The wave equations for Dunkl operators

Salem Ben Said, Bent Orsted

THE WAVE EQUATION FOR DUNKL OPERATORS

SALEM BEN SAÏD AND BENT ØRSTED

ABSTRACT. Let $k = (k_\alpha)_{\alpha\in\mathscr{R}}$ be a positive-real valued multiplicity function related to a root system $\mathscr{R}$, and Δ_k be the Dunkl-Laplacian operator. For $(x,t)\in\mathbb{R}^N\times\mathbb{R}$, denote by $u_k(x,t)$ the solution to the deformed wave equation $\Delta_k u_k(x,t) = \partial_{tt}u_k(x,t)$, where the initial data belong to the Schwartz space on $\mathbb{R}^N$. We prove that for $k\geq 0$ and $N\geq 1$, the wave equation satisfies a weak Huygens' principle, while a strict Huygens' principle holds if and only if $(N-3)/2+\sum_{\alpha\in\mathscr{R}^+}k_\alpha\in\mathbb{N}$. Here $\mathscr{R}^+\subset\mathscr{R}$ is a subsystem of positive roots. As a particular case, if the initial data are supported in a closed ball of radius $R>0$ about the origin, the strict Huygens principle implies that the support of $u_k(x,t)$ is contained in the conical shell $\{(x,t)\in\mathbb{R}^N\times\mathbb{R}\mid |t|-R\leq\|x\|\leq|t|+R\}$. Our approach uses the representation theory of the group $SL(2,\mathbb{R})$, and Paley-Wiener theory for the Dunkl transform. Also, we show that the (t-independent) energy functional of u_k is, for large $|t|$, partitioned into equal potential and kinetic parts.

1. Introduction

In a series of lectures at Yale University, J. Hadamard formulated two different meanings of Huygens' principle which are nowadays known as Hadamard's major and minor premises [19]. A typical statement of the major premise is "every point on a wave front acts as a source of a new wave front, propagating radially outward". This statement is mainly the original principle proposed by Christian Huygens in the 17th century [28], and it holds for a general class of wave propagations. In contrast to the major premise, the minor premise is a remarkable phenomena, that is valid only for very special equations, and never happens in even dimensional spaces. Mathematically, a second order hyperbolic equation satisfies Huygens' principle in the narrow sense ("minor premise"), if the solution of the corresponding Cauchy problem at some point x depends not on all the Cauchy data, but only on its part on the intersection of the characteristic conoid with vertex x with the Cauchy surface. This means that the fundamental solution of the corresponding Cauchy problem vanishes outside and inside the characteristic conoid, and thus must be located on it. Indeed, because we are living in a three-dimensional word we can hear each other clearly; one has a pure propagation without residual waves. This is not the case in the two dimensional space: when a pebble falls in water at a certain point x, the initial ripple on a circle around x will be followed by subsequent ripples. Thus a given point y will be hit by residual waves.

The problem of classifying all second order hyperbolic differential operators which obey Huygens' principle in the narrow sense, is known as the Hadamard problem. This problem has received a good deal of attention and the literature is extensive [41, 31, 10, 18, 32, 42, 36, 33, 22, 1, 3, 11, 6]. (Of course, this list of references is not complete.) Nevertheless, this problem is still far from being fully solved. In the present paper,

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we shall treat a natural differential-difference operator of a similar hyperbolic nature, namely one with the same leading symbol, but with additional reflection terms.

Henceforth, we will use the terminology "weak Huygens' principle" for Hadamard's major premises, and "strict Huygens' principle" for Hadamard's minor premises.

The propagation of waves in $\mathbb{R}^N$ is governed by the wave equation

$$(\mathbf{L}) \qquad \Delta^x u(x,t) = \partial_{tt} u(x,t), \qquad \text{for } (x,t) \in \mathbb{R}^N \times \mathbb{R}.$$

Here Δ^x denotes the usual Laplacian operator in the x-variable, and the subscript t indicates differentiation in the t-variable. It is a well known fact that **(L)** satisfies the weak Huygens principle for all $N \geq 1$, while the strict Huygens principle holds in all odd dimensions starting from 3 and never holds in even dimensions [10]. In this paper, we will investigate the validity of the weak and the strict Huygens principle for **(L)** when the Laplacian Δ is replaced by the Dunkl-Laplacian operator associated with Coxeter groups [12]. The main tools are the representation theory of the group $SL(2,\mathbb{R})$, and the Paley-Wiener theory for the Dunkl transform (or the generalized Fourier transform) [30, 45].

To be more specific, let G be a finite reflection group on $\mathbb{R}^N$ with root system $\mathscr{R}$, and choose a positive subsystem $\mathscr{R}^+$ in $\mathscr{R}$. Let $k : \mathscr{R} \to \mathbb{R}^+$, $\alpha \mapsto k_\alpha$, be a multiplicity function. The Dunkl-Laplacian operator is given by

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in \mathscr{R}^+} k_\alpha \left\{ \frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(r_\alpha x)}{\langle \alpha, x \rangle^2} \right\},$$

where Δ and ∇ are the usual Laplacian and gradient operators, $\langle \cdot, \cdot \rangle$ is the standard Euclidean scalar product in $\mathbb{R}^N$, and r_α is the reflection in the hyperplane orthogonal to the root α.

Consider the following Cauchy problem

$$(\mathbf{D}) \qquad \Delta_k u_k(x,t) = \partial_{tt} u_k(x,t), \qquad u_k(x,0) = f(x), \quad \partial_t u_k(x,0) = g(x),$$

where $u_k(x,t)$ is a function of $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, and the Cauchy data f and g are two Schwartz functions on $\mathbb{R}^N$. The main results of this paper are:

Claim 1. (Weak Huygens' principle) Assume that $k \geq 0$ and $N \geq 1$. For a given $x \in \mathbb{R}^N$, the solution $u_k(x,t)$ depends only on the values of $\tau_x(k)f(y)$ and $\tau_x(k)g(y)$ for $\|y\| \leq |t|$. Here $\tau_x(k)$ is a generalized translation operator. We emphasize that $\tau_x(k)$ is not defined on the space itself, but for functions living on it.

Claim 2. (Strict Huygens' principle) Assume that $k \geq 0$ and $N \geq 1$. For a given $x \in \mathbb{R}^N$, the solution $u_k(x,t)$ depends only on the values of $\tau_x(k)f(y)$ and $\tau_x(k)g(y)$ (and their derivatives) for $\|y\| = |t|$ if and only if

$$\frac{N-3}{2} + \sum_{\alpha \in \mathscr{R}^+} k_\alpha \in \mathbb{N}.$$

Here $\mathbb{N} = \{0, 1, 2, \ldots\}$.

These claims correspond to Theorem 3.3 and Theorem 3.15, respectively. In the classical case $k \equiv 0$, these two claims can be found, for instance, in [10, 32].

In particular, if $x = 0$, then in Claim 1 (resp. Claim 2) the solution $u_k(0,t)$ will depend only on the values of $f(y)$ and $g(y)$ for $\|y\| \leq |t|$ (resp. $\|y\| = |t|$).

In [1], and for integer-valued k, Berest and Veselov gave a necessary and sufficient condition for which the wave operator $\Delta_k^G - \partial_{tt}$ satisfies the strict Huygens principle. Here Δ_k^G denotes the G-invariant part of the Dunkl-Laplacian operator Δ_k. Notice that Claim 2 above is an extension of Berest-Veselov's result under a weaker condition.

Now let $\mathscr{C}_R^\infty(\mathbb{R}^N)$ be the space of smooth functions with compact support contained in the closed ball of radius $R > 0$ about the origin. If we assume that the Cauchy data f and g belong to $\mathscr{C}_R^\infty(\mathbb{R}^N)$, Claim 2 reads:

Claim 3. Assume that $k \geq 0$ and $N \geq 1$. For all possible Cauchy data $f, g \in \mathscr{C}_R^\infty(\mathbb{R}^N)$, the support of the solution $u_k(x,t)$ is contained in the conical shell

$$\{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid |t| - R \leq \|x\| \leq |t| + R\}$$

if and only if $(N-3)/2 + \sum_{\alpha \in \mathscr{R}^+} k_\alpha \in \mathbb{N}$.

The claim above corresponds to Theorem 3.17 below. In the classical case $k \equiv 0$, this claim was proved, for instance, in [23].

Here is the outline of our approach. We start by proving that there exist two tempered distributions $P_{k,t}^{(1)}$ and $P_{k,t}^{(2)}$ on $\mathbb{R}^N$, such that the solution u_k to the Cauchy problem **(D)** is uniquely given by

$$u_k(x,t) = (P_{k,t}^{(1)} *_k f)(x) + (P_{k,t}^{(2)} *_k g)(x). \tag{1.1}$$

Here $*_k$ is a Dunkl-type convolution. Based on a Paley-Wiener theorem [45], we show that $P_{k,t}^{(\ell)}$, for $\ell = 1, 2$, is supported inside the light cone $\mathscr{C} := \{(y,t) \mid \|y\| = |t|\}$, i.e. in the set $\{(y,t) \mid \|y\| \leq |t|\}$. To prove the strict Huygens principle, we use the representation theory of the group $SL(2,\mathbb{R})$. In the classical case, this approach goes back to R. Howe [24]. We show that $P_{k,t}^{(1)}$ and $P_{k,t}^{(2)}$ are supported on the light cone $\mathscr{C}$ if and only if $P_{k,t}^{(\ell)}$, for $\ell = 1, 2$, generates a finite-dimensional $\mathfrak{sl}(2,\mathbb{R})$-module of dimension

$$d_{k,\ell} = \frac{N+3}{2} - \ell + \sum_{\alpha \in \mathscr{R}^+} k_\alpha.$$

We can also give a different proof for Claim 3 using other techniques based only on de Jeu's Paley-Wiener theorem for the Dunkl transform [30]. See the end of Section 3 for a sketch of this approach; note that the details of this argument can be found in the last section, which deals with the principle of energy equipartition of a solution to **(D)**.

On the other hand, for $f \in \mathscr{C}^\infty(\mathbb{R}^N)$, denote by M_f the spherical mean operator, as first introduced in [34]

$$M_f(x,r) = d_k^{-1} \int_{S^{N-1}} \tau_x(k) f(ry) \upsilon_k(y) d\omega(y), \qquad x \in \mathbb{R}^N,\ r \geq 0.$$

Here d_k is a normalization constant, and υ_k is the G-invariant weight function given by $\upsilon_k(x) = \prod_{\alpha \in \mathscr{R}^+} |\langle \alpha, x \rangle|^{2k_\alpha}$, for $x \in \mathbb{R}^N$. A key result in Rösler's paper [38], is that the spherical mean operator is positivity-preserving. Keeping in mind (1.1), and using the spherical mean operator for the Cauchy data (f,g), we prove that

$$\begin{aligned} u_k(x,t) &= d_k\frac{\sqrt{\pi}}{\Gamma(\gamma_k+N/2)}\int_0^{|t|} r^{2\gamma_k+N-1}\frac{d}{dt}\left(\mathbb{S}_{-\gamma_k-\frac{N-3}{2}}(t^2-r^2)\right)M_f(x,r)dr \\ &\quad +\mathrm{sign(t)}d_k\frac{\sqrt{\pi}}{\Gamma(\gamma_k+N/2)}\int_0^{|t|} r^{2\gamma_k+N-1}\mathbb{S}_{-\gamma_k-\frac{N-3}{2}}(t^2-r^2)M_g(x,r)dr. \end{aligned} \tag{1.2}$$

Here $\gamma_k:=\sum_{\alpha\in\mathscr{R}^+}k_\alpha$, and $\mathbb{S}_\lambda(x):=x_+^{\lambda-1}/\Gamma(\lambda)$ is the Riemann-Liouville distribution. In the light of this integral representation of u_k, and Rösler results on the spherical mean operator, the claims 1 and 2 are, respectively, equivalent to:

Claim 4. (Weak Huygens' principle) Assume that $k\geq 0$ and $N\geq 1$. For a given $x\in\mathbb{R}^N$, the solution $u_k(x,t)$ depends only on the values of $f(y)$ and $g(y)$ for $\|x\|-|t|\leq\|y\|\leq\|x\|+|t|$.

Claim 5. (Strict Huygens' principle) Assume that $k\geq 0$ and $N\geq 1$. For a given $x\in\mathbb{R}^N$, the solution $u_k(x,t)$ depends only on the values of $f(y)$ and $g(y)$ (and their derivatives) for $\|y\|\geq\big|\|x\|-|t|\big|$ if and only if $(N-3)/2+\sum_{\alpha\in\mathscr{R}^+}k_\alpha\in\mathbb{N}$.

The two claims 4 and 5 correspond to Corollary 3.20 and Theorem 3.21, respectively.

Implicitly, the integral representation (1.2) of the solution u_k yields another proof of the weak and the strict Huygens principle.

In the last section we prove the conservation of the total energy, and the energy equipartition theorem for the solution u_k under suitable conditions on N and γ_k. In this part we choose to work with smooth Cauchy data (f,g) supported in the closed ball of radius $R>0$ about the origin. The advantage of this choice is to investigate, via Paley-Wiener theory for the Dunkl transform, the behavior of the difference between the kinetic and potential energy of the solution u_k to **(D)**. Indeed, if we denote by $\mathscr{K}_k[u_k](t)$ the kinetic energy, and by $\mathscr{P}_k[u_k](t)$ the potential energy, then the following claim holds:

Claim 6. For $k\geq 0$ and $N\geq 1$, assume that $(N-1)/2+\gamma_k\in\mathbb{N}$. Let u_k be the solution to the Cauchy problem (**D**), where the Cauchy data (f,g) are supported in the closed ball of radius $R>0$ about the origin.

(i) For fixed $s>0$, there exists a constant c depending on N, k and (f,g) but not on s, such that

$$|\mathscr{K}_k[u_k](t)-\mathscr{P}_k[u_k](t)|\leq ce^{-2s(|t|-R)},$$

for all $t\in\mathbb{R}$.

(ii) The principle of energy equipartition holds for all $|t|>R$.

The statements (i) and (ii) correspond to Theorem 4.3 and Theorem 4.4, respectively.

However, if the Cauchy data (f,g) belong to the Schwartz space, then the principle of energy equipartition reads

$$\lim_{|t|\to\infty}\mathscr{K}_k[u_k](t)=\lim_{|t|\to\infty}\mathscr{P}_k[u_k](t)=\frac{\text{The total (t-independent) energy of } u_k}{2},$$

for all $k\in\mathscr{K}^+$ and $N\geq 1$.

In the classical case $k\equiv 0$, the energy equipartition theorem can be found, for instance, in [32, 4].

This paper is organized as follows: In Section 2 we give an abbreviated background on the Dunkl theory. Section 3 is devoted to prove the main results, that is Claim 1, Claim 2, Claim 3, Claim 4, and Claim 5. In Section 4 we turn our attention to the proof of Claim 6, i.e. the energy conservation and equipartition theorems.

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2. Background

Throughout the paper, $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean scalar product in $\mathbb{R}^N$ as well as its bilinear extension to $\mathbb{C}^N \times \mathbb{C}^N$. For $x \in \mathbb{R}^N$, let $\|x\| = \langle x, x\rangle^{1/2}$. Denote by $\mathscr{S}(\mathbb{R}^N)$ the Schwartz space of rapidly decreasing functions equipped with the usual Fréchet space topology.

Let G be a finite reflection group on $\mathbb{R}^N$ with root system $\mathscr{R}$, and fix a positive subsystem $\mathscr{R}^+$ of $\mathscr{R}$. We will assume that $\langle\alpha,\alpha\rangle = 2$ for all $\alpha \in \mathscr{R}^+$.

For $\alpha \in \mathbb{R}^N \setminus \{0\}$, let r_α be the reflection in the hyperplane $\langle\alpha\rangle^\perp$ orthogonal to α

$$r_\alpha(x) := x - \langle\alpha, x\rangle\alpha, \qquad x \in \mathbb{R}^N.$$

Then G is the subgroup of the orthogonal group $O(N)$ which is generated by the reflections $\{r_\alpha \mid \alpha \in \mathscr{R}\}$. A multiplicity function on $\mathscr{R}$ is a G-invariant function $k : \mathscr{R} \to \mathbb{C}$. Setting $k_\alpha := k(\alpha)$ for $\alpha \in \mathscr{R}$, we have $k_{h\alpha} = k_\alpha$ for all $h \in G$. The $\mathbb{C}$-vector space of multiplicity functions on $\mathscr{R}$ is denoted by $\mathscr{K}$. If $m := \sharp\{G\text{-orbits in } \mathscr{R}\}$, then $\mathscr{K} \cong \mathbb{C}^m$.

For $\xi \in \mathbb{C}^N$ and $k \in \mathscr{K}$, in [12], Dunkl defined a family of first order differential-difference operators $T_\xi(k)$ that play the role of the usual partial differentiation. Dunkl's operators are defined by

$$T_\xi(k)f(x) := \partial_\xi f(x) + \sum_{\alpha\in\mathscr{R}^+} k_\alpha \langle\alpha,\xi\rangle \frac{f(x) - f(r_\alpha x)}{\langle\alpha, x\rangle}, \qquad f \in \mathscr{C}^1(\mathbb{R}^N).$$

Here ∂_ξ denotes the directional derivative corresponding to ξ. The definition of $T_\xi(k)$ is independent of the choice of $\mathscr{R}^+$, and these operators mutually commute, i.e. $T_\xi(k)T_\eta(k) = T_\eta(k)T_\xi(k)$. Further, if f and g are in $\mathscr{C}^1(\mathbb{R}^N)$, and at least one of them is G-invariant, then

$$T_\xi(k)[fg] = gT_\xi(k)f + fT_\xi(k)g. \tag{2.1}$$

We refer to [12, 15] for more details on the theory of Dunkl's operators.

