A resultant approach to detect intersecting curves in the projective space

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A RESULTANT APPROACH TO DETECT INTERSECTING CURVES IN $\mathbb{P}^3$

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ABSTRACT. Given two curves in $\mathbb{P}^3$, either implicitly or by a parameterization, we want to check if they intersect. For that purpose, we present and further develop generalized resultant techniques. Our aim is to provide a closed formula in the inputs which vanishes if and only if the two curves intersect. This could be useful in Computer Aided Design, for computing the intersection of algebraic surfaces.

1. Introduction

Resultants have a long history going back to Bézout in the eighteenth century. This algebraic construction received a renewed interest in the last decade with many new contributions and extensions. Generalized resultant is now a major tool in elimination theory, both for theoretical and practical purposes [GKZ94, CLO98].

In the homogeneous case, a generalized resultant is a closed formula which answer a problem of the following kind : let X be an irreducible projective variety in $\mathbb{P}^n$, $\underline{a} = (a_1, \ldots, a_N)$ be parameters, and $f_1(x, \underline{a}), \ldots, f_s(x, \underline{a})$ be s homogeneous polynomials in $x \in X$. A resultant, if it exists, is a polynomial in $\underline{a}$, called $\mathrm{Res}(a)$, such that

$$\mathrm{Res}(a) = 0 \Leftrightarrow \{x \in X \ / \ f_i(x, \underline{a}) = 0\} \neq \emptyset.$$

Most of the works in the subject focus on the case $s = \dim(X) + 1$. However the first author studied in its PhD thesis [Bus01a] (see also [Bus02]) *determinantal* resultants which can be used for some cases with $s > \dim(X) + 1$. Motivated by applications in CAD (Computer Aided Design) we now go further in this direction.

Indeed, a major problem in CAD is the efficient computation of the intersection of two algebraic surface patches in $\mathbb{R}^3$ given by parameterizations. A popular strategy is to first determine the implicit representation of one of the two surfaces, therefore implicitization techniques based on generalized resultants have been developed, including sparse resultants. Unfortunately, they bring up to the difficult question of the description of the base points of the parameterization [Bus01b, Cox01, BJ02].

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The second author proposed in [Gal03] to compute only a semi-implicit representation of a parameterized surface, which is much easier. The next step in this approach requires an algebraic closed formula condition for the intersection of two space curves, one given implicitly and the other one given by a parameterization.

In this paper we build resultant techniques to get original and quite complete answers to that question and to similar ones. We think that some of our results can lead to efficient and precise procedures of practical interest, other ones have only a theoretical interest.

We tried to keep the paper understandable by a large audience of readers. However the general idea developed here is more easily expressed using homological algebra and algebraic geometry. We have explicited the computations in important special cases, and we provided bounds on size and degrees of the expressions.

The paper is divided in four sections organized as follows. Section 2 describes the application from which emerged our question. In section 3 one recalls some basic definitions and the homological technique we will use for our developments. We present our main new tool the determinantal resultant in section 4. In section 5, we prove several propositions and theorems which provide an answer, for each case, to our computational question; this last section ends with a simple example on which we illustrate our different algorithms.

2. Motivations : semi-implicit representation of surfaces

An important question in Computer Geometric Aided Design is to compute precisely and efficiently the intersection of two patches of algebraic surfaces given by rational parametrizations. In [Gal03], we noted that it is much easier to represent one of the two patches by a family of implicitly defined curves than to compute an implicitization of the whole surface. This technique has for instance the advantage, among others, to avoid the difficulties coming from the base points when implicitizing. By generic flatness, except for few curves, this implicitization step can be done in an uniform way for all curves. We called that process semi-implicit representation of a parameterized surface. Moreover in many cases of interest for the applications, these curves have extra algebraic properties such as being determinantal. So we are led to an apparently simpler question but in a relative setting (i.e. on top of a product of one dimensional spaces): detect when two space curves intersect.

The answer to this question defines a curve in a two dimensional space, so in theory it can be given by a generalized resultant. Providing such a resultant is the target of this paper.

The fact that we obtain this equation via a resultant is a guaranty of a good numerical stability of the output. Moreover we can keep the result as a compact straight line program made by computation of some minors. Once we have this equation, the work for solving the CAGD problem can be finished by (certified) numerical computations. These numerical computations will lift the obtained real curve to a real curve in the four dimensional space where live the parameters of the input parametrized surfaces.

3. Preliminaries and previous results

This section is devoted to some known results about multigraded resultants. Our aim is here to recall the (co)homological techniques, that we will heavily use in section 4; they were introduced in [KSZ92] to study these resultants via determinants of complexes. As particular cases we recover the well-known Sylvester and Dixon resultants.

3.1. **Determinants of complexes.** Given a finite complex of free R-modules, where R is a factorial regular noetherian integral domain, which is generically exact (that is exact after tensorization by the field of fractions of R), we can associate to it a *determinant* (also called MacRae's invariant) which measures the part of its homology supported in codimension 1. We refer the reader to [KM76] and [GKZ94] appendix A for the general definition and properties. In this paper we will only need to compute determinants of complexes in the simple case of four-term exact complexes of finite-dimensional vector spaces. We recall a known method, going back to Cayley [Cay48], to do this.

Suppose that we have a four-term exact complex of vector spaces

$$0 \to V_3 \xrightarrow{\partial_2} V_2 \xrightarrow{\partial_1} V_1 \xrightarrow{\partial_0} V_0 \to 0. \tag{1}$$

Since ∂_0 is surjective, V_1 decomposes into $V_0 \oplus V_1'$ and $\partial_1 = \begin{pmatrix} \phi_0 & m_0 \end{pmatrix}$ with $\det(\phi_0) \neq 0$. Now, since $\mathrm{Im}(\partial_1) = \ker(\partial_0)$, V_2 decomposes into $V_1' \oplus V_3$ and $\partial_1 = \begin{pmatrix} m_1 & m_2 \\ \phi_1 & m_3 \end{pmatrix}$ with $\det(\phi_1) \neq 0$. Finally since ∂_2 is injective and $\mathrm{Im}(\partial_2) = \ker(\partial_1)$, we have $\partial_2 = \begin{pmatrix} m_4 \\ \phi_2 \end{pmatrix}$ with $\det(\phi_2) \neq 0$. The determinant of the complex (1) is then obtained as the quotient $\frac{\det(\phi_0)\det(\phi_2)}{\det(\phi_1)}$, and is independent of choices we made.

Note that if $V_3 = 0$, we can make the same decomposition which shows that the determinant of (1) is a quotient of the form $\frac{\det(\phi_0)}{\det(\phi_1)}$. Similarly, when moreover $V_2 = 0$, we recover the standard notion of determinant since the determinant of (1) is then the determinant of the map ∂_0.

3.2. **Classical multigraded resultants.** We recall results about classical resultants in the context of multigraded homogeneous polynomials, that is to say polynomials which are homogeneous in sets of variables. Let $l_1, \ldots, l_r$ be positive integers, and let $\mathbf{x}_1, \ldots, \mathbf{x}_r$ be r sets of variables consisting respectively in $l_t + 1$ variables $\mathbf{x}_t = (x_{t,0}, x_{t,1}, \ldots, x_{t,l_t})$, for all $t = 1, \ldots, r$. We denote by $S(\mathbf{d})$, where $\mathbf{d} = (d_1, \ldots, d_r)$, the vector space of multi-homogeneous polynomials defined over the variety $X = \mathbb{P}^{l_1} \times \ldots \times \mathbb{P}^{l_r}$ of multi-degree

$d_1, \ldots, d_r$, that is to say polynomials in the ring $S = \mathbb{K}[\mathbf{x}_1, \ldots, \mathbf{x}_r]$ which are homogeneous of degree d_t in the set of variables $\mathbf{x}_t$ for all $t = 1, \ldots, r$.

Let $s = \dim(X) = l_1 + \ldots + l_r$, and fix $s+1$ sequences $\mathbf{d}_0, \ldots, \mathbf{d}_s$, each consisting of r strictly positive integers. There exists an irreducible polynomial in $\mathbb{Z}[S(\mathbf{d}_0) \oplus \ldots \oplus S(\mathbf{d}_s)]$ (unique up to sign), called the *multigraded resultant* on X of type $(\mathbf{d}_0, \ldots, \mathbf{d}_s)$ and denoted by $\mathrm{Res}_{X,\mathbf{d}_0,\ldots,\mathbf{d}_s}$, such that, for all $s+1-$uple $(f_0, \ldots, f_s) \in S(\mathbf{d}_0) \oplus \ldots \oplus S(\mathbf{d}_s)$,

$$\mathrm{Res}_{X,\mathbf{d}_0,\ldots,\mathbf{d}_s}(f_0, \ldots, f_s) = 0 \Leftrightarrow \exists x \in X \ : \ f_0(x) = \cdots = f_s(x) = 0.$$

The existence of this polynomial follows from [KSZ92] section 5 (see also [Stu93] for a different presentation). It is known to be multi-homogeneous of degree N_i in each vector space $S(\mathbf{d}_i)$, $i = 0, \ldots, s$. The integers N_i are determined by the intersection formula $\int_X \prod_{j \neq i} c_1(\mathcal{O}_X(\mathbf{d}_j))$, where c_1 denotes the first Chern class (explicit formulas are given in [SZ94]).

