Using semi-implicit representation of algebraic surfaces

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Abstract— **A new general representation of algebraic surfaces, called semi-implicit, was introduced in [1]. Here we specialize this notion in order to apply it in Solid Modeling: we view a surface in $\mathbb{R}^3$ as a one-parameter (algebraic) family of algebraic low-degree curves. We present in more details the case where these curves are planar and the coefficients of the equations of the family of planes are low-degree polynomials. This provides an algebraic realization with interesting features of the paradigm of *active contour*. We provide examples showing that this definition encapsulates both usual and less known surfaces. The semi-implicit representation can be used for surface interpolation. The aim is to form new models, with diverse geometry, from scattered data with extra geometric information, and to be combined with other techniques.**

The paper mainly addresses the topic of performing the usual CAD operations with semi-implicit representations of surfaces. We derive formulae for computing the normal and the curvatures at a regular point. We provide exact algorithms for computing self-intersections of a surface and more generally its singular locus. We also present a surface/surface intersection algorithm relying on generalized resultant calculations.

This new approach is creating opportunities to explore and analyze rich and complex geometry of algebraic surfaces through families of models depending on a reduced number of parameters.

I. INTRODUCTION

Evolving curves occur in a wide variety of settings and were also used to describe boundaries of volumes. This gave rise in shape description to the natural idea of active contours. This paradigm has received various mathematical interpretation ranging from meshes to level set methods. We aim to develop an algebraic and geometric interpretation of this paradigm in order to contribute with new models having a rich geometry and on which one can perform efficiently the usual CAD (Computer Aided Design) operations.

In [1] we introduced a new general representation of algebraic surfaces that we called semi-implicit. Here we specialize this notion in order to apply it in Solid Modeling: we view a surface in $\mathbb{R}^3$ as a one-parameter (algebraic) family of algebraic low-degree curves. We present in more details the case where these curves are planar and the coefficients of the equations of the family of planes are low-degree polynomials.

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curvatures at a regular point. We provide exact algorithms for computing self-intersections of a surface and more generally its singular locus. We also present a surface/surface intersection algorithm relying on generalized resultant calculations.

Now we briefly enumerate the content of the following sections. In section II we show that most usual algebraic surfaces already enter in the category of algebraic surfaces we study. In section III we give precise definitions and basic properties of ours semi-implicit surfaces. We also provide an algorithm to compute their implicit representation. In section IV we address the approximation and interpolation of a point cloud by a semi-implicit surface. We illustrate our approach on a piece of the classical example of Stanford's bunny. In section V we give an example of a semi-implicit surface with a skeleton in the spirit of H. Blum shape descriptor. In section VI we provide formulae for computing the equation of the tangent plane together with the second fundamental form of a semi-implicit surface at a regular point. In section VII we give an algorithm to compute the self-intersection and singularity locus of a semi-implicit surface, and in section VIII we describe resultant-based methods in order to solve intersection problems.

II. EXAMPLES

Before writing down a precise definition of what we call a semi-implicit representation of an algebraic surface we would like to illustrate it through some simple examples. Roughly speaking, a semi-implicit representation of a surface consists in representing the surface as a *parameterized* family of *implicit* space curves.

A. Implicit algebraic surfaces

All algebraic surfaces S in $\mathbb{R}^3$ can be seen as a family of curves, i.e. admits a semi-implicit representation. Indeed, if $F(x, y, z)$ is a polynomial such that S corresponds to the zero locus of F (i.e. F is an implicit equation of S) then the map $z \to F(x, y, z)$ defines a family of curves (in the sense that for each $z_0 \in \mathbb{R}$ the intersection of S with the plane of equation $z = z_0$ is the curve defined by the polynomial $F(x, y, z_0)$) describing S.

B. Revolution surfaces

A cone $\mathcal{C}$ can be seen "semi-implicitly" by considering it as the union of some lines contained in $\mathcal{C}$. Indeed, we can build $\mathcal{C}$ by rotating the line intersection of both planes $x = z$ and $y = 0$; thus we have a family of lines parameterized by an angle $\theta \in [0; 2\pi[$:

$$\cos(\theta)y + \sin(\theta)x = 0, \ \cos(\theta)x - \sin(\theta)y - z = 0.$$

We can then move to an algebraic system by introducing the parameter $t = \tan(\frac{\theta}{2})$. Both equations defining $\mathcal{C}$ are then linear in x, y, z (because they are planes) and of degree 2 in t.

Given a revolution surface (with axe $z = 0$) represented implicitly by a polynomial $F(x, y, z)$, we can obtain a semi-implicit representation by cutting it with planes $y = tx$, where t is a parameter; the family is given by

$$y - tx = 0, \ F(x, xt, z) = 0.$$

For instance our previous cone can be represented by the family

$$y - tx = 0, \ (1 + t^2)x^2 - z^2 = 0.$$

Similarly we can semi-implicitly represent a torus using its implicit representation

$$(x^2 + y^2 + z^2 - (R^2 + r^2))^2 = 4R^2(r^2 - z^2),$$

where R and r are respectively the radius of the major and minor circles.

