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# ON THE DISTRIBUTION OF THE SUMMANDS OF PARTITIONS IN RESIDUE CLASSES

BY

CÉCILE DARTYGE, ANDRÁS SÁRKÖZY  
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ABSTRACT. It is proved that the summands of almost all partitions of  $n$  are well-distributed modulo  $d$  for  $d$  up to  $d = n^{1/2-\varepsilon}$ .

## 1. INTRODUCTION AND THE RESULTS

The Erdős–Lehner Theorem [3] asserts that almost all the  $p(n)$  unrestricted partitions of  $n$  contain

$$(1 + o(1)) \frac{\sqrt{6n}}{2\pi} \log n$$

parts (and the same expression approximates the maximal summand in almost all partitions of  $n$ ).

In [1] and [2] Dartyge and Sárközy studied the distribution of the summands of (unrestricted) partitions in residue classes and, in particular, they showed that if  $d \in \mathbb{N}$  is *fixed* and  $n \rightarrow +\infty$ , then the summands of almost all partitions of  $n$  are well-distributed modulo  $d$ . Moreover, they applied this result to study the rate of the square-free summands to all summands in a “random” partition of  $n$ . (See [4] and [6] for further related results.)

If we also want to study deeper arithmetic properties of random partitions, then we have to go further, and we also have to study the distribution of the summands modulo  $d$  for  $d$  tending to infinity possibly fast in terms of  $n$ . In [1] and [2] the crucial tool was a classical theorem of Meinardus, which cannot be used in the case  $d \rightarrow +\infty$ . Instead, we will use here a probabilistic argument which will enable us to show that the summands themselves are well-distributed modulo  $d$  for  $d$  up to  $d = n^{1/2-\varepsilon}$ , and the sums of the summands belonging to the different modulo  $d$  residue classes are also nearly equal for  $d$  up to  $d = n^{1/6}$ . (We remark that the saddle point method also could be used and, indeed, with a slightly more work some of our error terms could be improved upon. However, for our purposes it suffices to use the more elementary approach presented here. As about the bound  $n^{1/6}$ , it is certainly not sharp. E.g., our proof also yields that each sum in question

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is  $(1 + o(1))n/d$  for  $d = o(n^{1/4})$ .) Recall that in a “random” partition of  $n$  the  $O(1)$  summands occur with frequency of order of magnitude  $\sqrt{n}$ . This fact explains certain conditions in the results (in particular, in Corollary 1) below.

Consider a general unrestricted partition of the positive integer  $n$ :

$$\Pi = \Pi(n) = \left\{ \begin{array}{l} \lambda_1 + \lambda_2 + \cdots + \lambda_m = n, \lambda_1 \geq \cdots \geq \lambda_m (\geq 1) \\ \lambda_\mu \text{'s integers, } m = m(\Pi) \end{array} \right\}.$$

Let  $1 \leq r \leq d$ ,  $\Lambda \geq 1$  be integers, and in  $\Pi(n)$  we investigate the number [resp. the sum] of summands  $\equiv r \pmod{d}$  and  $\geq \Lambda$ . We define  $\Lambda' = \lceil \frac{\Lambda-r}{d} \rceil$ . (Here,  $\Lambda' \geq 0$ ;  $\Lambda' = 0 \Leftrightarrow 1 \leq \Lambda \leq r$ .) As a bound for  $\Lambda$  we shall use the double of the value of the main term in the Erdős–Lehner Theorem.

**THEOREM 1.** *Let  $\omega(n) \nearrow \infty$  arbitrarily slowly,  $d \leq \sqrt{n}$ ,  $\Lambda \leq \frac{\sqrt{6}}{\pi} \sqrt{n} \log n$ . Then, for all but  $p(n)/\omega(n)$  partitions of  $n$ , we have*

$$\begin{aligned} \sum_{\substack{\mu \\ \lambda_\mu \equiv r(d) \\ \lambda_\mu \geq \Lambda}} 1 &= \frac{\sqrt{6n}}{\pi d} \log \frac{1}{1 - \exp\left(-\frac{\pi(r+(\Lambda'+1)d)}{\sqrt{6n}}\right)} + \\ &+ O\left(\sqrt{\frac{\sqrt{n}\omega(n)}{\left(\exp\left(\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}\right) - 1\right) \cdot \min(d, r + \Lambda'd)}}\right). \end{aligned}$$

**COROLLARY 1 OF THEOREM 1** (with  $\Lambda = r$ ). *Let  $0 < \varepsilon < \frac{1}{2}$  (fixed),  $1 \leq d \leq n^{\frac{1}{2}-\varepsilon}$ ,  $\omega(n) \nearrow \infty$ ,  $\frac{\omega(n)}{\log n} d \leq r \leq d$ . Then, for all but  $p(n)/\omega(n)$  partitions of  $n$ , we have*

$$\sum_{\lambda_\mu \equiv r(d)} 1 = (1 + o(1)) \frac{\sqrt{6n}}{\pi d} \log\left(\frac{\sqrt{n}}{d}\right), \quad \text{as } n \rightarrow \infty.$$

**COROLLARY 2 OF THEOREM 1** (with  $\Lambda = d + 1$ ). *Let  $0 < \varepsilon < \frac{1}{2}$  (fixed),  $1 \leq d \leq n^{\frac{1}{2}-\varepsilon}$ . Then, for almost all partitions of  $n$ ,*

$$\sum_{\substack{\mu \\ \lambda_\mu \equiv r(d) \\ \lambda_\mu > r}} 1 = (1 + o(1)) \frac{\sqrt{6n}}{\pi d} \log\left(\frac{\sqrt{n}}{d}\right), \quad \text{as } n \rightarrow \infty.$$

**THEOREM 2.** *Let  $\omega(n) \nearrow \infty$  arbitrarily slowly,  $d \leq \sqrt[6]{n}$ ,  $\Lambda \leq \frac{\sqrt{6}}{\pi} \sqrt{n} \log n$ . Then, for all but  $p(n)/\omega(n)$  partitions of  $n$ , we have*

$$\sum_{\substack{\mu \\ \lambda_\mu \equiv r(d) \\ \lambda_\mu \geq \Lambda}} \lambda_\mu = \frac{n}{d} + O\left(\frac{\sqrt{n}}{d}(r + \Lambda'd) \log n\right) + O\left(\frac{n^{3/4} \log n}{\sqrt{d}} \sqrt{\omega(n)}\right).$$

**COROLLARY OF THEOREM 2** (with  $\Lambda = r$ ). *Let  $\omega(n) \nearrow \infty$  arbitrarily slowly, and  $d \leq n^{1/6}$ . Then, for all but  $p(n)/\omega(n)$  partitions of  $n$ , we have*

$$\sum_{\substack{\mu \\ \lambda_\mu \equiv r(d)}} \lambda_\mu = \frac{n}{d} + O\left(\frac{n^{3/4} \log n}{\sqrt{d}} \sqrt{\omega(n)}\right).$$

## 2. A PROBABILISTIC APPROACH

For  $1 \leq r \leq d$ ,  $\Lambda \geq 1$  (integers), let

$$S_0(n, \Pi, \Lambda; d, r) \stackrel{\text{def}}{=} \sum_{\substack{\mu \\ \lambda_\mu \equiv r(d) \\ \lambda_\mu \geq \Lambda}} 1$$

and

$$S_1(n, \Pi, \Lambda; d, r) \stackrel{\text{def}}{=} \sum_{\substack{\mu \\ \lambda_\mu \equiv r(d) \\ \lambda_\mu \geq \Lambda}} \lambda_\mu.$$

We investigate  $S_0(n, \Pi, \Lambda; d, r)$  and  $S_1(n, \Pi, \Lambda; d, r)$  together as

$$S_\alpha(n, \Pi, \Lambda; d, r) = \sum_{\substack{\mu \\ \lambda_\mu \equiv r(d) \\ \lambda_\mu \geq \Lambda}} \lambda_\mu^\alpha.$$

Let us consider the random field consisting of all possible choices of partitions of  $n$  with equal probability. For  $\alpha \in \{0, 1\}$ , let  $\xi_{n,\alpha}$  denote a random variable which assigns the nonnegative integer  $S_\alpha(n, \Pi, \Lambda; d, r)$  to a partition  $\Pi(n)$ .

