Mathematical and Numerical Analysis of a Class of Non-linear Elliptic Equations in the Two Dimensional Case

Noureddine Alaa, Jean Rodolphe Roche, A. Cheggour

Mathematical and Numerical Analysis of a Class of Non-linear Elliptic Equations in the Two Dimensional Case.

Nour Eddine Alaa[1], Abderrahim Cheggour[1] and Jean R. Roche[2]

[1] Département de Mathématiques et Informatique, Université des Sciences et Techniques Cadi Ayyad, B.P. 618, Guéliz, Marrakech, Maroc
alaa@fstg-marrakech.ac.ma

[2] I.E.C.N., Université Henri Poincaré, B.P. 239, 54506 Vandoeuvre lès Nancy, France roche@iecn.u-nancy.fr

Summary. The aim of this paper is to show the existence and present a numerical analysis of weak solutions for a quasi-linear elliptic problem with Dirichlet boundary conditions in a domain Ω and data belonging to $L^1(\Omega)$. A numerical algorithm to compute a numerical approximation of the weak solution is described and analyzed. Numerical examples are presented and commented.

1 Introduction

The principal objective of this work is to give a result of existence and present a numerical analysis of weak solutions for the following quasi-linear elliptic problem:

$$\begin{cases} -\Delta u(x) + G(x, \nabla u(x)) = F(x, u(x)) + f(x) \text{ in } \Omega\,, \\ u(x) = 0 \text{ on } \partial\Omega \end{cases} \tag{1}$$

where G, F are Caratheodory non negative functions. The function $f \in L^1(\Omega)$ is given finite non negative. The domain $\Omega \subset \mathbb{R}^N$ is open and bounded. Such problems arise from biological, chemical and physical systems.

The two essential ingredients to the analysis of this problem are the convexity of $s \to G(x, s)$ and that $G(x, s)$ is sub-quadratic w.r.t. s namely:

$$G(x, s) \leq C(k(x) + \|s\|^2), \quad \text{where} \quad k(x) \in L^1(\Omega) \text{ and } C > 0\,. \tag{2}$$

Then the problem (1) has a solution in $W_0^{1,q}(\Omega)$ where $1 \leq q < N/(N-1)$, $N \geq 2$, provided that (1) has a super-solution in $W_0^{1,1}(\Omega)$.

In previous work [4] the authors show the existence of a weak solution in the one-dimensional case and with arbitrary growth of the non linearity and data measure.

We study a numerical method to compute the solution of the problem (1). In the first step we compute a super solution using a domain decomposition method. In the second step we compute a sequence of solutions of an intermediate problem obtained by using the Yosida approximation of G. This sequence converges to the weak solution of the problem (1).

2 Statement of the Main Result

Throughout this paper we suppose

$$f \in L^1(\Omega),\ f \geq 0\,. \tag{3}$$

The functions $G : \Omega \times \mathbb{R}^N \to [0, +\infty[$ and $F : \Omega \times \mathbb{R} \to [0, +\infty[$ are such that:

$$G, F \text{ are measurable},\ r \to G(x, r) \text{ and } u \to F(x, u) \text{ are continuous.} \tag{4}$$

$$G \text{ is convex in } r \text{ and } F \text{ is nondecreasing in } u\,, \tag{5}$$

$$G(x, 0) = \min\left\{G(x, r),\ r \in \mathbb{R}^N\right\} = 0\,, \text{ and } \quad F(x, 0) = 0\,, \tag{6}$$

$$G(x, r) \leq C(|r|^2 + K(x))\,, \tag{7}$$

$$F(x, u) \in L^1(\Omega) \text{ for every } u \in \mathbb{R} \tag{8}$$

with a constant $C > 0$ and $K \in L^1(\Omega)$.

We introduce now the notion of weak solutions of problem (1).

Definition 1. *A function u is said to be a weak solution of the problem (1), if*

$$\begin{cases} u \in W_0^{1,1}(\Omega),\ G(x, \nabla u) \text{ and } F(x, u) \in L^1(\Omega), \\ -\Delta u + G(x, \nabla u) = F(x, u) + f \quad \text{in } \mathcal{D}'(\Omega). \end{cases} \tag{9}$$

We will be interested in proving the existence of weak positive solutions of problem (1).

Theorem 1. *Under hypotheses (3)—(7), and assuming that there exists w such that*

$$\begin{cases} w \in W_0^{1,1}(\Omega), \quad F(x, w) \in L^1(\Omega), \\ -\Delta w = F(x, w) + f \quad \text{in } \mathcal{D}'(\Omega), \end{cases} \tag{10}$$

the problem (1) has a positive weak solution.

