A character-automorphic Hardy spaces approach to discrete-time scale-invariant systems

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Abstract

We define the scale translation in discrete-time via the action of the group of automorphisms of the disk. Two important tools that we will use are the theory of automorphic functions and the theory of reproducing kernel Hilbert spaces. When the group is Fuchsian and of Widom type, we present a class of signals and systems which are both discrete-scale and discrete-time stationary. Finally a class of digital self-similar signals and systems is presented.

1 Introduction

A natural question which may arise from the analysis of a time signal is whether or not a change of observation scale will reveal a structure, as an indication of a hidden information. In this work, we will focus on the case where such a change in scale results in an abscence of any new structure. In other words, we will be interested to a class of signals and systems presenting the same informative structure for all observation scales. Such a signal or system is said to be scale-invariant or more generally, self-similar. Formally, a continuous-time stochastic process $y(t)$ is said to be wide-sense self-similar if

$$\begin{aligned} E(y(t)) &= \alpha^{-H} E(y(\alpha t)) \\ E\left\{y(t)\overline{y(s)}\right\} &= \alpha^{-2H} E\left\{y(\alpha t)\overline{y(\alpha s)}\right\}, \qquad t, s \in \mathbb{R}, \end{aligned} \tag{1.1}$$

for all *scale factor* $\alpha \in \mathbb{R}_+$, where H is the so-called Hurst parameter. The fractional Brownian motion of Mandelbrot and Van Ness [1] and a number of models derived from it (see *e.g.* [2], [3], [4], [5], [6]) are well-known examples of such processes. Another common definition is given in terms of system theory and input-output signals. From this point of view, a given system $\mathcal{S}$ is termed self-similar with parameter ν if, for a causal input $u(t)$, the relation

$$\mathcal{S}\{u(t)\} = \int_0^\infty K(t,\lambda)u(\lambda)d\lambda = y(t) \Longrightarrow \mathcal{S}\{u(\alpha t)\} = \alpha^{-\nu} y(\alpha t) \tag{1.2}$$

holds for any positive real number α. The self-similar property is then translated into some conditions on the kernel $K(t,\lambda)$. As an example, the kernel

$$K(t,\lambda) = t^{\nu-1}\frac{t}{\lambda}h\left(\frac{t}{\lambda}\right)$$

where $h(\cdot)$ represents a (pseudo) impulse response, was introduced by Yazıcı and Kashyap [7] (see also [8]).

Note that in any cases, the use of Lamperti's transformation [9], which maps "time-shift-invariance" to "scale-shift-invariance" and vice-versa, is a common key feature to the bulk of continuous-time self-similar process studies [7], [8], [10].

In the discrete-time setting however, the self-similar property does not have a clear cut definition, because the scaling operation is not well defined. Though the principle of invariance upon aggregation (see [11] and references therein) seems to attract the widest approval, it appears to us to be more a salient property of the fractional Gaussian noise than a proper definition of discrete-time self-similarity. In this context, the inherently scale-dependant representations such as wavelet-based transforms (see *e.g.* [12], [13]) are undoubtedly among the more successful ones, especially when approximation and/or synthesis is in concern.

In this work, however, our intent is to propose a discrete-time self-similar study, which does not dodge the question of time scaling. Our approach relies on the frequency domain (see also [14]), as was initiated in a previous work [15] for the continuous-time case. We have shown therein that the Laplace transform of the ouput of the model in [7] belongs to the fractional Hardy space with index $\nu > -1$, that is to the reproducing kernel Hilbert space with reproducing kernel

$$\mathcal{K}(w,s) = \frac{\Gamma(1+\nu)}{(s+\overline{w})^{1+\nu}}.$$

This space is isometrically invariant [15] under the transformation which to

f associates the function

$$s \mapsto \alpha^{\frac{1+\nu}{2}} f(\alpha s), \quad \alpha > 0. \tag{1.3}$$

One recognizes the same form as the definition of the self–similarity in the time domain. Our program may now be stated as follows. We discretize (1.3) and verify in section 2 that the scaling operation corresponds, in the discrete-time, to the action of a group of automorphisms of the disk. This then allows us to propose our definition of the scaling operator for discrete-time signals. The linear continuous-scale filtering counterparts of the basic concepts of linear continuous-time filtering are presented. In section 3, the discrete-scale transform is introduced within the framework of Fuchsian groups and character-automorphic Hardy spaces theory. White noise driven signal models, both discrete-time and discrete-scale wide-sense stationary, are described. Finally, we present in section 4 a class of linear discrete-time self-similar systems and signals.