To prove Theorems 1 and 2 we will use the Chebyshev inequality. We will need estimates for the mean value  $M(\xi_{n,\alpha})$  and the variance  $D^2(\xi_{n,\alpha})$  of  $\xi_{n,\alpha}$ .

In the next paragraphs we will prove the following lemmas:

**LEMMA 1.** *For  $n \geq n_0$ ,  $\Lambda \leq \frac{\sqrt{6}}{\pi} \sqrt{n} \log n$ ,  $\Lambda' = \left\lceil \frac{\Lambda-r}{d} \right\rceil$ , and  $d \leq n$ , we have*

$$\begin{aligned} M(\xi_{n,0}) = & \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \left\{ \frac{\sqrt{6n}}{\pi d} \log \frac{1}{1 - \exp\left(-\frac{\pi}{\sqrt{6n}}(r + (\Lambda' + 1)d)\right)} + \right. \\ & \left. + \frac{1}{\exp\left(\frac{\pi}{\sqrt{6n}}(r + \Lambda'd)\right) - 1} \right\} + \\ & + O(1) \frac{1}{\exp\left(\frac{\pi}{\sqrt{6n}}(r + (\Lambda' + 1)d)\right) - 1} + O(n^{-3}) \end{aligned}$$

and

$$\begin{aligned}
& M(\xi_{n,1}) = \\
& = \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \left\{ \frac{n}{d} - \frac{6n}{\pi^2 d} \sum_{t=1}^{\infty} \frac{1}{t^2} \left(1 - \exp\left(-\frac{\pi(r + \Lambda'd)t}{\sqrt{6n}}\right)\right) \right\} + \\
& + (r + \Lambda'd) \frac{\sqrt{6n}}{\pi d} \log \frac{1}{1 - \exp\left(-\frac{\pi(r + \Lambda'd)}{\sqrt{6n}}\right)} + \frac{O((\Lambda' + 1)d)}{\exp\left(\frac{\pi(r + \Lambda'd)}{\sqrt{6n}}\right) - 1} \Big\} + O(n^{-3}).
\end{aligned}$$

**LEMMA 2.** (i) For  $n \geq n_0$ ,  $d \leq \sqrt{n}$ ,  $\Lambda \leq \frac{\sqrt{6}}{\pi} \sqrt{n} \log n$ , and  $\Lambda' = \lceil \frac{\Lambda - r}{d} \rceil$ , we have

$$D^2(\xi_{n,0}) = O\left(\frac{\sqrt{n}}{\left(\exp\left(\frac{\pi(r + \Lambda'd)}{\sqrt{6n}}\right) - 1\right) \cdot \min(d, r + \Lambda'd)}\right).$$

(ii) For  $n \geq n_0$ ,  $d \leq n^{5/12}$  and  $\Lambda \leq \frac{\sqrt{6}}{\pi} \sqrt{n} \log n$ ,

$$D^2(\xi_{n,1}) = O\left(\frac{n^{3/2} \log^2 n}{d}\right).$$

In the last paragraph, we will use these two lemmas to complete the proofs of Theorems 1 and 2.

### 3. A CONSEQUENCE OF A FORMULA OF HARDY AND RAMANUJAN

The proofs of Lemmas 1 and 2 rely on a precise estimate of the rate  $\frac{p(n-t)}{p(n)}$ . By [5] we have

$$p(n) = \frac{\exp\left(\frac{2\pi}{\sqrt{6}} \sqrt{n - \frac{1}{24}}\right)}{4(n - \frac{1}{24})\sqrt{3}} \left\{ 1 - \frac{\sqrt{6}}{2\pi \sqrt{n - \frac{1}{24}}} \right\} + O\left(\exp\left(0.51 \cdot \frac{2\pi}{\sqrt{6}} \sqrt{n}\right)\right).$$

This implies that

$$p(n) = \frac{1}{4n\sqrt{3}} \left\{ 1 - \left(\frac{\sqrt{6}}{2\pi} + \frac{\pi}{24\sqrt{6}}\right) \frac{1}{\sqrt{n}} + O\left(\frac{1}{n}\right) \right\} \exp\left(\frac{2\pi}{\sqrt{6}} \sqrt{n}\right).$$

**LEMMA 3.** For  $n \geq 2$  and  $1 \leq t \leq n^{5/8}$ , we have

$$\begin{aligned}
& \frac{p(n-t)}{p(n)} = \\
& = \left(1 + O(n^{-3/4})\right) \left(1 + \frac{t}{n} - \frac{\pi t^2}{4n\sqrt{6n}} - \frac{3\pi t^3}{8n^2\sqrt{6n}} + \frac{\pi^2 t^4}{192n^3}\right) \exp\left(-\frac{\pi t}{\sqrt{6n}}\right).
\end{aligned}$$

*Proof.* With  $c = \frac{\sqrt{6}}{2\pi} + \frac{\pi}{24\sqrt{6}}$ ,

$$\begin{aligned}
& \frac{p(n-t)}{p(n)} = \\
&= \frac{n}{n-t} \cdot \frac{1 - \frac{c}{\sqrt{n-t}} + O\left(\frac{1}{n}\right)}{1 - \frac{c}{\sqrt{n}} + O\left(\frac{1}{n}\right)} \exp\left(-\frac{2\pi}{\sqrt{6}}(\sqrt{n} - \sqrt{n-t})\right) = \\
&= \left(1 + \frac{t}{n} + O(n^{-3/4})\right) \frac{1 - \frac{c}{\sqrt{n}} + O(n^{-3/4})}{1 - \frac{c}{\sqrt{n}} + O\left(\frac{1}{n}\right)} \exp\left(-\frac{2\pi}{\sqrt{6}} \cdot \frac{t}{\sqrt{n} + \sqrt{n-t}}\right) = \\
&= \left(1 + O(n^{-3/4})\right) \left(1 + \frac{t}{n}\right) \exp\left(-\frac{\pi t}{\sqrt{6n}} - \frac{\pi t^2}{4n\sqrt{6n}} - \right. \\
&\quad \left. - \frac{\pi t^2}{\sqrt{6n}} \left\{ \frac{1}{(\sqrt{n} + \sqrt{n-t})^2} - \frac{1}{4n} \right\}\right) = \\
&= \left(1 + O(n^{-3/4})\right) \left(1 + \frac{t}{n}\right) \exp\left(-\frac{\pi t}{\sqrt{6n}} - \frac{\pi t^2}{4n\sqrt{6n}} - \frac{\pi t^3}{8n^2\sqrt{6n}} + O\left(\frac{t^4}{n^{7/2}}\right)\right) = \\
&= \left(1 + O(n^{-3/4})\right) \left(1 + \frac{t}{n}\right) \exp\left(-\frac{\pi t}{\sqrt{6n}}\right) \left\{1 - \frac{\pi t^2}{4n\sqrt{6n}} + \frac{1}{2} \frac{\pi^2 t^4}{16n^2 6n}\right\} \cdot \\
&\quad \cdot \left\{1 - \frac{\pi t^3}{8n^2\sqrt{6n}}\right\} = \\
&= \left(1 + O(n^{-3/4})\right) \left(1 + \frac{t}{n}\right) \left(1 - \frac{\pi t^2}{4n\sqrt{6n}} + \frac{1}{2} \frac{\pi^2 t^4}{16n^2 6n} - \frac{\pi t^3}{8n^2\sqrt{6n}}\right) \cdot \\
&\quad \cdot \exp\left(-\frac{\pi t}{\sqrt{6n}}\right) = \\
&= \left(1 + O(n^{-3/4})\right) \left(1 + \frac{t}{n} - \frac{\pi t^2}{4n\sqrt{6n}} - \frac{3\pi t^3}{8n^2\sqrt{6n}} + \frac{\pi^2 t^4}{192n^3}\right) \exp\left(-\frac{\pi t}{\sqrt{6n}}\right).
\end{aligned}$$

#### 4. ESTIMATIONS OF THE MEAN VALUE: PROOF OF LEMMA 1

We start from

$$S_\alpha(n, \Pi, \Lambda; d, r) = \sum_{\substack{\mu \\ \lambda_\mu \equiv r(d) \\ \lambda_\mu \geq \Lambda}} \lambda_\mu^\alpha = \sum_{k=\Lambda'}^{n'} (r+kd)^\alpha \cdot \text{mult}_{in \Pi(n)}(r+kd)$$

where  $\Lambda' = \lceil \frac{\Lambda-r}{d} \rceil$ ,  $n' = \lfloor \frac{n-r}{d} \rfloor$ .