3 Proof of Theorem 1

3.1 Approximation Scheme

We consider the sequence defined by $u_0 = w$ and for $n \geq 0$, u_{n+1} is the solution of the problem

$$\begin{cases} -\Delta u_{n+1} + G_{n+1}(x, \nabla u_{n+1}) = F(x, u_n) + f \text{ in } \mathcal{D}'(\Omega), \\ u_{n+1} \in W_0^{1,1}(\Omega), G_{n+1}(x, \nabla u_{n+1}) \in L^1(\Omega), \end{cases} \tag{11}$$

where $G_n(x, r)$ denotes the Yosida approximation of $G(x, r)$. The function $G_n(x, r)$ is convex in r, increases pointwise to $G(x, r)$ as n tends to ∞ and satisfies

$$G_n \leq G_{n+1} \leq G, \qquad \|G_{n,r}(x, r)\|_\infty \leq n, \tag{12}$$

where $G_{n,r}$ denotes a section of subdifferential of G_n with respect to r.

The classical works ([3], [6], [9]) combined with an induction argument can be applied to prove that (11) has a solution such that

$$0 \leq u_{n+1} \leq u_n \leq w. \tag{13}$$

3.2 Estimates and Convergence

Let $\{u_n\}_n$ be a sequence defined as above. By integrating (11) in Ω and using (13) we obtain

$$\int_\Omega G_{n+1}(x, \nabla u_{n+1}) dx \leq \int_\Omega F(x, w) dx + \int_\Omega f(x) dx. \tag{14}$$

Therefore $\|\Delta u_{n+1}\|_{L^1(\Omega)}$ is bounded. Then there exists a subsequence still denoted by u_n for simplicity, such that u_n converges strongly to some u in $W_0^{1,q}(\Omega), 1 \leq q < N/(N-1)$, and $(u_n, \nabla u_n)$ converges to $(u, \nabla u)$ almost everywhere in Ω (see [8]).

Let us prove that u is in fact a solution of problem (1). According to the definition 2.1, we only have to show that

$$-\Delta u + G(x, \nabla u) = F(x, u) + f \text{ in } \mathcal{D}'(\Omega). \tag{15}$$

We know that $F(x, u_n) \longrightarrow F(x, u)$ strongly in $L^1(\Omega)$ and, for almost every x in Ω, there holds $G_{n+1}(x, \nabla u_{n+1}(x)) \longrightarrow G(x, \nabla u(x))$.

Then there exists a non-negative measure μ (see [7]) such that

$$-\Delta u_n + G_{n+1}(\nabla u_{n+1}) - F(u_n) - f \longrightarrow -\Delta u + G(\nabla u) - F(u) - f + \mu \text{ in } \mathcal{D}'(\Omega),$$

as n goes to ∞.

On the other hand

$$-\Delta u_{n+1} + G_{n+1}(x, \nabla u_{n+1}) = F(x, u_n) + f \longrightarrow F(x, u) + f \text{ in } L^1(\Omega). \tag{16}$$

Consequently

$$-\Delta u + G(x, \nabla u) \le F(x, u) + f \text{ in } \mathcal{D}'(\Omega). \tag{17}$$

Therefore to conclude the proof of theorem 1, we must establish the opposite inequality. To this end we introduce the following test function

$$\psi \exp(-Cu_{n+1}) H(\frac{u_{n+1}}{k}), \tag{18}$$

where $H \in C^1(\mathbb{R})$, $0 \le H(s) \le 1$, $H(s) = 0$ if $|s| \ge 1$ and $H(s) = 1$ if $|s| \le \frac{1}{2}$, C is given by relation (7) and $\psi \le 0, \psi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. We multiply the equation satisfied by u_{n+1} in (11) by this test function and we integrate in Ω, to obtain

$$\int_\Omega (f_n + F(x, u_n)) \psi \exp(-Cu_{n+1}) H(\frac{u_{n+1}}{k})\, dx = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_\Omega \nabla u_{n+1} \nabla \psi \exp(-Cu_{n+1}) H(\tfrac{u_{n+1}}{k})\, dx, \\ I_2 &= \tfrac{1}{k} \int_\Omega |\nabla u_{n+1}|^2 \psi \exp(-Cu_{n+1}) H'(\tfrac{u_{n+1}}{k})\, dx, \\ I_3 &= \int_\Omega (G_{n+1}(x, \nabla u_{n+1}) - C |\nabla u_{n+1}|^2) \psi \exp(-Cu_{n+1}) H(\tfrac{u_{n+1}}{k})\, dx. \end{aligned} \tag{19}$$

