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Abstract: This paper addresses the design of a distribution network in which a single supplier ships products to a set of retailers facing random demands via a set of distribution centers. Distribution centers are not known a priori and are to be located at a set of retailer locations. Decisions include: retailer locations to be selected as distribution centers, assignment of retailers to the distribution centers, and inventory to keep at each distribution center. The goal is to minimize the total location, shipment, and inventory costs, while ensuring a given retailer service level. A Lagrangian relaxation heuristic is proposed. Computation results show the effectiveness of the proposed heuristic and the duality gap is less than 1.5% in all tested problem instances.

Keywords: Supply chain design, facility location, optimization, Lagrangian relaxation.

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Une approche par relaxation Lagrangienne pour la conception d'un réseau de distribution stochastique

Résumé: Dans cet article, nous nous intéressons à l'étude d'un problème de conception d'un réseau de distribution dans lequel un seul fournisseur assure les livraisons, en seul type de produit, des demandes aléatoires des différents détaillants via un certain nombre de centres de distribution à localiser. Ni le nombre ni les localisations des différents centres sont connues d'avance. Dans cette étude, chaque détaillant est identifié par la zone où il est localisé. De plus, chaque zone est candidate pour une localisation possible d'un centre de distribution. L'objectif principal concerne la recherche des meilleures: 1- localisations des différents centres de distribution, 2- affectations des différents détaillants aux centres de distribution localisés et 3- quantités à stocker sachant que chaque centre utilise par hypothèse *une politique de la quantité économique* comme politique de stockage et doit maintenir un certain niveau de stock de sécurité pour garantir un *niveau de service* donné. Les différents coûts à optimiser couvrent 1- les coûts de localisation des centres, 2- les coûts de commandes et transports dans le réseau et 3- les coûts de stockage. Une approche par relaxation Lagrangienne est proposée et des expériences numériques sont réalisées. Les résultats obtenus attestant de l'efficacité de l'approche avec un cap de dualité inférieur à 1.5%.

Mots clés: Conception des chaînes logistiques, problème de localisation, optimisation, relaxation Lagrangienne.

1 Introduction

Location decisions are one of the most critical and most difficult decisions to design an efficient supply chain network. Locating distribution centers (DCs) in a supply chain network and assigning retailers/customers to them is a complex decision involving all members in the supply chain network. Furthermore, supply chain network design decisions are by nature costly and difficult to reverse, and hence their impact spans a long time horizon. For this reason, there is a rich literature dedicated to the deterministic and stochastic location problem. We summarize in the following major literature reviews and existing models and methods for location problem.

The classical model is the fixed charge facility location problem. This model forms the basis of many of the location models that have been used in supply chain design. All parameters are deterministic and the problem is to find the locations of the facilities and the shipment pattern between the facilities and the retailers in order to minimize the combined facility location and shipment cost subject to a requirement that all retailers' demands are met.

A number of solutions approaches have been proposed for the fixed charge facility location (FCFL). Simple heuristic typically begins by constructing a feasible solution and then by greedily adding or dropping facilities from the solution until no further improvement can be obtained. A Tabu search method was proposed in (*Al-Sultan and Fawzan, 1999*) to solve the FCFL problem and tested successfully on small and moderate size problems. A variable neighborhood search algorithm was proposed in (*Hansen and Mladenovic, 1997*) to solve both the FCFL problem and *P*-median problem. (*Geoffrion, 1974*) showed that when embedded in branch and bound, Lagrangian relaxation is powerful for identifying the optimal solutions of the fixed charge facility location model. In (*Geoffrion and Graves, 1974*) authors extended the traditional fixed charge facility location problem to include shipments from plant to distribution center and multiple commodities. (*Daskin, 1995*) and (*Galvao, 1993*) reviewed Lagrangian relaxation approaches for deterministic location problems.

(*Snyder, 2003*) and (*Snyder, 2004*) presented a rich state of the art on existing stochastic models for the facility location problem. Many of these models have as an objective to minimize the expected cost or maximize the expected profit of the system. Others take a probabilistic approach-for example, maximizing the probability that the solution is in some sense "good". Some models are solved using algorithms designed specifically for the problem, where others are solved using more general stochastic programming techniques.

In (*Louveaux, 1986*) stochastic versions of the capacitated *P*-median problem (CPMP) and capacitated fixed-charge location problem (CFLP) are presented, where customer's demands, production costs, transportation costs and selling prices are random variables. The goal is to choose facility locations, determine their capacities and decide which customers to serve and from which facilities to maximize the expected utility of profit.

In (*Ricciardi et al., 2002*) a facility location model with random throughput costs at the DCs is considered. The objective is to minimize the deterministic transportation cost (plant-DC and DC-customer) plus the expected throughput cost at the DCs. The authors first consider the network flow aspect of the problem (assuming the DC locations are given). They then embed the expected cost model into a non-linear integer program (NLIP). This model is solved heuristically since for each candidate solution to the location problem, a Lagrangian problem must be solved to compute the expected flows.

For joint transportation-location problem, (*França et al., 1982*) used Benders decomposition to solve a problem that is a combination of the CFLP and the stochastic transportation problem with random customers demands.