C. Linear families of plane conics

A linear family of conics is a surface obtained as the image of a regular map (without *base points*)

$$\begin{array}{ccc} \mathbb{P}^1 \times \mathbb{P}^1 & \overset{\phi}{\longrightarrow} & \mathbb{P}^3 \\ (s:t) \times (u:v) & \mapsto & (f_0(s,t;u,v): \cdots : f_3(s,t;u,v)), \end{array}$$

where polynomials $f_i(s, t; u, v)$ are bi-homogeneous of bi-degree $(1, 2)$. For all fixed $(s_0 : t_0) \in \mathbb{P}^1$ the image of $\phi_{|(s_0:t_0)}$ is a conic $C_{(s_0:t_0)}$ in $\mathbb{P}^3$ which is, as all conic in $\mathbb{P}^3$, contained in a plane that we denote $H_{(s_0:t_0)}$. We thus have a family H of planes parameterized by $\mathbb{P}^1$; it corresponds to a bi-homogeneous polynomial $L(x, y, z, w; s, t)$, linear in x, y, z, w and of degree ≤ 3 in s, t (this follows immediately from the definition of ϕ).

Consequently linear families of conics are contained in a larger class of surfaces which are semi-implicitly represented by a family of planes H and a family of surfaces of degree 2 given by a bi-homogeneous polynomial $C(x, y, z, a; s, t)$ of bi-degree (2,2). Observe that $C_{(s_0:t_0)} = C(x, y, z, w; s_0, t_0) \cap L(x, y, z, w; s_0, t_0)$ (set-theoretically at least). Let us comment how the degree of the family H in variables s, t affects the geometry of the associated surface. We have a map

$$\theta : \mathbb{P}^1 \rightarrow \mathbb{P}^{3\star} : (s:t) \mapsto H_{(s_0:t_0)}$$

whose image is a curve Γ in $\mathbb{P}^{3\star}$ (where $\star$ stands for the dual), assuming that L does not have an irreducible factor independent of x, y, z, w. Thus if $\deg(\Gamma) = 1$ then we deduce that H has a fixed line, and if $\deg(\Gamma) = 2$ then H as a fixed point.

Of course all the surfaces are not obtained as linear families of conics. To consider more general surfaces we have to increase the number of parameters, i.e. the degree of the algebraic families considered; e.g linear families of cubics in $\mathbb{P}^3$ should be the next case after the families of plane conics.

III. DEFINITION AND FIRST PROPERTIES

An implicit representation of a surface S in $\mathbb{P}^3$ consists in viewing it as a closed subvariety of $\mathbb{P}^3$, i.e. described as the zero locus of a non-zero homogeneous polynomial in $\mathbb{C}[x, y, z, w]$. In this section we give the definition (in a restricted case) of another way to represent surfaces in $\mathbb{P}^3$; we represent them as *parameterized* families of *implicitly* represented space curves. We call such a representation a *semi-implicit* representation. It basically consists in viewing a surface $S \subset \mathbb{P}^3$ as the projection on the second factor of a certain closed subvariety $\mathcal{Z}$ of $\mathbb{P}^3 \times \mathbb{P}^1$.

Definition 3.1: We call a semi-implicit representation of an algebraic surface $S \subset \mathbb{P}^3$ a couple of bi-homogeneous polynomials $F(x, y, z, w; s, t)$ and $G(x, y, z, w; s, t)$ defining a closed subvariety $\mathcal{Z} \subset \mathbb{P}^3 \times \mathbb{P}^1$ such that its projection on the first factor is surjective and is S on the first factor. If F is linear in the homogeneous variables x, y, z, w then the semi-implicit representation is called linear.

Remark 3.2: It is possible to give more general definition of semi-implicit representations involving more than two equations. Here we prefer to restrict ourselves to the setting of the previous definition.

Also observe that the hypothesis requiring that $\mathcal{Z}$ is a surface in $\mathbb{P}^3 \times \mathbb{P}^1$ is very important whereas the hypothesis asking that its projection on $\mathbb{P}^3$ is onto can be avoid. Indeed, the projection of $\mathcal{Z}$ on $\mathbb{P}^3$ is always a surface which is the union of S and other surfaces corresponding to parameters $(s : t) \in \mathbb{P}^1$ such that F and G equals (up to a multiplicative constant) as homogeneous polynomials in x, y, z, w (actually asking that $\mathcal{Z}$ projects onto S means that we assume that such other surfaces do not exist).

Going from a semi-implicit representation to an implicit representation of S is a useful operation, especially for intersection algorithms. One can complete it as follows.