By investigating separately each term, we get

$$\lim_{n\to\infty} I_1 = \int_\Omega \nabla u \nabla \psi \exp(-Cu) H(\frac{u}{k})\, dx$$
$$\text{and } \lim_{k\to\infty} I_2 = 0 \text{ uniformly on } n.$$

Now we investigate the remaining term I_3. Since G_{n+1} satisfies the inequality (7), $\psi \le 0$, and by applying Fatou's lemma, we obtain

$$\lim_{n\to\infty} I_3 \ge \int_\Omega (G(x, \nabla u) - C |\nabla u|^2) \psi \exp(-Cu) H(\frac{u}{k}) dx. \tag{20}$$

Finally we have shown

$$\int_\Omega \nabla u \nabla \psi \exp(-Cu) H(\frac{u}{k}) dx + \int_\Omega \psi (G(x, \nabla u) - C |\nabla u|^2) \exp(-Cu) H(\frac{u}{k}) dx$$
$$+\omega(\frac{1}{k}) \le \int_\Omega (F(x, u) + f) \psi \exp(-Cu) H(\frac{u}{k}) dx,$$

where $\omega(\varepsilon)$ denotes a quantity that tends to 0 when ε tends to 0. Now we choose $\psi = -\varphi \exp(Cu) H(\frac{u}{k})$, where $\varphi \ge 0, \varphi \in \mathcal{D}(\Omega)$ and we replace ψ by this value in the previous inequality to get after appropriate calculations and using that the third term is equivalent to $\omega(\frac{1}{k})$

$$-\int_\Omega \nabla u \nabla\varphi H(\frac{u}{k})^2 dx - \int_\Omega \varphi\, G(x,\nabla u) H(\frac{u}{k})^2 dx + \omega(\frac{1}{k})$$
$$\leq -\int_\Omega (F(x,u)+f)\varphi H(\frac{u}{k})^2 dx.$$

We finally pass to the limit as k tends to infinity and we use the fact that $\lim_{k\to\infty} H(\frac{u}{k}) = 1$, to conclude for every $\varphi \geq 0, \varphi \in \mathcal{D}(\Omega)$ that

$$\int_\Omega \nabla u \nabla \varphi dx + \int_\Omega \varphi G(x,\nabla u) dx \geq \int_\Omega (F(x,u)+f)\varphi dx.$$

This finishes the proof of the theorem 1.

4 Numerical Method

4.1 Introduction

In this section we present the numerical method to solve the equation (1) in $\mathbb{R}^2$. Formally the algorithm can be formulated in the following way:

1) Find $\overline{w} \in H_0^1(\Omega)$ such that:

$$-\Delta\overline{w}(x) \;\geq\; F(x,\overline{w}) \;+\; f \;\text{ in } \Omega. \tag{21}$$

2) Given $u_0 = \overline{w}$ we compute a sequence, $\{u_n\}_n$, solution in $H_0^1(\Omega)$ of the non linear equation:

$$-\,\Delta u_{n+1}(x) \;+\; G_{n+1}(x,\nabla u_{n+1}) \;=\; F(x,u_n) \;+\; f \;\text{ in } \Omega. \tag{22}$$

Both problems (21) and (22) are non-linear, and if (21) has a solution, in theorem 1 we have shown that (22) then also has a solution.

4.2 Numerical Algorithm

This subsection summarizes the algorithm introduced in the previous subsection.

1) First step: given $\overline{w}^0 = 0$, iteratively for $k = 1$ until convergence we compute $\overline{w}^{k+1} = \overline{w}^k + \delta$ where at each iteration δ is the solution of the linear problem:

$$\begin{cases} -\Delta\delta(x) - \frac{\partial F(x,\overline{w}^k)}{\partial r}\delta(x) = \Delta\overline{w}^k(x) + F(x,\overline{w}^k) + f \text{ in } \Omega, \\ \delta(x) = 0 \text{ on } \partial\Omega. \end{cases} \tag{23}$$

To solve at each iteration the linear problem (23) we consider the domain decomposition method which will be introduced as follows:

a) We compute $c_\infty = \left\| \frac{\partial F(\overline{w}^k)}{\partial r} \right\|_\infty$. Determine an overlapping subdomain decomposition Ω_i, $i = 1, \ldots, m$ such that $\Omega = \cup_{i=1}^m \Omega_i$ and satisfies:

$$\max\{\mathrm{mes}(\Omega_i), i = 1, \ldots, m\} < \min(\frac{c_0 \pi^2}{c_\infty}, \frac{\pi}{2\sqrt{c_\infty}}). \qquad (24)$$