Let  $s = \left\lfloor 6 \frac{\sqrt{6}}{\pi} \sqrt{n} \log n \right\rfloor + 1$  (and  $p(0) \doteq 1$ ).

$$\begin{aligned}
M(\xi_{n,\alpha}) &= \frac{1}{p(n)} \sum_{\Pi(n)} S_\alpha(n, \Pi, \Lambda; d, r) = \\
&= \frac{1}{p(n)} \sum_{\Pi(n)} \sum_{k=\Lambda'}^{n'} (r+kd)^\alpha \text{mult}_{in \Pi(n)}(r+kd) =
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{p(n)} \sum_{k=\Lambda'}^{n'} (r+kd)^\alpha \sum_{\substack{\Pi(n) \\ \text{in } \Pi(n)}} \text{mult} (r+kd) = \\
&= \frac{1}{p(n)} \sum_{k=\Lambda'}^{n'} (r+kd)^\alpha \sum_{t=1}^{\lfloor \frac{n}{r+kd} \rfloor} p(n - (r+kd)t) = \\
&= \sum_{\substack{\Lambda' \leq k \leq n' \\ 1 \leq t < \frac{s}{r+kd}}} (r+kd)^\alpha \frac{p(n - (r+kd)t)}{p(n)} + \\
&\quad + \sum_{k=\Lambda'}^{n'} (r+kd)^\alpha \sum_{\substack{1 \leq t \leq \frac{n}{r+kd} \\ (r+kd)t \geq s}} \frac{O(p(n-s))}{p(n)} \text{ by Lemma 3} \\
&= \sum_{\substack{\Lambda' \leq k \leq n' \\ 1 \leq t < \frac{s}{r+kd}}} (r+kd)^\alpha \left(1 + O(n^{-3/4})\right) \left\{ 1 + \frac{(r+kd)t}{n} - \frac{\pi((r+kd)t)^2}{4n\sqrt{6n}} - \right. \\
&\quad \left. - \frac{3\pi((r+kd)t)^3}{8n^2\sqrt{6n}} + \frac{\pi^2((r+kd)t)^4}{192n^3} \right\} \exp\left(-\frac{\pi}{\sqrt{6n}}(r+kd)t\right) + \\
&\quad + O(n) \cdot n^\alpha \cdot n \cdot O(n^{-6}) = \\
&= \left(1 + O(n^{-3/4})\right) \sum_{\substack{k \geq \Lambda' \\ t \geq 1 \\ (r+kd)t < s}} (r+kd)^\alpha \left\{ 1 + \frac{(r+kd)t}{n} - \frac{\pi((r+kd)t)^2}{4n\sqrt{6n}} \right\} \cdot \\
&\quad \cdot \exp\left(-\frac{\pi}{\sqrt{6n}}(r+kd)t\right) + O(n^{-3}) \quad \text{if } n \text{ is large enough}
\end{aligned}$$

since

$$\frac{s^3}{n^2\sqrt{n}} = O\left(\frac{\log^3 n}{n}\right), \quad \frac{s^4}{n^3} = O\left(\frac{\log^4 n}{n}\right), \quad \frac{s}{n} = o(1), \quad \frac{s^2}{n^{3/2}} = o(1),$$

moreover,  $r+kd < \frac{s}{t}$  implies that  $k \leq n'$  if  $n \geq n_0$ . Thus we have proved the following

**LEMMA 4.** For  $n \geq n_0$ ,  $\Lambda \leq \frac{\sqrt{6}}{\pi}\sqrt{n} \log n$ ,  $d \leq n$ , and  $s = \lfloor 6\frac{\sqrt{6}}{\pi}\sqrt{n} \log n \rfloor + 1$ , we have

$$\begin{aligned}
&M(\xi_{n,0}) = \\
&= \left(1 + O(n^{-3/4})\right) \sum_{\substack{k \geq \Lambda' \\ t \geq 1 \\ (r+kd)t < s}} \left(1 + \frac{(r+kd)t}{n} - \frac{\pi((r+kd)t)^2}{4n\sqrt{6n}}\right) \cdot \\
&\quad \cdot \exp\left(-\frac{\pi}{\sqrt{6n}}(r+kd)t\right) + O(n^{-3})
\end{aligned}$$

and

$$\begin{aligned}
M(\xi_{n,1}) &= \\
&= \left(1 + O(n^{-3/4})\right) \sum_{\substack{k \geq \Lambda' \\ t \geq 1 \\ (r+kd)t < s}} (r+kd) \left\{ 1 + \frac{(r+kd)t}{n} - \frac{\pi((r+kd)t)^2}{4n\sqrt{6n}} \right\} \\
&\quad \cdot \exp\left(-\frac{\pi}{\sqrt{6n}}(r+kd)t\right) + O(n^{-3}).
\end{aligned}$$

Again, let  $n \geq n_0$ ,  $\Lambda \leq \frac{\sqrt{6}}{\pi}\sqrt{n} \log n$ ,  $d \leq n$ ,  $s = \left\lfloor 6\frac{\sqrt{6}}{\pi}\sqrt{n} \log n \right\rfloor + 1$ ,  $\alpha \in \{0, 1\}$ . Since

$$\frac{s}{n} = O\left(\frac{\log n}{\sqrt{n}}\right) \quad \text{and} \quad \frac{s^2}{n^{3/2}} = O\left(\frac{\log^2 n}{\sqrt{n}}\right),$$

Lemma 4 yields that

$$\begin{aligned}
M(\xi_{n,\alpha}) &= \\
&= \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{\substack{k \geq \Lambda' \\ t \geq 1 \\ (r+kd)t < s}} (r+kd)^\alpha \exp\left(-\frac{\pi}{\sqrt{6n}}(r+kd)t\right) + O(n^{-3}) = \\
&= \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{\substack{k \geq \Lambda' \\ t \geq 1}} (r+kd)^\alpha \exp\left(-\frac{\pi}{\sqrt{6n}}(r+kd)t\right) + \\
&\quad + O(1) \sum_{\substack{k \geq \Lambda' \\ t \geq 1 \\ (r+kd)t \geq s}} (r+kd)^\alpha \exp\left(-\frac{\pi}{\sqrt{6n}}(r+kd)t\right) + O(n^{-3}) = \\
&= \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{k=\Lambda'}^{\infty} \sum_{t=1}^{\infty} (r+kd)^\alpha \exp\left(-\frac{\pi}{\sqrt{6n}}(r+kd)t\right) + \\
&\quad + O(1) \sum_{j=s}^{\infty} \exp\left(-\frac{\pi}{\sqrt{6n}}j\right) \sum_{\substack{(r+kd)t=j \\ t \geq 1, k \geq \Lambda'}} (r+kd)^\alpha + O(n^{-3}) = \\
&= \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{k=\Lambda'}^{\infty} \sum_{t=1}^{\infty} (r+kd)^\alpha \exp\left(-\frac{\pi}{\sqrt{6n}}(r+kd)t\right) + \\
&\quad + O(1) \sum_{j=s}^{\infty} \exp\left(-\frac{\pi}{\sqrt{6n}}j\right) O(j \cdot j^\alpha) + O(n^{-3}).
\end{aligned}$$