2 Scaling operator for discrete-time signals

2.1 Scale transform in frequency domain

If $F(s)$, $\Re(s) \geqslant 0$, denotes the Laplace transform of a continuous-time signal $f(t)$, $t \geqslant 0$, then it is well-known that, for any $\alpha = 1/\beta > 0$, $\sqrt{\alpha}F(\alpha s)$ is the Laplace transform of $f(\beta t)$. The time scale transform thus reads in a similar form as in the frequency domain. Let $\lambda = e^{i\theta}, |\theta| < \frac{\pi}{2}$ and consider the map $G(s) = \frac{\lambda - s}{\overline{\lambda} + s}$ which sends the right-half plane $\mathbb{C}_+$, conformally onto the unit-disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In all the sequel, we will be interested in the hyperbolic group of linear transformations

$$\Gamma = \left\{ \gamma_{\{\alpha\}}(z) = \frac{\gamma_1 z + \gamma_2}{\overline{\gamma}_2 z + \overline{\gamma}_1}, \quad \gamma_1 = \lambda + \overline{\lambda}\alpha, \quad \gamma_2 = \lambda(1-\alpha); \quad \alpha > 0 \right\} \tag{2.1}$$

As it is well-known, each element of Γ maps $\mathbb{D}$ and the unit-circle $\mathbb{T}$ onto themselves. We subsequently normalize each element $\gamma_{\{\alpha\}} \in \Gamma$ by $|\gamma_1|^2 - |\gamma_2|^2 = 1$ (Γ is a subgroup of $SU(1,1)$).

Lemma 2.1. *Consider the scale transforms of* $\mathbb{C}_+$ *onto itself:* $\alpha \mapsto S_\alpha(s) = \alpha s, \alpha > 0$. *Then any element* $\gamma_{\{\alpha\}} \in \Gamma$ *expresses as:*

$$\gamma_{\{\alpha\}} = G \circ S_\alpha \circ G^{-1} \tag{2.2}$$

Proof. The points $z_1 = \lambda^2$ and $z_2 = -1$ are the two common fixed points of all elements of Γ. Now, any point z is related to its transform $\gamma_{\{\alpha\}}(z)$, $\alpha \neq 1$, by

$$\frac{\gamma_{\{\alpha\}}(z) - z_1}{\gamma_{\{\alpha\}}(z) - z_2} = m_\alpha \frac{z - z_1}{z - z_2} \quad \text{where } m_\alpha = \frac{\gamma_1 - \overline{\gamma}_2 z_1}{\gamma_1 - \overline{\gamma}_2 z_2} \tag{2.3}$$

Multiplying both sides of (2.3) by $-\overline{\lambda}$ yields $G^{-1}(\gamma_{\{\alpha\}}(z)) = m_\alpha G^{-1}(z)$. A direct inspection shows that m_α, the multiplier [16] of $\gamma_{\{\alpha\}}$, is $m_\alpha = \alpha$ and the rest is plain. □

The lemma then shows that the action of Γ on $\mathbb{D}$ is equivalent to the scale operator on $\mathbb{C}_+$. As a direct consequence of (2.2) we have:

$$\gamma_{\{1\}} = id \quad \text{and for all } \alpha, \beta \in \mathbb{R}_+, \gamma_{\{\alpha\}} \circ \gamma_{\{\beta\}} = \gamma_{\{\alpha\beta\}}.$$

The group Γ is therefore Abelian and $\gamma_{\{\alpha\}}^{-1} = \gamma_{\{\frac{1}{\alpha}\}}$.

2.2 Scaling operator for discrete-time signals

We now use the equivalence stated above to define the scaling operation for discrete-time signals. Let $\{x_n\}_{n \geqslant 0}$ be a causal discrete-time signal and let the series $X(z) = \sum_{n \geqslant 0} x_n z^n$ stands for its formal Z transform. We assume that $\{x_n\}_{n \geqslant 0} \in \ell_2$ so that the series $X(z)$ converges to an element of the Hardy class $\mathcal{H}_2(\mathbb{D})$. For each $\alpha > 0$, set

$$X_\alpha(z) = \frac{1}{-\overline{\gamma}_2 z + \gamma_1} X\left(\gamma_{\{\frac{1}{\alpha}\}}(z)\right) = \sum_{n \in \mathbb{Z}} x_n^{\{\alpha\}} z^n. \tag{2.4}$$

We are now ready to give the

Definition 2.2. For a scale $\alpha > 0$, we define the scaling operation on a causal discrete-time signal $\{x_n\}_{n \geqslant 0}$ as the map $\{x_n\}_{n \geqslant 0} \mapsto \left\{x_n^{\{\alpha\}}\right\}_{n \in \mathbb{Z}}$ given above.