The counterpart of the usual Laplacian is the Dunkl-Laplacian defined by

$$\Delta_k := \sum_{j=1}^N T_{\xi_j}(k)^2,$$

where $\{\xi_1, \ldots, \xi_N\}$ is an arbitrary orthonormal basis of $(\mathbb{R}^N, \langle\cdot,\cdot\rangle)$. For the j-th basis vector ξ_j, we will use the abbreviation $T_{\xi_j}(k) = T_j(k)$. By the normalization $\langle\alpha,\alpha\rangle = 2$, we can rewrite Δ_k as

$$\Delta_k f(x) = \Delta f(x) + 2\sum_{\alpha\in\mathscr{R}^+} k_\alpha \left\{ \frac{\langle\nabla f(x), \alpha\rangle}{\langle\alpha, x\rangle} - \frac{f(x) - f(r_\alpha x)}{\langle\alpha, x\rangle^2} \right\},$$

where Δ and ∇ are the usual Laplacian and gradient operators, respectively.

Henceforth, $\mathscr{K}^+$ denotes the set of multiplicity functions $k = (k_\alpha)_{\alpha\in\mathscr{R}}$ such that $k_\alpha \geq 0$ for all $\alpha \in \mathscr{R}$. For $k \in \mathscr{K}^+$, there exists a generalization of the usual exponential kernel $e^{\langle\cdot,\cdot\rangle}$ by means of the Dunkl system of differential equations.

Theorem 2.1. *For $k \in \mathscr{K}^+$, the following hold:*

(i) (cf. [13, 35]) *There exists a unique holomorphic function E_k on $\mathbb{C}^N \times \mathbb{C}^N$ characterized by*

$$T_\xi(k)E_k(z, w) = \langle \xi, w\rangle E_k(z, w) \qquad \text{for all } \xi \in \mathbb{C}^N, \qquad E_k(0, w) = 1. \tag{2.2}$$

Further, the kernel E_k is symmetric in its arguments, and

$$E_k(\lambda z, w) = E_k(z, \lambda w), \quad E_k(hz, hw) = E_k(z, w)$$

for $z, w \in \mathbb{C}^N$, $\lambda \in \mathbb{C}$, and $h \in G$.

(ii) (cf. [29]) *For $x \in \mathbb{R}^N$ and $w \in \mathbb{C}^N$, we have*

$$|E_k(x, w)| \leq \sqrt{|G|}\, e^{\|x\|\,\|\operatorname{Re}(w)\|}.$$

For complex-valued k, there is a detailed investigation of (2.2) by Opdam [35]. Theorem 2.1(i) is a weak version of Opdam's result. The constant $\sqrt{|G|}$ in the statement (ii) above can be improved to 1, as a consequence of the Rösler's integral representation of Bochner-type of E_k [37]. For integral multiplicity function, another proof for Theorem 2.1 can be found in [7], by means of a contraction procedure. The function E_k is the so-called Dunkl kernel. When $k \equiv 0$, we have $E_0(z, w) = e^{\langle z,w\rangle}$ for $z, w \in \mathbb{C}^N$.

Let υ_k be the weight function on $\mathbb{R}^N$ defined by

$$\upsilon_k(x) := \prod_{\alpha\in\mathscr{R}^+} |\langle\alpha, x\rangle|^{2k_\alpha}, \qquad x \in \mathbb{R}^N.$$

It is G-invariant and homogeneous of degree $2\gamma_k$, with the index

$$\gamma_k := \sum_{\alpha\in\mathscr{R}^+} k_\alpha.$$

Notice that by G-invariance of k, the definition of υ_k does not depend on the special choice of $\mathscr{R}^+$.

Denote by dx the Lebesgue measure corresponding to $\langle\cdot,\cdot\rangle$. The Dunkl transform on the space $L^1(\mathbb{R}^N, \upsilon_k(x)dx)$ of integrable functions on $\mathbb{R}^N$ with respect to $\upsilon_k(x)dx$, is defined by

$$\mathscr{D}_k f(\xi) := \int_{\mathbb{R}^N} f(x)E_k(x, -i\xi)\upsilon_k(x)dx, \qquad \xi \in \mathbb{R}^N. \tag{2.3}$$

We set c_k to be the Mehta-type constant

$$c_k := \int_{\mathbb{R}^N} e^{-\|x\|^2/2}\upsilon_k(x)dx. \tag{2.4}$$

Many properties of the Euclidean Fourier transform carry over to the Dunkl transform.

Theorem 2.2. (cf. [14, 29]) *Let $k \in \mathscr{K}^+$. If $\mathscr{E}_k(f)(\xi) := \mathscr{D}_k(f)(-\xi)$, then the following hold:*

(i) *The transforms $\mathscr{D}_k$ and $\mathscr{E}_k$ are homeomorphisms of $\mathscr{S}(\mathbb{R}^N)$ and $\mathscr{D}_k\circ\mathscr{E}_k = \mathscr{E}_k\circ\mathscr{D}_k = c_k^2\mathbf{1}_{\mathscr{S}}$.*

(ii) (L^1-inversion) *If $f \in L^1(\mathbb{R}^N, \upsilon_k(x)dx)$, with $\mathscr{D}_k(f) \in L^1(\mathbb{R}^N, \upsilon_k(x)dx)$, then $\mathscr{D}_k(\mathscr{E}_k(f)) = \mathscr{E}_k(\mathscr{D}_k(f)) = c_k^2 f$ a.e.*

(iii) (Plancherel formula) *If* $f \in L^1(\mathbb{R}^N, \upsilon_k(x)dx) \cap L^2(\mathbb{R}^N, \upsilon_k(x)dx)$, *then* $\mathscr{D}_k(f) \in L^2(\mathbb{R}^N, \upsilon_k(x)dx)$ *and* $\|\mathscr{D}_k(f)\|_2 = c_k\|f\|_2$. *Furthermore,* $c_k^{-1}\mathscr{D}_k$ *extends uniquely from* $L^1(\mathbb{R}^N, \upsilon_k(x)dx) \cap L^2(\mathbb{R}^N, \upsilon_k(x)dx)$ *to a unitary operator on* $L^2(\mathbb{R}^N, \upsilon_k(x)dx)$.

In what follows we shall need a generalized translation operator. In [13], Dunkl proved that for $k \in \mathscr{K}^+$, there exists a linear isomorphism V_k that intertwines the algebra generated by the Dunkl operators with the algebra of partial differential operators. The intertwining operator V_k is determined uniquely by

$$T_\xi(k)V_k = V_k\partial_\xi \quad \text{for all } \xi \in \mathbb{R}^N, \quad V_k\mathscr{P}_m(\mathbb{R}^N) \subset \mathscr{P}_m(\mathbb{R}^N), \quad V_k(1) = 1,$$

where $\mathscr{P}_m(\mathbb{R}^N)$ denotes the space of homogeneous polynomials of degree m. In [45], Trimèche extended V_k from the polynomials to the space of smooth functions, and then used it to define a generalized translation operator on $\mathscr{C}^\infty(\mathbb{R}^N)$ by

$$\tau_x(k)f(y) := V_k^x V_k^y (V_k^{-1}f)(x-y), \qquad x, y \in \mathbb{R}^N.$$

Here the superscript denotes the relevant variable. When $k \equiv 0$, $\tau_x(0)f(y) = f(x-y)$. (In [45], Trimèche writes $x+y$, instead of $x-y$ as the argument of f. We mention that the generalized translation operator appeared for the first time in [39, p. 535] for Schwartz functions.) In particular, the operator $\tau_x(k)$ satisfies

$$\tau_0(k)f(y) = f(-y), \quad \Delta_k^x\big(\tau_x(k)f\big) = \tau_x(k)\big(\Delta_k f\big), \quad \tau_x(k)f(y) = \tau_{-y}(k)f(-x).$$

The following lemma collects some of the elementary properties of the translation operator; we refer to [44, 45] for more details.

Lemma 2.3. (i) *For every* $x \in \mathbb{R}^N$, $\tau_x(k)$ *is a continuous linear mapping from* $\mathscr{C}^\infty(\mathbb{R}^N)$ *into* $\mathscr{C}^\infty(\mathbb{R}^N)$.

(ii) *The function* $x \mapsto \tau_x(k)(f)$ *is of class* $\mathscr{C}^\infty$ *from* $\mathbb{R}^N$ *to* $\mathscr{C}^\infty(\mathbb{R}^N)$.

(iii) *For all* $z \in \mathbb{C}^N$,

$$\tau_x(k)\big(E_k(\cdot, z)\big)(y) = E_k(x, z)E_k(-y, z). \tag{2.5}$$

(iv) *For* $f \in \mathscr{S}(\mathbb{R}^N)$ *and for fixed* $x \in \mathbb{R}^N$, *the function* $\tau_x(k)f \in \mathscr{S}(\mathbb{R}^N)$. *Further,*

$$\mathscr{D}_k(\tau_x(k)f)(\xi) = E_k(x, -i\xi)\mathscr{D}_k(f)(\xi).$$

By means of the generalized translation operator $\tau_x(k)$, in [45], Trimèche defined the Dunkl convolution $*_k$ by

$$(f *_k g)(x) := \int_{\mathbb{R}^N} f(y)\tau_x(k)g(y)\upsilon_k(y)dy, \tag{2.6}$$

for $f, g \in \mathscr{S}(\mathbb{R}^N)$. It can then be proved that

$$\mathscr{D}_k(f *_k g)(\xi) = \mathscr{D}_k f(\xi)\mathscr{D}_k g(\xi) \qquad \text{and} \qquad f *_k g = g *_k f \tag{2.7}$$

(cf. [45, Theorem 7.2]). In Trimèche's paper, (2.7) is shown only for compactly supported test functions; in [38, Lemma 2.2], Rösler extended these properties to Schwartz function, using a simple density argument. We refer to [45, 44, 5] for more details on the Dunkl convolution.

Next we turn our attention to the Dunkl convolution of two distributions. Denote by $\mathscr{D}(\mathbb{R}^N)$ the space of smoothly compact supported functions on $\mathbb{R}^N$, and set $\mathscr{D}'(\mathbb{R}^N)$ to be its dual.

Let $\varphi_{x,y} \in \mathscr{D}(\mathbb{R}^N \times \mathbb{R}^N)$, $\mathcal{S}_x \in \mathscr{D}'(\mathbb{R}^N)$ and $\mathcal{T}_y \in \mathscr{D}'(\mathbb{R}^N)$. The tensor product $\mathcal{S}_x \otimes \mathcal{T}_y$ is a distribution defined on $\mathscr{D}(\mathbb{R}^N \times \mathbb{R}^N)$ by either one of the following equations

$$\langle \mathcal{S}_x \otimes \mathcal{T}_y, \varphi_{x,y} \rangle := \langle \mathcal{S}_x, \langle \mathcal{T}_y, \varphi_{x,y} \rangle \rangle,$$

$$\langle \mathcal{S}_x \otimes \mathcal{T}_y, \varphi_{x,y} \rangle := \langle \mathcal{T}_y, \langle \mathcal{S}_x, \varphi_{x,y} \rangle \rangle,$$

where $\langle \mathcal{T}_y, \varphi_{x,y} \rangle$ is the function defined by $x \mapsto \langle \mathcal{T}_y, \varphi_{x,y} \rangle$, and $\langle \mathcal{S}_x, \varphi_{x,y} \rangle$ is the function defined by $y \mapsto \langle \mathcal{S}_x, \varphi_{x,y} \rangle$. These two functions are in $\mathscr{D}(\mathbb{R}^N)$. Indeed, $x \mapsto \langle \mathcal{T}_y, \varphi_{x,y}(x, \cdot) \rangle$ is the composite mapping of

$$x \mapsto \varphi_{x,y}(x, \cdot) \tag{2.8}$$

and

$$\varphi_{x,y}(x, \cdot) \mapsto \langle \mathcal{T}_y, \varphi_{x,y}(x, \cdot) \rangle. \tag{2.9}$$

The map (2.8) is $\mathscr{C}^\infty$, while the map (2.9) is linear. Thus $x \mapsto \langle \mathcal{T}_y, \varphi_{x,y}(x, \cdot) \rangle$ is $\mathscr{C}^\infty$. Further, $\varphi_{x,y}$ having compact support K in $\mathscr{D}(\mathbb{R}^N \times \mathbb{R}^N)$ implies that $\langle \mathcal{T}_y, \varphi_{x,y} \rangle$ has compact support in $\mathbb{R}^N$ since $\langle \mathcal{T}_y, \varphi_{x,y}(x, \cdot) \rangle$ vanishes, as $\varphi_{x,y}(x, \cdot)$ does, when x does not belong to the compact projection of K on $\mathbb{R}^N$. The equivalence of the two definitions above follows from the fact that:

(i) If $\varphi_{x,y} = \Phi_x \Psi_y$ with $\Phi_x \in \mathscr{D}(\mathbb{R}^N)$ and $\Psi_y \in \mathscr{D}(\mathbb{R}^N)$, then $\langle \mathcal{S}_x \otimes \mathcal{T}_y, \varphi_{x,y} \rangle = \langle \mathcal{S}, \Phi \rangle \langle \mathcal{T}, \Psi \rangle$ and the two definitions coincide for pure tensors.

(ii) If φ belongs to the algebra tensor product $\mathscr{D}(\mathbb{R}^N) \otimes \mathscr{D}(\mathbb{R}^N)$, then $\varphi_{x,y}$ can be represented as finite sums

$$\varphi_{x,y} = \sum_j \Phi_x^{(j)} \Psi_y^{(j)}$$

where $\Phi_x^{(j)} \in \mathscr{D}(\mathbb{R}^N)$ and $\Psi_y^{(j)} \in \mathscr{D}(\mathbb{R}^N)$. Thus, by means of (i), the two definitions coincide on $\mathscr{D}(\mathbb{R}^N) \otimes \mathscr{D}(\mathbb{R}^N)$.

(iii) Finally, let $\varphi \in \mathscr{D}(\mathbb{R}^N \times \mathbb{R}^N)$. Using the well known fact that $\mathscr{D}(\mathbb{R}^N) \otimes \mathscr{D}(\mathbb{R}^N)$ is dense in $\mathscr{D}(\mathbb{R}^N \times \mathbb{R}^N)$ and in the light of (ii), the two definitions coincide on $\mathscr{D}(\mathbb{R}^N \times \mathbb{R}^N)$.

Convention. Let $f \in L^1(\mathbb{R}^N, \upsilon_k(x)dx)$ and $\varphi \in \mathscr{D}(\mathbb{R}^N)$. Set $\mathcal{T}_f$ to be the linear form on $\mathscr{D}(\mathbb{R}^N)$ defined by

$$\langle \mathcal{T}_f, \varphi \rangle = \int_{\mathbb{R}^N} f(x)\varphi(x)\upsilon_k(x)dx.$$

We may call $\mathcal{T}_f$ the distribution associated (or equivalent) to the function f, and we may write $\mathcal{T}_f = f$. (Hence the name "generalized functions" sometimes is given to distributions.)

The convolution $\mathcal{S} *_k \mathcal{T}$ of two distributions on $\mathbb{R}^N$, if it is defined, is a distribution on $\mathbb{R}^N$ such that

$$\langle \mathcal{S} *_k \mathcal{T}, \psi \rangle := \langle \mathcal{S}_x \otimes \mathcal{T}_y, \tau_x(k)\psi(-y) \rangle, \qquad \text{for all } \psi \in \mathscr{D}(\mathbb{R}^N).$$

Observe that, when defined, $\mathcal{S} *_k \mathcal{T}$ is commutative and the Dirac measure δ is the unit element of this convolution. To see the latter fact notice that $\langle \delta *_k \mathcal{T}, \psi \rangle = \langle \delta_x \otimes \mathcal{T}_y, \tau_x(k)\psi(-y) \rangle = \langle \mathcal{T}_y, \langle \delta_x, \tau_y(k)\psi(-x) \rangle \rangle = \langle \mathcal{T}, \psi \rangle$. Further, if one assumes that the support of $\mathcal{S}$ or $\mathcal{T}$ is compact, then the Dunkl convolution $\mathcal{S} *_k \mathcal{T}$ is well defined. For instance, if $\mathcal{T}$ belongs to the space $\mathscr{E}'(\mathbb{R}^N)$ of distributions on $\mathbb{R}^N$ with compact support, then the function $x \mapsto \langle \mathcal{T}_y, \tau_x(k)\psi(-y)\rangle$ is an element in $\mathscr{D}(\mathbb{R}^N)$ and the Dunkl convolution can be written as

$$\langle \mathcal{S} *_k \mathcal{T}, \psi\rangle = \langle \mathcal{S}_x, \langle \mathcal{T}_y, \tau_x(k)\psi(-y)\rangle\rangle. \tag{2.10}$$

Further, since $\langle \mathcal{S}_x, \tau_y(k)\psi(-x)\rangle \in \mathscr{C}^\infty(\mathbb{R}^N)$ and a distribution of compact support is well defined on smooth functions which are not necessarily with compact support, one may rewrite (2.10) as

$$\langle \mathcal{S} *_k \mathcal{T}, \psi\rangle = \langle \mathcal{T}_y, \langle \mathcal{S}_x, \tau_y(k)\psi(-x)\rangle\rangle.$$

Now, let $\mathcal{S} \in \mathscr{D}'(\mathbb{R}^N)$ and $\varphi \in \mathscr{D}(\mathbb{R}^N)$. We claim that $\mathcal{S} *_k \varphi$ is a $\mathscr{C}^\infty$ function on $\mathbb{R}^N$ such that

$$(\mathcal{S} *_k \varphi)(x) = (\varphi *_k \mathcal{S})(x) = \langle \mathcal{S}, \tau_x(k)\varphi\rangle. \tag{2.11}$$

One can see this as follows:

$$\begin{aligned}
\langle \mathcal{S} *_k \varphi, \psi\rangle &= \langle \mathcal{S}_x \otimes \varphi(y), \tau_x(k)\psi(-y)\rangle\\
&= \langle \mathcal{S}_x, \langle \varphi(y), \tau_x(k)\psi(-y)\rangle\rangle\\
&= \langle \mathcal{S}_x, \langle \psi(y), \tau_y(k)\varphi(x)\rangle\rangle\\
&= \langle \mathcal{S}_x \otimes \psi(y), \tau_y(k)\varphi(x)\rangle\\
&= \langle \psi(y), \langle \mathcal{S}_x, \tau_y(k)\varphi(x)\rangle\rangle.
\end{aligned}$$

Above we used the fact that $\int_{\mathbb{R}^N} \varphi(y)\tau_x(k)\psi(-y)\upsilon_k(y)dy = \int_{\mathbb{R}^N} \tau_y(k)\varphi(x)\psi(y)\upsilon_k(y)dy$ (see for instance [45]). Therefore

$$\mathcal{S} *_k \varphi(x) = \langle \mathcal{S}, \tau_x(k)\varphi\rangle. \tag{2.12}$$

Now, Lemma 2.3(ii) finishes the proof. Clearly, $\mathcal{S} *_k \varphi = \varphi *_k \mathcal{S}$.