Here, with  $x = \exp\left(-\frac{\pi}{\sqrt{6n}}\right)$ ,

$$\begin{aligned}
\sum_{j=s}^{\infty} j \exp\left(-\frac{\pi}{\sqrt{6n}}j\right) &= \sum_{j=s}^{\infty} j x^j = x \left( \sum_{j=s}^{\infty} x^j \right)' = x \left( \frac{x^s}{1-x} \right)' = \\
&= x \frac{s x^{s-1} (1-x) + x^s}{(1-x)^2} = \frac{s x^s}{1-x} + \frac{x^{s+1}}{(1-x)^2} = x^s \left( \frac{s}{1-x} + \frac{x}{(1-x)^2} \right)
\end{aligned}$$



and

$$\begin{aligned}
\sum_{j=s}^{\infty} j^2 \exp\left(-\frac{\pi}{\sqrt{6n}}j\right) &= \sum_{j=s}^{\infty} j^2 x^j = x \left( \sum_{j=s}^{\infty} j x^j \right)' = \\
&= x \left( \frac{s x^s}{1-x} + \frac{x^{s+1}}{(1-x)^2} \right)' = \\
&= x \left( \frac{s^2 x^{s-1} (1-x) - s \cdot x^s \cdot (-1)}{(1-x)^2} + \frac{(s+1)x^s (1-x)^2 - x^{s+1} 2(1-x)(-1)}{(1-x)^4} \right) = \\
&= x^s \left( \frac{s^2}{1-x} + \frac{s x}{(1-x)^2} + \frac{(s+1)x}{(1-x)^2} + \frac{2x^2}{(1-x)^3} \right) = \\
&= x^s \left( \frac{s^2}{1-x} + \frac{(2s+1)x}{(1-x)^2} + \frac{2x^2}{(1-x)^3} \right) = \\
&= x^s \left( O(n^{3/2} \log^2 n) + O(n^{3/2} \log n) + O(n^{3/2}) \right) = \\
&= x^s O(n^{3/2} \log^2 n) = O(n^{-6} n^{3/2} \log^2 n).
\end{aligned}$$

Consequently,

$$\sum_{j=s}^{\infty} j^{\alpha+1} \exp\left(-\frac{\pi}{\sqrt{6n}}j\right) = O(n^{-4.5} \log^2 n),$$

and

$$\begin{aligned}
M(\xi_{n,\alpha}) &= \\
&= \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{k=\Lambda'}^{\infty} \sum_{t=1}^{\infty} (r+kd)^{\alpha} \exp\left(-\frac{\pi}{\sqrt{6n}}(r+kd)t\right) + O(n^{-3}).
\end{aligned}$$

$[\alpha = 0 :]$

The following separation of  $k = \Lambda'$  plays a role for  $\Lambda' = 0$ :

$$\begin{aligned}
M(\xi_{n,0}) &= \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{t=1}^{\infty} \exp\left(-\frac{\pi}{\sqrt{6n}}(r+\Lambda'd)t\right) + \\
&\quad + \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{t=1}^{\infty} \exp\left(-\frac{\pi r t}{\sqrt{6n}}\right) \sum_{k=\Lambda'+1}^{\infty} \left(\exp\left(-\frac{\pi d t}{\sqrt{6n}}\right)\right)^k + \\
&\quad \quad \quad + O(n^{-3}) = \\
&= \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \frac{1}{\exp\left(\frac{\pi}{\sqrt{6n}}(r+\Lambda'd)\right) - 1} + O(n^{-3}) + \\
&\quad + \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{t=1}^{\infty} \exp\left(-\frac{\pi}{\sqrt{6n}}(r+(\Lambda'+1)d)t\right) \frac{1}{1 - \exp\left(-\frac{\pi d t}{\sqrt{6n}}\right)}.
\end{aligned}$$

For real  $\beta > 0$ , we have

$$\frac{1}{2} < \frac{1}{1 - e^{-\beta}} - \frac{1}{\beta} < 1$$

since

$$1 > 1 + \frac{1}{e^\beta - 1} - \frac{1}{\beta} = 1 - \frac{\frac{\beta}{2!} + \frac{\beta^2}{3!} + \dots}{e^\beta - 1} > 1 - \frac{\frac{\beta}{2} + \frac{\beta^2}{2 \cdot 2!} + \dots}{e^\beta - 1} = \frac{1}{2}.$$

Thus, by using the formula  $\sum_{t=1}^{\infty} \frac{e^{-at}}{t} = \log \frac{1}{1-e^{-a}}$  with  $a = \frac{\pi}{\sqrt{6n}}(r + (\Lambda' + 1)d)$ ,

$$\begin{aligned} M(\xi_{n,0}) &= \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \frac{1}{\exp\left(\frac{\pi}{\sqrt{6n}}(r + \Lambda'd)\right) - 1} + O(n^{-3}) + \\ &+ \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{t=1}^{\infty} \exp\left(-\frac{\pi}{\sqrt{6n}}(r + (\Lambda' + 1)d)t\right) \left\{\frac{\sqrt{6n}}{\pi dt} + O(1)\right\} = \\ &= \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \left\{\frac{1}{\exp\left(\frac{\pi}{\sqrt{6n}}(r + \Lambda'd)\right) - 1} + \right. \\ &\quad \left. + \frac{\sqrt{6n}}{\pi d} \log \frac{1}{1 - \exp\left(-\frac{\pi}{\sqrt{6n}}(r + (\Lambda' + 1)d)\right)}\right\} + \\ &+ O(1) \frac{1}{\exp\left(\frac{\pi}{\sqrt{6n}}(r + (\Lambda' + 1)d)\right) - 1} + O(n^{-3}). \end{aligned}$$

[ $\alpha = 1$  :]

$$\begin{aligned} M(\xi_{n,1}) &= \\ &= \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{t=1}^{\infty} \exp\left(-\frac{\pi rt}{\sqrt{6n}}\right) \sum_{k=\Lambda'}^{\infty} (r + kd)^1 \left(\exp\left(-\frac{\pi dt}{\sqrt{6n}}\right)\right)^k + \\ &\quad + O(n^{-3}) = \\ &= \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{t=1}^{\infty} \exp\left(-\frac{\pi rt}{\sqrt{6n}}\right) \left\{r \sum_{k=\Lambda'}^{\infty} \left(\exp\left(-\frac{\pi dt}{\sqrt{6n}}\right)\right)^k + \right. \\ &\quad \left. + d \sum_{k=\Lambda'}^{\infty} k \left(\exp\left(-\frac{\pi dt}{\sqrt{6n}}\right)\right)^k\right\} + O(n^{-3}). \end{aligned}$$

Since, for  $y = \exp\left(-\frac{\pi dt}{\sqrt{6n}}\right)$ ,

$$\sum_{k=\Lambda'}^{\infty} ky^k = y^{\Lambda'} \left(\frac{\Lambda'}{1-y} + \frac{y}{(1-y)^2}\right),$$

we obtain that

$$\begin{aligned} M(\xi_{n,1}) &= \\ &= \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{t=1}^{\infty} \exp\left(-\frac{\pi rt}{\sqrt{6n}}\right) \left\{r \frac{\exp\left(-\frac{\pi dt \Lambda'}{\sqrt{6n}}\right)}{1 - \exp\left(-\frac{\pi dt}{\sqrt{6n}}\right)} + \right. \end{aligned}$$

$$\begin{aligned}
& + d \cdot \exp\left(-\frac{\pi dt \Lambda'}{\sqrt{6n}}\right) \cdot \left\{ \frac{\Lambda'}{1 - \exp\left(-\frac{\pi dt}{\sqrt{6n}}\right)} + \frac{\exp\left(-\frac{\pi dt}{\sqrt{6n}}\right)}{\left(1 - \exp\left(-\frac{\pi dt}{\sqrt{6n}}\right)\right)^2} \right\} + \\
& \quad + O(n^{-3}) = \\
& = \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{t=1}^{\infty} \exp\left(-\frac{\pi(r + \Lambda'd)t}{\sqrt{6n}}\right) \left\{ \frac{r + \Lambda'd}{1 - \exp\left(-\frac{\pi dt}{\sqrt{6n}}\right)} + \right. \\
& \quad \left. + d \frac{\exp\left(-\frac{\pi dt}{\sqrt{6n}}\right)}{\left(1 - \exp\left(-\frac{\pi dt}{\sqrt{6n}}\right)\right)^2} \right\} + O(n^{-3}).
\end{aligned}$$