Although location decisions are strongly linked to tactical and operational decisions, traditionally, facility location decisions are made without taking into account the operational performances of the related supply chain. For example, while the contribution of inventory to distribution cost has been recognized for many years, only recently, (*Daskin et al. 2001*) addressed the so-called inventory-location model by incorporating inventory decisions in facility locations models. Hence there is a need for realistic yet tractable facility location decision models. In the following, we review some recent efforts to meet this need.

(*Erlebacher and Meller, 2000*) formulated a highly non-linear integer inventory-location model. The customers' demands are stochastic and rectilinear distances are used to represent the distances among the locations. Each DC operates under a continuous review inventory system. The problem consists in the determination of the number of DCs and their locations, as well as the customers they serve in order to minimize the fixed costs of operating the DCs, total DCs inventory holding costs and total transportation costs. Since the general version of the problem is NP-Hard, they developed analytical models and proposed heuristic procedures for special cases obtained under some simplified assumptions.

One of the early papers modifies the uncapacitated facility location problem to implicitly consider limited inventory levels (*Barahona and Jensen, 1998*). (*Nozick and Turnquist, 1998*) approximate inventory costs as part of the fixed facility costs assuming a linear relationship between inventory and the number of open facilities, and propose a model that takes into account constant service coverage. (*Nozick and Turnquist, 2001*) extend this model and treat demand coverage as part of the objective function.

The paper by (*Daskin et al. 2001*) is probably the first study that explicitly includes inventory costs as part of a simple, uncapacitated facility location model. Their model assumes economic order quantity based ordering and constant fill rate-based safety stocks across all facilities. The total cost function including the inventory costs makes the overall model a nonlinear integer program which is then solved using Lagrangian relaxation.

Stochastic versions of the joint inventory-location model are presented in (*Shen et al., 2003*) and (*Snyder, 2004*). The models choose DCs locations to minimize fixed costs (investment costs), transportation costs, and inventory costs at the DCs in the face of stochastic customer's demands. This leads to a difficult non-linear combinatorial problem. They considered the case when variance-to-mean ratio at each retailer is identical for all retailers, i.e. $\omega^2_i / \sigma_i = constant$ where ω_i is standard deviation of the demand at retailer i, and σ_i mean its demand. They formulated the problem as a non-linear integer program and presented a Branch and Price approach for it. Several computational experiments attested the effectiveness of the proposed approach.

This paper extends the inventory-location model proposed in (*Daskin et al., 2001*) and (*Shen et al., 2003*) by relaxing the constant variance-to-mean assumption. More specifically, we consider the design of a distribution network in which a single supplier ships products to a set of retailers facing random demands via a set of distribution centers with constant supply lead-times. The central issues of our problem are: how many and which retailer locations should be selected as the distribution centers, how to assign retailers to the distribution center, and how to manage the inventory at each distribution center. The goal is to minimize total location, shipment, and inventory cost, while ensuring a specified level of service at each distribution center. This leads to a difficult non-linear combinatorial problem.

We propose a Lagrangian relaxation approach to solve the problem. A relaxed problem is obtained by relaxing the assignment constraints of the inventory-location problem. A polynomial algorithm that takes into account the special structures of the relaxed problem is proposed to solve the relaxed problem. The dual problem is solved using a sub-gradient method to determine a lower bound of the inventory-location problem. Efficient feasible solutions are derived from the solutions of the relaxed problem during the search process of the dual solution. Computational experiments show efficiency of the proposed method and the duality gap is less than 1.5% in all test problems.

Note that the sub-problems resulting from this relaxation are similar to the pricing problems considered in (*Daskin et al., 2001*). Hence polynomial algorithm of this paper can also be used in the Branch & Price algorithm of for optimally solving the inventory-location problem without the assumption of constant variance-to-mean ratio.

The rest of the paper is organized as follow. Section 2 describes in details the problem under consideration. Section 3 presents a Lagrangian relaxation approach for solving the problem. Section 4 proposes a polynomial algorithm for solving the relaxed problem. Computational results and analyses are given in Section 5. Section 6 concludes the paper.

2 Problem setting

2.1 Problem

This paper addresses the design of a stochastic distribution network in which a single supplier ships product (single product type) to a set of retailers via a set of Distribution Centers (DC) to locate (Figure 1). Each retailer faces random demand. Each DC serves a set of retailers. The number and location of the DCs are not given a priori. They are chosen from a set of retailer locations. Following the principle of risk pooling, inventories are kept at selected DCs in order to cope against random demand.

Our model is an extension of the inventory-location model proposed in (*Daskin et al., 2001*). (*Daskin et al., 2001*) developed a location model with risk pooling that explicitly considers expected inventory cost when making facility locations decisions, thus combining strategic and tactical decisions into a single model. This leads to a difficult non-linear combinatorial problem. They considered the case when variance-to-mean ratio at each retailer is identical for all retailers (i.e. ϖ^2_i/σ_i = ***constant,*** where ϖ^2_i (*resp.* σ_i) is the variance (*resp.* mean) of the demand at retailer *i*), and where the supply lead-time is constant.