Comment. In view of the convention above, we see that (2.12) agrees with (2.6) in the case where $\mathcal{S}$ is defined by a function.

Since the mapping $\varphi \mapsto \mathscr{D}_k(\varphi)$ of $\mathscr{S}(\mathbb{R}^N)$ onto $\mathscr{S}(\mathbb{R}^N)$ is linear and continuous in the topology of $\mathscr{S}(\mathbb{R}^N)$, we can now define the Dunkl transform of a tempered distribution $\mathcal{T}$ as the tempered distribution $\mathscr{D}_k(\mathcal{T})$ defined through

$$\langle \mathscr{D}_k(\mathcal{T}), \varphi\rangle = \langle \mathcal{T}, \mathcal{D}_k(\varphi)\rangle, \qquad \varphi \in \mathscr{S}(\mathbb{R}^N).$$

Comment. Let $f \in L^1(\mathbb{R}^N, \upsilon_k(x)dx)$ and $\mathcal{T}_f$ be the distribution associated (or equivalent) to f. Obviously $\mathscr{D}_k(\mathcal{T}_f) = \mathcal{T}_{\mathscr{D}_k(f)}$, where $\mathscr{D}_k(f)$ is the Dunkl transform of f as defined in (2.3). This can be seen by changing the order of integration in $\langle \mathscr{D}_k(\mathcal{T}_f), \varphi\rangle = \int_{\mathbb{R}^N} f(x)\mathscr{D}_k(\varphi)(x)\upsilon_k(x)dx = \int_{\mathbb{R}^N} f(x)\big[\int_{\mathbb{R}^N} \varphi(y)E_k(y,-ix)\upsilon_k(y)dy\big]\upsilon_k(x)dx$. Thus the Dunkl transform of a tempered distribution is a generalization of the ordinary Dunkl transform of functions.

Notice that, if $\mathcal{T} \in \mathscr{E}'(\mathbb{R}^N)$, then, using the tensor product, its Dunkl transform can be written as

$$\langle \mathscr{D}_k(\mathcal{T}), \varphi\rangle = \langle\langle \mathcal{T}_y, E_k(-ix, y)\rangle, \varphi(x)\rangle.$$

That is

$$\mathscr{D}_k(\mathcal{T})(\xi) = \langle \mathcal{T}_x, E_k(-i\xi, x)\rangle, \qquad \forall\, \mathcal{T} \in \mathscr{E}'(\mathbb{R}^N).$$

The following elementary properties can be derived from the definition of the Dunkl transform of tempered distributions:

(i) $\mathscr{D}_k$ is a topological isomorphism of $\mathscr{S}'(\mathbb{R}^N)$ onto itself;
(ii) $\mathscr{D}_k(T_j(k)\mathcal{T})(\xi) = i\xi_j \mathscr{D}_k(\mathcal{T})(\xi)$; and
(iii) $T_j(k)(\mathscr{D}_k\mathcal{T}) = \mathscr{D}_k(-ix_j\mathcal{T})$.

Comment. In the statement (ii) above, the distribution $T_j(k)\mathcal{T}$ is defined by

$$\langle T_j(k)\mathcal{T}, \varphi\rangle := -\langle \mathcal{T}, T_j(k)\varphi\rangle, \qquad \forall\, \varphi \in \mathscr{D}(\mathbb{R}^N).$$

This definition makes sense since if $\varphi \in \mathscr{D}(\mathbb{R}^N)$, then $T_j(k)\varphi \in \mathscr{D}(\mathbb{R}^N)$ and $\varphi \mapsto -\langle \mathcal{T}, T_j(k)\varphi\rangle$ is linear and continuous on $\mathscr{D}(\mathbb{R}^N)$. If $\mathcal{T}$ is equivalent to a $\mathscr{C}^1$ function f, then $T_j(k)\mathcal{T}$ is equivalent to $T_j(k)f$:

$$\begin{aligned}\langle T_j(k)f, \varphi\rangle &= \int_{\mathbb{R}^N} \big[T_j(k)f(x)\big]\varphi(x)\upsilon_k(x)dx \\ &= -\int_{\mathbb{R}^N} f(x)\big[T_j(k)\varphi(x)\big]\upsilon_k(x)dx = -\langle f, T_j(k)\varphi\rangle.\end{aligned}$$

Next we turn our attention to the behavior of the convolution of two distributions under the Dunkl transform. We claim that

$$\mathscr{D}_k(\mathcal{S} *_k \mathcal{T}) = \mathscr{D}_k(\mathcal{S})\mathscr{D}_k(\mathcal{T}), \qquad \mathcal{S}, \mathcal{T} \in \mathscr{E}'(\mathbb{R}^N). \tag{2.13}$$

holds if one of the distributions is of compact support and the other one is a tempered distribution. Indeed, if $\varphi \in \mathscr{S}(\mathbb{R}^N)$, $\mathcal{S} \in \mathscr{E}'(\mathbb{R}^N)$ and $\mathcal{T} \in \mathscr{S}'(\mathbb{R}^N)$, then the following

$$\begin{aligned}\langle \mathscr{D}_k(\mathcal{S} *_k \mathcal{T}), \varphi\rangle = \langle \mathcal{S} *_k \mathcal{T}, \mathscr{D}_k(\varphi)\rangle &= \langle \mathcal{S}_x \otimes \mathcal{T}_y, \tau_x(k)\big(\mathscr{D}_k(\varphi)\big)(-y)\rangle \\ &= \langle \mathcal{T}_y, \langle \mathcal{S}_x, \tau_x(k)\big(\mathscr{D}_k(\varphi)\big)(-y)\rangle\rangle\end{aligned}$$

is well defined. This fact can be seen as follows: Since

$$\tau_x(k)f(y) = \int_{\mathbb{R}^N} \mathscr{D}_k(f)(\xi)E_k(ix,\xi)E_k(-iy,\xi)\upsilon_k(\xi)d\xi,$$

then

$$\tau_x(k)\big(\mathscr{D}_k(\varphi)\big)(-y) = \int_{\mathbb{R}^N} \varphi(\xi)E_k(-ix,\xi)E_k(-iy,\xi)\upsilon_k(\xi)d\xi = \mathscr{D}_k(E_k(-iy,\cdot)\varphi)(x).$$

Hence

$$\begin{aligned}\mathcal{X}(y) &:= \langle \mathcal{S}_x, \tau_x(k)\big(\mathscr{D}_k(\varphi)\big)(-y)\rangle \\ &= \langle \mathcal{S}_x, \mathscr{D}_k\big(E(-iy,\cdot)\varphi\big)(x)\rangle \\ &= \langle \mathscr{D}_k(\mathcal{S})(\xi), E_k(-iy,\xi)\varphi(\xi)\rangle.\end{aligned}$$

On the other hand, since $\mathcal{S} \in \mathscr{E}'(\mathbb{R}^N)$, $\mathscr{D}_k(\mathcal{S})$ is a $\mathscr{C}^\infty$ slowly increasing function (cf. [45, Theorem 5.2]), and

$$\mathcal{X}(y) = \int_{\mathbb{R}^N} \mathscr{D}_k(\mathcal{S})(\xi)E_k(-iy,\xi)\varphi(\xi)\upsilon_k(\xi)d\xi = \mathscr{D}_k(\mathscr{D}_k(\mathcal{S})\cdot\varphi)(y).$$

Thus, $\mathcal{X}(y)$ is in $\mathscr{S}(\mathbb{R}^N)$, and the mapping $\mathscr{S}(\mathbb{R}^N) \to \mathscr{S}(\mathbb{R}^N)$, defined by $\varphi \mapsto \mathcal{X}$ is continuous; also, if $\mathcal{T}$ is a tempered distribution, then

$$\langle \mathscr{D}_k(\mathcal{S} *_k \mathcal{T}), \varphi\rangle = \langle \mathcal{T}, \mathcal{X}\rangle$$

is well defined, depends continuously on φ, and

$$\langle \mathscr{D}_k(\mathcal{S} *_k \mathcal{T}), \varphi\rangle = \langle \mathcal{T}, \mathscr{D}_k(\mathscr{D}_k(\mathcal{S})\cdot\varphi)\rangle = \langle \mathscr{D}_k(\mathcal{T}), \mathscr{D}_k(\mathcal{S})\cdot\varphi\rangle = \langle \mathscr{D}_k(\mathcal{S})\mathscr{D}_k(\mathcal{T}), \varphi\rangle.$$

That is

$$\mathscr{D}_k(\mathcal{S} *_k \mathcal{T}) = \mathscr{D}_k(\mathcal{S})\mathscr{D}_k(\mathcal{T}), \qquad \text{for } \mathcal{S} \in \mathscr{E}'(\mathbb{R}^N),\ \mathcal{T} \in \mathscr{S}'(\mathbb{R}^N). \tag{2.14}$$

Remark 2.4. (i) Alternatively, one may prove (2.14) using the fact that $\mathscr{D}_k = \mathscr{D}_0 \circ {}^tV_k$, together with

$${}^tV_k(\mathcal{S} *_k \mathcal{T}) = {}^tV_k(\mathcal{S}) *_0 {}^tV_k(\mathcal{T}). \tag{2.15}$$

Here $\mathscr{D}_0$ denotes the Euclidean Fourier transform, ${}^tV_k$ is the transpose of the intertwining operator V_k, and $*_0$ is the classical convolution. Now, (2.15) was given in [5, Proposition 2.10] only for $\mathcal{S}, \mathcal{T} \in \mathscr{E}'(\mathbb{R}^N)$; an approximation argument gives the result if one of the distributions is a tempered distribution. We mention that later we will need (2.14) only for $\mathcal{S} \in \mathscr{E}'(\mathbb{R}^N)$ and $\mathcal{T} \in \mathscr{S}(\mathbb{R}^N)$.

(ii) Let $\mathcal{S}, \mathcal{T} \in \mathscr{E}'(\mathbb{R}^N)$. We claim that $\mathcal{S} *_k \mathcal{T}$ is a distribution with compact support. Thus, its Dunkl transform is a continuous function such that

$$\begin{aligned}\mathscr{D}_k(\mathcal{S} *_k \mathcal{T})(\xi) &= \langle \mathcal{S} *_k \mathcal{T}, E_k(-i\xi, \cdot)\rangle = \langle \mathcal{S}_x \otimes \mathcal{T}_y, \tau_x\big(E_k(-i\xi, \cdot)\big)(-y)\rangle \\ &= \langle \mathcal{S}_x, E_k(-i\xi, x)\rangle\langle \mathcal{T}_y, E_k(-i\xi, y)\rangle = \mathscr{D}_k(\mathcal{S})(\xi)\mathscr{D}_k(\mathcal{T})(\xi)\end{aligned}$$

(recall (2.5)). To prove the claim above, we shall argue as follows: On one hand we have the equation (2.15) above. On the other hand, by [45, Theorem 5.1] ${}^tV_k$ is a topological isomorphism from $\mathscr{E}'(\mathbb{R}^N)$ onto itself. Now using the well know fact that the classical convolution of two distributions with compact supports is again a compactly supported distribution, we can deduce that the right hand side of (2.15) belongs to $\mathscr{E}'(\mathbb{R}^N)$. Applying again [45, Theorem 5.1], we conclude that $\mathcal{S} *_k \mathcal{T} \in \mathscr{E}'(\mathbb{R}^N)$.

We close this section by recalling a Paley-Wiener theorem for the Dunkl transform. For $R > 0$, denote by $\mathscr{C}_R^\infty(\mathbb{R}^N)$ the space of smooth functions on $\mathbb{R}^N$ with support contained in the closed metric ball of radius R about the origin. Denote by $\mathscr{H}_R(\mathbb{C}^N)$ the space of entire functions f on $\mathbb{C}^N$ with the property that for each integer $M > 0$, there exists a constant α_M such that

$$|f(z)| \le \alpha_M(1 + \|z\|)^{-M} e^{R\|\mathrm{Im}(z)\|}.$$

Further, let $\mathscr{E}'_R(\mathbb{R}^N)$ be the space of distributions on $\mathbb{R}^N$ with support contained in the closed ball of radius R about the origin, and let $\mathcal{H}_R(\mathbb{C}^N)$ be the space of entire functions on $\mathbb{C}^N$ such that

$$|f(z)| \le C(1 + \|z\|)^M e^{R\|\,\mathrm{Im}(z)\|},$$

for some positive constants C and M.

Theorem 2.5. (Paley-Wiener Theorem) *Let G be a finite reflection group and suppose that $k \in \mathscr{K}^+$.*

(i) (cf. [30]) *The Dunkl transform $\mathscr{D}_k$ is a linear isomorphism between $\mathscr{C}_R^\infty(\mathbb{R}^N)$ and $\mathscr{H}_R(\mathbb{C}^N)$, for all $R > 0$.*

(ii) (cf. [45]) *The Dunkl transform $\mathscr{D}_k$ is a linear isomorphism between $\mathscr{E}'_R(\mathbb{R}^N)$ and $\mathcal{H}_R(\mathbb{C}^N)$, for all $R > 0$.*

Finally, let us point out the following fact regarding the Dunkl convolution. Equation (2.14) shows that the Dunkl transform of $\mathcal{S} *_k \mathcal{T}$, with $\mathcal{S} \in \mathscr{E}'(\mathbb{R}^N)$ and $\mathcal{T} \in \mathscr{S}'(\mathbb{R}^N)$, equals the product $\mathscr{D}_k(\mathcal{S})\mathscr{D}_k(\mathcal{T})$. Since $\mathcal{S} \in \mathscr{E}'(\mathbb{R}^N)$, by the (easy half of the) Paley-Wiener Theorem (ii), $\mathscr{D}_k(\mathcal{S})$ belongs to the space of smooth slowly increasing functions. Hence, for $\mathcal{T} \in \mathscr{S}'(\mathbb{R}^N)$, $\mathscr{D}_k(\mathcal{S})\mathscr{D}_k(\mathcal{T}) \in \mathscr{S}'(\mathbb{R}^N)$. This shows that if $\mathcal{S} \in \mathscr{E}'(\mathbb{R}^N)$ and

$\mathcal{T} \in \mathscr{S}'(\mathbb{R}^N)$, then $\mathcal{S} *_k \mathcal{T} \in \mathscr{S}'(\mathbb{R}^N)$. Similarly, one can prove that $\mathcal{S} *_k \mathcal{T} \in \mathscr{S}(\mathbb{R}^N)$ if $\mathcal{S} \in \mathscr{E}'(\mathbb{R}^N)$ and $\mathcal{T} \in \mathscr{S}(\mathbb{R}^N)$.

3. The wave equation for Dunkl operators

Except in a few places, most of the results below hold for complex-valued multiplicity functions k such that $\mathrm{Re}(k) \geq 0$. However, for the reader's convenience, we will restrict ourselves to multiplicity functions $k \in \mathscr{K}^+$.

For k in $\mathscr{K}^+$, consider the following Cauchy problem for the wave equation associated with the Dunkl-Laplacian operator

$$\begin{aligned} &\Delta_k u_k(x,t) = \partial_{tt} u_k(x,t), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ &u_k(x,0) = f(x), \;\; \partial_t u_k(x,0) = g(x). \end{aligned} \tag{3.1}$$

Here the functions f and g belong to $\mathscr{S}(\mathbb{R}^N)$. The subscript t indicates differentiation in the t-variable. Next, we will prove the following statements:

($\mathcal{S}_1$) Let $k \in \mathscr{K}^+$ and $N \geq 1$. For a given $x \in \mathbb{R}^N$, the solution $u_k(x,t)$ depends only on the values of $\tau_x(k)f(y)$ and $\tau_x(k)g(y)$ for $\|y\| \leq |t|$.

($\mathcal{S}_2$) Let $k \in \mathscr{K}^+$ and $N \geq 1$. For a given $x \in \mathbb{R}^N$, the solution $u_k(x,t)$ depends only on the values of $\tau_x(k)f(y)$ and $\tau_x(k)g(y)$ (and their derivatives) for $\|y\| = |t|$ if and only if $(N-3)/2 + \gamma_k \in \mathbb{N}$.