The technique used in [KSZ92] to define and study this resultant is very powerful to derive explicit formulas, it can be summarized as follows. The vanishing of the multigraded resultant expresses a failure of a complex of sheaves to be exact. This allows to construct a class of complexes of finite-dimensional vector spaces whose determinants are the resultant. To every collection of polynomials $f_0, f_1, \ldots, f_s$ in $S(\mathbf{d}_0) \oplus \ldots \oplus S(\mathbf{d}_s)$, we associate its Koszul complex of sheaves $\mathcal{K}_\bullet$ on X :

$$\mathcal{K}_{s+1} \xrightarrow{\partial_{s+1}} \mathcal{K}_s \xrightarrow{\partial_s} \cdots \xrightarrow{\partial_2} \mathcal{K}_1 \xrightarrow{\partial_1} \mathcal{K}_0 = \mathcal{O}_X,$$

where $\mathcal{K}_p = \wedge^p(\oplus_{i=0}^s \mathcal{O}_X(-\mathbf{d}_i))$, $p = 0, \ldots, s+1$, and $\partial_1 = (f_0, \ldots, f_s)$. This complex is known to be exact if and only if $f_0, \ldots, f_s$ have no common zero in X; it is hence generically exact. To go from $\mathcal{K}_\bullet$ to complexes of finite-dimensional vector spaces, we tensor $\mathcal{K}_\bullet$ by an invertible sheaf $\mathcal{M} = \mathcal{O}_X(m_1, \ldots, m_r)$ on X, where $m_1, \ldots, m_r$ are integers, then take global sections. In order to preserve the generic exactness of $\mathcal{K}_\bullet$ when taking global sections, the set of integers $m_1, \ldots, m_r$ have to satisfy the following conditions.

Proposition 3.1. *Let $\mathcal{M} = \mathcal{O}_X(m_1, \ldots, m_r)$ be any invertible sheaf on X. If for all $p = 0, \ldots, s+1$ and all integer $j > 0$ the cohomology groups $H^j(X, \mathcal{K}_p \otimes_{\mathcal{O}_X} \mathcal{M})$ vanish, then $\mathrm{Res}_{X,\mathbf{d}_0,\ldots,\mathbf{d}_s}(f_0, \ldots, f_s)$ equals the determinant of the complex of global sections of $\mathcal{K}_\bullet \otimes_{\mathcal{O}_X} \mathcal{M}$.*

Let us denote by $\mathbf{m}$ a sequence $(m_1, \ldots, m_r)$, and by $K_\bullet(\mathbf{m}; f_0, \ldots, f_s)$ the complex of global sections of $\mathcal{K}_\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_X(\mathbf{m})$. Its first differential on the far right is naturally identified with the (generically surjective) map ∂_1 :

$$\begin{array}{rcl} \oplus_{i=0}^s S(\mathbf{m} - \mathbf{d}_i) & \xrightarrow{\partial_1} & S(\mathbf{m}) \\ (g_0, \ldots, g_s) & \mapsto & f_0 g_0 + f_1 g_1 + \ldots + f_s g_s. \end{array} \tag{2}$$

As a consequence of being the determinant of $K_\bullet(\mathbf{m}; f_0, \ldots, f_s)$, we have the corollary (see [GKZ94] theorem 34) :

Corollary 3.2. *Under the hypothesis of proposition 3.1, the multigraded resultant* $\mathrm{Res}_{X,\mathbf{d}_0,\ldots,\mathbf{d}_s}(f_0,\ldots,f_s)$ *equals the gcd of all the determinants of the square minors of size* $\dim_{\mathbb{K}}(S(\mathbf{m}))$ *of the map* (2).

In fact the hypothesis needed in proposition 3.1 (and corollary 3.2) is purely combinatorial, for we know (see for instance [Har77] theorem III.5.1) that for all $j \geq 0$, $k \geq 1$ and any $n \in \mathbb{Z}$, the cohomology group

$$H^j(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(n)) \neq 0 \Leftrightarrow (n \geq 0 \text{ and } j = 0), \text{ or } (n < -k \text{ and } j = k). \tag{3}$$

This property gives a systematic way to deduce which integers $m_1,\ldots,m_r$ are valid for a given multigraded resultant type. We illustrate this technique with the two following examples of the well-known Sylvester and Dixon resultants and refer to [SZ94] for a more systematic and detailed use of these tools to find formulas for general multigraded resultants.

Example 3.3. We suppose here $r = 1$ and $l_1 = 1$, that is $X = \mathbb{P}^1$, and consider the resultant of two bivariate homogeneous polynomials f_0, f_1 of respective degree $\mathbf{d}_0 := d_1$ and $\mathbf{d}_1 := d_1$ in variables $\mathbf{x}_1 = (x_0, x_1)$. Such a resultant was first studied by Sylvester, it corresponds to the polynomial in $\mathbb{Z}[S(d_0) \oplus S(d_1)]$ we have denoted $\mathrm{Res}_{\mathbb{P}^1,d_0,d_1}(f_0,f_1)$, and is called the Sylvester resultant.

The Koszul complex associated to f_0 and f_1 tensorized by an invertible sheaf $\mathcal{M} = \mathcal{O}_X(n)$ on X, with $n \in \mathbb{Z}$, is of the form :

$$\mathcal{O}_X(n-d_0-d_1) \xrightarrow{\binom{f_1}{-f_0}} \mathcal{O}_X(n-d_0) \oplus \mathcal{O}_X(n-d_1) \xrightarrow{(f_0,f_1)} \mathcal{O}_X(n). \tag{4}$$

By proposition 3.1 we know that $\mathrm{Res}_{X,d_0,d_1}(f_0,f_1)$ is the determinant of the complex of global sections of (4), providing n is chosen such that the cohomology groups $H^1(X, \mathcal{O}_X(n-d_0))$, $H^1(X, \mathcal{O}_X(n-d_1))$ and $H^1(X, \mathcal{O}_X(n-d_0-d_1))$ vanish. By (3) we deduce easily that n has to be greater or equal to $d_0 + d_1 - 1$, and hence that $\mathrm{Res}_{X,d_0,d_1}(f_0,f_1)$ can be computed as the determinant of each complex

$$S(n-d_0-d_1) \xrightarrow{\binom{f_1}{-f_0}} S(n-d_0) \oplus S(n-d_1) \xrightarrow{(f_0,f_1)} S(n),$$

with $n \geq d_0 + d_1 - 1$. In particular, if we choose $n = d_0 + d_1 - 1$, then $S(n-d_0-d_1) = \emptyset$. Thus we deduce that $\mathrm{Res}_{X,d_0,d_1}(f_0,f_1)$ is the determinant of the well-known Sylvester's matrix

$$S(d_1-1) \oplus S(d_0-1) \xrightarrow{(f_0,f_1)} S(d_0+d_1-1).$$

Notice that $\mathrm{Res}_{\mathbb{P}^1,d_0,d_1}$ is homogeneous in $S(d_0)$ of degree $\int_X c_1(\mathcal{O}_X(d_1)) = d_1$, and homogeneous in $S(d_1)$ of degree d_0.

Example 3.4. Here we consider the case of three bi-homogeneous polynomials of same bi-degree $\mathbf{d} = (d_1, d_2)$ in both sets of variables $\mathbf{x}_1 = (x_{1,0}, x_{1,1})$ and $\mathbf{x}_2 = (x_{2,0}, x_{2,1})$, defined on $X = \mathbb{P}^1 \times \mathbb{P}^1$. This case was originally studied by A.L. Dixon in its famous paper [Dix08]. The corresponding bigraded resultant $\mathrm{Res}_{\mathbb{P}^1\times\mathbb{P}^1,\mathbf{d},\mathbf{d},\mathbf{d}}(f_0,f_1,f_2)$, where f_0, f_1, f_2 are polynomials in $S(d_1, d_2)$, is called the Dixon resultant. To compute it we begin with the Koszul complex associated to these polynomials, tensorized by an invertible sheaf $\mathcal{M} = \mathcal{O}_X(m, n)$, where $m, n \in \mathbb{Z}$:

$$0 \to \mathcal{O}_X(m - 3d_1, n - 3d_2) \to \mathcal{O}_X(m - 2d_1, n - 2d_2)^3$$
$$\to \mathcal{O}_X(m - d_1, n - d_2)^3 \to \mathcal{O}_X(m, n).$$

Going to global sections, $\mathrm{Res}_{\mathbb{P}^1\times\mathbb{P}^1,\mathbf{d},\mathbf{d},\mathbf{d}}(f_0, f_1, f_2)$ is then obtained as the determinant of this complex if the couple (m, n) is chosen such that

$$(5) \qquad H^j(X, \mathcal{O}_X(m - id_1, n - id_2)) = 0 \ \text{ for all } j = 1, 2 \text{ and } i = 0, 1, 2, 3.$$

By *Künneth formula* (see [Wei94] section 3.6) we know that

$$H^2(X, \mathcal{O}_X(m-id_1, n-id_2)) = H^1(\mathbb{P}^1, \mathcal{O}_X(m-id_1))\otimes H^1(\mathbb{P}^1, \mathcal{O}_X(n-id_2)),$$

and thus, using (3), we see that condition (5) is satisfied when $j = 2$ if either $m \geq 3d_1 - 1$ or $n \geq 3d_2 - 1$. Always by Künneth formula, the first cohomology group $H^1(X, \mathcal{O}_X(m - id_1, n - id_2))$ is the direct sum of

$$H^1(\mathbb{P}^1, \mathcal{O}_X(m - id_1)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_X(n - id_2))$$

and

$$H^0(\mathbb{P}^1, \mathcal{O}_X(m - id_1)) \otimes H^1(\mathbb{P}^1, \mathcal{O}_X(n - id_2)).$$

Using this equality it follows that (5) is satisfied for all couples such that $(m = 3d_1-1, n \geq 2d_2-1)$, $(m \geq 3d_1, n \geq 3d_2-1)$, $(m \geq 2d_1-1, n = 3d_2-1)$, and $(m \geq 3d_1 - 1, n \geq 3d_2)$. For all these couples $\mathrm{Res}_{X,\mathbf{d},\mathbf{d},\mathbf{d}}(f_0, f_1, f_2)$ is obtained as the determinant of the complex

$$S(m-3d_1, n-3d_2) \xrightarrow{\partial_3} S(m-2d_1, n-2d_2)^3 \xrightarrow{\partial_2} S(m-d_1, n-d_2)^3 \xrightarrow{\partial_1} S(m, n),$$

where the differentials $\partial_1, \partial_2, \partial_3$ are given by the matrices :

$$\partial_1 = \begin{pmatrix} f_0 & f_1 & f_2 \end{pmatrix}, \partial_2 = \begin{pmatrix} 0 & f_2 & f_1 \\ f_2 & 0 & -f_0 \\ -f_1 & -f_0 & 0 \end{pmatrix}, \partial_3 = \begin{pmatrix} f_2 \\ -f_1 \\ f_0 \end{pmatrix}.$$

If we want to obtain $\mathrm{Res}_{X,\mathbf{d},\mathbf{d},\mathbf{d}}(f_0, f_1, f_2)$ as the determinant of a single matrix of type (2), we have to find, if it exists, a couple (m, n) satisfying (5) and also such that $S(m - 2d_1, n - 2d_2)$ is empty. It is a straightforward computation to see that this happens for both couple $(2d_1 - 1, 3d_2 - 1)$ and $(3d_1 - 1, 2d_2 - 1)$. It follows that $\mathrm{Res}_{X,\mathbf{d},\mathbf{d},\mathbf{d}}(f_0, f_1, f_2)$ is the determinant of both maps

$$S(d_1 - 1, 2d_2 - 1)^3 \xrightarrow{\partial_1} S(2d_1 - 1, 3d_2 - 1),$$

and

$$S(2d_1 - 1, d_2 - 1)^3 \xrightarrow{\partial_1} S(3d_1 - 1, 2d_2 - 1),$$

which was given by A.L. Dixon in [Dix08], section 9.

