{R}'_2$ and the Hardy space $H^2(\Gamma(C)_2)$ splits into two parts, namely, even and odd part,

$$\begin{aligned} H^2(\Gamma(C)_2) &= H^2_{\text{even}}(\Gamma(C)_2) \oplus H^2_{\text{odd}}(\Gamma(C)_2) \\ &= \Big(\bigoplus_{\lambda\in\mathcal{R}'_{2,\text{even}}} \pi_\lambda \otimes \pi^*_\lambda\Big) \oplus \Big(\bigoplus_{\lambda\in\mathcal{R}'_{2,\text{odd}}} \pi_\lambda \otimes \pi^*_\lambda\Big). \end{aligned}$$

The even part

$$\begin{aligned} H^2(\Gamma(C)_2)_{\text{even}} &= \bigoplus_{\lambda\in\mathcal{R}'_{2,\text{even}}} \pi_\lambda \otimes \pi^*_\lambda \\ &= \bigoplus_{\substack{(\lambda_1,\dots,\lambda_r)\in\mathbb{Z}^r \\ -r>\lambda_1\geq\dots\geq\lambda_r}} \pi_\lambda \otimes \pi^*_\lambda \end{aligned}$$

coincides with the Hardy space $H^2(\Gamma(C))$ on the semigroup $\Gamma(C)$. Our goal in now is to identify the odd part

$$\begin{aligned} H^2(\Gamma(C)_2)_{\text{odd}} &= \bigoplus_{\lambda\in\mathcal{R}'_{2,\text{odd}}} \pi_\lambda \otimes \pi^*_\lambda \\ &= \bigoplus_{\substack{(\lambda_1,\dots,\lambda_r)\in\mathbb{Z}^r+\frac{1}{2} \\ -(r+\frac{1}{2})\geq\lambda_1\geq\dots\geq\lambda_r}} \pi_\lambda \otimes \pi^*_\lambda \end{aligned}$$

with the classical Hardy space $H^2(Sp(2r,\mathbb{R})/U(2r))$.

THEOREM 16.4 ([**K–Ø**$_1$], [**K–Ø**$_3$]). *The operator* $\mathtt{C}_{r+\frac{1}{2}}$ *given by (24) induces a unitary isomorphism*

$$H^2(\Gamma(C)_2)_{\text{odd}} \simeq H^2(Sp(2r,\mathbb{R})/U(2r)).$$

COROLLAIRY 16.5 ([**K–Ø**$_1$], [**K–Ø**$_3$]). *Under the action of* $Mp(r,\mathbb{R})\times Mp(r,\mathbb{R})$ *the Hardy space* $H^2(T_\Omega)$ *can be decomposed into a direct sum of the 'odd' holomorphic discrete series representations of* $Mp(r,\mathbb{R})$, *i.e.*

$$H^2(Sp(2r,\mathbb{R})/U(2r))_{|Mp(r,\mathbb{R})\times Mp(r,\mathbb{R})} = \bigoplus_{\lambda\in\mathcal{R}'_{2,\text{odd}}} \pi_\lambda \otimes \pi^*_\lambda.$$

COROLLAIRY 16.6 ([**K–Ø**$_1$], [**K–Ø**$_3$]). *Let* K_{odd} *be the kernel corresponding to* $H^2(\Gamma(C)_2)_{\text{odd}}$. *Then for every* $\gamma_1,\gamma_2 \in \Gamma(C)_2$

$$K_{\text{odd}}(\gamma_1,\gamma_2) = \operatorname{Det}(J - \gamma_2^* J\gamma_1)^{-(r+1/2)}.$$

COROLLAIRY 16.7 ([**K–Ø**$_1$], [**K–Ø**$_3$]). *On the interior of the metaplectic semigroup the distribution* $\operatorname{Det}(I-\gamma)^{-(r+1/2)}$ *has the following expansion*

$$\operatorname{Det}(I-\gamma)^{-(r+1/2)} = \sum_{\lambda\in\mathcal{R}'_{2,\text{odd}}} d_\lambda\, \mathbf{tr}(\pi_\lambda(\gamma)),$$

where d_λ is the formal dimension of π_λ.