Proposition 2.3. The discrete-time scaling operator is a causal unitary map from ℓ_2 onto itself.

The proposition derives from the following lemma.

Lemma 2.4. *To each $\gamma_{\{\alpha\}} \in \Gamma$, let us associate the functions*

$$\Phi_n^{\{\alpha\}}(z) = \frac{1}{\overline{\gamma}_2 z + \overline{\gamma}_1} \left[\gamma_{\{\alpha\}}(z)\right]^n, \quad n = 0, 1, \ldots \tag{2.5}$$

Then, the set $\left\{\Phi_n^{\{\alpha\}}(z)\right\}_{n\geqslant 0}$ *forms a complete orthornormal system in* $\mathcal{H}_2(\mathbb{D})$*. Moreover, for* $\alpha, \beta \in \mathbb{R}_+$*, we have:*

$$\left[\Phi_n^{\{\alpha\}}\right]^{\{\beta\}}(z) = \Phi_n^{\{\alpha\beta\}}(z), \quad \textit{for all } n \geqslant 0 \tag{2.6}$$

Proof of Lemma 2.4. For $\omega, z \in \mathbb{D}$, one may readily check the following identity:

$$1 - \overline{\gamma_{\{\alpha\}}(\omega)}\gamma_{\{\alpha\}}(z) = (1 - \overline{\omega}z)\overline{\Phi_0^{\{\alpha\}}(\omega)}\Phi_0^{\{\alpha\}}(z).$$

Using this identity, we have:

$$\mathcal{K}(\omega, z) = \sum_{n\geqslant 0} \overline{\Phi_n^{\{\alpha\}}(\omega)}\Phi_n^{\{\alpha\}}(z) = \frac{1}{1-\overline{\omega}z}$$

which is the reproducing kernel of $\mathcal{H}_2(\mathbb{D})$. The statement of the first part of the lemma[1] is then proved since $\mathcal{H}_2(\mathbb{D})$ is seperable. For the second statement of the lemma, observe that $\Phi_n^{\{\alpha\}}(z)$ is simply the image of the function z^n under the map $X(z) \mapsto X_{1/\alpha}(z)$ defined in (2.4). Therefore, we have:

$$\begin{aligned}\left[\Phi_n^{\{\alpha\}}\right]^{\{\beta\}}(z) &= \Phi_0^{\{\beta\}}(z)\Phi_n^{\{\alpha\}}(\gamma_{\{\beta\}}(z)) \\ &= \Phi_0^{\{\beta\}}(z)\Phi_0^{\{\alpha\}}(\gamma_{\{\beta\}}(z))[\gamma_{\{\alpha\}}(\gamma_{\{\beta\}}(z))]^n\end{aligned}$$

Now, it is easy to see that for all $\beta > 0$, $[\Phi_0^{\{\beta\}}(z)]^2$ is the derivative of $\gamma_{\{\beta\}}(z)$ and therefore

$$[\Phi_0^{\{\beta\}}(z)\Phi_0^{\{\alpha\}}(\gamma_{\{\beta\}}(z))]^2 = \frac{d}{dz}(\gamma_{\{\beta\}} \circ \gamma_{\{\alpha\}})(z).$$

The rest is plain. □

Proof of Proposition 2.3. Let us write more explicitly the map $\{x_n\}_{n\geqslant 0} \mapsto \left\{x_n^{\{\alpha\}}\right\}_{n\in\mathbb{Z}}$. With $X(z) = \sum_{n\geqslant 0} x_n z^n$, (2.4) reads as:

$$X_{1/\alpha}(z) = \Phi_0^{\{\alpha\}}(z) \sum_{n\geqslant 0} x_n [\gamma_{\{\alpha\}}]^n = \sum_{n\geqslant 0} x_n \Phi_n^{\{\alpha\}}(z)$$

Expressing $\Phi_n^{\{\alpha\}}(z) = \sum_{k\geqslant 0} \phi_n^{\{\alpha\}}(k) z^k$ in the standard basis of $\mathcal{H}_2(\mathbb{D})$ yields

$$x_n^{\{\alpha\}} = \sum_{k\geqslant 0} x_k \phi_k^{\{\alpha\}}(n) \tag{2.7}$$

[1]Note that this statement is valid for any $\gamma \in SU(1,1)$.