We can also compute the multi-degree of $\mathrm{Res}_{X,\mathbf{d},\mathbf{d},\mathbf{d}}(f_0, f_1, f_2)$. It is homogeneous in each of the three spaces $S(d_1, d_2)$ of degree $\int_X c_1(\mathcal{O}_X(d_1, d_2))^2 = d_1d_2$, and hence of total degree $3d_1d_2$.

4. Some determinantal resultants in low dimension

In this section we consider another construction of resultants which generalizes the preceding one and was called *determinantal resultants* by the first author in [Bus02] (see also [Bus01a]). We first present it, and then we further study it for the special purposes of this article. In particular, we generalize the Sylvester resultant to the determinantal situation that we will use in section 5. We also generalize the Dixon resultant to the determinantal situation and study a determinantal resultant to detect the intersection of two parameterized space curves.

4.1. **Determinantal multigraded resultants.** Let $X = \mathbb{P}^{l_1} \times \ldots \mathbb{P}^{l_r}$, where $l_1, \ldots, l_r$ are positive integers, and denote by $\mathbf{x}_1, \ldots, \mathbf{x}_r$ the sets of the corresponding homogeneous variables. For any sequence of r integers $\mathbf{d} := (d_1, \ldots, d_r)$, we denote by $S(\mathbf{d})$ the vector space of polynomials in $S = \mathbb{K}[\mathbf{x}_1, \ldots, \mathbf{x}_r]$ which are homogeneous of degree d_t in variables $\mathbf{x}_t$ for all $t = 1, \ldots, r$. We can view the classical resultant as a tool which provides a necessary and sufficient condition so that the matrix $(f_0, \ldots, f_s)$, defined over X of dimension s, is of rank lower or equal to zero in at least one point of X. This interpretation leads to determinantal resultants which give a necessary and sufficient condition so that a given polynomial matrix, defined over X, is of rank lower or equal to a given integer p. In what follows we focus on determinantal resultants in the particular case where $X = \mathbb{P}^{l_1} \times \ldots \times \mathbb{P}^{l_r}$, and where p is the greatest possible integer.

Let $\mathbf{k}_1, \ldots, \mathbf{k}_n$ be $n \geq 1$ sequences of r integers and $\mathbf{d}_1, \ldots, \mathbf{d}_{n+s}$ be $n+s$ other sequences of r integers. We can consider the vector space H of all the homogeneous map $\oplus_{i=1}^{n+s} \mathcal{O}_X(-\mathbf{d}_i) \to \oplus_{i=1}^{n} \mathcal{O}_X(-\mathbf{k}_i)$. It is the vector space of all the polynomial matrices

$$\begin{pmatrix} h_{1,1} & h_{1,2} & \cdots & h_{1,n+s} \\ h_{2,1} & h_{2,2} & \cdots & h_{2,n+s} \\ \vdots & \vdots & & \vdots \\ h_{n,1} & h_{n,2} & \cdots & h_{n,n+s} \end{pmatrix}, \tag{6}$$

where $h_{i,j} \in S(\mathbf{d}_j - \mathbf{k}_i)$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, n+s$. If all the $n(n+s)$ sequences $\mathbf{d}_j - \mathbf{k}_i$, with $i = 1, \ldots, n$ and $j = 1, \ldots, n+s$, are sequences of strictly positive integers, then there exists a polynomial in $\mathbb{Z}[H]$ (defined up to a non zero constant multiple), called the determinantal resultant on $X = \mathbb{P}^{l_1} \times \ldots \times \mathbb{P}^{l_r}$ of type $(\mathbf{d}_1, \ldots, \mathbf{d}_{n+s}; \mathbf{k}_1, \ldots, \mathbf{k}_n)$ and denoted by $\mathrm{Res}_{X,(\mathbf{d}_1,\ldots,\mathbf{d}_{n+s};\mathbf{k}_1,\ldots,\mathbf{k}_n)}$, such that, for all homogeneous map $\phi : \oplus_{i=1}^{n+s} \mathcal{O}_X(-\mathbf{d}_i) \to \oplus_{i=1}^{n} \mathcal{O}_X(-\mathbf{k}_i)$,

$$\mathrm{Res}_{X,(\mathbf{d}_1,\ldots,\mathbf{d}_{n+s};\mathbf{k}_1,\ldots,\mathbf{k}_n)}(\phi) = 0 \Leftrightarrow \exists x \in X : \mathrm{rank}(\phi(x)) \leq n-1.$$

This polynomial is multi-homogeneous in the coefficients of each column of the matrix (6), its multidegree is computed using the Thom-Porteous intersection formula in [Bus02], proposition 2.5.

Remark 4.1. This determinantal resultant depends on the sequences $\mathbf{d}_j - \mathbf{k}_i$, that is to say on H. It is possible to change the $\mathbf{d}_j$'s and the $\mathbf{k}_i$'s without changing the resultant. Note also that the case $n = 1$ and $\mathbf{k}_1 = (0, \ldots, 0)$ gives the classical multigraded resultant recalled in 3.2.

A technique similar to the one exposed in 3.2 can be used to compute this determinantal multigraded resultant. It involves a generalization of the Koszul complex, the so-called Eagon-Northcott complex (see e.g. [BV80]). To every homogeneous map $\phi \in H$ we associate its Eagon-Northcott complex of sheaves $\mathcal{E}_\bullet$ on X which is of the form

$$\mathcal{E}_{s+1} \xrightarrow{\partial_{s+1}} \mathcal{E}_s \xrightarrow{\partial_s} \cdots \xrightarrow{\partial_2} \mathcal{E}_1 \xrightarrow{\partial_1} \mathcal{E}_0 = \mathcal{O}_X,$$

where $\mathcal{E}_p = \wedge^{n+p-1} E \otimes S^{p-1} F^* \otimes \wedge^n F^*$, for all $p = 1, \ldots, s+1$, with the notations $E = \oplus_{i=1}^{n+s} \mathcal{O}_X(-\mathbf{d}_i)$ and $F = \oplus_{i=1}^{n} \mathcal{O}_X(-\mathbf{k}_i)$.

Proposition 4.2. *Let $\mathcal{M} = \mathcal{O}_X(m_1, \ldots, m_r)$ be any invertible sheaf on X. If for all $p = 0, \ldots, s+1$ and all integer $j > 0$ the cohomology groups $H^j(X, \mathcal{E}_p \otimes_{\mathcal{O}_X} \mathcal{M})$ vanish, then $\mathrm{Res}_{X,(\mathbf{d}_1,\ldots,\mathbf{d}_n+s;\mathbf{k}_1,\ldots\mathbf{k}_n)}(\phi)$ equals the determinant of the complex of global sections of $\mathcal{E}_\bullet \otimes_{\mathcal{O}_X} \mathcal{M}$.*

Let us denote by $E_\bullet(\mathbf{m}; \phi)$ the complex of global sections of the complex of sheaves $\mathcal{E}_\bullet \otimes_{\mathcal{O}_X} \mathcal{O}_X(\mathbf{m})$, where $\mathbf{m}$ is a sequence of r integers $(m_1, \ldots, m_r)$. Its first differential on the far right is naturally identified with the (generically surjective) map ∂_1 :

$$\begin{array}{lccc} (7) & \displaystyle\bigoplus_{1 \le i_1 < \ldots < i_n \le n+s} S(\mathbf{m} - \mathbf{e}_{i_1,\ldots,i_n}) & \xrightarrow{\partial_1} & S(\mathbf{m}) \\ & (\ldots, g_{i_1,\ldots,i_n}, \ldots) & \mapsto & \displaystyle\sum_{1 \le i_1 < \ldots < i_n \le n+s} g_{i_1,\ldots,i_n} \Delta_{i_1,\ldots,i_n}, \end{array}$$

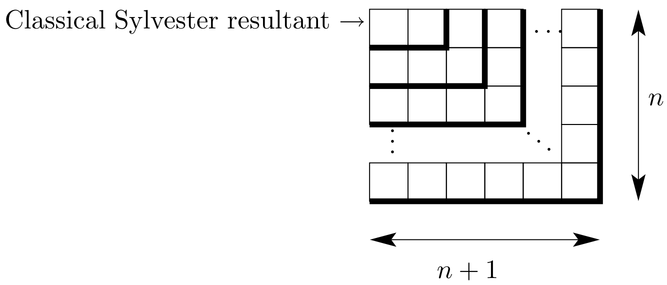
where $\Delta_{i_1,\ldots,i_n}$ is the determinant of the submatrix of (6) corresponding to columns $i_1, \ldots, i_n$; the sequence $\mathbf{e}_{i_1,\ldots,i_n}$ denotes its multi-degree. We have the corollary :

Corollary 4.3. *Under the hypothesis of proposition 4.2, the determinantal multigraded resultant $\mathrm{Res}_{X,(\mathbf{d}_1,\ldots,\mathbf{d}_{n+s};\mathbf{k}_1,\ldots\mathbf{k}_n)}(\phi)$ equals the gcd of all the square minors of size $\dim_{\mathbb{K}}(S(\mathbf{m}))$ of the map (7).*

We now present in more detail two particular situations which correspond to generalizations of the Sylvester resultant and of the Dixon resultant. We will end this section with a determinantal resultant specially designed to detect the intersection of two parameterized space curves.