The Bergman space on $\Gamma(C)$ is $\mathcal{H}_{2r+1}(\Gamma(C))$ and its reproducing kernel is given by

$$K_B(\gamma_1, \gamma_2) = \mathrm{Det}(J - \gamma_2^* J \gamma_1)^{-(2r+1)}.$$

Corollairy 16.8 ([**K–Ø**$_3$]). *The Bergman kernel K_B on the semigroup $\Gamma(C)$ is the square of the odd part K_{odd} of the Cauchy-Szegö kernel for $\Gamma(C)_2$.*

17. The case of $G = SO^*(2l)$

We study the case $G = SO^*(2l)$ as we did in section 16. We prove that the classical Hardy space is a proper subspace of odd part of the Olshanskiĭ Hardy space

Let $G = SO^*(2l)$ realized as a subgroup of $U(l,l)$,

$$G = \{g \in SO^*(2l, \mathbb{C}) \mid g^* J g = J\}, \;\; J = \begin{pmatrix} -I_l & 0 \\ 0 & I_l \end{pmatrix}$$

The Hardy parameter in this case is $N/R = l(2l-1)/l = 2l-1$ and the Koecher norm "**det**" is the square root of the usual determinant "Det" ($\mathbf{det} = \mathrm{Det}^{1/2}$). Thus the operator $\mathtt{C}_{\frac{N}{R}} = \mathtt{C}_{2l-1}$,

$$f = \mathtt{C}_{2l-1}(F) \;\; : \;\; f(Z) = \mathrm{Det}(Z + iJ)^{-(l-1/2)} F(\gamma)$$

provides an equivariant embedding of the classical Hardy space $H^2(SO^*(4l)/U(2l))$ into the odd part of the Hardy space $H^2(\Gamma(C)_2)_{\mathrm{odd}}$ on the double covering semigroup $\Gamma(C)_2$ of the minimal semigroup

$$\Gamma(C) = \{\gamma \in SO^*(2l, \mathbb{C}) \mid J - \gamma^* J \gamma \geq 0\}$$

(because of the square root in $\mathrm{Det}(Z + iJ)^{(l-1/2)}$). This is just like the symplectic case. We will identify the maximal compact subgroup with $U(l)$ as in the above section. The determinant factor is again exactly the Jacobian to a power such that we have preservation of L^2-norms on the respective boundaries. Then $\mathfrak{t}_{\mathbb{R}}$ is given by the same formula as in $Sp(r, \mathbb{R})$ case. Let $\epsilon_1, \epsilon_2, \ldots, \epsilon_l$ be the canonical basis of $\mathfrak{t}_{\mathbb{R}}^* = \mathfrak{t}_{\mathbb{R}} = \mathbb{R}^l$. The root system $\Delta = \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is of type D_l :

$$\begin{aligned} \Delta &= \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq l\}, \\ \Delta^+ &= \{\epsilon_i \pm \epsilon_j \mid 1 \leq i < j \leq l\}, \\ \Delta^+_{\mathfrak{k}} &= \{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq l\}, \\ \Delta^+_{\mathfrak{p}} &= \{\epsilon_i + \epsilon_j \mid 1 \leq i < j \leq l\}, \\ \rho &= (l-1)\epsilon_1 + (l-2)\epsilon_2 + \ldots + \epsilon_{l-1} \\ &\simeq (l-1, l-2, \ldots, 1, 0). \end{aligned}$$

The set of highest weights relative to the positive roots of $SO^*(2l)$ is

$$\mathcal{R} = \{\lambda = (\lambda_1 \ldots, \lambda_l) \in \mathbb{Z}^l \mid \lambda_1 \geq \ldots \geq \lambda_l\},$$

and $\lambda \in \mathcal{R}$ satisfies to the Harish-Chandra condition if and only if

$$-2l + 3 > \lambda_1 + \lambda_2.$$

Therefore, the odd holomorphic discrete series representations of the double covering group G_2 of $SO^*(2l)$ are those irreducible unitary representations π_λ, square-integrable with a highest weight $\lambda = (\lambda_1, \ldots, \lambda_l) \in \mathbb{Z}^l - \frac{1}{2}$ such that $0 \geq \lambda_1 \geq \ldots \geq \lambda_l$ and satisfying

$$-2l + 2 \geq \lambda_1 + \lambda_2.$$

Let $\mathcal{R}'_{2,\mathrm{odd}}$ denotes the set of these λ's. The Hardy space $H^2(\Gamma(C)_2)$ on the minimal cone $\Gamma(C)_2$ has then the following decomposition

$$H^2(\Gamma(C)_2)_{\mathrm{odd}} = \bigoplus_{\lambda \in \mathcal{R}'_{2,\mathrm{odd}}} \pi_\lambda \otimes \pi^*_\lambda.$$

Theorem 17.1 ([**K–Ø**$_3$]). *The classical Hardy space $H^2(SO^*(4l)/U(2l))$ is a proper invariant subspace of the "non–classical" Hardy space $H^2(\Gamma(C)_2)_{\mathrm{odd}}$.*

Corollairy 17.2 ([**K–Ø**$_3$]). *The representation of $SO^*(2l) \times SO^*(2l)$ in the Hardy space $H^2(\Gamma(C))$ cannot be obtained by a restriction of a representation of the holomorphic discrete series of $SO^*(4l)$ nor any continuation of this, such as the Hardy space.*

Let $\mathcal{H}^2(\Gamma(C))$ the conformal image of $H^2(SO^*(4l)/U(2l))$ via the operator $\mathtt{C}_{2l-1}$.