Proposition 3.3: Let S be a surface semi-implicitly represented by both bi-homogeneous polynomials $F(x, y, z, w; s, t)$ and $G(x, y, z, w; s, t)$, then the Sylvester resultant of F and G with respect to the homogeneous variables s, t is an implicit representation of S.

Let us recall how the Sylvester matrix, whose determinant is the so-called Sylvester resultant, is constructed. First write polynomials F and G as

$$F(x, y, z, w; s, t) = \sum_{i=0}^{d_1} c_{1,i}(x, y, z, w)s^i t^{d_1 - i},$$

$$G(x,y,z,w;s,t) = \sum_{i=0}^{d_2} c_{2,i}(x,y,z,w)s^i t^{d_2-i},$$

where the "coefficients" $c_{i,j}(x,y,z,w)$ are homogeneous polynomials in x,y,z,w. The entries of the Sylvester matrix are either 0 or a coefficient $c_{i,j}$, i.e. a homogeneous polynomial in the variables x,y,z,w. This matrix is constructed as follows:

$$
\overbrace{
\begin{array}{c}
\overbrace{s^{d_1-1}F \;\cdots\; t^{d_1-1}F}^{} \quad \overbrace{s^{d_2-1}G \;\cdots\; t^{d_2-1}G}^{}
\end{array}
}^{d_1+d_2}
$$

$$
\left[
\begin{array}{cccc|cccc}
c_{1,0} & & & \mathbf{0} & c_{2,0} & & & \mathbf{0} \\
\vdots & \ddots & & & \vdots & \ddots & & \\
\vdots & & c_{1,0} & & \vdots & & c_{2,0} & \\
c_{1,d_1} & & \vdots & & c_{2,d_2} & & \vdots & \\
& \ddots & \vdots & & & \ddots & \vdots & \\
\mathbf{0} & & c_{1,d_1} & & \mathbf{0} & & c_{2,d_2} &
\end{array}
\right]
\left.
\begin{array}{l}
s^{d_1+d_2-1} \\
s^{d_1+d_2-2}t \\
\vdots \\
s^{d_1-1}t^{d_2-1} \\
\vdots \\
\vdots \\
t^{d_1+d_2-1}
\end{array}
\right\} d_1+d_2
$$

The Sylvester resultant is hence a homogeneous polynomial in x,y,z,w and we know that it vanishes at a point $(x_0:y_0:z_0:w_0) \in \mathbb{P}^3$ if and only if there exists a point $(s_0:t_0) \in \mathbb{P}^1$ such that $F(x_0,y_0,z_0,w_0;s_0,t_0) = G(x_0,y_0,z_0,w_0;s_0,t_0) = 0$.

As a corollary we are able to compute the degree of a surface, an important invariant, from any of its semi-implicit representation.

Corollary 3.4: Suppose that S is semi-implicitly represented by two bi-homogeneous polynomials of respective bi-degree (k_1,d_1) and (k_2,d_2), then S is of degree $k_1d_2 + k_2d_1$.

Proof: This follows from standard properties of the Sylvester resultant (see e.g. [2]). ∎

Remark 3.5: In $\mathbb{R}^3$ one obtains all algebraic surfaces but with different degrees. However only rational surfaces may be parameterized (that is only surfaces with zero genus). Thus we can handle more general surfaces with semi-implicit representations than parameterized representations.

IV. POINT DATA APPROXIMATION

In this section we focus on the problem of approximating a 3D point cloud by a semi-implicit surface. Our approach can also be used to interpolate a 3D point cloud but seems to be less adapted for this purpose. Consequently we are going to describe the approximation process. The main idea is to decompose a 3D-data approximation problem into a 2D-data followed by a 1D-data approximation problems. To do this we consider particular semi-implicit surfaces, the *linear* ones. The following algorithm basically consists in approximating a 3D point cloud by a finite number of plane curves which are approximated (or interpolated) by a linear semi-implicit surface.

We start from a 3D point cloud in $\mathbb{P}^3$ and e.g. we choose the family H of planes parallel to the (x,y)-coordinate plane; it corresponds to the polynomial $L(x,y,z,w;s,t) = sz - tw$.
Step 1. Choose a finite number n of planes in the family H well distributed with respect to the given point cloud.

They correspond to the choice of n values $t_1,\ldots,t_n$ of the parameter t; we denote them $H_{t_1},\ldots,H_{t_n}$.
Step 2. Project each point on the nearest plane H_{t_1}, H_{t_2}, $\ldots$, or H_{t_n}.
Step 3. In each plane H_{t_i}, $i = 1,\ldots,n$, approximate all the projected points by an implicit curve C_i. In order to be able to put all these implicit plane curves C_i is an algebraic family we require that they have the same *shape*, i.e. a fixed monomial support in a given degree d. It follows that each curve C_i, which is contained in the plane H_{t_i}, has equation $\sum_{i+j+k=d} a_{i,j}x^j y^j w^k$.
Step 4. Approximate (or interpolate) all the plane curves C_i, $i = 1,\ldots,n$, in a parameterized algebraic family. To do this, choose a degree l and then interpolate all the coefficients $a_{i,j}$, j fixed and $i = 1,\ldots,n$, by a homogeneous polynomial $a_j(s,t)$ of degree l.