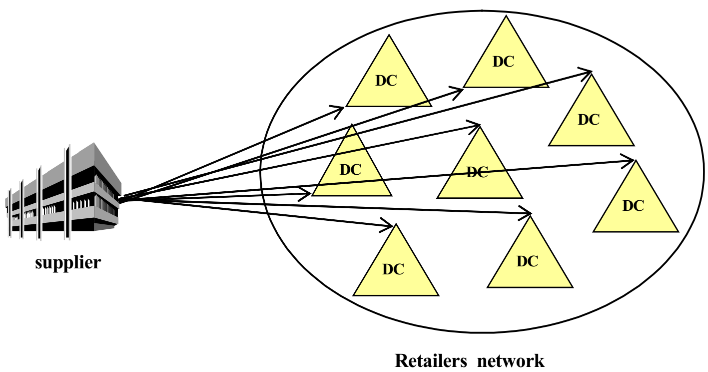

Figure 1. Structure of the studded supply chain

In this study, we relax the assumption of identical variance-to-mean ratio at each retailer. Each retailer location can be selected to host a DC. Each DC orders inventory from the supplier using an economic order quantity model (*EOQ*). The frequency of orders and the order quantity at each DC depend on the mean demand served by the DC which, in turn, is a function of the assignment of retailers to the DC. To this working inventory, each DC keeps a safety stock to protect against the possibility of stock-outs during the supply lead-time. The supply lead-time for deliveries from the supplier to a DC is constant and is different for different DC. The daily demand of each retailer is assumed to be normally distributed with a mean and variance. It is assumed that there is a transportation link between each pair of retailers. The transportation times between DCs and retailers are neglected.

The problem considered in this paper is the following: given a set I of retailers each facing an independent random demand, we must decide how many distribution centers to locate, where to locate them, which retailers to assign to each distribution center in other to minimize the total location, procurement, working-inventory and safety stock inventory costs.

2.2 Mathematical Model

The following notations are used to define the stochastic inventory-location problem under consideration.

I set of retailers indexed by i

DC_j distribution center located at retailer j

Demand

σ_i mean daily demand of retailer i

ω_i^2 variance of the daily demand of retailer I

Costs

f_j fixed annual cost of locating a DC_j

d_{ij} per-unit shipment cost from a DC_j to retailer i

K_j fixed cost per order placed to the supplier by a DC_j

κ_j fixed cost per shipment from the supplier to a DC_j

a_j per-unit shipment cost from the supplier to a DC_j

h_j inventory holding cost per unit per year in a DC_j

Others parameters

L_j lead-time in days from the supplier to a DC_j

ζ desired percentage of not stocking out at a DC during a retailer lead-time

z_ζ standard normal deviate such that $P(Z \, \Omega \, z_\zeta) = \zeta$

θ number of working days per year

Decision Variables

$$X_j \mid \begin{cases} 1 & \text{if we locate the } DC_j \\ 0 & \text{otherwise} \end{cases}$$

$$Y_{ij} \mid \begin{cases} 1 & \text{if retailer } i \text{ is served by a } DC_j \\ 0 & \text{otherwise} \end{cases}$$

Before formally formulating the problem, we outline the different components of the function to minimize. Consider first the cost of working inventory, fixed ordering cost and shipment cost related to each DC_j. Let the expected annual demand of DC_j be D_j units and let Q_j be the order quantity of the DC_j. The total annual fixed ordering cost is given by:

$$\frac{K_j D_j}{Q_j} \tag{1}$$

the total annual shipment cost is given by:

$$\frac{(\kappa_j + a_j Q_j) D_j}{Q_j} \tag{2}$$

and the total inventory-holding cost is given by:

$$\frac{h_j Q_j}{2} \tag{3}$$

Then, the sum of order, shipment, and inventory-carrying costs at the DC_j given by expression (1)-(3) is given by:

$$\frac{K_j D_j}{Q_j} + \frac{\kappa_j D_j}{Q_j} + \frac{h_j Q_j}{2} + a_j D_j \tag{4}$$

By taking the derivative of (4) with respect to Q_j we obtain

$$Q_j = \sqrt{\frac{2(K_j + \kappa_j) D_j}{h_j}} \tag{5}$$

Substituting Q_j into the cost function (4), the annual cost for ordering, shipment and working inventory is:

$$\sqrt{2 h_j D_j (K_j + \kappa_j)} + a_j D_j \tag{6}$$

The expected annual demand assigned to a regional DC located at retailer j is $D_j = \theta \sum_{i \in I} \sigma_i Y_{ij}$. As a result, (6) becomes:

$$\sqrt{2 h_j (K_j + \kappa_j) \theta \sum_{i \in I} \sigma_i Y_{ij}} + a_j \theta \sum_{i \in I} \sigma_i Y_{ij} \tag{7}$$

Consider now the safety stock cost related to DC_j. This depends on the distribution of the lead-time demand (LTD)$_j$, i.e. the demand arriving at DC_j while it is waiting for the delivery from the supplier. Since the supply lead-time (L_j) is a constant, *(LTD)$_j$* is a random variable with mean $L_j \sum_{i \in I} \sigma_i Y_{ij}$ and variance $L_j \sum_{i \in I} \omega_i^2 Y_{ij}$. The safety stock level (SS_j) to maintain at the DC_j required to ensure that stock-outs occur at supply delivery with a probability of Ǔ or less is equal to:

$$SS_j = z_\zeta \sqrt{L_j \sum_{i \in I} \omega_j^2 Y_{ij}}$$

The related safety stock cost at DC_j is

$$h_j z_\zeta \sqrt{L_j \sum_{i \in I} \omega_j^2 Y_{ij}} \tag{8}$$

Using (7) and (8) our problem is formulated as a non-linear combinatorial optimization problem given by:

(P)
$$J^* = \min_{X,Y} J(X,Y) \tag{9}$$

with

$$J(X,Y) = \sum_{j\in I} f_j X_j + \sum_{j\in I}\sum_{i\in I} D_{ij}Y_{ij} + \sum_{j\in I}\sqrt{\sum_{i\in I} c_{ij}Y_{ij}} + \sum_{j\in I}\sqrt{\sum_{i\in I} \zeta_{ij}Y_{ij}} \tag{10}$$

Subject to:

$$\sum_{j\in I} Y_{ij} = 1, \qquad \forall i\in I \tag{11}$$

$$Y_{ij} \le X_j, \qquad \forall i,j\in I \tag{12}$$

$$X_j, Y_{ij} \in \{0,1\}, \qquad \forall i\in I \tag{13}$$

where $D_{ij} = \theta\sigma_i(d_{ij}+a_j)$, $c_{ij} = 2h_j\theta(K_j+\kappa_j)\sigma_i$, $\zeta_{ij} = L_j\,{}_i^{2}(h_j z_\zeta)^2$.

The objective function minimizes the sum of the following costs: the first term corresponds to the fixed cost of locating facilities, the second term is the variable transportation cost from the suppliers to the DCs as well as the variable shipment cost from the DCs to the retailers, the third term represent the expected working inventory cost plus fixed order and fixed shipment cost at the DCs (assuming that each DC uses an economic order quantity policy), and finally the last term represents the cost of holding safety stock at the DCs to maintain a service level of ζ. Constraint (11) assumes that each retailer is assigned to exactly one DC. Constraint (12) states that retailers can only be assigned (Y_{ij} = 1) to opened distribution center (X_j = 1). Constraints (13) are standard integrity constraints.

The above integer programming model is non linear and the determination of exact solutions is a NP-hard problem. The main objective of this paper is to propose a Lagrangian relaxation-based method for solving the above optimization problem subject to constraints (12)-(13).

3 Lagrangian relaxation approach

The Lagrangian relaxation approach that we propose in this paper consists in (i) relaxing some constraints and introducing the corresponding terms into the cost function via Lagrangian multipliers, (ii) solving the relaxed problem for each setting of Lagrangian multipliers to obtain a lower bound, (iii) deriving a feasible solution to obtain an upper bound, (iv) maximizing the lower bound using a sub-gradient method. Each of the components of the Lagrangian relaxation heuristic will be explained. The efficiency of the Lagrangian relaxation heuristic is ensured by (i) the tight lower bound, and (ii) the feasible solutions derived from the solutions of the relaxed problem that capture most important features of the optimal solution of the initial problem **(P).**

3.1 Lagrangian relaxation

The Lagrangian relaxation method proposed in this paper consists in relaxing constraints (11) by introducing the Lagrangian multipliers ς_i, the relaxed problem is the following one:

(RP)
$$L(\varsigma) = \min_{X,Y} \sum_{j\in I} f_j X_j + \sum_{j\in I}\sum_{i\in I}(D_{ij} - \varsigma_i)Y_{ij} + \sum_{j\in I}\sqrt{\sum_{i\in I} c_{ij}Y_{ij}} + \sum_{j\in I}\sqrt{\sum_{i\in I} \zeta_{ij}Y_{ij}} + \sum_{i\in I}\varsigma_i \tag{14}$$

subject to constraints (12) and (13).

The relaxation of constraint (11) makes the location decisions of different DCs independent. The relaxed problem **(RP)** is equivalent to:

$$L(\varsigma) = \sum_{j\in I} L_j(\varsigma) + \sum_{i\in I}\varsigma_i \tag{15}$$

where $L_j(\varsigma)$ is the sub-problem defined by:

(SP$_j$) $$L_j(\varsigma) = \min_{X,Y} f_j X_j + \sum_{i\in I} (D_{ij} - \varsigma_i) Y_{ij} + \sqrt{\sum_{i\in I} c_{ij} Y_{ij}} + \sqrt{\sum_{i\in I} \zeta_{ij} Y_{ij}} \qquad (16)$$

subject to constraints (12)-(13).

Property 1:
(a) $L(\varsigma) \le J^*$ for all Lagrangian multiplier ς
(b) If the solution $(X_{\varsigma}, Y_{\varsigma})$ of the relaxed problem **(RP)** is a feasible solution of the initial problem **(P)**, then $(X_{\varsigma}, Y_{\varsigma})$ is an optimal solution to problem **(P)**.

3.2 Solving the dual problem

As mentioned above, the relaxed problem $L(\varsigma)$ gives a lower bound of the original problem, i.e., $L(\varsigma) \le J^*$ where J^* is the optimal cost of the original problem. The dual problem consists in determining Lagrangian multipliers ς_i to obtain the best lower bound, i.e.