Corollairy 17.3 ([**K–Ø**$_3$]). *$\mathcal{H}^2(\Gamma(C))$ is a reproducing kernel Hilbert space and its reproducing kernel K is the pre-image of the Cauchy–Szegö kernel of $H^2(SO^*(4l)/U(2l))$, i.e.*

$$K(\gamma_1, \gamma_2) = \mathrm{Det}(J - \gamma_2^* J \gamma_1)^{-(l-1/2)}.$$

Corollairy 17.4 ([**K–Ø**$_3$]). *On $\Gamma(C)^\circ$ the holomorphic function $\mathrm{Det}(I-\gamma)^{-(l-1/2)}$ has the following expansion*

$$\mathrm{Det}(I-\gamma)^{-(l-1/2)} = \sum_{-l+1/2 \geq \lambda_1 \geq \ldots \geq \lambda_l} d_\lambda \, \mathbf{tr}(\pi_\lambda(\gamma)),$$

where d_λ is the formal dimension of π_λ.

18. The case of $G = U(p, q)$

We study the case $G = U(p, q)$ as we did in section 16. Here we do not need double covering semigroup. We prove that the classical Hardy space is a proper subspace of of the Olshanskiĭ Hardy space

Wee fix $G = U(p, q)$ realized by

$$G = U(p, q) = \{g \in GL(n, \mathbb{C}) \mid g^* J g = J\}, \ J = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}$$

where $n = p + q$. In this case the Hardy parameter is $N/R = n^2/n = n$ and the Koecher norm "**det**" is the usual determinant "Det". Therefore the operator $\mathtt{C}_{\frac{N}{R}}$ given by

$$f = \mathtt{C}_n(F) \ : \ f(Z) = \mathrm{Det}(Z + iJ)^{-n} F(\gamma)$$

may be an intertwining operator between the classical Hardy space $H^2(SU(n, n)/S\big(U(n) \times U(n)\big)$ and the Hardy space $H^2(\Gamma(C))$ over the semigroup

$$\Gamma(C) = \{\gamma \in GL(n, \mathbb{C}) \mid J - \gamma^* J \gamma \geq 0\}.$$

To study unitary representations of $G = U(p, q)$ we identify it with $(U(1) \times SU(p, q))/\mathbb{Z}_{p+q}$. Thus the unitary irreducible representations of G are those of $U(1) \times SU(p, q)$ that are trivial on $(\zeta, \zeta^{-1} I_n)$ as in Remark 14.1, where I_n is the identity matrix and $\zeta^n = 1$. Therefore, the holomorphic discrete series representations of G that we are interested in are

$$\pi_{\lambda,k}(e^{i\theta} g) = e^{ik\theta} \pi_\lambda(g) \ , \ g \in SU(p, q), \ \theta \in \mathbb{R}$$

where $k \in \mathbb{Z}$ and π_λ are the holomorphic discrete series representations of $SU(p, q)$, realized on $\mathcal{D} = SU(p, q)/S\big(U(p) \times U(q)\big)$, for example in the scalar case:

$$\big(\pi_\lambda(g) f\big)(Z) = \mathrm{Det}(CZ + D)^{-\lambda} f\big((AZ + B)(CZ + D)^{-1}\big), \ g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

with λ an integer, and in general $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$. There will an underlying parity condition to make the representations trivial on $\mathbb{Z}_n$ as above; for example in the scalar case we must have that $k - q\lambda$ is divisible by n.