Another way of stating ($\mathcal{S}_1$) is that u_k is expressed as a sum of $*_k$-convolutions of f and g with distributions that vanish outside the ball of radius $|t|$ about the origin. Similarly, ($\mathcal{S}_2$) is equivalent to the fact that the distributions we convolve f and g with, also vanish inside the ball of radius $|t|$. In analogy with the classical case, i.e. when $k \equiv 0$, we shall say that (3.1) satisfies the weak Huygens principle if u_k satisfies ($\mathcal{S}_1$), and (3.1) satisfies the strict Huygens principle if u_k satisfies ($\mathcal{S}_2$).

For the time being, we only assume $k \in \mathscr{K}^+$ and $N \geq 1$. For $t \in \mathbb{R}$, denote by $P_{k,t}$ the 2×2 matrix of tempered distributions on $\mathbb{R}^N$

$$P_{k,t} = \begin{bmatrix} P_{k,t}^{11} & P_{k,t}^{12} \\ P_{k,t}^{21} & P_{k,t}^{22} \end{bmatrix} := \begin{bmatrix} \mathscr{D}_k^{-1}[\cos(t\|\cdot\|)] & \mathscr{D}_k^{-1}[\sin(t\|\cdot\|)/\|\cdot\|] \\ \mathscr{D}_k^{-1}[-\|\cdot\|\sin(t\|\cdot\|)] & \mathscr{D}_k^{-1}[\cos(t\|\cdot\|)] \end{bmatrix}. \tag{3.2}$$

In Theorem 3.2 below we shall prove that the $P_{k,t}^{ij}$'s are compactly supported distributions, which justifies the following operations with convolutions in view of (2.14). Put $U_k(x,0) := \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}$, where the Cauchy data $(f,g) \in \mathscr{S}(\mathbb{R}^N) \times \mathscr{S}(\mathbb{R}^N)$. Thus, we may define the vector column $U_k(x,t)$ by

$$\begin{aligned} U_k(x,t) &:= \{P_{k,t} *_k U_k(\cdot,0)\}(x) \\ &= \left\{ \begin{bmatrix} P_{k,t}^{11} & P_{k,t}^{12} \\ P_{k,t}^{21} & P_{k,t}^{22} \end{bmatrix} *_k \begin{bmatrix} f \\ g \end{bmatrix} \right\}(x). \end{aligned} \tag{3.3}$$

By applying the Dunkl transform $\mathscr{D}_k$ to (3.3), in the x-variable, we get

$$\mathscr{D}_k(U_k(\cdot,t))(\xi) = e^{t\mathbb{A}} \mathscr{D}_k(U_k(\cdot,0))(\xi), \tag{3.4}$$

where

$$\mathbb{A} := \begin{bmatrix} 0 & 1 \\ -\|\xi\|^2 & 0 \end{bmatrix}. \tag{3.5}$$

That is $\mathscr{D}_k(U_k(\cdot,t))(\xi)$ is a solution to the following ordinary differential equation

$$\partial_t \mathscr{D}_k(U_k(\cdot,t))(\xi) = \mathbb{A}\mathscr{D}_k(U_k(\cdot,t))(\xi) = \begin{bmatrix} 0 & 1 \\ -\|\xi\|^2 & 0 \end{bmatrix} \mathscr{D}_k(U_k(\cdot,t))(\xi). \tag{3.6}$$

Using the fact that $-\|\xi\|^2\mathscr{D}_k(f)(\xi) = \mathscr{D}_k(\Delta_k f)(\xi)$, and the injectivity of the Dunkl transform, we deduce that

$$\partial_t U_k(x,t) = \begin{bmatrix} 0 & 1 \\ \Delta_k & 0 \end{bmatrix} U_k(x,t). \tag{3.7}$$

Thus, if we write $U_k(x,t) = \begin{bmatrix} u_k(x,t) \\ v_k(x,t) \end{bmatrix}$, then $u_k(x,t)$ satisfies the following wave equation

$$\partial_{tt} u_k(x,t) = \Delta_k u_k(x,t).$$

Moreover, from (3.3) and in the light of the very last fact pointed out in the previous section regarding $*_k$, $u_k(\cdot,t) \in \mathscr{S}(\mathbb{R}^N)$ for each $t \in \mathbb{R}$.

Furthermore, $u_k(x,t) \to f(x)$ as $t \to 0$. Indeed, if δ denotes the Dirac functional, then, as $t \to 0$, $\mathscr{D}_k^{-1}(\cos(t\|\cdot\|)) \to \delta$ in $\mathscr{S}'(\mathbb{R}^N)$ and thus in $\mathscr{D}'(\mathbb{R}^N)$. On the other hand $\mathscr{D}_k^{-1}(\sin(t\|\cdot\|)/\|\cdot\|) \to 0$ as $t \to 0$. Using the continuity of the Dunkl convolution $*_k$, we deduce that

$$u_k(x,t) \to (\delta *_k f)(x) = f(x) \qquad \text{as} \quad t \to 0.$$

Similarly, one can prove that $(\partial_t u_k)(x,t) \to g(x)$ as $t \to 0$.

We mention that the solution u_k constructed above is unique. This claim is a consequence of the energy conservation theorem, which we will prove in the last section (see Theorem 4.1 below). Indeed, if we denote by

$$\mathscr{E}_k[u_k](t) := \frac{1}{2}\int_{\mathbb{R}^N} \Big(|\partial_t u_k(x,t)|^2 + \sum_{j=1}^N |T_j^x(k) u_k(x,t)|^2 \Big) v_k(x)dx$$

the total energy of the solution $u_k(x,t)$ at time t, then Theorem 4.1 below shows that $\mathscr{E}_k[u_k](t)$ is independent of t, and

$$\mathscr{E}_k[u_k](t) = \frac{c_k^{-2}}{2}\int_{\mathbb{R}^N} \Big(\|\xi\|^2 |\mathscr{D}_k f(\xi)|^2 + |\mathscr{D}_k g(\xi)|^2 \Big) v_k(\xi)d\xi.$$

Thus, if we suppose that $u_k^{(1)}$ and $u_k^{(2)}$ are two solutions of the wave equation with the same initial data, then $u_k^{(1)} - u_k^{(2)}$ is a solution of the wave equation with zero initial data. Therefore, the energy for the solution $u_k^{(1)} - u_k^{(2)}$ is zero. This implies that $\partial_t\big(u_k^{(1)} - u_k^{(2)}\big)(x,t) = 0$ for every $t \in \mathbb{R}$. That is $t \mapsto \big(u_k^{(1)} - u_k^{(2)}\big)(x,t)$ is a constant function, so $\big(u_k^{(1)} - u_k^{(2)}\big)(x,t) = \big(u_k^{(1)} - u_k^{(2)}\big)(x,0) = 0$. This proves that the solutions of the wave equation are uniquely determined by the initial Cauchy data. In the classical case $k \equiv 0$, the reader is referred to [32].

The following theorem collects all the above facts and discussions.

Theorem 3.1. *The solution to the Cauchy problem (3.1) is given uniquely by*

$$u_k(x,t) = (P_{k,t}^{11} *_k f)(x) + (P_{k,t}^{12} *_k g)(x),$$

where, for a fixed t, $P^{11}_{k,t}$ and $P^{12}_{k,t}$ are the tempered distributions on $\mathbb{R}^N$ given by

$$P^{11}_{k,t} = \mathscr{D}_k^{-1}\left[\cos(t\|\cdot\|)\right], \quad P^{12}_{k,t} = \mathscr{D}_k^{-1}\left[\sin(t\|\cdot\|)/\|\cdot\|\right].$$

We shall call the distributions $P^{ij}_{k,t}$ the propagators of the deformed wave equation.

Before investigating the support of the solution u_k and of the propagators, let us make some observations regarding the estimate and the limit of $u_k(\cdot,t)$ in $L^2(\mathbb{R}^N, \upsilon_k(x)dx)$. We restrict our attention to the L^2-behaviors because these are the most physically interesting quantities. First, for all $t \in \mathbb{R}$, we have the following Strichartz-type inequality

$$\|u_k(\cdot,t)\|_k \leq \|f\|_k + \|(-\Delta_k)^{-1/2} g\|_k. \tag{3.8}$$

Here $\|\cdot\|_k$ denotes the norm in $L^2(\mathbb{R}^N, \upsilon_k(x)dx)$. Secondly, as $|t| \to \infty$, the function $t \mapsto \|u_k(\cdot,t)\|_k$ has a finite limit depending on the initial data

$$\lim_{|t|\to\infty} \|u_k(\cdot,t)\|_k^2 = \frac{1}{2}\|f\|_k^2 + \frac{1}{2}\|(-\Delta_k)^{-1/2} g\|_k^2. \tag{3.9}$$

It follows that, if $\|u_k(\cdot,t)\|_k \to 0$ as $|t| \to \infty$, then

$$u_k \equiv 0.$$

To prove (3.8) and (3.9), we express $\int_{\mathbb{R}^N} |u_k(x,t)|^2 \upsilon_k(x)dx$ in terms of $\mathscr{D}_k(u_k(\cdot,t))(\xi)$ by means of the Plancherel formula. In view of

$$\mathscr{D}_k(u_k(\cdot,t))(\xi) = \cos(t\|\xi\|)\mathscr{D}_k f(\xi) + \frac{\sin(t\|\xi\|)}{\|\xi\|}\mathscr{D}_k g(\xi), \tag{3.10}$$

we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} |u_k(x,t)|^2 \upsilon_k(x)dx &= \frac{c_k^{-2}}{2}\int_{\mathbb{R}^N}\left\{|\mathscr{D}_k f(\xi)|^2 + \frac{|\mathscr{D}_k g(\xi)|^2}{\|\xi\|^2}\right\}\upsilon_k(\xi)d\xi \\
&\quad + \frac{c_k^{-2}}{2}\int_{\mathbb{R}^N} |\mathscr{D}_k f(\xi)|^2 \cos(2t\|\xi\|)\upsilon_k(\xi)d\xi \\
&\quad - \frac{c_k^{-2}}{2}\int_{\mathbb{R}^N} \frac{|\mathscr{D}_k g(\xi)|^2}{\|\xi\|^2}\cos(2t\|\xi\|)\upsilon_k(\xi)d\xi \\
&\quad + \frac{c_k^{-2}}{2}\int_{\mathbb{R}^N} \frac{\mathscr{D}_k f(\xi)\overline{\mathscr{D}_k g(\xi)} + \overline{\mathscr{D}_k f(\xi)}\mathscr{D}_k g(\xi)}{\|\xi\|}\sin(2t\|\xi\|)\upsilon_k(\xi)d\xi.
\end{aligned}$$

Above we used the familiar trigonometric identities for double angles. Now the Strichartz inequality is clear. Equation (3.9) follows by using the classical Riemann-Lebesgue lemma for the Euclidean Fourier sine and cosine transforms.

Now we turn our attention to the statements $(\mathcal{S}_1)$ and $(\mathcal{S}_2)$, stated at the beginning of this section. Recall that

$$u_k(x,t) = \langle P^{11}_{k,t}(y), \tau_x(k)f(y)\rangle + \langle P^{12}_{k,t}(y), \tau_x(k)g(y)\rangle.$$

The statement $(\mathcal{S}_1)$ claims that $u_k(x,t)$ depends only on the values of $\tau_x(k)f(y)$ and $\tau_x(k)g(y)$ for $\|y\| \leq |t|$. In other words, $P^{ij}_{k,t}$ is supported in the set $\{y \in \mathbb{R}^N \mid \|y\| \leq |t|\}$. On the other hand, the statement $(\mathcal{S}_2)$ claims that $u_k(x,t)$ depends only on the values of $\tau_x(k)f(y)$ and $\tau_x(k)g(y)$ for $\|y\| = |t|$ if and only if $(N-3)/2 + \gamma_k \in \mathbb{N}$. In other words, $P^{ij}_{k,t}$ is supported on the set $\{y \in \mathbb{R}^N \mid \|y\| = |t|\}$ if and only if $(N-3)/2 + \gamma_k \in \mathbb{N}$.

To prove $(\mathcal{S}_1)$, our method uses the Paley-Wiener Theorem 2.5(ii) for the Dunkl transform.

The first key observation is that the functions $\cos(t\|x\|)$ and $\sin(t\|x\|)/\|x\|$ can be extended to entire functions on $\mathbb{C}^N$. Indeed, for $z \in \mathbb{C}$, the functions $\cos z$ and $\sin z/z$ are both even, and thus we may consider the functions $\cos(\sqrt{z})$ and $\sin(\sqrt{z})/\sqrt{z}$ which are entire analytic functions of z (even though $\sqrt{z}$ is not single-valued). Thus, the analytic extensions of $\cos(t\|x\|)$ and $\sin(t\|x\|)/\|x\|$, respectively, are

$$\cos(t\langle z, z\rangle^{1/2}), \qquad \frac{\sin(t\langle z, z\rangle^{1/2})}{\langle z, z\rangle^{1/2}}.$$

In order to apply the Paley-Wiener theorem, we need to show that

$$\big|\cos(t\langle z, z\rangle^{1/2})\big|, \quad \left|\frac{\sin(t\langle z, z\rangle^{1/2})}{\langle z, z\rangle^{1/2}}\right| \leq c\, e^{|t|\,\|\mathrm{Im}(z)\|}, \tag{3.11}$$

for some constant c. We believe that the above two inequalities are proved somewhere in the literature. However, in order to be self-contained, we shall give a proof: If we write $\langle z, z\rangle^{1/2} = u + iv$ and use the fact that $|\cos(u+iv)|$ and $|\sin(u+iv)/(u+iv)|$ are both bounded by a constant c times $e^{|v|}$, we obtain

$$\big|\cos(t\langle z, z\rangle^{1/2})\big|, \quad \left|\frac{\sin(t\langle z, z\rangle^{1/2})}{\langle z, z\rangle^{1/2}}\right| \leq c\, e^{|t|\,|v|}.$$

Further, as $\langle z, z\rangle = (u+iv)^2$, we have $u^2 - v^2 = \|\mathrm{Re}(z)\|^2 - \|\mathrm{Im}(z)\|^2$ and $uv = \langle \mathrm{Re}(z), \mathrm{Im}(z)\rangle$. Thus, by Cauchy-Schwartz-Buniakowsly inequality, it follows that $u^2v^2 \leq \|\mathrm{Re}(z)\|^2\|\mathrm{Im}(z)\|^2$, which is equivalent to $v^2(v^2+\|\mathrm{Re}(z)\|^2-\|\mathrm{Im}(z)\|^2) \leq \|\mathrm{Re}(z)\|^2\|\mathrm{Im}(z)\|^2$. This amounts to

$$\left(v^2 + \frac{\|\mathrm{Re}(z)\|^2 - \|\mathrm{Im}(z)\|^2}{2}\right)^2 \leq \left(\frac{\|\mathrm{Re}(z)\|^2 + \|\mathrm{Im}(z)\|^2}{2}\right)^2,$$

which yields $v^2 \leq \|\mathrm{Im}(z)\|^2$. Now, applying the Paley-Wiener Theorem 2.5(ii), we conclude that the distributions $\mathscr{D}_k^{-1}[\cos(t\|\cdot\|)]$ and $\mathscr{D}_k^{-1}[\sin(t\|\cdot\|)/\|\cdot\|]$ are supported in the set $\|x\| \leq |t|$. We have proved:

Theorem 3.2. *For all $k \in \mathscr{K}^+$ and $N \geq 1$, the propagators $P_{k,t}^{11}$ and $P_{k,t}^{12}$ are supported in the set $\{y \in \mathbb{R}^N \mid \|y\| \leq |t|\}$.*

Thus, the following weak Huygens principle holds.

Theorem 3.3. (Weak Huygens' Principle) *Assume that $k \in \mathscr{K}^+$ and $N \geq 1$. For a given point $x \in \mathbb{R}^N$, the solution $u_k(x,t)$ to the Cauchy problem (3.1) depends only on the values of $\tau_x(k)f(y)$ and $\tau_x(k)g(y)$ for $\|y\| \leq |t|$.*

Notice that the above theorem holds in all dimensions N.

We shall now discuss the strict Huygens principle which will hold only under a condition involving N and the multiplicity function k. Our approach uses the representation theory of the group $SL(2,\mathbb{R})$, following [26].

We start by investigating certain symmetries and invariance of the deformed wave equation, which are reflected in symmetries and invariance of the propagators. To see this, we define the 2×2 matrix $P_k = \begin{bmatrix} P_k^{11} & P_k^{12} \\ P_k^{21} & P_k^{22} \end{bmatrix}$ of entrywise distributions on $\mathbb{R}^{N+1}$, where

$$P_k^{ij}(\psi_1 \otimes \psi_2) := \int_{\mathbb{R}} P_{k,t}^{ij}(\psi_1)\psi_2(t)dt, \qquad i,j = 1,2,$$

for $\psi_1 \in \mathscr{S}(\mathbb{R}^N)$ and $\psi_2 \in \mathscr{S}(\mathbb{R})$. Here we used the fact that $\mathscr{S}(\mathbb{R}^{N+1}) \simeq \mathscr{S}(\mathbb{R}^N)\widehat{\otimes}\mathscr{S}(\mathbb{R})$ is the unique topological tensor product of $\mathscr{S}(\mathbb{R}^N)$ and $\mathscr{S}(\mathbb{R})$ as nuclear spaces. From the constructive proof of theorem 3.1, it follows that

$$\Delta_k P_k^{ij} = \partial_{tt} P_k^{ij}, \qquad i,j = 1,2.$$

For $h \in G$, $\psi \in \mathscr{S}(\mathbb{R}^{N+1})$, and for each $t \in \mathbb{R}$, denote by π_x the unitary action of G on $\psi(\cdot, t)$ given by

$$\pi_x(h)\psi(x,t) := \psi(h^{-1} \cdot x, t).$$

By duality, we have the action π_x^* of G on tempered distributions by the rule

$$\pi_x^*(h)(T)(\psi) = T(\pi_x(h)^{-1}\psi),$$

for $\psi \in \mathscr{S}(\mathbb{R}^{N+1})$ and $T \in \mathscr{S}'(\mathbb{R}^{N+1})$. Further, let τ be the operation of time-reflection $\tau(x,t) = (x,-t)$, and denote by

$$\pi_t(\tau)\psi(x,t) := \psi(x,-t).$$

Similarly as for π_x^*, we obtain the action π_t^* on distributions.