(DP) $$L^* = \max_{\varsigma} L(\varsigma) \qquad (17)$$

$L(\varsigma)$ is a piece-wise linear concave function and Problem **(DP)** is a non differential optimization problem which can be solved using the standard subgradient optimization procedure developed by (*Fisher, 1981*):

$$\varsigma_i^{n+1} = \varsigma_i^n + \chi \frac{L^* - L(\varsigma)}{\left\langle \nabla L(\varsigma_i^n), \nabla L(\varsigma_i^n) \right\rangle} \nabla L(\varsigma_i^n) \qquad (18)$$

where ∇ is the gradient defined as follows :

$$\nabla L(\varsigma_i) = 1 - \sum_{j\in I} Y_{ij} \qquad (19)$$

As L^* is unknown, we replace L^* by the best-so-far upper bound in (18). Parameter χ enables us to control the variation of Lagrangian multiplier ς. We take $\chi = 2$ and reduce it if no improvement of $L(\varsigma)$ is observed after a certain number of iterations.

3.3 Derive a feasible solution

At each iteration of the sub-gradient algorithm for solving the dual problem, a feasible solution and an upper bound of the original problem can be derived from the solution of the relaxed problem **(RP)**. For each solution (*X, Y*) of problem **(RP)**, the relaxed constraints (11) are checked. If (11) are verified, then the solution is an optimal solution. Otherwise, the solution is modified as follows to derive a feasible solution.

Case I: $X_j = 0$, $\forall j \in I$, i.e. no DC is open. Then, we choose to open one DC at some location and assign all retailers to it. The location of the DC is chosen in order to minimize the overall cost.

Case II: $\exists j \in I$ such that $X_j = 1$, i.e. at least one DC is opened. Constraint (11) is checked sequentially for each retailer *i*. If constraint (11) does not hold, the following two cases are considered.

- If $\sum_{j\in I} Y_{ij} > 1$ (retailer assigned to more than one DC), the retailer *i* is assigned to the DC *j* such that $Y_{ij} = 1$ and that the overall cost *J(X, Y)* of the modified solution is minimal.
- If $\sum_{j\in I} Y_{ij} = 0$ (retailers without DC assignment), the retailer *i* is assigned to an opened DC_j such that the overall cost *J(X, Y)* of the modified solution is minimal.

3.4 Lagrangian relaxation heuristic

Algorithm 1. Lagrangian relaxation heuristic

Initialization : Select $\kappa_0 > 0$ (precision), set $n = 0$, $\varsigma = 0$.

Repeat Steps 1-5

Step1. Solve all relaxed sub-problems **(SP$_j$)** and compute $L_j(\varsigma)$, $\forall j \in I$

Step2. Compute $L(\varsigma)$ using equation (15)

Step3. Derive a feasible solution (X, Y) and compute the related upper bound $J(X, Y)$

Step4. Compute $J^* = MIN\{J^*, J(X, Y)\}$ and update the best-so-far solution

Step5. Update the Lagrangian multiplier ς_i^n using equation (18)

Until $\left\| \varsigma_i^n - \varsigma_i^{n-1} \right\| \le \kappa_0$

4 Solving the sub-problems

The main objective of this section is to propose a polynomial algorithm for solving sub-problems **(SP$_j$)** of the relaxed problem and for determining $L_j(\varsigma)$.

For each sub-problem **(SP$_j$)**, two cases need to be considered:

- If $X_j = 0$, constraint (12) implies that $Y_{ij} = 0$ for all I
- If $X_j = 1$ (i.e. the DC_j is open), constraint (12) becomes redundant and the selection of retailers to be served by the DC_j is obtained by solving the second sub-problem:

(SPP$_j$)
$$V_j(\varsigma) = \min_{Z_i \in \{0,1\}} \sum_{i \in I} b_{ij} Z_i + \sqrt{\sum_{i \in I} c_{ij} Z_i} + \sqrt{\sum_{i \in I} \zeta_{ij} Z_i} \tag{20}$$

with Y_{ij} replaced by Z_i and $b_{ij} = D_{ij} - \varsigma_i$.

To summarize, the solution of the sub-problem **(SP$_j$)** is :

- $X_j = 0$, $Y_{ij} = 0$, $L_j(\varsigma) = 0$ if $f_j + V_j(\varsigma) \ge 0$,
- $X_j = 1$, $Y_{ij} = Z_i$, $L_j(\varsigma) = f_j + V_j(\varsigma)$ if $f_j + V_j(\varsigma) < 0$.

That is

$$L_j(\varsigma) = Min\{0, f_j + V_j(\varsigma)\} \tag{21}$$

The remaining part of this subsection determines $L_j(\varsigma)$ via solution of problem **(SPP$_j$)**. Similar to the proof in (*Shen et al,* 2003), it can be shown that **(SPP$_j$)** can be transformed into a sub-modular function minimization problem and hence can be solved polynomially by general sub-modular function minimization algorithms. In (*Shen et al,* 2003), an efficient construction algorithm was proposed in the case when the objective function cost of (20) has only one square root. In our case, minimizing (20) is not trivial since $\zeta_{ij} \neq 0$. In this paper, we propose a polynomial algorithm for solving **(SPP$_j$)** that takes into account the special structures of problem **(SPP$_j$).**

Property 2: If $b_{ij} \ge 0$, then the optimal solution of problem **(SPP$_j$)** is such that $Z_i = 0$.

Consequently, without loss of generality, we assume that $b_{ij} = D_{ij} - \varsigma_i < 0$.