4.2. **Determinantal Sylvester resultants.** Let n be a positive integer, we consider matrices with polynomial entries in $\mathbb{P}^1$ (more precisely homogeneous matrices defined in $\mathbb{P}^1$) of size $n \times (n+1)$, and give a necessary and sufficient condition on the coefficients of these polynomial entries so that the rank drops. The case $n = 1$, that is a 1×2 matrix, corresponds to the classical Sylvester resultant. If we add one line and one column we obtain a new

Sylvester resultant corresponding to the vanishing of all the 2×2 minors of a 2×3 matrix. This is graphically illustrated by the following figure



Theorem 4.4. *Let n be a positive integer, and $(d_1, \ldots, d_{n+1})$, $(k_1, \ldots, k_n)$ be two sequences of integers such that $d_i - k_j > 0$ for all i, j. For any morphism $\phi : \oplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^1}(-d_i) \to \oplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(-k_i)$ and for any integer*

$$m \geq \sum_{j=1}^{n+1} d_j - \sum_{j=1}^{n} k_j - \min_{j=1,\ldots,n} k_j - 1,$$

the determinantal resultant $\mathrm{Res}_{\mathbb{P}^1,(d_1,\ldots,d_{n+1};k_1,\ldots,k_n)}(\phi)$ is the determinant of the finite-dimensional vector spaces

$$\bigoplus_{i=1}^{n} S(m+k_i - \sum_{j=1}^{n+1} d_j + \sum_{j=1}^{n} k_j) \xrightarrow{\phi^*} \bigoplus_{i=1}^{n+1} S(m+d_i - \sum_{j=1}^{n+1} d_j + \sum_{j=1}^{n} k_j) \xrightarrow{\wedge^n \phi} S(m).$$

Moreover, if $k := k_1 = \ldots = k_n$ then $\mathrm{Res}_{\mathbb{P}^1,(d_1,\ldots,d_{n+1};k_1,\ldots,k_n)}(\phi)$ is the determinant of the square matrix of size $d_1 + \ldots + d_{n+1} - (n+1)k$,

$$\bigoplus_{i=1}^{n+1} S(d_i - k - 1) \xrightarrow{\wedge^n \phi} S\Big(\sum_{j=1}^{n+1} (d_j - k) - 1\Big).$$

Before giving the proof of this theorem, we recall that $\wedge^n \phi$ is defined, for all integer m, by

$$\bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^1}(d_i - \sum_{j=1}^{n+1} d_j + \sum_{j=1}^{n} k_j) \xrightarrow{\wedge^n \phi} \mathcal{O}_{\mathbb{P}^1} \; : \; (g_1, \ldots, g_{n+1}) \mapsto \sum_{i=1}^{n+1} (-1)^{i-1} g_i \Delta_i,$$

where Δ_i, for $i = 1, \ldots, n+1$, denotes the determinant of the matrix ϕ without its i^{th} column, and that ϕ^* denotes the dual of ϕ, that is

$$\bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^1}(k_i) \xrightarrow{\phi^*} \bigoplus_{i=1}^{n+1} \mathcal{O}_{\mathbb{P}^1}(d_i) \; : \; (g_1, \ldots, g_n) \mapsto (\ldots, \sum_{i=1}^{n} g_i \phi_{i,k}, \ldots)_{k=1,\ldots,n+1}.$$

Proof. Let m be any integer and $\phi : \oplus_{i=1}^{n+1}\mathcal{O}_{\mathbb{P}^1}(-d_i) \to \oplus_{i=1}^{n}\mathcal{O}_{\mathbb{P}^1}(-k_i)$. The Eagon-Northcott complex of ϕ, tensorized by the invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(m)$ is of the form :

$$\bigoplus_{i=1}^{n}\mathcal{O}_{\mathbb{P}^1}(m+k_i-\sum_{j=1}^{n+1}d_j+\sum_{j=1}^{n}k_j) \to \bigoplus_{i=1}^{n+1}\mathcal{O}_{\mathbb{P}^1}(m+d_i-\sum_{j=1}^{n+1}d_j+\sum_{j=1}^{n}k_j) \to \mathcal{O}_{\mathbb{P}^1}(m).$$

From proposition 4.2 and (3) it comes easily that the determinant of the complex of global sections of this complex is the determinantal resultant of ϕ for all $m \geq \sum_{j=1}^{n+1} d_j - \sum_{j=1}^{n} k_j - \min_{j=1,\ldots,n} k_j - 1$, and this proves the first claim. Taking for instance the lowest possible value m_0 of m, we obtain that $\mathrm{Res}_{\mathbb{P}^1,(d_1,\ldots,d_{n+1};k_1,\ldots,k_n)}(\phi)$ is the determinant of the following finite-dimensional vector spaces complex

$$\bigoplus_{i=1}^{n} S(k_i - \min_j k_j - 1) \xrightarrow{\phi^*} \bigoplus_{i=1}^{n+1} S(d_i - \min_j k_j - 1) \xrightarrow{\wedge^n\phi} S(m_0).$$

But the vector space $\bigoplus_{i=1}^{n} S(k_i - \min_j k_j - 1)$ is zero if and only if $k_1 = \ldots = k_n$, which proves the second claim. □

Remark 4.5. The first statement of this theorem implies that such a resultant is obtained either as the quotient of two determinants, or as the gcd of all minors of size $\dim(S(m))$ of the map $\wedge^n\phi$.
Note also that, by remark 4.1, we can suppose that the minimum of the integers $k_1, \ldots, k_n$ is zero without changing the resultant. For instance we can simplify the second statement of the theorem by taking $d_i - k$ instead of d_i for all $i = 1, \ldots, n+1$, and then $k = 0$. Thus the square matrix is of size $d_1 + \ldots + d_{n+1}$ (to be compared with the classical Sylvester matrix which is of size $d_1 + d_2$).

We can also give the multidegree of the determinantal Sylvester resultant.

Proposition 4.6. *Let n be a positive integer, and let $(d_1, \ldots, d_{n+1})$, $(k_1, \ldots, k_n)$ be two sequences of integers such that $d_i - k_j > 0$ for all i, j. The determinantal resultant $\mathrm{Res}_{\mathbb{P}^1,(d_1,\ldots,d_{n+1};k_1,\ldots,k_n)}$ is an irreducible polynomial in $\mathbb{Z}[S(d_i - k_j); \forall i, j]$ which is, for all $i = 1, \ldots, n+1$, homogeneous in $\oplus_{j=1}^{n} S(d_i - k_j)$ of degree*

$$N_i := \sum_{j=1}^{n+1} d_j - \sum_{j=1}^{n} k_j - d_i.$$

It is of total degree

$$\deg(\mathrm{Res}_{\mathbb{P}^1,(d_1,\ldots,d_{n+1};k_1,\ldots,k_n)}) = n\sum_{j=1}^{n+1} d_j - (n+1)\sum_{j=1}^{n} k_j.$$

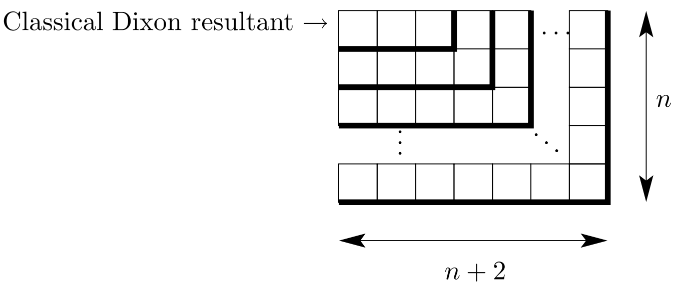
Proof. We use the result of [Bus02], section 5.1, which says that the resultant $\mathrm{Res}_{\mathbb{P}^1,(d_1,\ldots,d_{n+1};k_1,\ldots,k_n)}$ is homogeneous in the coefficients of the i^{th}

column of the generic map $\phi : \oplus_{i=1}^{n+1}\mathcal{O}_{\mathbb{P}^1}(-d_i) \to \oplus_{i=1}^{n}\mathcal{O}_{\mathbb{P}^1}(-k_i)$, that is in $\oplus_{j=1}^{n}S(d_i - k_j)$, of degree N_i. Each integer N_i, for all $i = 1,\dots,n+1$, is obtained as the coefficient of α_i of the multivariate polynomial (in variables $\alpha_1,\dots,\alpha_{n+1}$) computed itself as the coefficient of the monomial t^2 in the univariate polynomial (in variable t) :

$$\frac{\prod_{i=1}^{n+1}(1-(d_i+\alpha_i)t)}{\prod_{i=1}^{n}(1-k_it)}.$$

By a straightforward computation one finds $\sum_{j=1}^{n+1} d_j - \sum_{j=1}^{n} k_j - d_i$. $\square$

4.3. **Determinantal Dixon resultants.** In this subsection we consider determinantal resultants in $X = \mathbb{P}^1 \times \mathbb{P}^1$, assuming that $\mathbf{d}_1 = \mathbf{d}_2 = \ldots = \mathbf{d}_{n+2} = (-d_1, -d_2)$, with n, d_1 and d_2 positive integers, and $\mathbf{k}_1 = \mathbf{k}_2 = \ldots = \mathbf{k}_n = (0,0)$. These determinantal resultants, that we denote by $\mathrm{Res}_{\mathbb{P}^1\times\mathbb{P}^1,(d_1,d_2)^n}$ for simplicity, can be seen as generalizations of the well-known Dixon resultant (see example 3.4), obtained for $n = 1$:



Theorem 4.7. *Let n, d_1, d_2 be three positive integers. For any morphism $\phi : \oplus_{i=1}^{n+2}\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-d_1,-d_2) \to \oplus_{i=1}^{n}\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$, the resultant $\mathrm{Res}_{\mathbb{P}^1\times\mathbb{P}^1,(d_1,d_2)^n}(\phi)$ is the determinant of both square matrices*

$$\bigoplus_{i=1}^{\frac{(n+2)(n+1)}{2}} S(d_1-1, 2d_2-1) \xrightarrow{\wedge^n\phi} S((n+1)d_1-1,(n+2)d_2-1),$$

$$\bigoplus_{i=1}^{\frac{(n+2)(n+1)}{2}} S(2d_1-1, d_2-1) \xrightarrow{\wedge^n\phi} S((n+2)d_1-1,(n+1)d_2-1),$$

of size $(n+2)(n+1)d_1d_2$.