Let $\mathfrak{t} \subset \mathfrak{k}$ be a Cartan subalgebra consisting of diagonal matrices with purely imaginary values and $\mathfrak{t}_\mathbb{R} = i\mathfrak{t}$. Then the root system $\Delta =$

$\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{t}_{\mathbb{C}})$ is of type A_{n-1} :

$$\begin{aligned}
\Delta &= \{\epsilon_i - \epsilon_j \mid 1 \le i \ne j \le n\}, \\
\Delta^+ &= \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le n\}, \\
\Delta^+_{\mathfrak{k}} &= \{\epsilon_i - \epsilon_j \mid 1 \le i < j \le p \text{ or } p+1 \le i < j \le n\}, \\
\Delta^+_{\mathfrak{p}} &= \{\epsilon_i - \epsilon_j \mid 1 \le i \le p \text{ and } p+1 \le j \le n\}, \\
2\rho &= (n-1)\epsilon_1 + (n-3)\epsilon_2 + \ldots - (n-3)\epsilon_{n-1} - (n-1)\epsilon_n \\
&\simeq (n-1, n-3, n-5, \ldots, -n+3, -n+1),
\end{aligned}$$

where $\epsilon_1, \ldots, \epsilon_n$ is the canonical basis of $\mathfrak{t}^*_{\mathbb{R}} \simeq \mathfrak{t}_{\mathbb{R}} \simeq \mathbb{R}^n$. Then the holomorphic discrete series representations of $SU(p,q)$ are the above representations π_λ with $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ satisfying

$$\lambda_i - \lambda_{i+1} \ge 0, \ \ i \ne p, \ \ 1 \le i \le n-1,$$

and the Harish-Chandra condition

$$\lambda_n - \lambda_1 > n - 1.$$

Let $\pi_{\lambda,k}$ be an irreducible unitary representation of G with highest weight (λ, k), $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then we prove that $\pi_{\lambda,k}$ is a $C-$dissipative representation of the holomorphic discrete series if and only if (λ, k) belongs to the set $\mathcal{R}_{\text{diss}}$ of $(\lambda_1, \ldots, \lambda_n, k) \in \mathbb{Z}^{n+1}$ such that

$$\begin{cases} \lambda_n - \lambda_1 > n - 1 \\ 0 \ge \lambda_1 \ge \ldots \ge \lambda_p, \ \lambda_{p+1} \ge \ldots \ge \lambda_n \ge 0 \\ [\lambda] - n\lambda_n \le k \le [\lambda] - n\lambda_1. \end{cases}$$

Hence the Hardy space on the semigroup $\Gamma(C)$ has the following decomposition

$$H^2(\Gamma(C)) = \bigoplus_{(\lambda,k)\in\mathcal{R}_{\text{diss}}} \pi_{\lambda,k} \otimes \pi^*_{\lambda,k}.$$

Theorem 18.1 ([**K–Ø**$_3$]). *The classical Hardy space $H^2\big(SU(n,n)/S\big(U(n)\times U(n)\big)\big)$ is a proper invariant subspace of the "non–classical" Hardy space $H^2(\Gamma(C))$.*

Corollairy 18.2 ([**K–Ø**$_3$]). *The representation of $S\big(U(p,q) \times U(p,q)\big)$ in the Hardy space $H^2(\Gamma(C))$ cannot be obtained by a restriction of a representation of the holomorphic discrete series of $SU(n,n)$ nor any analytic continuation of this, such as the Hardy space.*

Let $\mathcal{H}^2(\Gamma(C))$ be the conformal image of $H^2\big(SU(n,n)/S\big(U(n) \times U(n)\big)\big)$ via the operator $\mathtt{C}_n$.

COROLLAIRY 18.3 ([**K–Ø**$_3$]). *$\mathcal{H}^2(\Gamma(C))$ is a reproducing Hilbert space and its reproducing kernel K is the pre-image of the Cauchy-Szegö kernel of $H^2\big(SU(n,n)/S\big(U(n)\times U(n)\big)\big)$, i.e.*

$$K(\gamma_1,\gamma_2) = \mathrm{Det}(J - \gamma_2^* J\gamma_1)^{-n}.$$

COROLLAIRY 18.4 ([**K–Ø**$_3$]). *On $\Gamma(C)^\circ$ the holomorphic function $\mathrm{Det}(I-\gamma)^{-n}$ has the following expansion*

$$\mathrm{Det}(I-\gamma)^{-n} = \sum_{(\lambda,k)\in\mathcal{R}_{\mathrm{decay}}} d_{\lambda,k}\,\mathbf{tr}(\pi_{\lambda,k}(\gamma)),$$

where $d_{\lambda,k}$ is the formal dimension of $\pi_{\lambda,k}$ and $\mathcal{R}_{\mathrm{decay}}$ is the set of $(\lambda_1,\dots,\lambda_n,k)\in\mathbb{Z}^{n+1}$ such that

$$\begin{cases} \lambda_n - \lambda_1 > n-1 \\ 0 \geq \lambda_1 \geq \ldots \geq \lambda_p,\ \lambda_{p+1} \geq \ldots \geq \lambda_n \geq 0 \\ [\lambda] - n(\lambda_n + n) \leq k \leq [\lambda] - n(\lambda_1 + n) \end{cases}$$

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