Instead of using the optimal economic order quantity determined in Section II, our basic idea of solving (21) is to use fixed order quantity Q of DC_j. We solve then the related sub-problem (21) by searching simultaneously optimal values of Z and Q.

The term of (20) depending on order quantity Q is

$$\sqrt{\sum_{i\in I} c_{ij} Z_i} = \sqrt{2h_j\theta(K_j+\kappa_j)\sum_{i\in I}\sigma_i Z_i} \tag{22}$$

which is the total inventory running cost and fixed order/shipment cost. We replace this term by the following cost:

$$\sqrt{\sum_{i\in I} c_{ij} Z_i} = \inf_{Q>0}\left\{\frac{(K_j+\kappa_j)\theta\sum_{i\in I}\sigma_i Z_i}{Q} + \frac{h_j Q}{2}\right\} \tag{23}$$

Replacing this expression into (20) leads to:

$$V_j(\varsigma) := \inf_{Q>0}\left[\min_{Z_i\in\{0,1\}} \sum_{i\in I} B_i(Q) Z_i + \sqrt{\sum_{i\in I}\zeta_{ij} Z_i} + \frac{h_j Q}{2}\right] \tag{24}$$

with $B_i(Q) = b_{ij} + \frac{k_{ij}}{Q}$ and $k_{ij} = (K_j+\kappa_j)\theta\sigma_i$

With this new formulation, the problem is not only to compute the decision variable Z_i (retailer assignment decision), but also to compute the order quantity Q (inventory decision) of each located DC_j.

To solve problem (24), we define

$$U_j(\varsigma, Q) = \min_{Z_i\in\{0,1\}} \sum_{i\in I} B_i(Q) Z_i + \sqrt{\sum_{i\in I}\zeta_{ij} Z_i} \tag{25}$$

The following polynomial algorithm for solving problem (25) is from (*Shen et al., 2003*).

Algorithm 2 (computation $U_j(\zeta,Q)$)

Step 1. Partitioning the set I into three subsets:

$I^+(Q) = \{i : B_i(Q) \geq 0\}$, $I^0(Q) = \{i : B_i(Q) < 0 \text{ and } \zeta_{ij} = 0\}$, $I^-(Q) = \{i : i \notin I^+(Q) \cup I^0(Q)\}$

Without loss of generality, let $I^-(Q)$ = {1, 2, …, N} with N = $|I^-(Q)|$.

Step 2. Sort the elements of $I^-(Q)$ as follows:

$$\frac{B_1(Q)}{\zeta_{1j}} \leq \frac{B_2(Q)}{\zeta_{2j}} \leq \dots \leq \frac{B_N(Q)}{\zeta_{Nj}} \text{ and } \frac{k_{ij}}{\zeta_{ij}} \geq \frac{k_{(i+1)j}}{\zeta_{(i+1)j}} \text{ if } \frac{B_i(Q)}{\zeta_{ij}} = \frac{B_{(i+1)}(Q)}{\zeta_{(i+1)j}}$$

Step 3. Compute $U_j(\varsigma, Q) = \min_{Z\in T(Q)} \sum_{i\in I} B_i(Q) Z_i + \sqrt{\sum_{i\in I}\zeta_{ij} Z_i}$

where T(Q) is the set of solutions Z such that $Z_i = 0, \forall i\in I^+(Q)$, $Z_i = 1, \forall i\in I^0(Q)$,

$Z_1 = \dots = Z_k = 1$, and $Z_{k+1} = \dots = Z_N = 0$ for some $k \geq 0$.

From Algorithm 2, $V_j(\varsigma)$ can be determined by (i) screening for all possible value of the order quantity Q, (ii) determining the set $\mathrm{T}(Q)$ of possible solutions Z for each Q, (iii) determining $V_j(\varsigma)$ by using relation (20) but restricting to the set of Z identify in (ii), i.e. the order quantity Q is replaced by the optimal order quantity for each Z. Combining Algorithm 2, (24), (25) and (20),

$$V_j(\varsigma) = \inf_{Q \geq 0}\left\{ \min_{Z \in \mathrm{T}(Q)} \sum_{i \in I} b_{ij} Z_i + \sqrt{\sum_{i \in I} c_{ij} Z_i} + \sqrt{\sum_{i \in I} \zeta_{ij} Z_i} \right\} \tag{26}$$

The following properties allow restricting the order quantity Q in a finite interval.

Property 3: For the sub-problem **(SP_j)**, if $Y_{ij} = 0$, then $X_j = 0$ and $L_j(\varsigma) = 0$.

Property 4: For the sub-problem **(SP_j)**, if $Y_{ij} \neq 0$, then $X_j = 1$ and the optimal order quantity Q* of DC_j is such that

$$Q_{\inf} = \sqrt{\frac{2 \min_{i \in I} k_{ij}}{h_j}} \leq Q^* \leq Q_{\sup} = \sqrt{\frac{2 \sum_{i \in I} k_{ij}}{h_j}}$$

The proofs of these two properties are obvious. For **(SP_j)**, if $Y_{ij} = 0$, then $X_j = 0$ minimize the expression of (16).

Property 4 is obvious as $Q^* = \sqrt{\frac{2 \sum_{i \in I} k_{ij} Y_{ij}}{h_j}}$.