Moreover $\mathrm{Res}_{\mathbb{P}^1\times\mathbb{P}^1,(d_1,d_2)^n}$ is homogeneous in the coefficients of each column of the morphism ϕ of degree $(n+1)nd_1d_2$; its total degree is $(n+2)(n+1)nd_1d_2$.

Proof. Let (p,q) be a couple of integers and $\phi : \oplus_{i=1}^{n+2}\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-d_1,-d_2) \to \oplus_{i=1}^{n}\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}$. The Eagon-Northcott of ϕ, tensorized by the invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(p,q)$ on $X = \mathbb{P}^1 \times \mathbb{P}^1$, is of the form :

$$\bigoplus_{i=1}^{\frac{n(n-1)}{2}} \mathcal{O}_X(p-(n+2)d_1, q-(n+2)d_2) \to \bigoplus_{i=1}^{n^2} \mathcal{O}_X(p-(n+1)d_1, q-(n+1)d_2)$$

$$\to \bigoplus_{i=1}^{\frac{(n+2)(n+1)}{2}} \mathcal{O}_X(p-nd_1, q-nd_2) \xrightarrow{\wedge^n\phi} \mathcal{O}_X(p,q).$$

To identify the possible values of integers p and q such that the determinant of the complex of global sections of this complex gives the resultant $\mathrm{Res}_{\mathbb{P}^1\times\mathbb{P}^1,(d_1,d_2)^n}(\phi)$, we have to check two conditions. First p and q must be such that

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(p-id_1)) \otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(q-id_2)) = 0, \text{ for } i = n, n+1, n+2.$$

This implies that $p \geq (n+2)d_1 - 1$ or $q \geq (n+2)d_2 - 1$. The second condition is the simultaneous vanishing of

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(p-id_1)) \otimes H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(q-id_2))$$

and

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(p-id_1)) \otimes H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(q-id_2))$$

for all $i = n, n+1, n+2$. Then, the following sets of couples

$$(p = (n+2)d_1 - 1, q \geq (n+1)d_2 - 1), (p \geq (n+2)d_1, q \geq (n+2)d_2 - 1),$$

$$(p \geq (n+1)d_1 - 1, q = (n+2)d_2 - 1), \text{ and } (p \geq (n+2)d_1, q \geq (n+2)d_2),$$

give complexes of finite-dimensional vector spaces whose determinant is exactly the determinantal resultant. In fact, among these couples of integers (p,q), two of them reduces to a single map. They correspond to integers p and q such that the cohomology group $H^0(X, \mathcal{O}_X(p-(n+1)d_1, q-(n+1)d_2))$ vanishes, that is such that $p \leq (n+1)d_1 - 1$ or $q \leq (n+1)d_2 - 1$. We deduce that our determinantal resultant is the determinant of both maps :

$$\bigoplus_{i=1}^{\frac{(n+2)(n+1)}{2}} S(d_1-1, 2d_2-1) \xrightarrow{\wedge^n\phi} S((n+1)d_1-1, (n+2)d_2-1),$$

$$\bigoplus_{i=1}^{\frac{(n+2)(n+1)}{2}} S(2d_1-1, d_2-1) \xrightarrow{\wedge^n\phi} S((n+2)d_1-1, (n+1)d_2-1),$$

which give square matrices of size $(n+2)(n+1)d_1d_2$.

The multidegree of $\mathrm{Res}_{\mathbb{P}^1\times\mathbb{P}^1,(d_1,d_2)^n}(\phi)$ is easily obtained from the determinant of both preceding maps; this determinantal resultant is homogeneous of degree $(n+1)nd_1d_2$ in the coefficients of each column of the matrix ϕ. It is hence of total degree $(n+2)(n+1)nd_1d_2$. $\square$

4.4. **Determinantal resultants of two parameterized space curves.** As we have seen in the previous subsection, determinantal Dixon resultants deal with matrices having bihomogeneous polynomials of bidegree (d_1, d_2) entries, where $d_1 \geq 1$ and $d_2 \geq 1$. The integers d_1 and d_2 are supposed to be positive in order to fulfill a very ampleness hypothesis used to define determinantal resultants in [Bus02], theorem 2.1. However, this theorem states the existence of determinantal resultants in the general setting of morphisms between two arbitrary vector bundles E and F on a projective and irreducible variety X. In some much more simple situations, the very ampleness hypothesis (which says that the vector bundle $\mathcal{H}om(E, F)$ is very ample on X) can be weakened. We now consider such a case, which geometrically corresponds to intersect two parameterized space curves.

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and consider the vector space H of all the homogeneous map $\mathcal{O}_X^4 \to \mathcal{O}_X(m, 0) \oplus \mathcal{O}_X(0, n)$, with $m \geq 1$ and $n \geq 1$. It is the vector space of all the polynomial matrices

$$\begin{pmatrix} f_0(s,t) & f_1(s,t) & f_2(s,t) & f_3(s,t) \\ g_0(u,v) & g_1(u,v) & g_2(u,v) & g_3(u,v) \end{pmatrix}, \tag{8}$$

where $f_i(s, t)$ are homogeneous polynomials of degree m in $\mathbb{P}^1$, and $g_i(u, v)$ are homogeneous polynomials of degree n in $\mathbb{P}^1$. We can think of both lines of these matrices as parameterized space curves.

As we just mentioned, denoting $E = \mathcal{O}_X^4$ and $F = \mathcal{O}_X(m, 0) \oplus \mathcal{O}_X(0, n)$, the vector bundle $\mathcal{H}om(E, F)$ is not very ample on $X = \mathbb{P}^1 \times \mathbb{P}^1$, but the determinantal resultant of morphisms from E to F is well defined; we will denote it by $\mathrm{Res}_{X,m,n}$.

Theorem 4.8. *Let m and n be two positive integers. The determinantal resultant $\mathrm{Res}_{X,m,n}$ is well defined, it is homogeneous of degree n in the coefficients of the first line, of degree m in the coefficients of the second line, and hence of total degree mn. For any morphism $\phi : \mathcal{O}_X^4 \to \mathcal{O}_X(m, 0) \oplus \mathcal{O}_X(0, n)$, it is the determinant of the following Eagon-Northcott finite-dimensional complexes of vector spaces*

$$0 \to S(p - 3m; q - n) \oplus S(p - 2m; q - 2n) \oplus S(p - m; p - 3n) \xrightarrow{\partial_2}$$

$$S(p - 2m; q - n)^4 \oplus S(p - m; q - 2n)^4 \xrightarrow{\partial_1} S(p - m; q - n)^6 \xrightarrow{\wedge^2 \phi} S(p; q),$$

for all $p \geq 3m - 1$ and $q \geq 3n - 1$.

In particular, it vanishes if and only if the rank of the $9mn \times 24mn$ matrix

$$S(2m - 1; 2n - 1)^6 \xrightarrow{\wedge^2 \phi} S(3m - 1, 3n - 1)$$

drops.

Proof. First we justify the existence of $\mathrm{Res}_{X,m,n}$. Recall that we denote $E = \mathcal{O}_X^4$, $F = \mathcal{O}_X(m, 0) \oplus \mathcal{O}_X(0, n)$ and $H = \mathrm{Hom}(E, F)$. Although the vector bundle $\mathcal{H}om(E, F)$ is not very ample on X, the proof of [Bus02] theorem 1 applies. In this proof the very ampleness hypothesis is used to show that the projection from the incidence variety W to the projectivized parameter space $\mathbb{P}(H)$

$$W = \{(x, \phi) \in X \times \mathbb{P}(H) : \mathrm{rank}(\phi(x)) \leq 1\} \to \mathbb{P}(H)$$

is birational onto its image (which is called the resultant variety). The argument is the following : given a zero-dimensional subscheme z of $\mathbb{P}^1 \times \mathbb{P}^1$ of degree two (that is two distinct points or a double point), the locus of matrices ϕ in $\mathbb{P}(H)$ of rank lower or equal to 1 on z is of codimension twice the codimension of matrices of rank lower or equal to 1 on only one smooth point (which is here 3). In our particular situation, this property remains true, even if $\mathcal{H}om(E, F)$ is not very ample.

The remaining of the proof is a standard use of techniques previously exposed. Choose a morphism $\phi : \mathcal{O}_X^4 \to \mathcal{O}_X(m, 0) \oplus \mathcal{O}_X(0, n)$. Its associated Eagon-Northcott complex is of the form

$$0 \to \mathcal{O}_X(-3m; -n) \oplus \mathcal{O}_X(-2m; -2n) \oplus \mathcal{O}_X(-m; -3n) \xrightarrow{\partial_2}$$

$$\mathcal{O}_X(-2m; -n)^4 \oplus \mathcal{O}_X(-m; -2n)^4 \xrightarrow{\partial_1} \mathcal{O}_X(-m; -n)^6 \xrightarrow{\wedge^2 \phi} \mathcal{O}_X.$$

After tensorization by an invertible sheaf $\mathcal{O}_X(p; q)$, it appears, after a straightforward computation, that the condition of proposition 3.1 to preserve generic exactness when taking global sections is $p \geq 3m - 1$ and $q \geq 3n - 1$.