We denote by m the number of subdomains Ω_i and $\partial\Omega_i$ is the boundary of Ω_i.

b) Iteratively:
for $l = 1$,..until convergence and for $i = 1, \ldots, m$ we solve the following subdomain problems:

$$\begin{cases} -\Delta \delta_i^l(x) - \frac{\partial F(x, \overline{w}^k)}{\partial r} \delta_i^l(x) = \Delta \overline{w}^k(x) + F(x, \overline{w}^k) + f \text{ in } \Omega_i, \\ \delta_i^l(x) = \delta_j^{l-1}(x), \text{ on } \partial\Omega_i \cap \Omega_j, \\ \delta_i^l(x) = 0, \text{ on } \partial\Omega \cap \partial\Omega_i. \end{cases} \qquad (25)$$

On each subdomain Ω_i we consider a finite element approximation method with N_i elements. At the end of the l-th loop we have computed an approximate discrete solution of the linear indefinite problem (23).

2) At this step for $u_0 = \overline{w}$, iteratively for $n = 1$, until convergence we solve the following non-linear problem

$$\begin{cases} -\Delta u_n(x) + G_n(x, \nabla u_n) = F(x, u_{n-1}) + f \text{ in } \Omega, \\ u_n(x) = 0 \text{ on } \partial\Omega. \end{cases} \qquad (26)$$

At each n-th step the problem (26) is solved by using a Newton method. The discrete approximation of the solution of (1) is obtained at the end of the n-th loop.

4.3 Convergence of the Domain Decomposition Method

To simplify, without lost of generality, we assume that we can consider a two-domain decomposition $\Omega = \Omega_1 \bigcup \Omega_2$ such that:

$$\max\{\mathrm{mes}(\Omega_i), i = 1, 2\} < \min(\frac{c_0 \pi^2}{c_\infty}, \frac{\pi}{2\sqrt{c_\infty}}). \qquad (27)$$

Now to prove the convergence of the Schwarz overlapping domain decomposition algorithm applied to problem (23), we consider two problems:

$$\begin{cases} -\Delta v_1(x) + c(x) v_1(x) = h(x) \text{ in } \Omega_1, \\ v_1(x) = 0 \text{ on } \partial\Omega \cap \partial\Omega_1; \ v_1(x) = v_2(x) \text{ on } \partial\Omega_1 \cap \Omega_2 \end{cases} \qquad (28)$$

and

$$\begin{cases} -\Delta v_2(x) + c(x) v_2(x) = h(x) \text{ in } \Omega_2, \\ v_2(x) = v_1(x) \text{ on } \partial\Omega_2 \cap \Omega_1; \ v_2(x) = 0 \text{ on } \partial\Omega \cap \Omega_2. \end{cases} \qquad (29)$$

Let v be

$$v = \begin{cases} v_1 & \text{in } \Omega_1, \\ v_2 & \text{in } \Omega_2, \end{cases} \tag{30}$$

$v_1 = v_2$ in $\Omega_1 \cap \Omega_2$.

With the restriction (27) we can suppose the existence of a solution of (28) in $W_0^{1,q}(\Omega_1)$ and a solution of (29) in $W_0^{1,q}(\Omega_2)$.

Then, if v^0 is an initialization function defined in Ω and vanishing in $\partial\Omega$, we define for $k \geq 0$ two sequences v_i^k, $i = 1, 2$ solving the following problems:

$$\begin{cases} -\Delta v_1^{k+1}(x) + c(x)\, v_1^{k+1}(x) = h(x) \quad \text{in } \Omega_1, \\ v_1^{k+1}(x) = 0 \text{ on } \partial\Omega \cap \partial\Omega_1;\ v_1^{k+1}(x) = v_2^k(x) \text{ on } \partial\Omega_1 \cap \Omega_2 \end{cases} \tag{31}$$

and

$$\begin{cases} -\Delta v_2^{k+1}(x) + c(x)\, v_2^{k+1}(x) = h(x) \quad \text{in } \Omega_2, \\ v_2^{k+1}(x) = v_1^k(x) \text{ on } \partial\Omega_2 \cap \Omega_1;\ v_2^{k+1}(x) = 0 \text{ on } \partial\Omega \cap \Omega_2. \end{cases} \tag{32}$$

Theorem 2. *Assume Ω_1 and Ω_2 with the restriction (27). Then the sequence v^k converges to v in $W_0^{1,q}(\Omega_1)$ and $W_0^{1,q}(\Omega_2)$.*

Proof. We give here an idea of the proof.