We thus obtain a family of surfaces

$$C(x,y,z;t) = \sum_{i+j+k=d} a_i(s,t)x^j y^j w^k$$

of bi-degree (d,l). This family and the family of planes $H(x,y,z,w;s,t) = sz - tw$ gives an *approximate* semi-implicit representation of the given point cloud. Note that the choice of the family of planes is very important, at least to well distribute all the points on the different planes H_{t_i}, $i = 1,\ldots,n$. Also the choice of the degrees d and l are key-ingredients for the quality of the approximation, these parameters influence directly the complexity and the performance of the algorithm.

The third step of the previous algorithm requires to approximate, or interpolate, a 2D point cloud (since points are assumed to be projected in a plane at this step) by an implicit plane curve. Also the last step requires to approximate, or interpolate, a 1D point cloud. Algorithms for solving such problems have been studied a lot, however especially in the case of parametrically represented curves and surfaces. A recent overview on this topic can be found in [3] and also in [4]. A more particular study in the case of implicitly represented curves and surfaces may be found in [5] and in [6] (and also in [7] where the author starts from a parametrically represented surface).

Example 4.1: As an application of this previous interpolation method we were interested in a semi-implicit representation of a rabbit ear. The ear we took is illustrated by figure 1 where one can see a meshed and a scattered data representation. Late is obtained with 927 points. Cutting by horizontal planes, that is to say of equations $z = c$ where c is constant we formed 20 sets of 45 points, each set being include in a horizontal plane. Notice that we choose, at a first attempt to this problem, to take equally distributed planes. Now in every plane we have a "slice" of the rabbit's ear (this is not really a slice since we have projected some points which were near this slice) which we have to interpolate. The slices number 9 and 17 are showed in figure 2 below.

The next step in the previous algorithm is to approximate the 45 points in each "slice" by an algebraic curve implicitly represented. This is the more time-consuming step of the process. A way to perform it is to use a particular family of planar quartic curves called *dinoid* which are studied in [8] and that we are going to see in the sequel.

V. AN EXAMPLE OF SURFACE WITH SKELETON

We consider a special kind of semi-implicit surfaces which could be useful to provide simple models for compression or animation.

We choose a semi implicit surface represented by a family of planar curves whose real part is formed by a multiple point and an oval. In [8], we call such a curve a dinoid. The oval will modelize an active contour and the multiple point its skeleton following the general idea of Blum [9].

To illustrate this approach, we consider a family of quartic dinoid. First we give the equation $f(x, y) = 0$ of such a curve when the singular point is at the origin. This imposes 3 conditions: the vanishing at the origin of f and of its two partial derivatives, so f must be the sum of 3 homogeneous polynomials of degree 2, 3 and 4. Setting $y = sx$, we get:

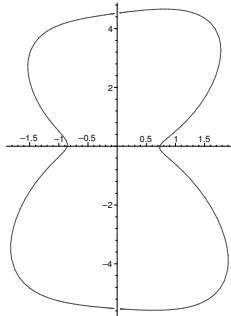$$g(x, s) = f(x, sx) = x^2(f_4(s)x^2 + f_3(s)x + f_2(s)) = 0,$$

where f_i are univariate polynomials of degree i. Then we see that for each value of the slope $s = y/x$ such that $f_4(s)f_2(s) < 0$, $f(x, y)$, $g(x, s)$ admits only two solutions surrounding the origin. For instance for

$$f(x, y) = 16\, x^4 + x^2 y^2 + y^4 + 2\, x^3 - 2\, xy^2$$
$$+y^3 - 10\, x^2 - 3\, xy - 25\, y^2$$

we have

$$g(x, s) = (s^4 + s^2 + 16)x^2 - 25s^2 + (s^3 - 2s^2 + 2)x - 3s - 10.$$

In that case the curve defined by f is drawn in the following picture.