We can also compute the degree of this resultant using the results of [Bus02]. Again a straightforward computation shows that $\mathrm{Res}_{X,m,n}$ is homogeneous in the coefficients of the first line of degree n, and homogeneous in the coefficients of the second line of degree m; it is hence of total degree mn. □

Note that other similar determinantal resultants exist. For instance we can consider the determinantal resultant of 2×5 matrices with one line corresponding to a parameterized curve from $\mathbb{P}^1$ to $\mathbb{P}^4$, and the second line corresponding to a parameterized surface from $\mathbb{P}^2$ (or $\mathbb{P}^1 \times \mathbb{P}^1$) to $\mathbb{P}^4$; this determinantal resultant is thus defined over $X = \mathbb{P}^1 \times \mathbb{P}^2$ (or, if the second line is defined on $\mathbb{P}^1 \times \mathbb{P}^1$, $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$).

5. Detecting space curves intersection

Given two families of space curves depending respectively on a parameter λ and a parameter μ, we would like to give a necessary and sufficient condition on these parameters so that two curves of each family intersect. Here by a parameter we mean either a single variable or a list of independent variables. There is naturally two ways for giving a space curve: a parametric and an implicit formulation. In what follows we discuss the three different cases consisting of detecting the intersection of two families of parameterized curves, or two families of implicit curves, or finally a parameterized curves family and an implicit curves family. We end with an expanded example.

Note that we will always suppose hereafter that the parameterized curves have no base points, but this is not restrictive since we can remove base points via gcd computations. Hereafter we denote by X, Y, Z, T the homogeneous coordinates of the projective space $\mathbb{P}^3$.

5.1. **Intersection of two families of implicit curves.** Let $\mathcal{C}_\lambda$ and $\mathcal{D}_\mu$ be two families of space curves that we suppose here given implicitly. We would like to yield a condition on the parameters λ and μ so that these two families intersect in $\mathbb{P}^3$. The simplest case is to suppose that $\mathcal{C}_\lambda$ and $\mathcal{D}_\mu$ are families of *complete intersection* space curves. This means that $\mathcal{C}_\lambda$ (resp. $\mathcal{D}_\mu$) is given by two polynomials, say $H_1(\lambda, X, Y, Z, T)$ and $H_2(\lambda, X, Y, Z, T)$ (resp. $H_3(\mu, X, Y, Z, T)$ and $H_4(\mu, X, Y, Z, T)$), homogeneous in variables X, Y, Z, T, which have no common factor for all possible values of λ (resp. μ). The condition we are looking for is then a polynomial in λ and μ which is nothing but the classical resultant over $\mathbb{P}^3$,

$$\mathrm{Res}_{\mathbb{P}^3}(H_1, H_2, H_3, H_4) \in \mathbb{K}[\lambda, \mu],$$

which eliminates the variables X, Y, Z, T. This resultant can be computed from all the graded part ν of the first map (and the whole) Koszul complex associated to the sequence H_1, H_2, H_3, H_4, if $\nu \geq \sum_{i=1}^4 \deg(H_i) - 3$. In the general situation a similar property holds, based on a known result of elimination theory that we now recall (see [Jou80] and [Laz77]). We provide a proof for the convenience of the reader, and also to shed light on the proof of theorem 5.4 to come.

Theorem 5.1. *Let A be a noetherian commutative ring, $n \geq 1$ be a given integer, and $C = A[X_1, \ldots, X_n]$, where $X_1, \ldots, X_n$ are indeterminates. We denote $\mathfrak{m} = (X_1, \ldots, X_n)$ the irrelevant ideal of C. Let $f_1, \ldots, f_r$ be $r \geq n$ homogeneous polynomials in C of respective degree $d_1 \geq d_2 \geq \ldots \geq d_r \geq 1$. Both following statements are equivalent :*

1) *$\exists\, n \in \mathbb{N}$ such that $\mathfrak{m}^n \subset (f_1, \ldots, f_r)$,*
2) *The map of free A-modules $\oplus_{i=1}^r C(-d_i)_\nu \xrightarrow{(f_1,\ldots,f_r)} C_\nu$ is surjective for all $\nu \geq \delta := d_1 + d_2 + \ldots + d_n - n + 1$.*

Proof. Let us denote by $K^\bullet(\mathbf{f}; C)$ the Koszul complex associated to the sequence $(f_1, \ldots, f_r)$ in C. We have $K^q(\mathbf{f}; C) = 0$ for all $q > 0$ and $q < -r$, and $K^q(\mathbf{f}; C) = \wedge^{-q}(C^r)$ for $-r \leq q \leq 0$ (we here adopt the cohomological notation). We denote $H^q(\mathbf{f}; C) := H^q(K^\bullet(\mathbf{f}; C))$ for all $q \in \mathbb{Z}$. Note that since the polynomials $f_1, \ldots, f_r$ are homogeneous, all the C-modules $K^q(\mathbf{f}; C)$ and $H^q(\mathbf{f}; C)$ are graded. From the exact sequence of A-modules

$$\oplus_{i=1}^r C(-d_i)_\nu \xrightarrow{(f_1,\ldots,f_r)} C_\nu \to H^0(\mathbf{f}; C)_\nu \to 0,$$

we deduce immediately that 2) implies 1), and also that we have to prove that $H^0(\mathbf{f}; C)_\nu$ for all $\nu \geq \delta$ to show that 1) implies 2).

First we denote by $C^\bullet_{\mathfrak{m}}(C)$ the standard Cech complex

$$0 \to C \to \oplus_{i=1}^n C_{X_i} \to \oplus_{1 \leq i < j \leq n} C_{X_i X_j} \to \ldots \to C_{X_1 \ldots X_n} \to 0,$$

and recall that $H^i_{\mathfrak{m}}(C) \simeq H^i(C^\bullet_{\mathfrak{m}}(C))$. We can now construct the bicomplex $K^{\bullet\bullet}(\mathbf{f};C) := K^\bullet(\mathbf{f};C) \otimes C^\bullet_{\mathfrak{m}}(C)$ which gives two spectral sequences having the same limit

$$\begin{cases} {}'E_1^{p,q} = H^q(\mathbf{f}; C^p_{\mathfrak{m}}(C)) \Rightarrow E_\infty^{p,q} \\ {}''E_2^{p,q} = H^q(\mathbf{f}; H^p_{\mathfrak{m}}(C)) \Rightarrow E_\infty^{p,q} \end{cases},$$

where, for any C-module M, $H^q(\mathbf{f};M)$ denotes $H^q(\mathbf{f};C) \otimes_C M$. On one hand, as we suppose 1), we deduce that the support of $H^\bullet(\mathbf{f};C)$ is contained in $\mathfrak{m}$, and hence that ${}'E_1^{p,q} = 0$ for all $p \neq 0$, and one the other hand ${}''E_2^{p,q} = 0$ for all $p \neq n$ since $H^i_{\mathfrak{m}}(C) = 0$ for all $i \neq n$. It follows in particular that we have an isomorphism of graded C-modules $H^0(\mathbf{f};C) \simeq H^{-n}(\mathbf{f}; H^n_{\mathfrak{m}}(C))$. Since $H^n_{\mathfrak{m}}(C)_\nu = 0$ for all $\nu \geq n+1$, we deduce that $H^0(\mathbf{f};C)_\nu = 0$ for all $\nu \geq \delta$. $\square$

Let us return to our problem. We suppose now that the family $\mathcal{C}_\lambda$ is given by n polynomials $H_i(\lambda, X, Y, Z, T)_{i=1,\dots,n}$, homogeneous in the variables X, Y, Z, W of respective degree $d_1 \geq d_2 \geq \dots \geq d_n \geq 1$, such that for all possible values of λ they define a space curve. Similarly we suppose that $\mathcal{D}_\mu$ is given by m polynomials $H_i(\mu, X, Y, Z, T)_{i=n+1,\dots,n+m}$, homogeneous in the variables X, Y, Z, W of respective degree $d_{n+1} \geq d_{n+2} \geq \dots \geq d_{n+m} \geq 1$.

Proposition 5.2. *With the above notations, let δ be the sum of the four greatest integers in the set $\{d_1, \dots, d_{n+m}\}$, minus 3. Then for all values of λ and μ, both space curves $\mathcal{C}_\lambda$ and $\mathcal{D}_\mu$ intersect if and only if the map*

$$\begin{array}{rcl} \oplus_{i=1}^{n+m} \mathbb{K}[X,Y,Z,T]_{\delta - d_i} & \to & \mathbb{K}[X,Y,Z,T]_\delta \\ (g_1, \dots, g_{n+m}) & \mapsto & \sum_{i=1}^{n+m} g_i H_i, \end{array}$$

is not surjective.

The matrices involved in this proposition are in general quite big, and almost never square. We know examine the case of two families of parameterized curves.

5.2. **Intersection of two families of parameterized curves.** Suppose given two families $\mathcal{C}_\lambda$ and $\mathcal{D}_\mu$ of *parameterized* space curves without base points. The family $\mathcal{C}_\lambda$ corresponds to four homogeneous polynomials of degree $m \geq 1$, $f_{0,\lambda}(s,t), f_{1,\lambda}(s,t), f_{2,\lambda}(s,t),\ f_{3,\lambda}(s,t)$ without common factor for all possible value of λ, and the family $\mathcal{D}_\mu$ to four homogeneous polynomials of degree $n \geq 1$, $g_{0,\mu}(u,v), g_{1,\mu}(u,v), g_{2,\mu}(u,v),\ g_{3,\mu}(u,v)$ without common factor for all possible value of μ. We can detect their intersection with the following proposition, using the results of subsection 4.4 :

Proposition 5.3. *With the above assumptions, the determinantal resultant*

$$\mathrm{Res}_{\mathbb{P}^1 \times \mathbb{P}^1, m, n} \begin{pmatrix} f_{0,\lambda}(s,t) & f_{1,\lambda}(s,t) & f_{2,\lambda}(s,t) & f_{3,\lambda}(s,t) \\ g_{0,\mu}(u,v) & g_{1,\mu}(u,v) & g_{2,\mu}(u,v) & g_{3,\mu}(u,v) \end{pmatrix}$$

vanishes at $\lambda_0, \mu_0 \in \mathbb{K}$ if and only if $\mathcal{C}_{\lambda_0}$ and $\mathcal{D}_{\mu_0}$ intersect in $\mathbb{P}^3$.