Begin with a solution $u_k(x,t)$ to the Cauchy problem (3.1) with Cauchy data (f,g). Then $\pi_x(h)u_k(x,t)$ solves the wave equation with initial data $(\pi_x(h)f, \pi_x(h)g)$. The analogue of (3.3) reads

$$\pi_x(h)U_k(x,t) = \{P_{k,t} *_k \pi_x(h)U_k(\cdot,0)\}(x).$$

This amounts to

$$U_k(x,t) = \pi_x^*(h)\{P_{k,t} *_k \pi_x(h)U_k(\cdot,0)\}(x) = \{\pi_x^*(h)P_{k,t} *_k U_k(\cdot,0)\}(x),$$

which implies

$$\pi_x^*(h)P_{k,t}^{ij} = P_{k,t}^{ij}, \qquad i,j = 1,2.$$

The G-invariance of $P_{k,t}^{ij}$ can also be observed directly from (3.2). Plugging this into the definition of P_k^{ij}, we conclude that

$$\pi_x^*(h)P_k^{ij} = P_k^{ij}, \qquad i,j = 1,2.$$

For the operation of time-reflection, clearly $\pi_t(\tau)u_k(x,t) = u_k(x,-t)$ solves the Cauchy problem (3.1) with Cauchy data $(f,-g)$. Thus, the analogue of (3.3) reads

$$\begin{bmatrix} u_k(x,-t) \\ -(\partial_t u_k)(x,-t) \end{bmatrix} = P_{k,t} *_k \begin{bmatrix} f \\ -g \end{bmatrix},$$

which we may rewrite as

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} U_k(x,-t) = P_{k,t} *_k \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} U_k(x,0). \tag{3.12}$$

On the other hand, from (3.3), it follows that $U_k(x,-t) = P_{k,-t} *_k U_k(x,0)$. Comparing this with equation (3.12), we obtain

$$P_{k,-t}^{ij} = (-1)^{i-j} P_{k,t}^{ij} \qquad \text{for } i,j = 1,2,$$

which implies

$$\pi_t^*(\tau)P_k^{ij} = (-1)^{i-j} P_k^{ij} \qquad \text{for } i,j = 1,2.$$

Remark 3.4. From the time-reflection action on the propagators, it is clear that time is reversible, except for a minus sign that may appear when the second Cauchy datum g or its Dunkl transform are involved. So the past is determined by the present as well as the future.

Next, we will investigate the symmetries of the propagators under a dilation operator. This will inform us on the degree of the homogeneity of the distributions P_k^{ij}, with $i, j = 1, 2$.

For $\lambda > 0$ and $\psi \in \mathscr{S}(\mathbb{R}^{N+1})$, denote by

$$S_\lambda^x \psi(x,t) := \psi(\lambda x, t), \qquad S_\lambda^t \psi(x,t) := \psi(x, \lambda t),$$

where the superscript denotes the relevant variable. Set $S_\lambda := S_\lambda^x \circ S_\lambda^t$. By duality, the operators S_λ^x, S_λ^t, and S_λ act on distributions in the standard way.

We begin by looking to the symmetry properties of $P_{k,t}^{ij}$ under the dilation S_λ. Observe that if $u_k(x,t)$ is a solution to (3.1) with initial data $(f(x), g(x))$, then $S_\lambda u_k(x,t)$ solves the wave equation with initial data $(S_\lambda^x f(x), \lambda S_\lambda^x g(x))$. Thus

$$S_\lambda U_k(x,t) = P_{k,t} *_k \begin{bmatrix} S_\lambda^x f \\ \lambda S_\lambda^x g \end{bmatrix}. \tag{3.13}$$

On the other hand

$$\begin{aligned} S_\lambda U_k(x,t) = \begin{bmatrix} S_\lambda u_k(x,t) \\ \partial_t\{S_\lambda u_k(x,t)\} \end{bmatrix} &= \begin{bmatrix} u_k(\lambda x, \lambda t) \\ \lambda\{\partial_t u_k\}(\lambda x, \lambda t) \end{bmatrix} \\ &= \begin{bmatrix} u_k \\ \lambda \partial_t u_k \end{bmatrix} (\lambda x, \lambda t) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} u_k \\ \partial_t u_k \end{bmatrix} (\lambda x, \lambda t) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \Big\{ P_{k,\lambda t} *_k \begin{bmatrix} f \\ g \end{bmatrix} \Big\} (\lambda x) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} S_\lambda^x \Big\{ P_{k,\lambda t} *_k \begin{bmatrix} f \\ g \end{bmatrix} \Big\} (x). \end{aligned}$$

Using the fact that if $f_\lambda(x) := \lambda^{\gamma_k + N/2} f(\lambda x)$, then $\mathscr{D}_k(f_\lambda)(\xi) = \lambda^{-\gamma_k - N/2} \mathscr{D}_k(f)(\lambda \xi)$, one can check that S_λ^x preserves the convolution $*_k$. Therefore

$$\begin{aligned} S_\lambda U_k(x,t) &= \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \Big\{ S_\lambda^x P_{k,\lambda t} *_k \begin{bmatrix} S_\lambda^x f \\ S_\lambda^x g \end{bmatrix} \Big\} (x) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \Big\{ S_\lambda^x P_{k,\lambda t} *_k \begin{bmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} S_\lambda^x f \\ \lambda S_\lambda^x g \end{bmatrix} \Big\} (x). \end{aligned} \tag{3.14}$$

Comparing (3.13) with (3.14) gives $S_\lambda^x P_{k,\lambda t}^{ij} = \lambda^{j-i} P_{k,t}^{ij}$, for $i, j = 1, 2$. Now one can obtain the symmetry properties of P_k^{ij} as follows: For $\psi_1 \in \mathscr{S}(\mathbb{R}^N)$ and $\psi_2 \in \mathscr{S}(\mathbb{R})$, we have

$$\begin{aligned} S_\lambda(P_k^{ij})(\psi_1\otimes\psi_2) &= P_k^{ij}(S^x_{\lambda^{-1}}(\psi_1)\otimes S^t_{\lambda^{-1}}(\psi_2)) \\ &= \int_{\mathbb{R}} P^{ij}_{k,t}(S^x_{\lambda^{-1}}(\psi_1))S^t_{\lambda^{-1}}(\psi_2)(t)dt \\ &= \lambda\int_{\mathbb{R}} P^{ij}_{k,\lambda t}(S^x_{\lambda^{-1}}(\psi_1))\psi_2(t)dt \\ &= \lambda\int_{\mathbb{R}} S^x_\lambda(P^{ij}_{k,\lambda t}(\psi_1))\psi_2(t)dt \\ &= \lambda^{1+j-i}\int_{\mathbb{R}} P^{ij}_{k,t}(\psi_1)\psi_2(t)dt \\ &= \lambda^{1+j-i}P_k^{ij}(\psi_1\otimes\psi_2). \end{aligned}$$

We summarize the above computations.

Proposition 3.5. *Let $k\in\mathscr{K}^+$ and $N\geq 1$.*

(i) *The distribution P_k^{ij} satisfies the deformed wave equation, i.e.*

$$\Delta_k P_k^{ij} = \partial_{tt}P_k^{ij}, \qquad i,j=1,2. \tag{3.15}$$

(ii) *If $h\in G$ and τ denotes the operation of time-reflection, then*

$$\pi_x^*(h)P_k^{ij}=P_k^{ij}, \qquad \pi_t^*(\tau)P_k^{ij}=(-1)^{i-j}P_k^{ij}, \qquad i,j=1,2.$$

(iii) *For $\lambda>0$*

$$S_\lambda P_k^{ij} = \lambda^{1+j-i}P_k^{ij}, \qquad i,j=1,2.$$

Next, we will prove similar statements for what we shall call the Dunkl-Fourier transform of P_k^{ij}. For $\psi\in\mathscr{S}(\mathbb{R}^{N+1})$, denote by

$$\mathscr{D}_k\mathscr{F}\psi(x,t) := \int_{\mathbb{R}^{N+1}} \psi(x',t')E_k(x',-ix)e^{itt'}\upsilon_k(x')dx'dt'.$$

For a distribution $\mathcal{T}$ of compact support, we write

$$\mathscr{D}_k\mathscr{F}(\mathcal{T}) = \widetilde{\mathscr{D}_k\mathscr{F}}(\mathcal{T})(x,t)\upsilon_k(x)dxdt,$$

where

$$\widetilde{\mathscr{D}_k\mathscr{F}}(\mathcal{T})(x,t) = \mathcal{T}(E_k(x',-ix)e^{itt'}).$$

Since $E_k(h\cdot x,x')=E_k(x,h^{-1}\cdot x')$, for $h\in G$, and υ_k is G-invariant, then in the light of Proposition 3.5(ii), it follows that

$$\pi_x^*(h)\mathscr{D}_k\mathscr{F}(P_k^{ij}) = \mathscr{D}_k\mathscr{F}(P_k^{ij}), \qquad \text{for all } \ h\in G,$$

and

$$\pi_t^*(\tau)\mathscr{D}_k\mathscr{F}(P_k^{ij}) = (-1)^{i-j}\mathscr{D}_k\mathscr{F}(P_k^{ij}).$$

A crucial observation regarding $\mathscr{D}_k\mathscr{F}(P_k^{ij})$ is that

$$(\|x\|^2-t^2)\mathscr{D}_k\mathscr{F}(P_k^{ij})=0, \qquad i,j=1,2. \tag{3.16}$$

This follows by taking the Dunkl-Fourier transform of (3.15) together with the fact that $\mathscr{D}_k\mathscr{F}(\Delta_k\psi)(x,t)=-\|x\|^2\mathscr{D}_k\mathscr{F}(\psi)(x,t)$ and $\mathscr{D}_k\mathscr{F}(\partial_{tt}\psi)(x,t)=-t^2\mathscr{D}_k\mathscr{F}(\psi)(x,t)$. Equation (3.16) says the distribution $\mathscr{D}_k\mathscr{F}(P_k^{ij})$ is supported on the light cone $\mathscr{C}=\left\{(x,t)\in\mathbb{R}^{N+1}\mid \|x\|-t^2=0\right\}$, for $i,j=1,2$.

Consider now the symmetry property of $\mathscr{D}_k\mathscr{F}(P_k^{ij})$. In view of Proposition 3.5(iii) and the fact that $E_k(\lambda x, x') = E_k(x, \lambda x')$, we have

$$\begin{aligned} S_\lambda\left[\mathscr{D}_k\mathscr{F}(P_k^{ij})\right] &= S_\lambda\left[\widetilde{\mathscr{D}_k\mathscr{F}}(P_k^{ij})(x,t)\upsilon_k(x)dxdt\right] \\ &= S_\lambda\left[\widetilde{\mathscr{D}_k\mathscr{F}}(P_k^{ij})\right](x,t)S_\lambda\left[\upsilon_k(x)dxdt\right] \\ &= \lambda^{2\gamma_k+N+1}\widetilde{\mathscr{D}_k\mathscr{F}}(P_k^{ij})(\lambda x,\lambda t)\upsilon_k(x)dxdt \\ &= \lambda^{2\gamma_k+N+1}P_k^{ij}(E_k(\lambda x,-ix')e^{i\lambda tt'})\upsilon_k(x)dxdt \\ &= \lambda^{2\gamma_k+N+1}P_k^{ij}(E_k(x,-i\lambda x')e^{it\lambda t'})\upsilon_k(x)dxdt \\ &= \lambda^{2\gamma_k+N+1}P_k^{ij}(S_\lambda\left[E_k(x,-ix')e^{itt'}\right])\upsilon_k(x)dxdt \\ &= \lambda^{2\gamma_k+N+1}\mathscr{D}_k\mathscr{F}(S_{\lambda^{-1}}P_k^{ij}) \\ &= \lambda^{2\gamma_k+N+i-j}\mathscr{D}_k\mathscr{F}(P_k^{ij}). \end{aligned}$$

Similarly to Proposition 3.5, we get:

Proposition 3.6. *Let $k \in \mathscr{K}^+$ and $N \geq 1$.*

(i) *The distribution $\mathscr{D}_k\mathscr{F}(P_k^{ij})$ is supported on the light cone $\mathscr{C}$, i.e.*

$$(\|x\|^2 - t^2)\mathscr{D}_k\mathscr{F}(P_k^{ij}) = 0, \qquad i,j = 1,2.$$

(ii) *If $h \in G$ and τ denotes the operation of time-reflection, then*

$$\pi_x^*(h)\mathscr{D}_k\mathscr{F}(P_k^{ij}) = \mathscr{D}_k\mathscr{F}(P_k^{ij}), \quad \pi_t^*(\tau)\mathscr{D}_k\mathscr{F}(P_k^{ij}) = (-1)^{i-j}\mathscr{D}_k\mathscr{F}(P_k^{ij}), \quad i,j = 1,2.$$

(iii) *For $\lambda > 0$*

$$S_\lambda\left[\mathscr{D}_k\mathscr{F}(P_k^{ij})\right] = \lambda^{2\gamma_k+N+i-j}\mathscr{D}_k\mathscr{F}(P_k^{ij}), \qquad i,j = 1,2.$$

Next we shall describe the structure of a representation of the universal covering group $\widetilde{SL(2,\mathbb{R})}$ of $SL(2,\mathbb{R})$ on $\mathscr{S}(\mathbb{R}^{N+1})$. This structure, together with Proposition 3.5 and Proposition 3.6, allows to prove that the Cauchy problem (3.1) satisfies the strict Huygens principle, under a condition involving N and k. We adapt the method of R. Howe for the classical wave equation, i.e. when $k \equiv 0$ (cf. [24, 27]).

Choose $x_1, x_2, \ldots, x_N$ as the usual system of coordinates on $\mathbb{R}^N$. Let

$$\mathbb{E}_{N,1} := \frac{1}{2}(\|x\|^2 - t^2), \quad \mathbb{F}_{N,1} := -\frac{1}{2}(\Delta_k - \partial_{tt}), \quad \mathbb{H}_{N,1} := \frac{N+1}{2} + \gamma_k + \sum_{j=1}^{N} x_j\partial_j + t\partial_t.$$

Using [21, Theorem 3.3], the following commutation relations hold

$$[\mathbb{E}_{N,1}, \mathbb{H}_{N,1}] = -2\mathbb{E}_{N,1}, \quad [\mathbb{F}_{N,1}, \mathbb{H}_{N,1}] = 2\mathbb{F}_{N,1}, \quad [\mathbb{E}_{N,1}, \mathbb{F}_{N,1}] = \mathbb{H}_{N,1}. \tag{3.17}$$

These are the commutation relations of a standard basis of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$. Equation (3.17) gives rise to a representation Ω_k of $\mathfrak{sl}(2,\mathbb{R})$. On $\mathscr{S}(\mathbb{R}^{N+1})$, the representation Ω_k can be described as

$$\Omega_k(\mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}}) = \mathfrak{sl}_2^+ \oplus \mathfrak{sl}_2^0 \oplus \mathfrak{sl}_2^-, \tag{3.18}$$

where

$$\mathfrak{sl}_2^+ = \mathrm{Span}\{\mathbb{E}_{N,1}\}, \quad \mathfrak{sl}_2^0 = \mathrm{Span}\{\mathbb{H}_{N,1}\}, \quad \mathfrak{sl}_2^- = \mathrm{Span}\{\mathbb{F}_{N,1}\}.$$

The decomposition (3.18) is an instance of the Cartan decomposition

$$\mathfrak{sl}(2,\mathbb{R})_{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}^-,$$

where $\mathfrak{sl}_2^+ \simeq \Omega_k(\mathfrak{p}^+)$, $\mathfrak{sl}_2^0 \simeq \Omega_k(\mathfrak{k}_{\mathbb{C}})$, and $\mathfrak{sl}_2^- \simeq \Omega_k(\mathfrak{p}^-)$. Here $\mathfrak{k} = \mathfrak{u}(1)$, the Lie algebra of the compact group $U(1)$. The integrated form of the Lie algebra representation Ω_k is an analogue of the metaplectic representation of the universal covering $\widetilde{SL(2,\mathbb{R})}$ of the group $SL(2,\mathbb{R})$. If $(N+1)/2 + \gamma_k \in \frac{1}{2}\mathbb{Z} \setminus \mathbb{Z}$, we obtain a representation of the double covering $Mp(2,\mathbb{R})$ of $SL(2,\mathbb{R})$, and if $(N+1)/2 + \gamma_k \in \mathbb{Z}$ we obtain a representation of $SL(2,\mathbb{R})$.

Remark 3.7. Following [8], we may rewrite the Dunkl-Fourier transform as

$$\mathscr{D}_k\mathscr{F} = e^{i\frac{\pi}{2}(\gamma_k+(N+1)/2)} e^{-i\frac{\pi}{2}(\mathbb{E}_{N,1}+\mathbb{F}_{N,1})}.$$

That is, up to a scalar factor, $\mathscr{D}_k\mathscr{F}$ is an element of the integrated form of the representation Ω_k, given by the formulas above.