For real  $\beta > 0$ , we have

$$\frac{1}{\beta} + \frac{1}{2} < \frac{1}{1 - e^{-\beta}} < \frac{1}{\beta} + 1.$$

Thus,

$$\frac{1}{\beta} - \frac{1}{2} < \frac{1}{e^{\beta} - 1} = \frac{1}{1 - e^{-\beta}} - 1 < \frac{1}{\beta}.$$

For  $0 < \beta < 2$  these imply that

$$\frac{1}{\beta^2} - \frac{1}{4} < \frac{1}{(e^{\beta} - 1)(1 - e^{-\beta})}$$

which is trivial for  $\beta \geq 2$ . Next,

$$\begin{aligned}
\frac{1}{(e^{\beta} - 1)(1 - e^{-\beta})} & = \left(\frac{e^{\beta/2}}{e^{\beta} - 1}\right)^2 = \left(\frac{e^{\beta/2}}{\beta \left(1 + \frac{\beta}{2!} + \frac{\beta^2}{3!} + \dots + \frac{\beta^k}{(k+1)!} + \dots\right)}\right)^2 < \\
& < \left(\frac{e^{\beta/2}}{\beta \left(1 + \frac{\beta/2}{1!} + \frac{(\beta/2)^2}{2!} + \dots + \frac{(\beta/2)^k}{k!} + \dots\right)}\right)^2 = \frac{1}{\beta^2}.
\end{aligned}$$

For  $\beta > 0$  we obtained that

$$\frac{1}{\beta^2} - \frac{1}{4} < \frac{e^{-\beta}}{(1 - e^{-\beta})^2} < \frac{1}{\beta^2}.$$

Therefore,

$$\begin{aligned}
& M(\xi_{n,1}) = \\
& = \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{t=1}^{\infty} \exp\left(-\frac{\pi(r + \Lambda'd)t}{\sqrt{6n}}\right) \left\{ (r + \Lambda'd) \left(\frac{\sqrt{6n}}{\pi dt} + O(1)\right) + \right. \\
& \quad \left. + d \cdot \left(\left(\frac{\sqrt{6n}}{\pi dt}\right)^2 + O(1)\right) \right\} + O(n^{-3}) =
\end{aligned}$$

$$\begin{aligned}
&= \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \left\{ \frac{6n}{\pi^2 d} \sum_{t=1}^{\infty} \frac{1}{t^2} \exp\left(-\frac{\pi(r + \Lambda'd)t}{\sqrt{6n}}\right) + \right. \\
&\quad + (r + \Lambda'd) \frac{\sqrt{6n}}{\pi d} \sum_{t=1}^{\infty} \frac{1}{t} \exp\left(-\frac{\pi(r + \Lambda'd)t}{\sqrt{6n}}\right) + \\
&\quad \left. + O((\Lambda' + 1)d) \sum_{t=1}^{\infty} \exp\left(-\frac{\pi(r + \Lambda'd)t}{\sqrt{6n}}\right) \right\} + O(n^{-3}) = \\
&= \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \left\{ \frac{n}{d} - \frac{6n}{\pi^2 d} \sum_{t=1}^{\infty} \frac{1}{t^2} \left(1 - \exp\left(-\frac{\pi(r + \Lambda'd)t}{\sqrt{6n}}\right)\right) + \right. \\
&\quad \left. + (r + \Lambda'd) \frac{\sqrt{6n}}{\pi d} \log \frac{1}{1 - \exp\left(-\frac{\pi(r + \Lambda'd)t}{\sqrt{6n}}\right)} + \frac{O((\Lambda' + 1)d)}{\exp\left(\frac{\pi(r + \Lambda'd)}{\sqrt{6n}}\right) - 1} \right\} + \\
&\quad + O(n^{-3}).
\end{aligned}$$

Thus we have proved our Lemma 1. Later we shall also use the following

*Remark 1.* Here, since for  $\beta \geq 0$ ,  $1 - e^{-\beta} \leq \beta$  holds,

$$\begin{aligned}
&\frac{6n}{\pi^2 d} \sum_{t=1}^{\infty} \frac{1}{t^2} \left(1 - \exp\left(-\frac{\pi(r + \Lambda'd)t}{\sqrt{6n}}\right)\right) = \\
&= \frac{6n}{\pi^2 d} \left( \sum_{t=1}^n \frac{1}{t^2} O\left(\frac{(r + \Lambda'd)t}{\sqrt{n}}\right) + \sum_{t=n+1}^{\infty} \frac{1}{t^2} O(1) \right) = \\
&= \frac{6n}{\pi^2 d} \left( O\left(\frac{r + \Lambda'd}{\sqrt{n}}\right) O(\log n) + O\left(\frac{1}{n}\right) \right) = O\left(\frac{\sqrt{n}}{d} (r + \Lambda'd) \log n\right) + \\
&\quad + O\left(\frac{1}{d}\right).
\end{aligned}$$

## 5. PROOF OF LEMMA 2

Since

$$\begin{aligned}
D^2(\xi_{n,\alpha}) &= M((\xi_{n,\alpha} - M(\xi_{n,\alpha}))^2) = \\
&= M(\xi_{n,\alpha}^2) - (M(\xi_{n,\alpha}))^2
\end{aligned}$$

we shall estimate  $M(\xi_{n,\alpha}^2)$  from above (for  $\alpha \in \{0, 1\}$ ).

$$\begin{aligned}
M(\xi_{n,\alpha}^2) &= \frac{1}{p(n)} \sum_{\Pi(n)} S_{\alpha}^2(n, \Pi, \Lambda; d, r) = \\
&= \frac{1}{p(n)} \sum_{\Pi(n)} \left( \sum_{k=\Lambda'}^{n'} (r + kd)^{\alpha} \text{mult}_{in \Pi(n)}(r + kd) \right) \left( \sum_{j=\Lambda'}^{n'} (r + jd)^{\alpha} \text{mult}_{in \Pi(n)}(r + jd) \right) = \\
&= \frac{1}{p(n)} \sum_{k=\Lambda'}^{n'} (r + kd)^{\alpha} \sum_{j=\Lambda'}^{n'} (r + jd)^{\alpha} \sum_{\Pi(n)} \text{mult}_{in \Pi(n)}(r + kd) \cdot \text{mult}_{in \Pi(n)}(r + jd) =
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{p(n)} \sum_{k=\Lambda'}^{n'} (r+kd)^\alpha \sum_{j=\Lambda'}^{n'} (r+jd)^\alpha \sum_{t_1=1}^n \sum_{t_2=1}^n \sum_{\substack{\Pi(n) \\ \text{mult}_{in \Pi(n)}(r+kd) \geq t_1 \\ \text{mult}_{in \Pi(n)}(r+jd) \geq t_2}} 1 = \\
&= \frac{1}{p(n)} \sum_{k=\Lambda'}^{n'} (r+kd)^\alpha \sum_{\substack{j=\Lambda' \\ j \neq k}}^{n'} (r+jd)^\alpha \sum_{\substack{t_1, t_2 \geq 1 \\ t_1(r+kd) + t_2(r+jd) \leq n}} p(n - t_1(r+kd) - \\
&\quad - t_2(r+jd)) + \frac{1}{p(n)} \sum_{k=\Lambda'}^{n'} (r+kd)^{2\alpha} \sum_{\substack{t \geq 1 \\ t(r+kd) \leq n}} (2t-1)p(n-t(r+kd)) = \\
&= \frac{1}{p(n)} \sum_{k=\Lambda'}^{n'} (r+kd)^\alpha \sum_{j=\Lambda'}^{n'} (r+jd)^\alpha \sum_{\substack{t_1, t_2 \geq 1 \\ t_1(r+kd) + t_2(r+jd) \leq n}} p(n - t_1(r+kd) - \\
&\quad - t_2(r+jd)) + \frac{1}{p(n)} \sum_{k=\Lambda'}^{n'} (r+kd)^{2\alpha} \sum_{\substack{t \geq 1 \\ t(r+kd) \leq n}} t \cdot p(n-t(r+kd)).
\end{aligned}$$