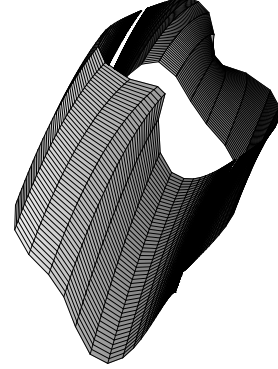

We now consider a family of such curves, its semi-implicit equations are:

$$L(x, y, z, t) = z - 5t + (1 + 2t)x - \frac{1}{40}y,$$

$$F(x, y, t) = 25\, y^2 + y^4 + y^3 + 16\, x^4 - y^2 xt + 1/4\, y^2 t^2 + 2\, x^3$$
$$-\frac{31}{4}\, t^3 + t^4 + 1/8\, t^5 - 8\, xt^3 + 24\, x^2 t^2 - 32\, x^3 t + \frac{63}{2}\, xt^2$$

$$-33\, x^2 t + y^2 x^2 - 5/2\, t^2 + 10\, xt - 10\, x^2 - xt^4 + 3\, x^2 t^3 - 4\, x^3 t^2$$
$$+2\, tx^4 + y^2 t - 2\, y^2 x + 3/2\, yt - 3\, yx.$$

For $t = 0$ we recover the curve whose equation is $f = 0$. The spanned surface S is drawn in the following picture.

In this case the skeleton of S is given by the parametric equations:

$$x = t/2 \quad y = 0 \quad z = t^2 + \frac{11}{2}t.$$

VI. USUAL DIFFERENTIAL GEOMETRIC INVARIANTS

Given a semi-implicitly represented surface, it is possible to compute at any regular point the usual differential geometric invariants such as :

- The equation of the tangent plane thus the normal,
- The second fundamental form thus the curvatures.

Let us do it for a surface S passing by the origin and represented by a family of plane curves parameterized by t (the origin being obtained for $t = 0$). S is given by the two equations: $L(t, x, y, z)$ of degree one in x, y, z and of possibly higher degree in t; $F(t, x, y, z)$ of any degree in x, y, z, t. With our hypothesis they satisfy $L(0, 0, 0, 0) = 0$, and $F(0, 0, 0, 0) = 0$.

The tangent space of S at the origin is generically the projection of the tangent space at the surface defined by L and F at $(0, 0, 0, 0)$ in $\mathbb{R}^4$. So in order to compute the tangent space we can truncate L and F and keep only their affine Taylor expansions, that we call L_1 and F_1.

To be more specific let

$$L_1 := lx + my + nz + pt, \quad F_1 := ax + by + cz + dt,$$

then the equation of the tangent space is:

$$Tg := (-pa + dl)x + (-pb + dm)y + (dn - pc)z.$$

If p and d are both zero, then L_1 and F_1 should be proportional in order that the origin is non singular on S, in that case we keep either equation.

The 3 coefficients of Tg define the coordinates of the normal at S at the origin.

The computation of the second fundamental form is more complicated. It amounts to compute an implicit equation of S near by the origin, truncated at orders greater than three. We take the Taylor expansions of L and F at order 3 in x, y, z
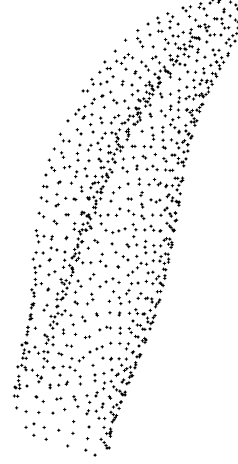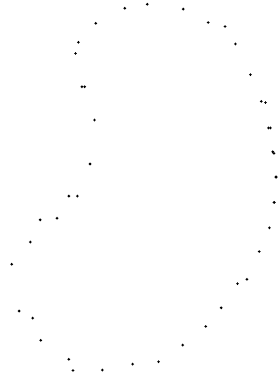
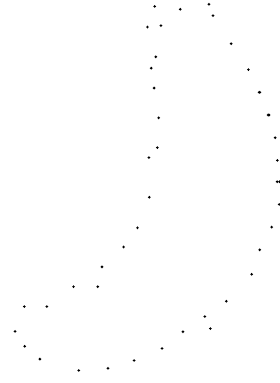(a) mesh   (b) scattered data

Fig. 1.   A rabbit ear

(a) "slice" number 9   (b) "slice" number 17

Fig. 2.   "Slices" of a rabbit ear

(we should not truncate also in t). Let us call them L_2 and F_2. They are two polynomials in x, y, z, t.

We use a resultant to eliminate t between L_2 and F_2 and we get a polynomial $G(x, y, z)$ in x, y, z whose degree depends on the degrees in t of L_2 and F_2. Then we compute a Taylor expansion at order 3 of G at the origin and get a polynomial of degree 2 which writes $Tg + Q_1$, with Q_1 a quadratic form in x, y, z. This provides a local equation of S at the origin.

Then it suffices to perform a change of coordinates (which preserves the metric), call X, Y, Z the new coordinates, so that the previous local equation of S at the origin becomes $Z + Q(X, Y, Z) = 0$, where Q is a quadratic form. Finally the second fundamental form for S at the origin is simply $Q(X, Y, 0)$.