Let $d^k = v_1^k - v$ in Ω_1 and $e^k = v_2^k - v$ in Ω_2 then $d^k \in L^\infty(\Omega_1)$ and $e^k \in L^\infty(\Omega_2)$.

Thanks to the maximum principle we prove the following inequalities:

$$||d^{k+2}||_\infty \leq \gamma\, ||d^k||_\infty \text{ and } ||e^{k+2}||_\infty \leq \gamma\, ||e^k||_\infty \tag{33}$$

where $\gamma < 1$.

But to be able to apply the maximum principle it will be necessary that the subdomains Ω_1 and Ω_2 verify the restriction (27).

4.4 Numerical Results

The algorithm introduced in the previous section has been implemented numerically for the model problem (1) where:
$G(x, r) = |r|^p = (r_1^2 + r_2^2)^{\frac{p}{2}}$ and $r = (r_1, r_2) \in \mathbb{R}^2$ for $1 < p < \infty$.
$F(x, s) = \eta\, s^q$ where $s \in \mathbb{R}^+$ and $1 < q < \infty$.
$f(x) = x_1^\alpha + x_2^\beta$ where $x = (x_1, x_2) \in \Omega$ and $-1 < \alpha, \beta < \infty$.

The number of subdomains is not fixed, it changes at each iteration according to the criterion (27). In figure 1 we can see the solution shape when the algorithm converges with $m = 36$ subdomains.

To study the convergence history of the numerical simulation plotted in figure 1 we consider two steps. In the first step, where we compute a super-solution, we observe the evolution of the number of subdomains: it goes from $m = 4$ subdomains to $m = 36$ subdomains in seven iterations according to criterion (27). Simulation stops after 34 iterations when the residual is less

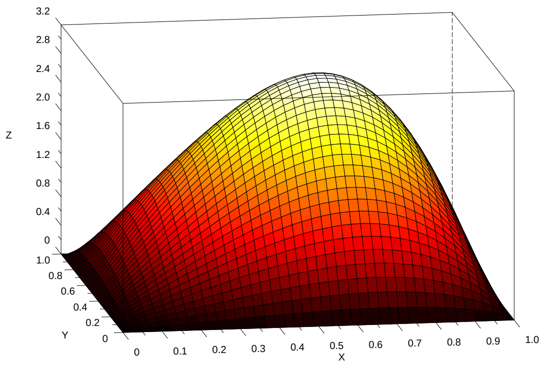

Fig. 1. $\eta = 45,\ p = q = 3,\ \alpha = \beta = 2,\ m = 36$

than 10^{-6}. In the second step, starting with the super-solution computed in the previous step we perform nine iterations of the Yosida approximation described in section 3 and the simulation stops when the correction computed is in uniform norm less than 10^{-6}.

References

1. Alaa, N., Maach, F., Mounir, I.: Existence for some quasilinear elliptic systems with critical growth nonlinearity and L^1data. Journal of Applied Analysis, **11**, 81–94 (2005)
2. Alaa, N., Mounir, I.: Global existence for reaction-diffusion systems with mass control and critical growth with respect to the gradient. Journal of Mathematical Analysis and Applications, **253**, 532–557 (2001)
3. Alaa, N., Pierre, M.: Weak solution of some quasilinear elliptic equations with measures. SIAM J. Math.Anal, **24**, 23–35 (1993)
4. Alaa, N., Roche, J.R.: Theoretical and numerical analysis of a class of nonlinear elliptic equations. Mediterr. J. Math, **2**, 327–344 (2005)
5. Amann, H., Crandall, M.G.: On some existence theorems for semi linear equations. Indiana Univ. Math. J, **7**, 779–790 (1978)
6. Boccardo, L., Murat, F., Puel, J.P.: Existence de solutions non bornées pour certaines équations quasi-linéaires. Portugaliae Math., **41**, 507–534 (1982)
7. Brezis, H.: Analyse fonctionnelle appliquée, Masson, Paris (1980)
8. Brezis, H., W. Strauss, W.: Semilinear elliptic equation in L^1. J. Math. Soc. Japan, **25**, 565–590 (1973)
9. Lions, P.L.: Résolution de problèmes elliptiques quasilinéaires. Arch. Ration. Mech. Analysis, **74**, 335–353 (1980)
10. Porretta, A.: Existence for elliptic equations in L^1 having lower order terms with natural growth. Portugaliae Math., **57**, 179–190 (2000)