By Lemma 3, with  $s = \left\lfloor 6 \frac{\sqrt{6}}{\pi} \sqrt{n} \log n \right\rfloor + 1$  and  $n \geq n'_0$ , since

$$\sum_{\substack{k \geq \Lambda' \\ t \geq 1 \\ t(r+kd) \leq n}} 1 \leq \sum_{m=1}^n \frac{n}{m} = O(n \log n),$$

$$\begin{aligned}
&M(\xi_{n,\alpha}^2) = \\
&= \sum_{\substack{k, j \geq \Lambda' \\ t_1, t_2 \geq 1 \\ t_1(r+kd) < s \\ t_2(r+jd) < s}} (r+kd)^\alpha (r+jd)^\alpha \frac{p(n - t_1(r+kd) - t_2(r+jd))}{p(n)} + \\
&\quad + O((n \log n)^2 n^{2\alpha} n^{-6}) + \\
&\quad + \sum_{\substack{k \geq \Lambda' \\ t \geq 1 \\ t(r+kd) < s}} (r+kd)^{2\alpha} t \frac{p(n - t(r+kd))}{p(n)} + O((n \log n) n^{2\alpha} \cdot n \cdot n^{-6}) = \\
&= \sum_{\substack{k, j \geq \Lambda' \\ t_1, t_2 \geq 1 \\ t_1(r+kd) < s \\ t_2(r+jd) < s}} (r+kd)^\alpha (r+jd)^\alpha \left(1 + O(n^{-3/4})\right) \left(1 + \frac{t_1(r+kd) + t_2(r+jd)}{n} - \right. \\
&\quad \left. - \frac{\pi(t_1(r+kd) + t_2(r+jd))^2}{4n\sqrt{6n}}\right) \exp\left(-\frac{\pi(t_1(r+kd) + t_2(r+jd))}{\sqrt{6n}}\right) +
\end{aligned}$$

$$\begin{aligned}
& + O(n^{-2} \log^2 n) + \\
& + \sum_{\substack{k \geq \Lambda' \\ t \geq 1 \\ t(r+kd) < s}} (r+kd)^{2\alpha} t \left(1 + O(n^{-3/4})\right) \left(1 + \frac{t(r+kd)}{n} - \right. \\
& \quad \left. - \frac{\pi(t(r+kd))^2}{4n\sqrt{6n}}\right) \exp\left(-\frac{\pi t(r+kd)}{\sqrt{6n}}\right).
\end{aligned}$$

In the first multisum, for sufficiently large  $n$ ,

$$\begin{aligned}
& 1 + \frac{t_1(r+kd) + t_2(r+jd)}{n} - \frac{\pi(t_1(r+kd) + t_2(r+jd))^2}{4n\sqrt{6n}} < \\
& < 1 + \frac{t_1(r+kd)}{n} + \frac{t_2(r+jd)}{n} - \frac{\pi(t_1(r+kd))^2}{4n\sqrt{6n}} - \frac{\pi(t_2(r+jd))^2}{4n\sqrt{6n}} = \\
& = \left(1 + \frac{t_1(r+kd)}{n} - \frac{\pi(t_1(r+kd))^2}{4n\sqrt{6n}}\right) \left(1 + \frac{t_2(r+jd)}{n} - \frac{\pi(t_2(r+jd))^2}{4n\sqrt{6n}}\right) + \\
& \quad + O\left(\frac{\log^4 n}{n}\right) = \\
& = \left(1 + O(n^{-3/4})\right) \left(1 + \frac{t_1(r+kd)}{n} - \frac{\pi(t_1(r+kd))^2}{4n\sqrt{6n}}\right) \cdot \\
& \quad \cdot \left(1 + \frac{t_2(r+jd)}{n} - \frac{\pi(t_2(r+jd))^2}{4n\sqrt{6n}}\right).
\end{aligned}$$

Consequently,

$$\begin{aligned}
& M(\xi_{n,\alpha}^2) \leq \\
& \leq (1 + O(n^{-3/4})) \left\{ \sum_{\substack{k \geq \Lambda' \\ t_1 \geq 1 \\ t_1(r+kd) < s}} (r+kd)^\alpha \left(1 + \frac{t_1(r+kd)}{n} - \frac{\pi(t_1(r+kd))^2}{4n\sqrt{6n}}\right) \cdot \right. \\
& \quad \left. \exp\left(-\frac{\pi t_1(r+kd)}{\sqrt{6n}}\right) \right\}^2 + O(n^{-2} \log^2 n) + \\
& + (1 + O(n^{-3/4})) \sum_{\substack{k \geq \Lambda' \\ t \geq 1 \\ t(r+kd) < s}} (r+kd)^{2\alpha} t \left(1 + \frac{t(r+kd)}{n} - \frac{\pi(t(r+kd))^2}{4n\sqrt{6n}}\right) \cdot \\
& \quad \cdot \exp\left(-\frac{\pi t(r+kd)}{\sqrt{6n}}\right).
\end{aligned}$$

*Remark 2.* It is supposed that  $d \leq \sqrt{n}$ . Then  $r + \Lambda'd \leq \Lambda + d < s$ , i.e.,  $1(r + \Lambda'd) < s$  and the error term in the term with  $k = \Lambda'$  and  $t = 1$  in the last sum is

$$O(n^{-3/4})(r + \Lambda'd)^{2\alpha} \exp\left(-\frac{\pi(r + \Lambda'd)}{\sqrt{6n}}\right)$$

which majorizes the error term  $O(n^{-2} \log^2 n)$  since

$$(r + \Lambda'd)^{2\alpha} \exp\left(-\frac{\pi(r + \Lambda'd)}{\sqrt{6n}}\right) \geq \exp\left(-\frac{\pi(\Lambda + d)}{\sqrt{6n}}\right) > \frac{1}{e^2 n}.$$

Therefore, the error term  $O(n^{-2} \log^2 n)$  can be dropped. The same argument can be applied to the error terms  $O(n^{-3})$  in Lemma 4. Consequently, each error term  $O(n^{-3})$  of Lemma 1 can be dropped for  $d \leq \sqrt{n}$ .

In this way, for  $\Lambda \leq \frac{\sqrt{6}}{\pi} \sqrt{n} \log n$ ,  $d \leq \sqrt{n}$  and  $n \geq n''_0$ , we obtain that

$$\begin{aligned} M(\xi_{n,\alpha}^2) &\leq (1 + O(n^{-3/4}))M^2(\xi_{n,\alpha}) + \\ &+ \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{\substack{k \geq \Lambda' \\ t \geq 1 \\ t(r+kd) < s}} (r + kd)^{2\alpha} t \exp\left(-\frac{\pi t(r + kd)}{\sqrt{6n}}\right). \end{aligned}$$

Finally,

$$\begin{aligned} D^2(\xi_{n,\alpha}) &= M(\xi_{n,\alpha}^2) - M^2(\xi_{n,\alpha}) \leq O(n^{-3/4})M^2(\xi_{n,\alpha}) + \\ &+ \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{\substack{k \geq \Lambda' \\ t \geq 1 \\ t(r+kd) < s}} (r + kd)^{2\alpha} t \exp\left(-\frac{\pi t(r + kd)}{\sqrt{6n}}\right). \end{aligned}$$