Example 6.1: Let

$$L = (1 + 2t)x - y + (1 + 5t)z + (-1 + 2t)t + t^6 x,$$

$$F = (-2 + t)x + (-3 + 4t)y + (-3 - 2t)z + (-4 + 2t)t$$

$$-2x^2 + 5y^2 + 2z^2 + 3xy + 2xz + 5yz + t^3 x^3.$$

Then the equation of the tangent plane at the origin is

$$6x - y + 7z = 0.$$

After formal computations described above we obtain the following local equation of S at the origin:

$$7z + 6x - y - 27yz + 12z^2 - 23xy + 69x^2 + 90xz + 3y^2 = 0.$$

We note that without the $t^6 x$ term in the equation L the local equation would have been different, namely:

$$6x - y + 7z + 3y^2 - 27yz + \frac{46}{3}xz - \frac{37}{3}xy + 12z^2 + 5x^2.$$

This shows that the intermediate resultant computation was actually useful.

VII. Singularities and self-intersection points

An important problem in Computer Aided Geometric Design is the detection of singularities and self-intersection points of a 3D-surface. We describe a method to complete such a detection in case the considered surface is semi-implicitly represented.

Let S be a surface semi-implicitly represented by both polynomials $F(x, y, z, w; s, t)$ and $G(x, y, z, w; s, t)$ of respective bi-degree (k_1, d_1) and (k_2, d_2). A given point $(x_0 : y_0 : z_0 : w_0)$ of $S \subset \mathbb{P}^3$ is a self-intersection point if there exist two distinct points $(s_1 : t_1)$ and $(s_2 : t_2)$ in $\mathbb{P}^1$ such that:

$$F(x_0, y_0, z_0, w_0; s_1, t_1) = G(x_0, y_0, z_0, w_0; s_1, t_1) = 0, \quad and$$

$$F(x_0, y_0, z_0, w_0; s_2, t_2) = G(x_0, y_0, z_0, w_0; s_2, t_2) = 0.$$

By proposition 3.3 we know that an implicit equation of S can be obtained as the determinant of the Sylvester matrix of F and G with respect to the homogeneous variables s, t. We denote by $R(x, y, z, w)$ this Sylvester matrix and take a given point $p = (x_0, y_0, w_0, z_0) \in \mathbb{P}^3$. If p is not on S then clearly the kernel of $R(p)$ is reduced to 0 since its determinant is non-zero. Now if p is on S then obviously the kernel of $R(p)^t$ (where t stands for transpose) is no reduced to zero since it contains a multiple of the vector of monomials $(s_0, t_0)^{d_1+d_2-1}$, where $(s_0 : t_0) \in \mathbb{P}^1$ is such that

$$F(x_0, y_0, z_0, w_0; s_0, t_0) = G(x_0, y_0, z_0, w_0; s_0, t_0) = 0$$

(observe that we can consequently compute $(s_0 : t_0)$). If now p is a self-intersection point of S then the dimension of the kernel of $R(p)^t$ is at least 2 since this kernel contains both non collinear vector of monomials $(s_1, t_1)^{d_1+d_2-1}$ and $(s_2, t_2)^{d_1+d_2-1}$. Thus a necessary condition for a point p to be a self-intersection point is that $\text{rank}(R(p)) \leq d_1 + d_2 - 3$. Similarly, if $(p; s_0, t_0)$ is a singular point of the semi-implicit representation such that

$$F(p; s_0, t_0) = G(p; s_0, t_0) = 0 \text{ and}$$

$$\partial_s F(p; s_0, t_0) = \partial_s G(p; s_0, t_0) = 0,$$

then both non-collinear monomial vectors $(s_0, t_0)^{d_1+d_2-1}$ and $\partial_s((s_0, t_0)^{d_1+d_2-1})$ and in the kernel of $R(p)$. In other words, singularities and self-intersection points of the semi-implicit representation of S are located on the zero locus of the $(d_1 + d_2 - 2) \times (d_1 + d_2 - 2)$ minors of the Sylvester matrix $R(x, y, z, w)$ in $\mathbb{P}^3$.

VIII. Intersecting a semi-implicit surface

In this section we investigate the intersection problems between different curves and surfaces. Our aim is to show that semi-implicit representations are well adapted to these operations; being an intermediate representation between parameterized and implicit representations, they gather their advantages. We illustrate it on the three main configurations, say the intersection between a semi-implicit surface and a parameterized curve, a parameterized surface a semi-implicit surface.

Hereafter S denote a surface semi-implicitly represented by both polynomials $F(x, y, z, w; s, t)$ and $G(x, y, z, w; s, t)$ of respective bi-degree (k_1, d_1) and (k_2, d_2).