As a result, combination of Properties 3-4 and relations (22) and (26) leads to:

$$L_j(\varsigma) = MIN\left\{ 0, \inf_{Q \in [Q_{\inf}, Q_{\sup}]} \left\{ \min_{Z \in \mathrm{T}(Q)} \sum_{i \in I} b_{ij} Z_i + \sqrt{\sum_{i \in I} c_{ij} Z_i} + \sqrt{\sum_{i \in I} \zeta_{ij} Z_i} \right\} \right\} \tag{27}$$

In the following, Algorithm 2 and relation (27) are used to determine $L_j(\varsigma)$. Continuous screening of Q is not necessary as, when increasing Q, the set $\mathrm{T}(Q)$ determined by Algorithm 2 does not change as long as the partition of the set I in Step 1 and the ordering in Step 2 do not change.

Property 5: $\mathrm{T}(q) = \mathrm{T}(Q)$, $\forall q$ such that $Q \leq q < H(Q)$ where set $\mathrm{T}(.)$ is determined by algorithm 2 and

$$H(Q) = MIN\left\{ q^+, \min_{i \in I^-(Q) / q_i \geq Q} q_i \right\}$$

$$q^+ \equiv \min_{i \in I^+(Q)} \{ q : B_i(q) = 0 \} = \min_{i \in I^+(Q)} \left\{ -\frac{k_{ij}}{b_{ij}} \right\}$$

$$q_i \equiv \inf\left\{ q : \frac{B_i(q)}{\zeta_{ij}} \geq \frac{B_{(i+1)}(q)}{\zeta_{(i+1)j}} \right\} = \begin{cases} \dfrac{k_{ij}\zeta_{(i+1)j} - k_{(i+1)j}\zeta_{ij}}{b_{(i+1)j}\zeta_{ij} - b_{ij}\zeta_{(i+1)j}} & \text{if } \dfrac{k_{ij}}{\zeta_{ij}} \leq \dfrac{k_{(i+1)j}}{\zeta_{(i+1)j}} \\ \infty & \text{otherwise} \end{cases}$$

Proof: Since $B_i(Q) = b_{ij} + \frac{k_{ij}}{Q}$, $B_i(Q)$ is strictly decreasing in Q. Hence the partition of the set I in Step 1 of Algorithm 2 does not change if $Q \geq q < q^+$. Consider the ordering of Step 2. For any couple $(i, i') \in I \times I$ such that

$$\frac{k_{ij}}{\varsigma_{ij}} \geq \frac{k_{i'j}}{\varsigma_{i'j}},$$

$$\frac{B_i(q)}{\varsigma_{ij}} \geq \frac{B_{i'}(q)}{\varsigma_{i'j}} \quad \text{if} \quad q \geq S$$

$$\frac{B_i(q)}{\varsigma_{ij}} \leq \frac{B_{i'}(q)}{\varsigma_{i'j}} \quad \text{if} \quad q \leq S$$

with $S = \dfrac{k_{i'j}\varsigma_{ij} - k_{ij}\varsigma_{i'j}}{b_{ij}\varsigma_{i'j} - b_{i'j}\varsigma_{ij}}$.

As a result:

If $\frac{B_i(Q)}{\varsigma_{ij}} = \frac{B_{(i+1)}(Q)}{\varsigma_{(i+1)j}}$, since $\frac{k_{ij}}{\varsigma_{ij}} \leq \frac{k_{(i+1)j}}{\varsigma_{(i+1)j}}$, $\frac{B_i(q)}{\varsigma_{ij}} \geq \frac{B_{(i+1)}(q)}{\varsigma_{(i+1)j}}$, $\forall q > Q$.

If $\frac{B_i(Q)}{\varsigma_{ij}} < \frac{B_{(i+1)}(Q)}{\varsigma_{(i+1)j}}$ and $\frac{k_{ij}}{\varsigma_{ij}} \leq \frac{k_{(i+1)j}}{\varsigma_{(i+1)j}}$, then $\frac{B_i(q)}{\varsigma_{ij}} < \frac{B_{(i+1)}(q)}{\varsigma_{(i+1)j}}$, $\forall q > Q$.

If $\frac{B_i(Q)}{\varsigma_{ij}} < \frac{B_{(i+1)}(Q)}{\varsigma_{(i+1)j}}$ and $\frac{k_{ij}}{\varsigma_{ij}} < \frac{k_{(i+1)j}}{\varsigma_{(i+1)j}}$, then the ordering between (i) and ($i+1$) will switch at $q = q_i$. This concludes the proof. Q.E.D.

***Algorithm 3* (*solving the sub-problems $L_j(\varsigma)$*)**

Step1. Initialize $Q \leftarrow Q_{\inf}$, $Y = 0$, $X = 0$ and $L_j(\varsigma) = 0$.

Step2. Solving problem $U_j(\varsigma, Q)$ to determine the set $T(Q)$.

Step3. For each $Z \in T(Q)$,

3.1. Compute $V(Z) = \sum_{i \in I} b_{ij} Z_i + \sqrt{\sum_{i \in I} c_{ij} Z_i} + \sqrt{\sum_{i \in I} \varsigma_{ij} Z_i}$

3.2. If $V(Z) + f_j < L_j(\varsigma)$, $L_j(\varsigma) = V(Z) + f_j$, $X = 1$, $Y = Z$

Step4. Compute the next $H(Q)$ such that the set $T(Q)$ changes as defined in Property 5.