Proof. This proposition is clear since we have supposed that our curves have no base points. $\square$

However, this resultant involves big matrices.

Another way to eliminate variables s, t, u, v from the six 2×2 minors of the matrix

$$\begin{pmatrix} f_{0,\lambda}(s,t) & f_{1,\lambda}(s,t) & f_{2,\lambda}(s,t) & f_{3,\lambda}(s,t) \\ g_{0,\mu}(u,v) & g_{1,\mu}(u,v) & g_{2,\mu}(u,v) & g_{3,\mu}(u,v) \end{pmatrix}$$

is to study, as in theorem 5.1, their associated Koszul complex. This is our next theorem. We have restricted ourselves to the case of bi-graded polynomials of same bi-degree, but the technique used in the proof applies similarly to multi-graded polynomials of any multi-degree.

Theorem 5.4. *Let A be a noetherian commutative ring. Let $C_1 = A[s,t]$ with irrelevant ideal $\mathfrak{m}_1 = (s,t) \subset C_1$, and $C_2 = A[u,v]$ with irrelevant ideal $\mathfrak{m}_2 = (u,v) \subset C_2$. Define the bi-graded ring $C = C_1 \otimes_A C_2$ with irrelevant ideal $\mathfrak{m} = \mathfrak{m}_1\mathfrak{m}_2$, and suppose given $f_1, \dots, f_r$ bi-homogeneous polynomials in C of the same bi-degree (m,n), such that $r \geq 3$, $m \geq 1$ and $n \geq 1$. Both following statements are equivalent :*

1) *$\exists\, n \in \mathbb{N}$ such that $\mathfrak{m}^n \subset (f_1, \dots, f_r)$,*

2) *The map of free A-modules $C(-m,-n)^r_{\nu_1,\nu_2} \xrightarrow{(f_1,\dots,f_r)} C_{\nu_1,\nu_2}$ is surjective for all (ν_1,ν_2) such that $\nu_1 \geq 3m-3$ and $\nu_2 \geq 3n-3$.*

Proof. As in the proof of theorem 5.1, let us denote by $K^\bullet(\mathbf{f};C)$ the Koszul complex associated to the sequence $(f_1,\dots,f_r)$ in C, and its cohomology $H^q(\mathbf{f};C) := H^q(K^\bullet(\mathbf{f};C))$ for all $q \in \mathbb{Z}$. Note that since the polynomials $f_1,\dots,f_r$ are bi-homogeneous, all the C-modules $K^q(\mathbf{f};C)$ and $H^q(\mathbf{f};C)$ are naturally bi-graded. From the exact sequence of A-modules

$$C(-m,-n)^r_{\nu_1,\nu_2} \xrightarrow{(f_1,\dots,f_r)} C_{\nu_1,\nu_2} \to H^0(\mathbf{f};C)_{\nu_1,\nu_2} \to 0,$$

we have 2) implies 1), and to prove 1) implies 2) we just have to show that $H^0(\mathbf{f};C)_{\nu_1,\nu_2} = 0$ for all (ν_1,ν_2) such that $\nu_1 \geq 3m-3$ and $\nu_2 \geq 3n-3$.

Always as theorem 5.1, we denote by $C^\bullet_{\mathfrak{m}}(C)$ the standard Cech complex

$$0 \to C \to \oplus_{i=1}^4 C_{m_i} \to \oplus_{1\leq i<j\leq 4} C_{m_i m_j} \to \dots \to C_{m_1m_2m_3m_4} \to 0,$$

where $\mathfrak{m} = (m_1,m_2,m_3,m_4)$, that is $m_1 = su$, $m_2 = sv$, $m_3 = tu$ and $m_4 = tv$. Recall also that $H^i_{\mathfrak{m}}(C) \simeq H^i(C^\bullet_{\mathfrak{m}}(C))$. We can now construct the bicomplex $K^{\bullet\bullet}(\mathbf{f};C) := K^\bullet(\mathbf{f};C) \otimes C^\bullet_{\mathfrak{m}}(C)$ which gives two spectral sequences having the same limit

$$\begin{cases} {}'E_1^{p,q} = H^q(\mathbf{f};C^p_{\mathfrak{m}}(C)) \Rightarrow E_\infty^{p,q} \\ {}''E_2^{p,q} = H^q(\mathbf{f};H^p_{\mathfrak{m}}(C)) \Rightarrow E_\infty^{p,q} \end{cases}.$$

First, as we suppose 1), we deduce that the support of $H^\bullet(\mathbf{f};C)$ is contained in $\mathfrak{m}$, and hence that ${}'E_1^{p,q} = 0$ for all $p \neq 0$. To study ${}''E_2^{p,q}$, we have to compute the local cohomology modules $H^i_{\mathfrak{m}}(C)$, which can be done (once again) with the Künneth formula.

First, clearly $\mathrm{depth}_{\mathfrak{m}}(C) \geq 2$, and hence $H^0_{\mathfrak{m}}(C) = H^1_{\mathfrak{m}}(C) = 0$. Now let $U_i = \mathrm{Spec}(C_i) \setminus V(\mathfrak{m}_i)$, $i = 1, 2$. The fibred product $U = U_1 \times_{\mathrm{Spec}(A)} U_2$ is nothing but $\mathrm{Spec}(C) \setminus V(\mathfrak{m})$, and the Künneth formula gives for all $p \geq 0$:

$$H^p(U, \mathcal{O}_U) = \bigoplus_{p_1+p_2=p} H^{p_1}(U_1, \mathcal{O}_{U_1}) \otimes H^{p_2}(U_2, \mathcal{O}_{U_2}).$$

As $H^{p+1}_{\mathfrak{m}}(C) = H^p(U, \mathcal{O}_U)$ for all $p \geq 1$, and also similarly $H^{p+1}_{\mathfrak{m}_i}(C_i) = H^p(U_i, \mathcal{O}_{U_i})$ for $i = 1, 2$ and $p \geq 1$, we deduce that $H^p_{\mathfrak{m}}(C) = 0$ for $p \geq 4$ and

$$H^2_{\mathfrak{m}}(C) = C_1 \otimes H^2_{\mathfrak{m}_2}(C_2) \oplus H^2_{\mathfrak{m}_1}(C_1) \otimes C_2, \; H^3_{\mathfrak{m}}(C) = H^2_{\mathfrak{m}_1}(C_1) \otimes H^2_{\mathfrak{m}_2}(C_2).$$

Returning to our bicomplex, we deduce that $''E^{p,q}_2 = 0$ for all $p \notin \{2, 3\}$, and obtain an isomorphism of bi-graded C-modules

$$H^0(\mathbf{f}; C) \simeq H^{-2}(\mathbf{f}; H^2_{\mathfrak{m}}(C)) \oplus H^{-3}(\mathbf{f}; H^3_{\mathfrak{m}}(C)).$$

A straightforward computation using our knowledge of the local cohomology of the bi-graded ring C shows that $H^0(\mathbf{f}; C)_{\nu_1,\nu_2} = 0$ for all (ν_1, ν_2) such that $\nu_1 \geq 3m - 3$ and $\nu_2 \geq 3n - 3$. □

5.3. **Intersection of a family of parameterized curves with a family of implicit curves.** Let $\mathcal{C}_\lambda$ be a family of *parameterized* space curves without base points, that is four homogeneous polynomials $f_{0,\lambda}(s, t), f_{1,\lambda}(s, t)$, $f_{2,\lambda}(s, t)$, $f_{3,\lambda}(s, t)$ without common factor for all possible value of λ.

Let $\mathcal{D}_\mu$ be another family of space curves given *implicitly*. The simplest situation is when $\mathcal{D}_\mu$ is a family of complete intersection implicit curves, that is represented by two homogeneous polynomials $H_{0,\mu}(X, Y, Z, T)$ and $H_{1,\mu}(X, Y, Z, T)$ depending on the parameter μ, without common factor for any value of μ. In this way, for any specialization of μ in $\mathbb{K}$, the intersection of both surfaces defined by $H_{0,\mu}$ and $H_{1,\mu}$ is a (complete intersection) curve in $\mathbb{P}^3$. It is then easy to see that the Sylvester resultant

$$\mathrm{Res}_{\mathbb{P}^1}\big(H_{0,\mu}(f_{0,\lambda}, f_{1,\lambda}, f_{2,\lambda}, f_{3,\lambda}), H_{1,\mu}(f_{0,\lambda}, f_{1,\lambda}, f_{2,\lambda}, f_{3,\lambda})\big) \in \mathbb{K}[\lambda, \mu]$$

vanishes at $\lambda_0, \mu_0 \in \mathbb{K}$ if and only if both space curves $\mathcal{C}_{\lambda_0}$ and $\mathcal{D}_{\mu_0}$ intersect in $\mathbb{P}^3$. This resultant being computed as the determinant of a square matrix (see 3.3), we deduce an easy algorithm to test the intersection of two such families of curves, but we have to suppose that the family $\mathcal{D}_\mu$ is a family of complete intersection curves, which is quite restrictive.

Using the generalized Sylvester resultant we have developed in 4.2, we can weaken the complete intersection hypothesis and obtain a similar algorithm for a wide range of families of implicit curves, namely the families of *determinantal* implicit curves.