A. With a parameterized space curve

Let g_0, g_1, g_2, g_3, be four homogeneous polynomials in both variables s, t of the same degree d, and let C be the parameterized curve (we write here, for simplicity, the affine version of this parameterization, i.e. set $t = 1$ and $w = 1$)

$$C : \quad \left(x = \frac{g_1(s)}{g_0(s)}, y = \frac{g_2(s)}{g_0(s)}, z = \frac{g_3(s)}{g_0(s)} \right).$$

We assume w.l.o.g. that there is no base point, i.e. that $\gcd(g_0, g_1, g_2, g_3)$ is a constant. Our goal is to compute the intersection of C and S. First by proposition 3.3 we know that there exists a resultant matrix $R(x, y, z)$ whose determinant is an implicit representation of S. Now substituting respectively x, y, z by $\frac{g_1(s)}{g_0(s)}, \frac{g_2(s)}{g_0(s)}$ and $\frac{g_3(s)}{g_0(s)}$ we obtain a matrix $R(s)$ depending on the alone variable s that we can decompose as

$$\text{R}(s) = \text{R}_d s^d + \cdots + \text{R}_0,$$

where the coefficients R_i are numerical matrices of the same size than $\text{R}(s)$. Now we are looking for the values of s such that this surface and the curve intersect at $C(s)$, that is such that the determinant of $\text{R}(s)$ vanishes. This is relate to known methods to solve such "equation" [10], [11]. We have to compute the vectors $\mathbf{v}$ (indexed by monomials in s) such that $\text{R}(s)^t \mathbf{v} = 0$. The intersection problem can thus be transformed into the following generalized eigenvector problem (solved by efficient and stable numerical algorithms):

$$\left[\begin{pmatrix} 0 & \mathbb{I} & & \\ & \ddots & \ddots & \\ & & 0 & \mathbb{I} \\ -\text{R}_0^t & \cdots & & -\text{R}_{d-1}^t \end{pmatrix} - \begin{pmatrix} \mathbb{I} & & & \\ & \ddots & & \\ & & \mathbb{I} & \\ & & & \text{R}_d^t \end{pmatrix} \right] \mathbf{w} = 0,$$

where $\mathbb{I}$ is the identity matrix and $\mathbf{w}$ denote the vector $(\mathbf{v}, s\mathbf{v}, \cdots, s^{d-1}\mathbf{v})^t$.

Such a tool can be useful in ray tracing techniques which involve the intersection of a surface with a line, and similarly to inside/outside positioning of a point with respect to a semi-implicit surface.

B. With a parameterized surface

Let f_0, f_1, f_2, f_3 be four homogeneous polynomials of the same degree d in the homogeneous variables t_0, t_1, t_2. They define a parameterized surface in $\mathbb{P}^3$ (here again we present the affine point of view, setting $t_0 = 1$ and $w = 1$):

$$S' : \quad \left(x = \frac{f_1(t_1, t_2)}{f_0(t_1, t_2)}, y = \frac{f_2(t_1, t_2)}{f_0(t_1, t_2)}, z = \frac{f_3(t_1, t_2)}{f_0(t_1, t_2)} \right).$$

Our goal is here again to represent the intersection curve $\mathcal{C}$ of S and S'. We assume for simplicity that the parameterization of S' is without base point (i.e. $f - 0, f_1, f_2, f_3$ have no common root in $\mathbb{P}^2$), and hence that S' is of degree d^2. By

Bezout's theorem we deduce that $\mathcal{C}$ is of degree $d^2(k_1 d_2 + k_2 d_1)$.

By proposition 3.3 we know that there exists a resultant matrix $\mathrm{R}(x,y,z)$ whose determinant is an implicit representation of S. Substituting respectively x, y, z by $\frac{f_1(t_1,t_2)}{f_0(t_1,t_2)}$, $\frac{f_2(t_1,t_2)}{f_0(t_1,t_2)}$ and $\frac{f_3(t_1,t_2)}{f_0(t_1,t_2)}$ we obtain a matrix $\mathrm{R}(t_1, t_2)$ depending only on both variables t_1 and t_2. Its determinant define a curve (implicitly represented) which is of degree $d(k_1 d_2 + k_2 d_1)$, that is to say of lower degree than $\mathcal{C}$. This curve is a *representation* of the intersection curve $\mathcal{C}$ since every point t_1, t_2 such that $\mathrm{R}(t_1, t_2) = 0$ can be sent on $\mathcal{C}$ by the parameterization of S'.

This method consisting of representing an intersection *space* curve by a birational *plane* curve is very useful in practice. It has been widely studied in the works of D. Manocha [10], [11] in the context of the intersection of two parameterized surfaces. We thus show here that we can also represent in this process one of the surface semi-implicitly instead of parametrically.

C. With a semi-implicit surface

In the case of the intersection of two semi-implicit surface we can, as in the previous paragraph, obtain a plane curve which is birational to the intersection curve. To be more precise let S' be a semi-implicit surface defined by both polynomials $F'(x, y, z, w; s', t')$ and $G'(x, y, z, w; s', t')$ of respective bi-degree (k'_1, d'_1) and (k'_2, d'_2). We are interesting in the intersection (space) curve of S and S' in $\mathbb{P}^3$ (with homogeneous variables x, y, z, w).