Recall that $(\mathcal{S}_2)$ is equivalent to the fact that the propagators P_k^{11} and P_k^{12} are supported on the light cone $\mathscr{C} = \{(x,t) \in \mathbb{R}^N \times \mathbb{R} \mid \|x\|^2 - t^2 = 0\}$. Next we will present our argument for the P_k^{ij}'s with $i,j = 1,2$. Since $\mathscr{C}$ is the locus of zeros of $\|x\|^2 - t^2$, then P_k^{ij} is supported on $\mathscr{C}$ if and only if

$$\mathbb{E}_{N,1}^m \cdot P_k^{ij} = 0 \tag{3.19}$$

for some positive integer m, or

$$\mathbb{F}_{N,1}^m \cdot \mathscr{D}_k\mathscr{F}(P_k^{ij}) = 0 \tag{3.20}$$

for some positive integer m (P_k^{ij} and $\mathscr{D}_k\mathscr{F}(P_k^{ij})$ are distributions of finite order. See, for instance, [43]). In the light of Proposition 3.5(i) (or Proposition 3.6(i)) together with homogeneity of P_k^{ij} (or $\mathscr{D}_k\mathscr{F}(P_k^{ij})$), i.e. it is a weight vector for $\mathbb{H}_{N,1}$, the equation (3.19) (or (3.20)) amounts to saying the distribution P_k^{ij} (or $\mathscr{D}_k\mathscr{F}(P_k^{ij})$) generates a finite-dimensional $\Omega_k^*(\mathfrak{sl}(2,\mathbb{R}))$-module. Thus, the qualitative part of the strict Huygens principle holds.

Theorem 3.8. *The strict Huygens principle holds if and only if P_k^{ij} (or $\mathscr{D}_k\mathscr{F}(P_k^{ij})$) is supported on the light cone $\mathscr{C}$, if and only if P_k^{ij} (or $\mathscr{D}_k\mathscr{F}(P_k^{ij})$) generates a finite-dimensional $\Omega_k^*(\mathfrak{sl}(2,\mathbb{R}))$-module. In this case, P_k^{ij} and $\mathscr{D}_k\mathscr{F}(P_k^{ij})$ belong to the same module.*

Claim 3.9. The strict Huygens principle cannot hold when

$$\frac{N+1}{2} + \gamma_k \notin \mathbb{Z}.$$

To prove the claim, we need the following branching decomposition of $\mathscr{S}(\mathbb{R}^N)$ under the action of $G \times \widetilde{SL(2,\mathbb{R})}$. Those readers who are familiar with the theory of Howe reductive dual pairs [24, 25] will find that our formulation can be thought of as an analogue of Howe's theory.

Recall that $x_1, \dots, x_N$ denotes the usual system of coordinates on $\mathbb{R}^N$. Set

$$\mathscr{H}_k := \frac{N}{2} + \gamma_k + \sum_{j=1}^{N} x_j \partial_j,$$
$$E := \frac{\mathscr{H}_k - \Delta_k/4 - \|x\|^2}{2}, \quad F := \frac{\mathscr{H}_k + \Delta_k/4 + \|x\|^2}{2}, \quad H := -\frac{\Delta_k}{4} + \|x\|^2.$$

Using again [21, Theorem 3.3], we can derive the following $\mathfrak{sl}(2,\mathbb{R})$-commutation relations

$$[E, H] = -2E, \qquad [F, H] = 2F, \qquad [E, F] = H. \tag{3.21}$$

What makes $\{E, F, H\}$ important is the fact that H is the infinitesimal generator of the maximal compact subgroup $SO(2,\mathbb{R})$ of $SL(2,\mathbb{R})$. Observe that $E^* = -F$ and $H^* = H$ in $L^2(\mathbb{R}^N, \upsilon_k(x)dx)$. This is a consequence of the fact that Δ_k is symmetric, while $\mathscr{H}_k^* = -\mathscr{H}_k$ as the below verification shows (you may require $k_\alpha \geq 1$, and after the formula is established, the restriction can be dropped, i.e. back to $k_\alpha \geq 0$, by analytic continuation)

$$\begin{aligned}\int_{\mathbb{R}^N} \mathscr{H}_k f(x) g(x) \upsilon_k(x) dx &= -\int_{\mathbb{R}^N} f(x) \Big\{ \sum_{j=1}^{N} x_j \partial_j g(x) \Big\} \upsilon_k(x) dx \\ &\quad + \Big(\gamma_k - \frac{N}{2}\Big) \int_{\mathbb{R}^N} f(x) g(x) \upsilon_k(x) dx \\ &\quad - \int_{\mathbb{R}^N} f(x) g(x) \Big\{ \sum_{j=1}^{N} x_j \partial_j \upsilon_k(x) \Big\} dx,\end{aligned}$$

where $\sum_{j=1}^N x_j \partial_j \upsilon_k(x) = 2\gamma_k \upsilon_k(x)$, since υ_k is homogeneous of degree $2\gamma_k$. Equation (3.21), together with the observation above, gives rise to an infinitesimally unitary representation ω_k of $\mathfrak{sl}(2,\mathbb{R})$. Similarly as for Ω_k, we may describe this representation as

$$\omega_k(\mathfrak{p}^+) = \mathrm{Span}\{E\}, \quad \omega_k(\mathfrak{k}_{\mathbb{C}}) = \mathrm{Span}\{H\}, \quad \omega_k(\mathfrak{p}^-) = \mathrm{Span}\{F\}.$$

Here $\mathfrak{k} = \mathfrak{so}(2)$, the Lie algebra of the compact group $SO(2,\mathbb{R})$.

For $h \in G$, denote by $\pi(h)$ the action of G on $\mathscr{S}(\mathbb{R}^N)$

$$\pi(h) f(x) = f(h^{-1}x).$$

The actions of G and $\mathfrak{sl}(2,\mathbb{R})$ on $\mathscr{S}(\mathbb{R}^N)$ commute.

To investigate the structure of the representation ω_k, note that for a polynomial $p \in \mathscr{P}(\mathbb{R}^N)$

$$e^{\nu\|x\|^2} p(-T_\xi(k)) e^{-\nu\|x\|^2} = p(2\nu\langle \xi, \cdot\rangle - T_\xi(k)), \qquad \text{for } \nu \in \mathbb{R}.$$

This follows from the product rule (2.1). In particular, if $p(x) = \sum_{j=1}^N x_j^2$, we obtain

$$e^{\nu\|x\|^2} \Delta_k e^{-\nu\|x\|^2} = 4\|x\|^2 + \Delta_k - 4\nu \mathscr{H}_k, \qquad \text{for } \nu \in \mathbb{R}.$$

Thus, we may rewrite the $\mathfrak{sl}(2)$-triple $\{E, F, H\}$ as

$$E = -\frac{1}{8} e^{\|x\|^2} \Delta_k e^{-\|x\|^2}, \tag{3.22}$$

$$F = \frac{1}{8} e^{-\|x\|^2} \Delta_k e^{\|x\|^2}, \tag{3.23}$$

$$H = e^{-\|x\|^2} \Big(-\frac{\Delta_k}{4} + \mathscr{H}_k \Big) e^{\|x\|^2}. \tag{3.24}$$

Next we shall investigate the lowest weight modules for the $\mathfrak{sl}(2)$-triple $\{E, F, H\}$. According to (3.23), the kernel of F consists of functions of the form $e^{-\|x\|^2} h(x)$ where h is harmonic, i.e. $\Delta_k h = 0$. Now by (3.24), we get $H(e^{-\|x\|^2} h(x)) = e^{-\|x\|^2} \mathscr{H}_k h(x)$. Thus, $e^{-\|x\|^2} h(x)$ is an eigenvector for H if and only if h is a homogeneous polynomial. In conclusion, h is a harmonic homogeneous polynomial. Further, if h has degree m, then

$$H\big(e^{-\|x\|^2} h(x)\big) = (m + \frac{N}{2} + \gamma_k) e^{-\|x\|^2} h(x).$$

Henceforth, for $m \in \mathbb{N}$, we set $\mathscr{H}_m(k)$ to be the space of harmonic homogeneous polynomials on $\mathbb{R}^N$ of degree m.

On the other hand, the vectors $v_s := E^s\big(e^{-\|x\|^2} h_m(x)\big)$, with $s \in \mathbb{N}$, are eigenvectors for H with eigenvalues $N/2 + \gamma_k + m + 2s$. Further, the vectors v_s form an orthonormal basis for the space of the representation. Denote by $\mathscr{W}_{N/2+\gamma_k+m}$ the $\mathfrak{sl}(2, \mathbb{R})$-representation with lowest weight $N/2 + \gamma_k + m$. Moreover, for $\psi \in \mathscr{S}(\mathbb{R}^+)$ and $h_m \in \mathscr{H}_m(k)$, one can check that

$$\mathscr{H}_k\Big(h_m(x)\psi(\|x\|^2)\Big) = \big\{(m + N/2 + \gamma_k)\psi(\|x\|^2) + 2\|x\|^2\psi'(\|x\|^2)\big\}\, h_m(x),$$

$$\Delta_k\Big(h_m(x)\psi(\|x\|^2)\Big) = 4\,\big\{\|x\|^2\psi''(\|x\|^2) + (m + N/2 + \gamma_k)\psi'(\|x\|^2)\big\}\, h_m(x).$$

Thus, for every $s \in \mathbb{N}$, E^s leaves the set $\mathscr{I} h_m := \{\psi(\|\cdot\|^2) h_m \mid \psi \in \mathscr{S}(\mathbb{R}^+)\}$ invariant. In particular, the vectors v_s belong to the space $e^{-\|x\|^2} \mathscr{P}(\mathbb{R}^N)$, which is dense in $\mathscr{S}(\mathbb{R}^N)$.

We summarize the consequences of the above computations.

Theorem 3.10. *Assume that $k \in \mathscr{K}^+$ and $N \geq 1$. Let $\mathfrak{k} = \mathfrak{so}(2)$, as before.*

(i) *The direct sum $\sum^{\oplus}_{m \in \mathbb{Z}^+} \mathscr{H}_m(k) \cdot \mathscr{I}(\mathbb{R}^N)$, where $\mathscr{I}(\mathbb{R}^N)$ denotes the space of $O(N)$-invariant Schwartz functions on $\mathbb{R}^N$, is dense in $\mathscr{S}(\mathbb{R}^N)$.*

(ii) *As a $G \times \mathfrak{sl}(2, \mathbb{R})$-module, the $G \times \mathfrak{k}$-finite vectors in the Schwartz space admit the following multiplicity-free decomposition*

$$\mathscr{S}(\mathbb{R}^N)_{G \times \mathfrak{k}} = \bigoplus_{m \in \mathbb{Z}^+} \widetilde{\mathscr{H}_m}(k) \otimes \mathscr{W}_{m + \frac{N}{2} + \gamma_k},$$

where $\mathscr{W}_{m+\frac{N}{2}+\gamma_k}$ is the $\mathfrak{sl}(2, \mathbb{R})$-representation of lowest weight $m + \frac{N}{2} + \gamma_k$, and $\widetilde{\mathscr{H}_m}(k) := e^{-\|x\|^2} \mathscr{H}_m(k)$. The summands are mutually orthogonal with respect to the inner product on $L^2(\mathbb{R}^N, \upsilon_k(x)dx)$. The representation $\mathscr{W}_{m+\frac{N}{2}+\gamma_k}$ integrates to an irreducible unitary representation of the universal covering $\widetilde{SL}(2, \mathbb{R})$.

Remark 3.11. The decomposition in (ii) could just as well be formulated for $L^2(\mathbb{R}^N, \upsilon_k(x)dx)$ as for the Schwartz space.

The following is then immediate.

Corollary 3.12. *Under the action of* $\mathfrak{sl}(2,\mathbb{R})$*, the* $\mathfrak{k}$*-finite vectors in the Schwartz space* $\mathscr{S}(\mathbb{R}^N)$ *decompose as*

$$\mathscr{S}(\mathbb{R}^N)_{\mathfrak{k}} = \bigoplus_{m\in\mathbb{Z}^+} \dim(\widetilde{\mathscr{H}_m}(k))\mathscr{W}_{m+\frac{N}{2}+\gamma_k},$$

where $\dim(\widetilde{\mathscr{H}_m}(k)) = \begin{pmatrix} m+N-1 \\ N-1 \end{pmatrix} - \begin{pmatrix} m+N-3 \\ N-1 \end{pmatrix}$. *If* $N > 1$, $\dim(\widetilde{\mathscr{H}_m}(k))$ *is always nonzero, but if* $N=1$*, it is zero for* $m \geq 2$.

Clearly now the Claim 3.9 holds, since the spectrum of $\omega_k(\mathfrak{k})$ (or its dual) acting on $\mathscr{S}(\mathbb{R}^{N+1})$ (or $\mathscr{S}'(\mathbb{R}^{N+1})$) is $(N+1)/2+\gamma_k+\mathbb{Z}^+$, whilst the spectrum of $\omega_k(\mathfrak{k})$ (or its dual) in finite dimensional modules is contained in $\mathbb{Z}$. Thus, the following is proved.

Theorem 3.13. *The strict Huygens principle cannot hold when*

$$\frac{N+1}{2}+\gamma_k \notin \mathbb{Z}.$$

The above theorem leaves the likelihood that the modified wave equation may satisfies Huygens' principle when $(N+1)/2+\gamma_k \in \mathbb{Z}$.

Using Proposition 3.5(iii) and Proposition 3.6(iii), we have

$$\begin{cases} \Big\{\sum_{\ell=1}^{N} x_\ell\partial_\ell + t\partial_t\Big\} P_k^{ij} = (1+j-i)P_k^{ij}, \\ \Big\{\sum_{\ell=1}^{N} x_\ell\partial_\ell + t\partial_t\Big\} \mathscr{D}_k\mathscr{F}(P_k^{ij}) = (2\gamma_k+N+i-j)\mathscr{D}_k\mathscr{F}(P_k^{ij}), \end{cases} \qquad i,j=1,2,$$

and therefore

$$\begin{cases} \mathbb{H}_{N,1}P_k^{ij} = -\left(\dfrac{N+1}{2}+\gamma_k+i-j-1\right)P_k^{ij}, \\ \mathbb{H}_{N,1}\mathscr{D}_k\mathscr{F}(P_k^{ij}) = \left(\dfrac{N+1}{2}+\gamma_k+i-j-1\right)\mathscr{D}_k\mathscr{F}(P_k^{ij}), \end{cases} \qquad i,j=1,2.$$

Thus, if we assume $(N-1)/2+\gamma_k+i-j\in\mathbb{N}$, with $i,j=1,2$, and keeping in mind that

$$\mathbb{F}_{N,1}\cdot P_k^{ij} = 0 \qquad \text{and} \qquad \mathbb{E}_{N,1}\cdot \mathscr{D}_k\mathscr{F}(P_k^{ij}) = 0,$$

we can conclude that each distribution P_k^{ij}, with $i,j=1,2$, generates a finite-dimensional $\Omega_k^*(\mathfrak{sl}(2,\mathbb{R}))$ on $\mathscr{S}'(\mathbb{R}^{N+1})$ of highest weight $(N-1)/2+\gamma_k+i-j$. It is worthwhile to recall that for a finite-dimensional representation $\mathbb{V}$ of $SL(2,\mathbb{R})$, the operator $\mathbb{F}_{N,1}^{(\dim\mathbb{V}-1)}$ converts a highest weight vector to a lowest weight, up to a constant [20, 46]. We now summarize all the above computations and discussions.

Proposition 3.14. *Under the assumption*

$$\frac{N-1}{2}+\gamma_k+i-j\in\mathbb{N}, \tag{3.25}$$

the tempered distribution P_k^{ij} *generates an* $\mathfrak{sl}(2,\mathbb{R})$*-module of dimension*

$$d_{i,j}(k) = \frac{N-1}{2}+\gamma_k+i-j+1, \qquad i,j=1,2,$$

with highest weight vector $\mathscr{D}_k\mathscr{F}(P_k^{ij})$ *of highest weight* $\left(\frac{N-1}{2}+\gamma_k+i-j\right)$. *Further, for each* i *and* j, *there exists a constant* $\alpha_{i,j}$ *such that*

$$P_k^{ij}=\alpha_{i,j}\mathbb{F}_{N,1}^{d_{i,j}(k)-1}\cdot\mathscr{D}_k\mathscr{F}(P_k^{ij}),$$

which is equivalent to

$$\mathscr{D}_k\mathscr{F}(P_k^{ij})=(-1)^{(N-1)/2+\gamma_k}\alpha_{i,j}\mathbb{E}_{N,1}^{d_{i,j}(k)-1}\cdot P_k^{ij}.$$

By taking into account the condition (3.25) for both P_k^{11} and P_k^{12}, we obtain:

Theorem 3.15. (Strict Huygens' Principle) *Assume that* $k\in\mathscr{K}^+$ *and* $N\geq 1$. *For a given* $x\in\mathbb{R}^N$, *the solution* $u_k(x,t)$ *to the Cauchy problem (3.1) depends only on the values of* $\tau_x(k)f(y)$ *and* $\tau_x(k)g(y)$ *(and their derivatives) for* $\|y\|=|t|$ *if and only if*

$$\frac{N-3}{2}+\gamma_k\in\mathbb{N}.$$

Remark 3.16. By now one can see that the representation theory of the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ can be used as a crucial (and surprising) tool to investigate problems in harmonic analysis. The paper [9] contains two other applications of the representation ω_k to analysis. The first application deals with a Bochner-type formula for the Dunkl transform. The second application releases the connection between the Fourier analysis on an arbitrary flat symmetric space $\mathfrak{p}$ and the Dunkl theory on a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. In particular, we show how the Bessel function $F(k,x,y):=\frac{1}{|W|}\sum_{w\in W}E_k(wx,y)$ is connected to the restriction of the spherical functions on $\mathfrak{p}$ to $\mathfrak{a}$. Here W denotes the Weyl group associated with $\mathfrak{a}$. This latter fact was proved earlier by de Jeu [30], using a different approach. The basis for all of these applications is that the Dunkl transform belongs to the integrated form of our metaplectic-type representation.