$[\alpha = 0 :]$

$$\begin{aligned} D^2(\xi_{n,0}) &\leq O(n^{-3/4})M^2(\xi_{n,0}) + \\ &+ \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{k=\Lambda'}^{\infty} \sum_{t=1}^{\infty} t \exp\left(-\frac{\pi t(r + kd)}{\sqrt{6n}}\right) = \\ &= O(n^{-3/4})M^2(\xi_{n,0}) + \\ &\left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{t=1}^{\infty} t \exp\left(-\frac{\pi t r}{\sqrt{6n}}\right) \cdot \frac{\exp\left(-\frac{\pi t \Lambda' d}{\sqrt{6n}}\right)}{1 - \exp\left(-\frac{\pi t d}{\sqrt{6n}}\right)}. \end{aligned}$$

Using again the inequalities  $\frac{1}{\beta} + \frac{1}{2} < \frac{1}{1-e^{-\beta}} < \frac{1}{\beta} + 1$  (for  $\beta > 0$ ) and the formula  $\sum_{t=1}^{\infty} t e^{-at} = \frac{1}{(e^a - 1)(1 - e^{-a})}$  we obtain

$$\begin{aligned} D^2(\xi_{n,0}) &\leq O(n^{-3/4})M^2(\xi_{n,0}) + \\ &+ \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{t=1}^{\infty} t \exp\left(-\frac{\pi t(r + \Lambda'd)}{\sqrt{6n}}\right) \left\{ \frac{\sqrt{6n}}{\pi t d} + 1 \right\} = \\ &= O(n^{-3/4})M^2(\xi_{n,0}) + \\ &+ \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \left\{ \frac{\sqrt{6n}}{\pi d} \cdot \frac{1}{\exp\left(\frac{\pi(r + \Lambda'd)}{\sqrt{6n}}\right) - 1} + \right. \end{aligned}$$

$$\begin{aligned}
& \left. + \frac{1}{\exp\left(\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}\right) - 1} \cdot \frac{1}{1 - \exp\left(-\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}\right)} \right\} = \\
= & \frac{1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)}{\exp\left(\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}\right) - 1} \left\{ \frac{\sqrt{6n}}{\pi d} + \frac{1}{1 - \exp\left(-\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}\right)} + \right. \\
& \left. + \left( \sqrt{\exp\left(\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}\right) - 1} \cdot n^{-3/8} |M(\xi_{n,0})| \right)^2 \right\}.
\end{aligned}$$

Let  $y \doteq \sqrt{\exp\left(\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}\right) - 1}$ . We have

$$\exp\left(\frac{\pi(r+(\Lambda'+1)d)}{\sqrt{6n}}\right) = (y^2 + 1) \exp\left(\frac{\pi d}{\sqrt{6n}}\right) \geq y^2 + 1$$

and, by Lemma 1,

$$\begin{aligned}
y \cdot n^{-3/8} |M(\xi_{n,0})| &= O(yn^{-3/8}) \left\{ \frac{\sqrt{n}}{d} \log\left(1 + \frac{1}{y^2}\right) + \frac{1}{y^2} \right\} = \\
&= O(n^{-3/8}) \left\{ \frac{\sqrt{n}}{d} y \log\left(1 + \frac{1}{y^2}\right) + \frac{1}{y} \right\}.
\end{aligned}$$

$$y \log\left(1 + \frac{1}{y^2}\right) \leq \begin{cases} y \cdot \frac{1}{y^2} = \frac{1}{y} \leq 1 & \text{if } y \geq 1; \\ y \log \frac{2}{y^2} = y \log 2 + 2y \log \frac{1}{y} \leq \\ \leq y \log 2 + 2y \frac{1}{y} \leq \log 2 + 2 & \text{if } 0 < y < 1. \end{cases}$$

Thus,

$$\begin{aligned}
\left(y \cdot n^{-3/8} |M(\xi_{n,0})|\right)^2 &= O(n^{-3/4}) \left\{ \frac{\sqrt{n}}{d} + \frac{1}{y} \right\}^2 = \\
&= O(n^{-3/4}) \left\{ \frac{\sqrt{n}}{d} + n^{1/4} \right\}^2
\end{aligned}$$

since  $y > \sqrt{\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}} \geq \sqrt{\frac{\pi}{\sqrt{6}}} n^{-1/4}$ . As  $d \leq \sqrt{n}$ , we have

$$\begin{aligned}
O(n^{-3/4}) \left\{ \frac{\sqrt{n}}{d} + n^{1/4} \right\}^2 &= O(n^{-3/4}) \left\{ \frac{n}{d^2} + \frac{2n^{3/4}}{d} + n^{1/2} \right\} = \\
&= O(n^{-3/4}) \frac{\sqrt{n}}{d} \left\{ \frac{\sqrt{n}}{d} + 2n^{1/4} + d \right\} = O(n^{-3/4}) \frac{\sqrt{n}}{d} \left\{ \sqrt{n} + 2n^{1/4} + \sqrt{n} \right\} = \\
&= \frac{\sqrt{n}}{d} O(n^{-1/4}).
\end{aligned}$$

Finally,

$$D^2(\xi_{n,0}) \leq$$



$$\begin{aligned}
&\leq \frac{1 + O(n^{-1/4})}{\exp\left(\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}\right) - 1} \left\{ \frac{\sqrt{6n}}{\pi d} + \frac{1}{1 - \exp\left(-\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}\right)} \right\} = \\
&= \frac{O(1)}{\exp\left(\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}\right) - 1} \left\{ \frac{\sqrt{n}}{d} + 1 + \frac{1}{\exp\left(\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}\right) - 1} \right\} = \\
&= \frac{O(1)}{\exp\left(\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}\right) - 1} \left\{ \frac{\sqrt{n}}{d} + \frac{\sqrt{n}}{r + \Lambda'd} \right\} = \\
&= \frac{O(1)}{\exp\left(\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}\right) - 1} \cdot \frac{\sqrt{n}}{\min(d, r + \Lambda'd)}.
\end{aligned}$$

$[\alpha = 1 :]$

For  $\Lambda \leq \frac{\sqrt{6}}{\pi} \sqrt{n} \log n$  and  $d \leq \sqrt{n}$ , using Lemma 1 and Remark 1,

$$\begin{aligned}
D^2(\xi_{n,1}) &\leq O(n^{-3/4})M^2(\xi_{n,1}) + \\
&\quad + \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \sum_{\substack{k \geq \Lambda' \\ t \geq 1 \\ t(r+kd) < s}} (r + kd)^2 t \exp\left(-\frac{\pi t(r + kd)}{\sqrt{6n}}\right) \leq \\
&\leq O(n^{-3/4})M^2(\xi_{n,1}) + \\
&\quad + O(s^2) \sum_{\substack{k \geq \Lambda' \\ t \geq 1 \\ t(r+kd) < s}} \frac{1}{t} \exp\left(-\frac{\pi t(r + kd)}{\sqrt{6n}}\right) \leq \\
&\leq O(n^{-3/4}) \left\{ \frac{n}{d} + (r + \Lambda'd) \frac{\sqrt{n}}{d} \log n + \sqrt{n} d \right\}^2 + \\
&\quad + O(n \log^2 n) \sum_{t=1}^{\infty} \frac{1}{t} \exp\left(-\frac{\pi t r}{\sqrt{6n}}\right) \sum_{k=\Lambda'}^{\infty} \exp\left(-\frac{\pi t d k}{\sqrt{6n}}\right) \leq \\
&\leq O(n^{-3/4}) \left\{ \frac{n}{d} + (\Lambda + d) \frac{\sqrt{n}}{d} \log n + \sqrt{n} d \right\}^2 + \\
&\quad + O(n \log^2 n) \sum_{t=1}^{\infty} \frac{1}{t} \exp\left(-\frac{\pi t(r + \Lambda'd)}{\sqrt{6n}}\right) \frac{1}{1 - \exp\left(-\frac{\pi t d}{\sqrt{6n}}\right)} \leq \\
&\leq O(n^{-3/4}) \left\{ \frac{n \log^2 n}{d} + \sqrt{n} d \right\}^2 + \\
&\quad + O(n \log^2 n) \sum_{t=1}^{\infty} \frac{1}{t} \exp\left(-\frac{\pi t(r + \Lambda'd)}{\sqrt{6n}}\right) \left\{ \frac{\sqrt{6n}}{\pi t d} + O(1) \right\} \leq \\
&\leq O(n^{1/4}) \left\{ \frac{\sqrt{n} \log^2 n}{d} + d \right\}^2 + \\
&\quad + O(n \log^2 n) \left\{ \frac{\sqrt{6n}}{\pi d} \sum_{t=1}^{\infty} \frac{1}{t^2} + O\left(\log \frac{1}{1 - \exp\left(-\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}\right)}\right) \right\} =
\end{aligned}$$