Step5. If $H(Q) > Q_{\sup}$ then STOP. Else set $Q = H(Q)$ and go to Step2.

In (*Shen et al., 2003*), it is proved that Algorithm 2 is polynomial. As a result, Algorithm 3 for computing $L_j(\varsigma)$ is polynomial if the number of iterations in it is polynomial. This is obvious as the number of sets $T(Q)$ to consider is equal to the number of changes of the set $I^+(Q)$ which is upper bounded by $|I|$ plus the number of changes of ordering in $I^-(Q)$ which is upper bounded by $|I|(|I|-1)$ as the ordering of any coupe $(i, i') \in I \times I$ switches only once as shown in the proof of Property 5.

Note that by extending the screening of Q to (0, ←), Algorithm 3 can be used to solve problem (SPP_j) to determine $V_j(\varsigma)$. Further, by appropriately defining the retailer average demand, Algorithm 3 can be used to solve problem (**SPP_j**) with arbitrary parameters b_{ij}, c_{ij} and ζ_{ij}.

5 Computational results and analysis

Series of computational experiments were carried out on a PC Pentium IV, 2.80 GHz and 512Mo of RAM. We tested our algorithm on networks composed by respectively 10, 25, 50, 80 and 100 retailers' locations (# Retailers) with respectively 10, 25, 50, 80 and 100 potential distribution centers locations.

The problem instances are randomly generated as follows. The average retailer demand is generated randomly from a uniform distribution between 2500 and 5000 units. The standard deviation of demand was randomly generated uniformly from 50 to 213. The supply lead-times were randomly generated from the integer interval [1, 7]. For each DC_j, h_j =50$, z_ζ = 1.96, and ζ=97.5%. The per-unit shipment costs (a_j) were randomly generated from the integer interval [1, 3]. For each DC_j, order cost plus fixed shipment cost $K_j+\kappa_j$ = 50. The fixed costs f_j of locating a DC at retailer j were generated uniformly from interval [25000, 45000]. The unit costs (d_{ij}) to ship products from the DC_j to the retailer i were uniformly generated from the integer interval (1, 3). We set θ =250 (number of working days). All costs are expressed in dollars.

Table 1 summarizes the lower and uppers bounds (*LB* and *UB*) of total costs expressed in millions of dollars, duality *GAP* which is equal to $(J^* - L^*)/J^*$, number of located DCs (#DCs) and computational time. The duality GAP for all the experiments was between 0.9% and 1.3%. 67 CPU seconds are needed to compute the case with 100 retailers. These results show the effectiveness of our approach.

Table 1. Performances of solution procedure vs. number of retailers

# Retailers	Lower bound	Upper bound	GAP (%)	# DCs	CPU (s)
10	25.55	25.80	0.94	3	4.2
25	58.44	59.00	1.01	7	6.12
50	115.69	116.97	1.09	13	7.42
80	168.96	171.24	1.30	15	27.42
100	210.19	213.05	1.30	17	67

For the case with 50 retailers, figure 2 shows the evolution of the Lagrangian bounds (LB and UB) at certain number of iterations and the convergence curves of the Lagrangian bounds vs. the number of iterations.

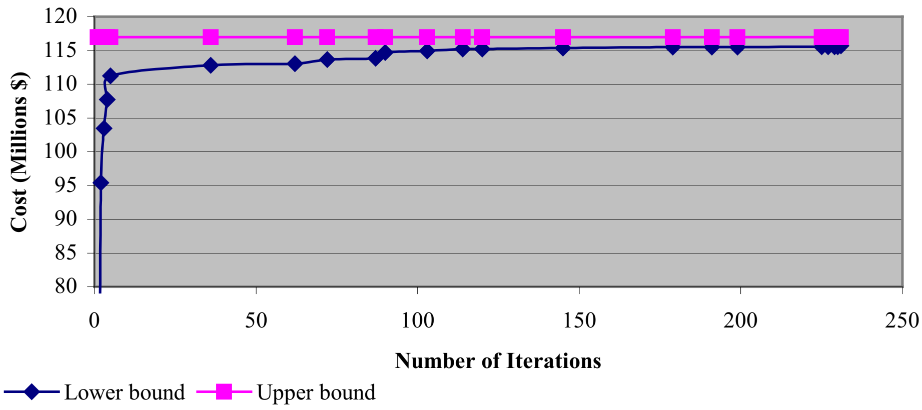

Figure 2. LB and UB evolutions vs. number of iterations

6 Conclusions and perspectives

In this paper, we have presented a Lagrangian relaxation technique to solve an inventory-location problem with random demands and constant supply lead times. The model determines the location of distribution centers and the assignment of retailers to the distribution centers to minimize the total fixed distribution centers location costs, running inventory costs at the DC, transportation cost and the safety stock cost at the distribution centers. Numerical results show the effectiveness of the algorithm.

This study can be extended in a number of important ways. First, we can consider the case where the system operates under other inventory policies such as base stock, *(R, Q)* or *(s, S)* and study the complexity of the model. The second extension concerns the multi-suppliers-multi-product supply chains scenario.

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