Now, let us consider the following Cauchy problem

(3.26)
$$\Delta_k u_k(x,t)=\partial_{tt}u_k(x,t),\qquad u_k(x,0)=f(x),\quad \partial_t u_k(x,0)=g(x),\quad f,g\in\mathscr{C}_R^\infty(\mathbb{R}^N),$$

where $\mathscr{C}_R^\infty(\mathbb{R}^N)$ stands for the set of smooth functions with support contained in the closed ball of radius $R>0$ about the origin. In these circumstances, Theorem 3.15 reads:

Theorem 3.17. *Assume that* $k\in\mathscr{K}^+$ *and* $N\geq 1$. *For all possible initial data* $f,g\in\mathscr{C}_R^\infty(\mathbb{R}^N)$, *the support of the solution* $u_k(x,t)$ *to the Cauchy problem (3.26) is contained in the conical shell*

$$\mathscr{C}=\left\{(x,t)\in\mathbb{R}^N\times\mathbb{R}\mid |t|-R\leq\|x\|\leq|t|+R\right\}\tag{3.27}$$

if and only if

$$(N-3)/2+\gamma_k\in\mathbb{N}.$$

The shell $\mathscr{C}$ *is the union*

$$\bigcup_{\|y\|\leq R}\mathscr{C}_y\tag{3.28}$$

where $\mathscr{C}_y$ *is the light cone*

$$\mathscr{C}_y=\left\{(x,t)\in\mathbb{R}^N\times\mathbb{R}\mid \|x-y\|=|t|\right\}.$$

We start with the proof of the right hand side inequality in (3.27). Recall that $\mathscr{D}_k(\tau_{\mathrm{y}}(k)f)(\xi) = E_k(\mathrm{y}, -i\xi)\mathscr{D}_k(f)(\xi)$. Using the fact that $|E_k(\mathrm{y}, -i\xi)| \leq \sqrt{|G|}e^{\|\mathrm{y}\|\,\|\mathrm{Im}(\xi)\|}$, and the Paley-Wiener Theorem 2.5(i) for the function f, we deduce that for each $M \in \mathbb{N}$ there exists a constant α_M such that the entire function $\xi \mapsto \mathscr{D}_k(\tau_{\mathrm{y}}(k)f)(\xi)$ satisfies

$$|\mathscr{D}_k(\tau_{\mathrm{y}}(k)f)(\xi)| \leq \alpha_M(1 + \|\xi\|)^{-M}e^{\|\mathrm{Im}(\xi)\|(R+\|\mathrm{y}\|)}.$$

Thus, $\tau_{\mathrm{y}}(k)f$ is supported in the closed ball of radius $R + \|\mathrm{y}\|$ about the origin. Similarly for $\tau_{\mathrm{y}}(k)g$. In view of Theorem 3.3, we conclude that for all $k \in \mathscr{K}^+$ and $N \geq 1$, the support of the solution $u_k(x,t)$ to (3.26) is contained in the set $\{\ (x,t) \mid \|x\| \leq R + |t|\ \}$. Next, we will prove the left hand side inequality in (3.27), which holds only if $(N-3)/2 + \gamma_k \in \mathbb{N}$. By Theorem 3.15, the solution $u_k(0,t)$ depends only the values of $f(y)$ and $g(y)$ for $\|y\| = |t|$. That is

$$u_k(0,t) = 0 \qquad \text{for } |t| > R. \tag{3.29}$$

We write $\tau_{\mathrm{y}}^{\vee}(k)f(x)$ for $\tau_{\mathrm{y}}(k)f(-x)$. If $k \equiv 0$, then $\tau_{\mathrm{y}}^{\vee}(0)f(x) = f(\mathrm{y}+x)$. One can check that $\tau_{\mathrm{y}}^{\vee}(k)$ commutes with $\Delta_k - \partial_{tt}$. Thus, if $u_k(x,t)$ is a solution to the Cauchy problem (3.26) with the Cauchy data (f,g), then $\tau_{\mathrm{y}}^{\vee}(k)u_k(x,t)$ solves (3.26) with initial data $(\tau_{\mathrm{y}}^{\vee}(k)f, \tau_{\mathrm{y}}^{\vee}(k)g)$. Since $\tau_{\mathrm{y}}^{\vee}(k)f$ and $\tau_{\mathrm{y}}^{\vee}(k)g$ have support contained in $\overline{B(\mathbf{0}, R + \|\mathrm{y}\|)}$, (3.29) implies that $\tau_{\mathrm{y}}^{\vee}(k)u_k(0,t) = 0$ for $|t| > R + \|\mathrm{y}\|$, i.e.

$$u_k(\mathrm{y},t) = 0 \qquad \text{for } |t| > R + \|\mathrm{y}\|.$$

Finally, the set (3.27) coincides with the union (3.28) since: if $(x,t) \in \mathscr{C}_y$ with $\|y\| \leq R$, then $\|x-y\| = |t|$ so $\|x\| \leq \|x-y\| + \|y\| \leq |t| + R$ and $|t| = \|x-y\| \leq \|x\| + R$, implies (3.27). Conversely, if (x,t) satisfies (3.27), then $(x,t) \in \mathscr{C}_y$ with $y = x - |t|\frac{x}{\|x\|} = \frac{x}{\|x\|}(\|x\| - |t|)$ which has norm less than or equal to R.

However, we can prove Theorem 3.17 by using a different approach involving only the Paley-Wiener Theorem 2.5(i). We shall sketch this approach at the end of this section, and its details will be illustrated in the next section to prove the principle of energy equipartition.

Now, let us go back to the Cauchy problem (3.1) where the Cauchy data $(f,g) \in \mathscr{S}(\mathbb{R}^N) \times \mathscr{S}(\mathbb{R}^N)$. It is natural to think about some connection between solutions to wave equations and spherical mean type operators. As in the classical case, we shall express the solution u_k to (3.1) in terms of what is commonly called the Dunkl-type spherical mean operator.

In [34], the authors defined the Dunkl-type spherical mean operator $f \mapsto M_f$ on $\mathscr{C}^\infty(\mathbb{R}^N)$ by

$$M_f(x,r) := \frac{1}{d_k}\int_{S^{N-1}} \tau_x(k)f(ry)\upsilon_k(y)d\omega(y), \qquad x \in \mathbb{R}^N, r \geq 0,$$

where $d_k := \int_{S^{N-1}} \upsilon_k(x)d\omega(x)$. According to [38, Theorem 4.1], there exists a unique compactly supported probability measure $\sigma_{x,r}^k$ such that

$$M_f(x,r) = \int_{\mathbb{R}^N} f(\xi)d\sigma_{x,r}^k(\xi),$$

and

$$\mathrm{supp}(\sigma_{x,r}^k) \subseteq \bigcup_{h\in G}\{\xi \in \mathbb{R}^N \mid \|\xi - hx\| \leq r\}.$$

A sharper statement on the support of $\sigma^k_{x,r}$ is given in [38, Corollary 5.2]

$$\operatorname{supp}(\sigma^k_{x,r}) \subseteq \{\xi \in \mathbb{R}^N \mid \|\xi\| \geq \big|\|x\| - r\big|\}. \tag{3.30}$$

Before expressing the solution u_k in terms of the spherical mean operator, let us recall few known facts about the Riemann-Liouville distributions on the real line [17].

Let $\Lambda = \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) > 0\}$. Consider the locally integrable function on $\mathbb{R}$ defined for $\lambda \in \mathbb{C}$ by

$$x_+^{\lambda-1} := \begin{cases} x^{\lambda-1} & x > 0, \\ 0 & x \leq 0. \end{cases}$$

For $\psi \in \mathscr{D}(\mathbb{R})$, the corresponding regular distribution

$$\langle x_+^{\lambda-1}, \psi \rangle = \int_0^\infty x^{\lambda-1}\psi(x)dx$$

is a holomorphic $\mathscr{D}'(\mathbb{R})$-valued function with respect to the variable $\lambda \in \Lambda$. It admits an analytic continuation into the domain $\Lambda' = \{\lambda \in \mathbb{C} \mid \lambda \neq 0, 1, 2, 3, \ldots\}$, where

$$\mathrm{Res}_{\lambda \to m}\, x_+^{\lambda-1} = \frac{(-1)^m}{m!}\delta^{(m)}(x), \qquad \text{for } m = 0, 1, 2, 3, \ldots.$$

To eliminate these poles, one can divide $x_+^{\lambda-1}$ by $\Gamma(\lambda)$. Therefore, we may define an entire $\mathscr{D}'(\mathbb{R})$-valued function by

$$\mathbb{C} \ni \lambda \mapsto \mathbb{S}_\lambda(x) := \frac{x_+^{\lambda-1}}{\Gamma(\lambda)} \in \mathscr{D}'(\mathbb{R}).$$

This distribution is nowadays known as the Riemann-Liouville distribution. In particular

$$\begin{aligned} &\mathbb{S}_{-m}(x) = \delta^{(m)}(x), \qquad \text{for all } m = 0, 1, 2, 3, \ldots \\ &\frac{d}{dx}\mathbb{S}_\lambda(x) = \mathbb{S}_{\lambda-1}(x). \end{aligned} \tag{3.31}$$

Next, we turn our attention to the relation between u_k and the spherical mean operator. By Theorem 3.1, we know that

$$u_k(x,t) = \int_{\mathbb{R}^N} P^{11}_{k,t}(y)\tau_x(k)f(y)\upsilon_k(y)dy + \int_{\mathbb{R}^N} P^{12}_{k,t}(y)\tau_x(k)g(y)\upsilon_k(y)dy. \tag{3.32}$$

Since $P^{ij}_{k,-t} = (-1)^{i-j}P^{ij}_{k,t}$, we shall present proofs valid for $t > 0$, and make the suitably altered statements for $t \in \mathbb{R}$ without further proof.

By [40], if $F(x) = F_0(\|x\|)$ where $F_0 : \mathbb{R}^+ \to \mathbb{C}$, then $\mathscr{D}_k F(\xi) = \mathcal{H}_{\gamma_k+N/2-1}F_0(\|\xi\|)$, where $\mathcal{H}_\alpha$ denotes the Hankel transform defined by

$$\mathcal{H}_\alpha F_0(r) := \frac{1}{2^\alpha\Gamma(\alpha+1)}\int_0^\infty F_0(s)\frac{J_\alpha(rs)}{(rs)^\alpha}s^{2\alpha+1}ds.$$

Here J_α denotes the Bessel function of the first kind. Thus, in terms of the spherical mean operator, we may rewrite (3.32) as

$$\begin{aligned} u_k(x,t) &= \int_0^\infty r^{2\gamma_k+N-1}\int_{S^{N-1}} P^{11}_{k,t}(ry')\tau_x(k)f(ry')\upsilon_k(y')d\omega(y')dr \\ &\quad + \int_0^\infty r^{2\gamma_k+N-1}\int_{S^{N-1}} P^{12}_{k,t}(ry')\tau_x(k)g(ry')\upsilon_k(y')d\omega(y')dr \\ &= d_k\int_0^\infty r^{2\gamma_k+N-1}\mathcal{H}_{\gamma_k+N/2-1}F_t(r)M_f(x,r)dr \\ &\quad + d_k\int_0^\infty r^{2\gamma_k+N-1}\mathcal{H}_{\gamma_k+N/2-1}G_t(r)M_g(x,r)dr, \end{aligned}$$

where $F_t(s) = \cos(ts)$ and $G_t(s) = \sin(ts)/s$. On the other hand, we have

$$\begin{aligned} \mathcal{H}_\alpha F_t(r) &= \frac{1}{2^\alpha\Gamma(\alpha+1)r^\alpha}\int_0^\infty \cos(ts)J_\alpha(rs)s^{\alpha+1}ds \\ &= \begin{cases} \dfrac{2\sqrt{\pi}}{\Gamma(\alpha+1)}t\dfrac{(t^2-r^2)^{-\alpha-\frac{3}{2}}}{\Gamma(-\alpha-\frac{1}{2})} & \text{if } 0<r<t \\ 0 & \text{if } 0<t<r \end{cases} \qquad \text{(cf. [16, p. 32, formula (4)])} \\ &= \frac{2\sqrt{\pi}}{\Gamma(\alpha+1)}t\mathbb{S}_{-\alpha-\frac{1}{2}}(t^2-r^2) \\ &= \frac{\sqrt{\pi}}{\Gamma(\alpha+1)}\frac{d}{dt}\left(\mathbb{S}_{-\alpha+\frac{1}{2}}(t^2-r^2)\right). \end{aligned}$$

Similarly for G_t, we have

$$\begin{aligned} \mathcal{H}_\alpha G_t(r) &= \begin{cases} \dfrac{\sqrt{\pi}}{\Gamma(\alpha+1)}\dfrac{(t^2-r^2)^{-\alpha-\frac{1}{2}}}{\Gamma(-\alpha+\frac{1}{2})} & \text{if } 0<r<t \\ 0 & \text{if } 0<t<r \end{cases} \qquad \text{(cf. [16, p. 36, formula (28)])} \\ &= \frac{\sqrt{\pi}}{\Gamma(\alpha+1)}\mathbb{S}_{-\alpha+\frac{1}{2}}(t^2-r^2). \end{aligned}$$

We summarize the above computations.

Theorem 3.18. *For all $(x,t)\in\mathbb{R}^N\times\mathbb{R}$*

$$\begin{aligned} u_k(x,t) &= d_k\frac{\sqrt{\pi}}{\Gamma(\gamma_k+N/2)}\int_0^{|t|} r^{2\gamma_k+N-1}\frac{d}{dt}\left(\mathbb{S}_{-\gamma_k-\frac{N-3}{2}}(t^2-r^2)\right)M_f(x,r)dr \\ &\quad + \mathrm{sign(t)}d_k\frac{\sqrt{\pi}}{\Gamma(\gamma_k+N/2)}\int_0^{|t|} r^{2\gamma_k+N-1}\mathbb{S}_{-\gamma_k-\frac{N-3}{2}}(t^2-r^2)M_g(x,r)dr. \end{aligned}$$

Keeping in mind Rösler's results on the support of the measure $\sigma^k_{x,r}$ associated with M_f and M_g, Theorem 3.3 implies the following:

Theorem 3.19. (Weak Huygens' Principle) *Let $k\in\mathscr{K}^+$, $N\geq 1$, and let a point $x\in\mathbb{R}^N$ be given. The solution $u_k(x,t)$ to the Cauchy problem (3.1) depends only on the values of $f(y)$ and $g(y)$ in the union*

$$\bigcup_{h\in G}\left\{\, y\in\mathbb{R}^N \;\mid\; \|y-hx\|\leq|t| \right\}.$$

A slightly weaker variant of the above theorem says:

Corollary 3.20. *Assume that $k \in \mathscr{K}^+$ and $N \geq 1$. For a given $x \in \mathbb{R}^N$, the solution $u_k(x,t)$ to the Cauchy problem (3.1) depends only on the values of $f(y)$ and $g(y)$ for $\|x\| - |t| \leq \|y\| \leq \|x\| + |t|$.*

Similarly, by (3.30), Theorem 3.15 yields:

Theorem 3.21. (Strict Huygens' Principle) *Let $k \in \mathscr{K}^+$ and $N \geq 1$. The solution $u_k(x,t)$ to the Cauchy problem (3.1) depends only on the values of $f(y)$ and $g(y)$ in the set*

$$\{y \in \mathbb{R}^N \mid \|y\| \geq \big|\|x\| - |t|\big|\}$$

if and only if

$$\frac{N-3}{2} + \gamma_k \in \mathbb{N}.$$

Remark 3.22. (i) Note that, if the initial data (f,g) are supported inside a closed ball of radius R about the origin, then, by means of Theorem 3.21, we recover Theorem 3.17.

(ii) Let G_1 and G_2 be two finite reflection groups on $\mathbb{R}^N$ and $\mathbb{R}^M$, with root systems $\mathscr{R}_1$ and $\mathscr{R}_2$, respectively. Set k_1 and k_2 to be the multiplicity functions on $\mathscr{R}_1$ and $\mathscr{R}_2$, respectively. Consider the generalized wave equation

$$\Delta_{k_1}^x u_{k_1,k_2}(x,y) = \Delta_{k_2}^y u_{k_1,k_2}(x,y) \qquad (x,y) \in \mathbb{R}^N \times \mathbb{R}^M,$$

where Δ_{k_1} (resp. Δ_{k_2}) denotes the Dunkl-Laplacian operator associated with G_1 (resp. G_2). Here the superscript indicates the relevant variable. If $\frac{N-M}{2} + \gamma_{k_1} - \gamma_{k_2} - 1 \in \mathbb{N}$, then there exists a distribution T on $\mathbb{R}^N \times \mathbb{R}^M$ with singular support, i.e. T is supported on the set $\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^M \mid \sum_{i=1}^N x_i^2 = \sum_{i=1}^M y_i^2\}$, so that $(\Delta_{k_1} - \Delta_{k_2})T = \delta$.

We close this section by making the following comment. As we mentioned before, we can prove Theorem 3.17 using another method involving only the Paley-Wiener Theorem 2.5(i). We sketch this approach and its details will be illustrated in the next section to prove the principle of energy equipartition.