$$\begin{aligned}
&= O(n^{1/4}) \left\{ \frac{\sqrt{n} \log^2 n}{d} + d \right\}^2 + O(n \log^2 n) \left\{ \frac{\sqrt{n}}{d} + \log n \right\} = \\
&= O(1) \left\{ \frac{n^{5/4} \log^4 n}{d^2} + n^{3/4} \log^2 n + n^{1/4} d^2 + \frac{n^{3/2} \log^2 n}{d} + n \log^3 n \right\} = \\
&= O(1) \left\{ \frac{n^{3/2} \log^2 n}{d} + n \log^3 n + n^{1/4} d^2 \right\} = \\
&= O\left(\frac{n^{3/2}}{d}\right) \left\{ \log^2 n + \frac{d \log^3 n}{\sqrt{n}} + \frac{d^3}{n^{5/4}} \right\}.
\end{aligned}$$

For  $d \leq n^{5/12}$ , we obtain that

$$D^2(\xi_{n,1}) = O\left(\frac{n^{3/2} \log^2 n}{d}\right).$$

## 6. COMPLETIONS OF THE PROOFS OF THE THEOREMS

*Case  $\alpha = 0$ .* Lemmas 1 and 2 for  $\alpha = 0$  give the assertion in our Theorem 1:

Let  $\omega(n) \nearrow \infty$ ,  $d \leq \sqrt{n}$ ,  $\Lambda \leq \frac{\sqrt{6}}{\pi} \sqrt{n} \log n$ . Applying Chebyshev's inequality and using Remark 2, we obtain that, for all but  $p(n)/\omega(n)$  partitions of  $n$ ,

$$\begin{aligned}
\sum_{\substack{\mu \\ \lambda_\mu \equiv r(d) \\ \lambda_\mu \geq \Lambda}} 1 &= M(\xi_{n,0}) + O\left(\sqrt{\frac{\sqrt{n} \omega(n)}{\left(\exp\left(\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}\right) - 1\right) \min(d, r + \Lambda'd)}}\right) = \\
&= \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \left\{ \frac{\sqrt{6n}}{\pi d} \log \frac{1}{1 - \exp\left(-\frac{\pi(r+(\Lambda'+1)d)}{\sqrt{6n}}\right)} + \right. \\
&\quad \left. + \frac{1}{\exp\left(\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}\right) - 1} \right\} + \\
&\quad + \frac{O(1)}{\exp\left(\frac{\pi(r+(\Lambda'+1)d)}{\sqrt{6n}}\right) - 1} + O\left(\sqrt{\frac{\sqrt{n} \omega(n)}{\left(\exp\left(\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}\right) - 1\right) \cdot \min(d, r + \Lambda'd)}}\right)
\end{aligned}$$

where  $\Lambda' = \lceil \frac{\Lambda-r}{d} \rceil$ ,  $1 \leq r \leq d$ . Again with  $y = \sqrt{\exp\left(\frac{\pi(r+\Lambda'd)}{\sqrt{6n}}\right) - 1}$ ,

$$\begin{aligned}
&O\left(\frac{\log^2 n}{\sqrt{n}}\right) \frac{\sqrt{6n}}{\pi d} \log\left(1 + \frac{1}{y^2}\right) + \frac{O(1)}{y^2} = \\
&= O\left(\frac{1}{y}\right) \left\{ \frac{\log^2 n}{d} y \log\left(1 + \frac{1}{y^2}\right) + \frac{1}{y} \right\} \leq \\
&\leq O\left(\frac{1}{y}\right) \left\{ \frac{\log^2 n}{d} O(1) + O(1) \sqrt{\frac{\sqrt{n}}{r + \Lambda'd}} \right\} = O\left(\frac{1}{y}\right) \sqrt{\frac{\sqrt{n}}{\min(d, r + \Lambda'd)}}
\end{aligned}$$

thus our Theorem 1 is proved.

Case  $\alpha = 1$ . Lemmas 1 and 2 for  $\alpha = 1$  give the assertion in our Theorem 2:

Let  $\omega(n) \nearrow \infty$ ,  $d \leq n^{5/12}$ ,  $\Lambda \leq \frac{\sqrt{6}}{\pi} \sqrt{n} \log n$ . Applying Chebyshev's inequality and using Remark 1, we obtain that, for all but  $p(n)/\omega(n)$  partitions of  $n$ ,

$$\begin{aligned}
& \sum_{\substack{\mu \\ \lambda_\mu \equiv r(d) \\ \lambda_\mu \geq \Lambda}} \lambda_\mu = \\
&= M(\xi_{n,1}) + O\left(\frac{n^{3/4}}{\sqrt{d}}(\log n)\sqrt{\omega(n)}\right) = \\
&= \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \left\{ \frac{n}{d} - \frac{6n}{\pi^2 d} \sum_{t=1}^{\infty} \frac{1}{t^2} \left(1 - \exp\left(-\frac{\pi(r + \Lambda'd)t}{\sqrt{6n}}\right)\right) \right\} + \\
&\quad + (r + \Lambda'd) \frac{\sqrt{6n}}{\pi d} \log \frac{1}{1 - \exp\left(-\frac{\pi(r + \Lambda'd)}{\sqrt{6n}}\right)} + \frac{O((\Lambda' + 1)d)}{\exp\left(\frac{\pi(r + \Lambda'd)}{\sqrt{6n}}\right) - 1} \left\} + \\
&\quad + O\left(\frac{n^{3/4} \log n}{\sqrt{d}} \sqrt{\omega(n)}\right) = \\
&= \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \left\{ \frac{n}{d} + O\left(\frac{\sqrt{n}}{d}(r + \Lambda'd) \log n\right) + O\left(\frac{1}{d}\right) + \right. \\
&\quad \left. + O\left((r + \Lambda'd) \frac{\sqrt{n}}{d} \log n\right) + O(\sqrt{nd}) \right\} + \\
&\quad + O\left(\frac{n^{3/4} \log n}{\sqrt{d}} \sqrt{\omega(n)}\right) = \\
&= \left(1 + O\left(\frac{\log^2 n}{\sqrt{n}}\right)\right) \left\{ \frac{n}{d} + O\left(\frac{\sqrt{n}}{d}(r + \Lambda'd) \log n\right) + O(\sqrt{nd}) \right\} + \\
&\quad + O\left(\frac{n^{3/4} \log n}{\sqrt{d}} \sqrt{\omega(n)}\right) = \\
&= \frac{n}{d} + O\left(\frac{\sqrt{n}}{d} \log^2 n\right) + O\left(\frac{\sqrt{n}}{d}(r + \Lambda'd) \log n\right) + O(\sqrt{nd}) + \\
&\quad + O\left(\frac{n^{3/4} \log n}{\sqrt{d}} \sqrt{\omega(n)}\right).
\end{aligned}$$

For  $d \leq n^{1/6}$   $\sqrt{nd} \leq \frac{n^{3/4}}{\sqrt{d}}$  so that we get

$$\frac{n}{d} + O\left(\frac{\sqrt{n}}{d}(r + \Lambda'd) \log n\right) + O\left(\frac{n^{3/4} \log n}{\sqrt{d}} \sqrt{\omega(n)}\right)$$

which completes the proof of Theorem 2.

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