As previously we are going to use a resultant. However we need a more general resultant that the Sylvester one. We are going to use the resultant of four homogeneous polynomials in four homogeneous variables. This resultant was introduced by Macaulay [12] and has been since widely studied [13]–[15] being very useful in a wide range of applications (see e.g. [15] for applications in Computer Aided Geometric Design). For a nice introduction to this topic we refer the reader to [2].

It appears that the resultants of F, G, F' and G' with respect to the homogeneous variables x, y, z, w is a polynomial in s and s'. It vanishes at a given point s_0, s'_0 if and only if both surfaces S and S' intersect with these parameters, i.e. there exists (at least) a point $\mathbf{x} \in \mathbb{P}^3$ such that

$$F(\mathbf{x}; s_0) = G(\mathbf{x}; s_0) = F'(\mathbf{x}; s'_0) = G'(\mathbf{x}; s'_0).$$

Let us denote by $R(s, s')$ this polynomial. It defines a plane curve in the plane of coordinates (s, s') which is in correspondence with the intersection (space) curve of S and S'. We can therefore, as in the previous paragraph, apply all the techniques developed by many authors on such a representation of the intersection curve.

IX. CONCLUSION

In this paper we explained how one can develop new algebraic models for representing shapes together with efficient algorithms to compute their local differential geometric invariants, their singularity locus and their intersections. We illustrated our approach with simple examples. To represent a complex shape in our context we will decompose it into smaller parts. Each of these parts should be sliced in order to get a family of curves having similar shapes. Therefore there are at least two main next tasks : one is a detection/recovery problem, and the other one is to reconstruct a model from a set of such parts. In a future work we will compare our algebraic approach which is rigid by nature (this allows to handle complex shapes with few parameters) with other more flexible techniques.

REFERENCES

[1] L. Busé and A. Galligo, "Semi-implicit representations of surfaces in $\mathbb{P}^3$, resultants and applications," to appear in J. of Symbolic Computation.

[2] D. Cox, J. Little, and D. O'Shea, *Using algebraic geometry*, ser. Graduate Texts in Mathematics. Springer, 1998.

[3] D. Hansford and G. Farin, "Curve and surface constructions," in *Handbook of computer aided geometric design*. Amsterdam: North-Holland, 2002, pp. 165–192.

[4] M. Gasca and T. Sauer, "On the history of multivariate polynomial interpolation," *J. Comput. Appl. Math.*, vol. 122, no. 1-2, pp. 23–35, 2000, numerical analysis 2000, Vol. II: Interpolation and extrapolation.

[5] V. Pratt, "Direct least-squares fitting of algebraic surfaces," *Comput. Graphics*, vol. 21, no. 4, pp. 145–152, 1987, SIGGRAPH '87 (Anaheim, CA, 1987).

[6] C. L. Bajaj and I. Ihm, "Algebraic surface design with hermite interpolation," *ACM Transactions on Graphics*, vol. 11, no. 1, pp. 61–91, 1992.

[7] T. Dokken, "Approximate implicitization," in *Mathematical methods for curves and surfaces (Oslo, 2000)*, ser. Innov. Appl. Math. Nashville, TN: Vanderbilt Univ. Press, 2001, pp. 81–102.

[8] A. Galligo and P. Hermunn, "Study of a family of planar quartic curves," in preparation.

[9] H. Blum, "A transformation for extracting new descriptors of shapes," in *Models for the Perception of Speech and Visual Form*, ser. Cambridge, vol. MIT Press, 1967, pp. 362–380.

[10] J. F. Canny and D. Manocha, "A new approach for surface intersection," *International Journal of Computational Geometry and Applications*, vol. 1, no. 4, pp. 491–516, 1991.

[11] S. Krishnan and D. Manocha, "An efficient surface intersection algorithm based on lower dimensional formulation," *ACM Transactions on Computer Graphics*, vol. 16, no. 1, pp. 74–106, 1997.

[12] F. Macaulay, "Some formulae in elimination," *Proc. London Math. Soc.*, vol. 33, pp. 3–27, 1902.

[13] J.-P. Jouanolou, "Le formalisme du résultant," *Adv. in Math.*, vol. 90, no. 2, pp. 117–263, 1991.

[14] I. Gelfand, M. Kapranov, and A. Zelevinsky, *Discriminants, Resultants and Multidimensional Determinants*. Birkhäuser, Boston-Basel-Berlin, 1994.

[15] L. Busé, M. Elkadi, and B. Mourrain, "Using projection operators in computer aided geometric design," in *Topics in Algebraic Geometry and Geometric Modeling*, ser. Contemporary Mathematics, vol. 334. AMS, 2003.