We suppose now that the family of implicit curves $\mathcal{D}_\mu$ is a flat family of arithmetically Cohen-Macaulay curves in $\mathbb{P}^3$. By the Hilbert-Burch theorem (see [Eis94] theorem 20.15), such a family corresponds to a homogeneous map

$$\phi_\mu : \oplus_{i=1}^{n+1}\mathbb{K}[X,Y,Z,T](-d_i) \to \oplus_{i=1}^{n}\mathbb{K}[X,Y,Z,T](-k_i),$$

where n is a positive integer and $d_i > k_j$ for all i,j, depending on the parameter μ such that the ideal of all its $n \times n$ minors, denoted $I_n(\phi_\mu)$, is of codimension 2 for all possible value of μ. The map ϕ_μ corresponds to a polynomial matrix

$$(9) \qquad \begin{pmatrix} \phi_{1,1,\mu} & \phi_{1,2,\mu} & \cdots & \phi_{1,n+1,\mu} \\ \phi_{2,1,\mu} & \phi_{2,2,\mu} & \cdots & \phi_{2,n+1,\mu} \\ \vdots & \vdots & & \vdots \\ \phi_{n,1,\mu} & \phi_{n,2,\mu} & \cdots & \phi_{n,n+1,\mu} \end{pmatrix}$$

where $\phi_{i,j,\mu}$ is a homogeneous polynomial in variable X,Y,Z,T of degree $d_j - k_i$, depending on the parameter μ.

Proposition 5.5. *With the above assumptions, the determinantal resultant*

$$\mathrm{Res}_{\mathbb{P}^1,(d_1,\ldots,d_{n+1};k_1,\ldots,k_n)}\big(\phi_\mu(f_{0,\lambda},f_{1,\lambda},f_{2,\lambda},f_{3,\lambda})\big)$$

vanishes at $\lambda_0,\mu_0 \in \mathbb{K}$ if and only if $\mathcal{C}_{\lambda_0}$ and $\mathcal{D}_{\mu_0}$ intersect in $\mathbb{P}^3$.

Notice that, as we have seen in 4.2, this determinantal resultant is the quotient of two determinants. If moreover we have $k_1 = \ldots = k_n$, then it is the determinant of a single matrix.

5.4. **An example.** Hereafter we apply all the techniques developed in this section to an explicit example involving a family of cubics and a family of conics, our aim being to compare the different formulations we have given. We made our computations with the software `Macaulay2` [GS], using a package[1] providing functions to compute different kinds of resultants.

Our first family is a family of cubics depending on a single parameter λ. The parametric formulation is given by

$$\begin{array}{rcl} \mathcal{C}_\lambda : \mathbb{P}^1 & \to & \mathbb{P}^3 \\ (s:t) & \mapsto & (s^3, s^2t - t^3, \lambda s^2 t + st^2, -s^3 + t^3). \end{array}$$

This family of cubics is in fact a determinantal family of cubics, their implicit equations in $\mathbb{P}^3$ are obtained as the 2×2 minors of the matrix

$$\begin{pmatrix} X & X+Y+T & -X\lambda - Y\lambda - T\lambda + Z \\ X+Y+T & -X\lambda - Y\lambda - T\lambda + Z & X+T \end{pmatrix}.$$

Our second family is a family of conics depending on a single parameter μ. The parameterization is

$$\begin{array}{rcl} \mathcal{D}_\mu : \mathbb{P}^1 & \to & \mathbb{P}^3 \\ (u:v) & \mapsto & (uv, \mu v^2 + u^2, \mu v^2, v^2), \end{array}$$

[1] available at `http://math.unice.fr/~lbuse/m2package.html`

and this family is implicitly a complete intersection given by both equations :

$$-X^2 + YT - ZT, \; -\mu T + Z.$$

Using the implicit/implicit technique to test the intersection of $\mathcal{C}_\lambda$ and $\mathcal{D}_\mu$ we obtain a 56×115 matrix. Computing a 56×56 minor we get the condition we are looking for :

$$\lambda^6\mu^3 + 3\lambda^6\mu^2 - 3\lambda^5\mu^3 + 3\lambda^6\mu - 6\lambda^5\mu^2 - \lambda^3\mu^4 + \lambda^6 - 3\lambda^5\mu - 4\lambda^4\mu^2 + 4\lambda^3\mu^3 + 10\lambda^2\mu^4 + 6\lambda\mu^5 + \mu^6 - 5\lambda^4\mu + 10\lambda^3\mu^2 + 14\lambda^2\mu^3 - 3\lambda\mu^4 - 3\mu^5 - \lambda^4 + 5\lambda^3\mu + 4\lambda^2\mu^2 - 3\lambda\mu^3 + 4\mu^4 + \lambda^3 - 2\lambda^2\mu + 8\lambda\mu^2 - \mu^3 + \lambda\mu + 2\mu^2 - \lambda + 4\mu + 1.$$

The parametric/parametric technique is better : we obtain a 28×48 matrix, whose a 28×28 minor gives also the desired condition. Both these results are however not satisfactory because we obtain quite big matrices for curves of such low degrees. We now investigate the mixed situation parametric/implicit. In our case, we can consider either $\mathcal{C}_\lambda$ or $\mathcal{D}_\mu$ as the implicit family. We begin by considering that $\mathcal{D}_\mu$ is the implcit family. Our condition is then given by a classical Sylvester resultant : it is obtained as the determinant of a 9×9 matrix. Now considering that $\mathcal{C}_\lambda$ is the implicit family the condition we are looking for is given in a very compact way by the following 6×6 matrix

$$\begin{pmatrix}
-1 & 0 & \lambda & 0 & -\lambda^2 & 0 \\
-\lambda-2 & -1 & 2\lambda & \lambda & -2\lambda^2+1 & -\lambda^2 \\
-\lambda-2\mu-3 & -\lambda-2 & 2\lambda\mu+3\lambda-\mu+1 & 2\lambda & -2\lambda^2\mu-3\lambda^2+2\lambda\mu+2 & -2\lambda^2+1 \\
-\lambda\mu-\lambda-\mu-2 & -\lambda-2\mu-3 & 2\lambda\mu+2\lambda-\mu+1 & 2\lambda\mu+3\lambda-\mu+1 & -2\lambda^2\mu-2\lambda^2+2\lambda\mu+\mu+2 & -2\lambda^2\mu-3\lambda^2+2\lambda\mu+2 \\
-\mu^2-2\mu-1 & -\lambda\mu-\lambda-\mu-2 & \lambda\mu^2+2\lambda\mu-\mu^2+\lambda-\mu & 2\lambda\mu+2\lambda-\mu+1 & -\lambda^2\mu^2-2\lambda^2\mu+2\lambda\mu^2-\lambda^2+2\lambda\mu-\mu^2+\mu+1 & -2\lambda^2\mu-2\lambda^2+2\lambda\mu+\mu+2 \\
0 & -\mu^2-2\mu-1 & 0 & \lambda\mu^2+2\lambda\mu-\mu^2+\lambda-\mu & 0 & -\lambda^2\mu^2-2\lambda^2\mu+2\lambda\mu^2-\lambda^2+2\lambda\mu-\mu^2+\mu+1
\end{pmatrix}$$

These experimental results show, in our point of view, how relevant the concept of semi-implicitization is in this kind of problems.

References

[BJ02] Laurent Busé and Jean-Pierre Jouanolou. On the closed image of a rational map and the implicitization problem. *Preprint math.AG/0210096*, 2002.

[Bus01a] Laurent Busé. *Étude du résultant sur une variété algébrique*. PhD thesis, University of Nice, 2001.

[Bus01b] Laurent Busé. Residual resultant over the projective plane and the implicitization problem. *proceedings ISSAC2001*, pages 48–55, 2001.

[Bus02] Laurent Busé. Determinantal resultant. *Preprint math.AG/0209404*, 2002.

[BV80] Winfried Bruns and Udo Vetter. Determinantal rings. *Lecture Notes in Mathematics*, 1327, 1980.

[Cay48] A. Cayley. On the theory of elimination. *Cambridge and Dublin Math. Journal*, 3:116–120, 1848.

[CLO98] David Cox, J. Little, and D. O'Shea. *Using algebraic geometry*. Graduate Texts in Mathematics. Springer, 1998.

[Cox01] David A. Cox. Equations of parametric curves and surfaces via syzygies. *Contemporary Mathematics*, 286:1–20, 2001.

[Dix08] A. L. Dixon. The eliminant of three quantics in two independent variables. *Proc. London Math. Soc.*, 7:49–69, 1908.

[Eis94] D. Eisenbud. *Commutative Algebra with a view toward Algebraic Geometry*, volume 150 of *Graduate Texts in Math.* Springer-Verlag, 1994.

[Gal03] André Galligo. Semi-implicit representation of parameterized bi-cubic surfaces and applications. *in preparation*, 2003.

[GKZ94] I.M. Gelfand, M.M. Kapranov, and A.V. Zelevinsky. *Discriminants, Resultants and Multidimensional Determinants*. Birkhäuser, Boston-Basel-Berlin, 1994.

[GS] Daniel R. Grayson and Michael E. Stillman. Macaulay 2, a software system for research in algebraic geometry. Available at http://www.math.uiuc.edu/Macaulay2.

[Har77] Robin Hartshorne. *Algebraic Geometry*. Springer-Verlag, 1977.

[Jou80] J.-P. Jouanolou. Idéaux résultants. *Adv. in Math.*, 37:212–238, 1980.

[KM76] F. Knudsen and D. Mumford. The projectivity of the moduli space of stable curves. i: Preliminaries on Det and Div. *Math. Scand.*, 39:19–55, 1976.

[KSZ92] M. M. Kapranov, B. Sturmfels, and A.V. Zelevinsky. Chow polytopes and general resultants. *Duke Math. Journal*, 67:189–218, 1992.

[Laz77] D. Lazard. Algèbre linéaire sur $k[x_1, \ldots, x_n]$ et élimination. *Bull. Soc. math. France*, 105:165–190, 1977.

[Stu93] Bernd Sturmfels. Sparse elimination theory. *Sympos. Math., XXXIV, Cambridge Univ. Press*, pages 264–298, 1993.

[SZ94] Bernd Sturmfels and Andrei Zelevinsky. Multigraded resultants of Sylvester type. *Sympos. Math., XXXIV, Cambridge Univ. Press*, 163:115–127, 1994.

[Wei94] Charles A. Weibel. *An introduction to homological algebra*. Cambridge studies in advanced mathematics : Cambridge University Press, 1994.

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