Using (3.4) and the inversion formula of the Dunkl transform, we may rewrite u_k as

$$u_k(x,t) = c_k^{-2} \int_0^\infty \left\{ \Phi_k(r,x)\cos(tr) + \frac{\Psi_k(r,x)}{r}\sin(tr) \right\} dr, \tag{3.33}$$

where

$$\Phi_k(r,x) = r^{2\gamma_k+N-1} \int_{S^{N-1}} \mathscr{D}_k f(r\xi') E_k(ix, r\xi') \upsilon_k(\xi') d\omega(\xi'),$$
$$\Psi_k(r,x) = r^{2\gamma_k+N-1} \int_{S^{N-1}} \mathscr{D}_k g(r\xi') E_k(ix, r\xi') \upsilon_k(\xi') d\omega(\xi').$$

If $(N-1)/2 + \gamma_k \in \mathbb{N}$, then, for fixed x, the integral formulas for $\Phi_k(r,x)$ and $\Psi_k(r,x)$ continue analytically to even functions for $r \in \mathbb{C}$. In these circumstances, (3.33) becomes

$$u_k(x,t) = \frac{c_k^{-2}}{2} \int_{\mathbb{R}} \left\{ \Phi_k(r,x) + \mathrm{sign}(t)\frac{\Psi_k(r,x)}{ir} \right\} e^{ir|t|} dr.$$

Let $r = a + ib \in \mathbb{C}$. The holomorphic extensions Φ_k and Ψ_k satisfy

$$|\Phi_k(r,x)| \leq c_0(k)|r|^{2\gamma_k+N-1} e^{|b|\|x\|} \sup_{\xi' \in S^{N-1}} |\mathscr{D}_k f(r\xi')|,$$
$$\left|\frac{\Psi_k(r,x)}{r}\right| \leq c_0(k)|r|^{2\gamma_k+N-2} e^{|b|\|x\|} \sup_{\xi' \in S^{N-1}} |\mathscr{D}_k g(r\xi')|.$$

If $(N-1)/2+\gamma_k=0$, the last estimate gives a problem at $r=0$. Thus we shall exclude this case, and the condition $(N-1)/2+\gamma_k\in\mathbb{N}$ becomes $(N-3)/2+\gamma_k\in\mathbb{N}$. Indeed, the condition $(N-1)/2+\gamma_k=0$ is equivalent to $N=1$ and $k\equiv 0$, which corresponds to the rank one classical wave equation, where the strict Huygens principle fails.

Applying the Paley-Wiener theorem to the Cauchy data (f,g), we conclude that, for fixed $s>0$, there exists a constant c depending only on N, k and the Cauchy data, such that

$$|u_k(x,t)|\leq ce^{-s(|t|-\|x\|-R)},\qquad \text{for all } (x,t)\in\mathbb{R}^N\times\mathbb{R}.$$

Now the left hand side inequality in (3.27) is rather clear.

4. Energy theorems

Energy is defined in physics as the ability to do work. "Kinetic energy" corresponds to energy in the form of motion, and "potential energy" corresponds to energy in a form stored for later use. These are defined below for our wave equation (we shall not comment on any physical significance).

In this section, we show that, under a condition involving k and N, the difference between the kinetic and potential energies of the solution to (3.1) decays like $e^{-2|t|s}$, for fixed $s>0$. Thus, the energy equipartition theorem holds. The equipartition says when $|t|$ is large, the kinetic and potential energies are both equal to the half of the (t-independent) total energy.

For the time being, we only assume $k\in\mathscr{K}^+$ and $N\geq 1$.

Let $u_k(x,t)$ be the solution to the Cauchy problem (3.1). Define the kinetic and potential energies by

$$\begin{aligned}\mathscr{K}_k[u_k](t) &:= \frac{1}{2}\int_{\mathbb{R}^N}|\partial_t u_k(x,t)|^2\upsilon_k(x)dx,\\ \mathscr{P}_k[u_k](t) &:= \frac{1}{2}\int_{\mathbb{R}^N}\sum_{j=1}^N|T_j^x(k)u_k(x,t)|^2\upsilon_k(x)dx.\end{aligned}$$

Here the superscript x denotes the relevant variable. The total energy of u_k is by definition $\mathscr{E}_k[u_k](t):=\mathscr{K}_k[u_k](t)+\mathscr{P}_k[u_k](t)$.

Before investigate the difference between the kinetic and potential energies, we notice that $\mathscr{E}_k[u_k](t)$ is a conserved quantity, i.e. $\mathscr{E}_k[u_k](t)$ is independent of t. To see this, we express the total energy in terms of $\mathscr{D}_k(u_k(\cdot,t))(\xi)$. Since

$$\mathscr{D}_k(T_j^x(k)u_k(\cdot,t))(\xi)=-i\xi_j\mathscr{D}_k(u_k(\cdot,t))(\xi),$$

by means of the Plancherel formula, we obtain

$$\mathscr{E}_k[u_k](t)=\frac{c_k^{-2}}{2}\int_{\mathbb{R}^N}\Big\{|\partial_t\mathscr{D}_k(u_k(\cdot,t))(\xi)|^2+\|\xi\|^2|\mathscr{D}_k(u_k(\cdot,t))(\xi)|^2\Big\}\upsilon_k(\xi)d\xi.$$

On the other hand, since

$$\mathscr{D}_k(u_k(\cdot,t))(\xi)=\cos(t\|\xi\|)\mathscr{D}_kf(\xi)+\frac{\sin(t\|\xi\|)}{\|\xi\|}\mathscr{D}_kg(\xi),\qquad \text{for all } t\in\mathbb{R},$$

we compute

$$
\begin{aligned}
(4.1) \qquad |\mathscr{D}_k(u_k(\cdot,t))(\xi)|^2 &= \cos^2(t\|\xi\|)|\mathscr{D}_k f(\xi)|^2 + \frac{\sin^2(t\|\xi\|)}{\|\xi\|^2}|\mathscr{D}_k g(\xi)|^2 \\
&\quad +2\frac{\cos(t\|\xi\|)\sin(t\|\xi\|)}{\|\xi\|}\mathrm{Re}\left(\mathscr{D}_k f(\xi)\overline{\mathscr{D}_k g(\xi)}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
(4.2) \qquad |\partial_t\mathscr{D}_k(u_k(\cdot,t))(\xi)|^2 &= \cos^2(t\|\xi\|)|\mathscr{D}_k g(\xi)|^2 + \|\xi\|^2\sin^2(t\|\xi\|)|\mathscr{D}_k f(\xi)|^2 \\
&\quad -2\|\xi\|\cos(t\|\xi\|)\sin(t\|\xi\|)\mathrm{Re}\left(\mathscr{D}_k f(\xi)\overline{\mathscr{D}_k g(\xi)}\right).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\mathscr{E}_k[u_k](t) &= \frac{c_k^{-2}}{2}\int_{\mathbb{R}^N}\Big\{\|\xi\|^2|\mathscr{D}_k f(\xi)|^2 + |\mathscr{D}_k g(\xi)|^2\Big\}\upsilon_k(\xi)d\xi \\
&= \frac{1}{2}\int_{\mathbb{R}^N}\Big\{\sum_{j=1}^N |T_j^x(k)f(x)|^2 + |g(x)|^2\Big\}\upsilon_k(x)dx.
\end{aligned}
$$

Hence, we established the following theorem.

Theorem 4.1. (Conservation of energy) *Let $k \in \mathscr{K}^+$, $N \geq 1$ and assume that the initial data $f, g \in \mathscr{S}(\mathbb{R}^N)$. Then the total energy $\mathscr{E}_k[u_k]$ is finite and independent of t.*

Consider now the mater of the energy equipartition. Using (4.2) and repeating the argument used above to prove the conservation of $\mathscr{E}_k[u_k]$, we may rewrite the kinetic energy as

$$
\begin{aligned}
\mathscr{K}_k[u_k](t) &= \frac{c_k^{-2}}{4}\|\mathscr{D}_k(g)\|_k^2 + \frac{c_k^{-2}}{4}\|\langle\cdot,\cdot\rangle^{1/2}\mathscr{D}_k(f)\|_k^2 \\
&\quad +\frac{c_k^{-2}}{4}\int_{\mathbb{R}^N}\Big[|\mathscr{D}_k g(\xi)|^2 - \|\xi\|^2|\mathscr{D}_k f(\xi)|^2\Big]\cos(2t\|\xi\|)\upsilon_k(\xi)d\xi \\
&\quad -\frac{c_k^{-2}}{4}\int_{\mathbb{R}^N}\Big[\overline{\mathscr{D}_k f(\xi)}\mathscr{D}_k g(\xi) + \overline{\mathscr{D}_k g(\xi)}\mathscr{D}_k f(\xi)\Big]\|\xi\|\sin(2t\|\xi\|)\upsilon_k(\xi)d\xi,
\end{aligned}
$$

using the familiar trigonometric identities for double angles. Here $\|\cdot\|_k$ denotes the norm in $L^2(\mathbb{R}^N, \upsilon_k(x)dx)$. Similarly, by (4.1) we obtain

$$
\begin{aligned}
\mathscr{P}_k[u_k](t) &= \frac{c_k^{-2}}{4}\|\mathscr{D}_k(g)\|_k^2 + \frac{c_k^{-2}}{4}\|\langle\cdot,\cdot\rangle^{1/2}\mathscr{D}_k(f)\|_k^2 \\
&\quad +\frac{c_k^{-2}}{4}\int_{\mathbb{R}^N}\Big[\|\xi\|^2|\mathscr{D}_k f(\xi)|^2 - |\mathscr{D}_k g(\xi)|^2\Big]\cos(2t\|\xi\|)\upsilon_k(\xi)d\xi \\
&\quad +\frac{c_k^{-2}}{4}\int_{\mathbb{R}^N}\Big[\overline{\mathscr{D}_k f(\xi)}\mathscr{D}_k g(\xi) + \overline{\mathscr{D}_k g(\xi)}\mathscr{D}_k f(\xi)\Big]\|\xi\|\sin(2t\|\xi\|)\upsilon_k(\xi)d\xi.
\end{aligned}
$$

Now the difference between the kinetic and potential energies is given by

$$
\begin{aligned}
\mathscr{K}_k[u_k](t) - \mathscr{P}_k[u_k](t) &= \frac{c_k^{-2}}{2}\int_{\mathbb{R}^N}\Big[|\mathscr{D}_k g(\xi)|^2 - \|\xi\|^2|\mathscr{D}_k f(\xi)|^2\Big]\cos(2t\|\xi\|)\upsilon_k(\xi)d\xi \\
(4.3) \qquad &\quad -\frac{c_k^{-2}}{2}\int_{\mathbb{R}^N}\Big[\overline{\mathscr{D}_k f(\xi)}\mathscr{D}_k g(\xi) + \overline{\mathscr{D}_k g(\xi)}\mathscr{D}_k f(\xi)\Big]\|\xi\|\sin(2t\|\xi\|)\upsilon_k(\xi)d\xi.
\end{aligned}
$$

Using the spherical-polar coordinates $\xi = r\xi'$, we have

$$\mathscr{K}_k[u_k](t) - \mathscr{P}_k[u_k](t) = \frac{c_k^{-2}}{2}\int_0^\infty \Big\{\Phi_k(r)\cos(2tr) - \Psi_k(r) r\sin(2tr)\Big\}dr,$$

where

$$\Phi_k(r) = r^{2\gamma_k+N-1}\int_{S^{N-1}}\Big\{|\mathscr{D}_k g(r\xi')|^2 - r^2|\mathscr{D}_k f(r\xi')|^2\Big\}\upsilon_k(\xi')d\omega(\xi')$$

$$\Psi_k(r) = r^{2\gamma_k+N-1}\int_{S^{N-1}}\Big\{\mathscr{D}_k f(r\xi')\overline{\mathscr{D}_k g(r\xi')} + \overline{\mathscr{D}_k f(r\xi')}\mathscr{D}_k g(r\xi')\Big\}\,\upsilon_k(\xi')d\omega(\xi').$$

Henceforth, we will choose to work with solutions to (3.1) where the Cauchy data (f, g) belong to $\mathscr{C}^\infty(\mathbb{R}^N)$ and supported in the closed ball of radius $R > 0$ about the origin. Further, by Remark 3.4, we shall often presenting proofs valid for $t > 0$, and formulate the suitably altered statement for all $t \in \mathbb{R}$, without comment.

Since $\overline{E_k(z,w)} = E_k(\overline{z},\overline{w})$, it follows that $\xi \mapsto \overline{\mathscr{D}_k f(-\xi)}$ is the Dunkl transform of $\overline{f}$. Thus $\overline{\mathscr{D}_k f(\xi)}$, similarly $\overline{\mathscr{D}_k g(\xi)}$, belongs to the Paley-Wiener space $\mathscr{H}_R(\mathbb{C}^N)$. In particular, they can be extended to entire analytic functions on $\mathbb{C}^N$. Since $\upsilon_k(\xi')d\omega(\xi')$ is (-1)-invariant, the following lemma holds.

Lemma 4.2. *If $\frac{N-1}{2} + \gamma_k \in \mathbb{N}$, the functions Φ_k and Ψ_k continue analytically to even functions of r.*

In the light of the above lemma, we may rewrite $\mathscr{K}_k[u_k](t) - \mathscr{P}_k[u_k](t)$ as

$$\frac{c_k^{-2}}{4}\int_{\mathbb{R}}\{\Phi_k(r) + ir\Psi_k(r)\}\,e^{2itr}dr. \tag{4.4}$$

Further, using the Paley-Wiener Theorem 2.5(i), and since S^{N-1} is compact, we conclude that for any $M \in \mathbb{N}$ there exist two constants α_M and β_M such that

$$\begin{aligned} |\Phi_k(z)| &\le c_0(N,k)\alpha_M(1+|z|)^{-M}e^{2R|\mathrm{Im}(z)|}, \\ |z\Psi_k(z)| &\le c_0(N,k)\beta_M(1+|z|)^{-M}e^{2R|\mathrm{Im}(z)|}, \end{aligned} \tag{4.5}$$

with $z \in \mathbb{C}$.

Fix $s > 0$. To find a bound for $\mathscr{K}_k[u_k](t) - \mathscr{P}_k[u_k](t)$, we shift the contour in the integral (4.4) from $\mathbb{R}$ to $\mathbb{R} + is$. This idea was inspired by [2, 3]. Thus

$$\begin{aligned} \mathscr{K}_k[u_k](t) - \mathscr{P}_k[u_k](t) &= \frac{c_k^{-2}}{4}\int_{\mathbb{R}}\{\Phi_k(r) + ir\Psi_k(r)\}\,e^{2irt}dr \\ &= c_k^{-2}\frac{e^{-2ts}}{4}\int_{\mathbb{R}}\{\Phi_k(r+is) + i(r+is)\Psi_k(r+is)\}\,e^{2irt}dr. \end{aligned}$$

In view of (4.5), there exists a constant $\chi_M(N,k)$ such that

$$\Big|\mathscr{K}_k[u_k](t) - \mathscr{P}_k[u_k](t)\Big| \le \chi_M(N,k)e^{-2ts}e^{2Rs}\int_{\mathbb{R}}(1+|r|)^{-M}dr,$$

and the following holds:

Theorem 4.3. *For $k \in \mathscr{K}^+$ and $N \ge 1$, assume that*

$$\frac{N-1}{2} + \gamma_k \in \mathbb{N}.$$

Let u_k be the solution to the Cauchy problem (3.1), where the Cauchy data (f,g) are supported in the closed ball of radius $R > 0$ about the origin. Fix $s > 0$. Then there exists a constant C depending on N, k and (f, g) but not on s, such that

$$\Big|\mathscr{K}_k[u_k](t) - \mathscr{P}_k[u_k](t)\Big| \leq Ce^{-2s(|t|-R)}, \qquad \textit{for all } t \in \mathbb{R}.$$

The following is then immediate.

Theorem 4.4. (Energy Equipartition Theorem) *Under the same assumptions as in the previous theorem, we have*

$$\mathscr{K}_k[u_k](t) = \mathscr{P}_k[u_k](t) = \frac{\mathscr{E}_k[u_k](R)}{2} \qquad \textit{for } |t| \geq R.$$

We close this section by making two comments. First, in the theorem above we did not exclude the case $N = 1$ if $k \equiv 0$, since the classical wave equation on $\mathbb{R} \times \mathbb{R}$ has an equipartitioned energy.

Second, it is possible to prove the energy equipartition theorem when the Cauchy data (f, g) are two Schwartz functions on $\mathbb{R}^N$. Actually, under the same assumptions as in Theorem 4.1, we have

$$\lim_{|t|\to\infty} \mathscr{K}_k[u_k](t) = \lim_{|t|\to\infty} \mathscr{P}_k[u_k](t) = \frac{\mathscr{E}_k[u_k](0)}{2}.$$

To see this one needs to show that the integrals in (4.3) tend to zero as $|t| \to \infty$. This follows by means of the classical Riemann-Lebesgue lemma for the Euclidean Fourier sine and cosine transforms. In the classical case $k \equiv 0$, the two limit formulas above can be found in [4].

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S. Ben Saïd: Université Henri Poincaré-Nancy 1, Institut Elie Cartan, Département de Mathématiques, B.P. 239, 54506 Vandoeuvre-Les-Nancy, Cedex, France
E-mail address: Salem.BenSaid@iecn.u-nancy.fr

B. Ørsted: University of Aarhus, Department of Mathematical Sciences, Building 530, Ny Munkegade, DK 8000, Aarhus C, Denmark
E-mail address: orsted@imf.au.dk