Since the functions $\Phi_n^{\{\alpha\}}(z), n \geqslant 0$ form an orthonormal basis of $\mathcal{H}_2(\mathbb{D})$, we deduce that the infinite matrix $\mathbf{\Phi}^{\{\alpha\}}$ whose $(n+1)^{th}$ row is formed by $[\phi_0^{\{\alpha\}}(n)\ \phi_1^{\{\alpha\}}(n)\ \cdots]$, is a unitary operator form ℓ_2 onto itself: $\mathbf{\Phi}^{\{\alpha\}}[\mathbf{\Phi}^{\{\alpha\}}]^* = [\mathbf{\Phi}^{\{\alpha\}}]^*\mathbf{\Phi}^{\{\alpha\}} = I$. The discrete-time scaling then appears as a simple change of basis. □

Proposition 2.3 and Lemma 2.4 tell us that every discrete-time stable and causal signal is a scale transform of another such signal:

$$\left[x_n^{\{\alpha\}}\right]^{\{\beta\}} = x_n^{\{\alpha\beta\}}, \quad \forall n \geqslant 0.$$

2.3 Scale-invariant filtering

The scale-invariant counterparts of the basic notions of linear time-invariant filtering are given here. We begin with a brief review of the continuous-time case and then show how the basic ideas naturally extend to the discrete-time setting.

2.3.1 For a continuous-time signal f, let $g(t,\alpha) = f(\alpha t)$ denotes its α-scale translation, at time t. Since we are interested in the dependance of g on α, that is the action of the multiplicative group of positive real numbers, let t be fixed and set $g_t(\alpha)$ for the corresponding function, defined on $\mathbb{R}_+$. If g_t and h_t are integrable, then their convolution product is defined by

$$y_t(\alpha) = (g_t \star h_t)(\alpha) = \int_{\mathbb{R}_+} g_t(\beta^{-1}\alpha)h_t(\beta)d\mu(\beta) \tag{2.8}$$

where $\mu(\cdot)$ is the (left) Haar measure on $\mathbb{R}_+$, given by $d\mu(\beta) = \beta^{-1}d\beta$. Now, (2.8) may be viewed as the input-output relation of a linear scale-invariant filter. In this case, g_t represents the input signal, y_t its corresponding output and h_t the impulse response of the filter that is the response to the neutral element $\tilde{\delta}$ of the group algebra of $\mathbb{R}_+$. The filter is BIBO-stable if h_t belongs to $L_1(d\mu)$, asymptotic stability is in the sense $L_2(d\mu)$ and causality means that $h_t(\alpha) = 0$ for $\alpha < 1$. We then recover the linear scale-invariant filtering model of [7] obtained from the Lamperti's transformation applied to a linear time-invariant filter.

For $w \in \mathbb{R}$, associate the function $\chi_w : \mathbb{R}_+ \to \mathbb{T}$, $\alpha \mapsto \chi_w(\alpha) = \alpha^{iw}$. It is easy to verify that:

$$\chi_w(1) = 1 \quad \text{and } \chi_w(\alpha\beta) = \chi_w(\alpha)\chi_w(\beta) \text{ for all } \alpha, \beta \in \mathbb{R}_+$$

The set of all such $\chi_w(\cdot)$ forms the dual group [17] of $\mathbb{R}_+$, subsequently denoted by $\widehat{\boldsymbol{R}}$: its element are called the characters of $\mathbb{R}_+$ and it allows one to define an abstract version of the Fourier transform $\mathcal{F}$ and its inverse $\mathcal{F}^{-1}$, by:

$$\hat{g}_t(\chi) = \mathcal{F}\{g_t\}(\chi) = \int_{\mathbb{R}_+} g_t(\alpha)\overline{\chi(\alpha)}d\mu(\alpha) \tag{2.9}$$

$$g_t(\alpha) = \mathcal{F}^{-1}\{\hat{g}_t\}(\alpha) = \int_{\widehat{\boldsymbol{R}}} \hat{g}_t(\chi)\chi(\alpha)d\hat{\mu}(\chi). \tag{2.10}$$

Here, $\hat{\mu}(\cdot)$ is the Haar measure on $\widehat{\boldsymbol{R}}$ and we have ignored the dependance of χ_w on w to ease the notations. Using the definition of $\chi(\alpha)$, we recognize the Mellin transforms. The transfer function of the linear scale-invariant filter is consequently given as the Mellin transform of its impulse reponse [7], [8].

2.3.2 It turns out that all these preceding ideas apply verbatim in the discrete-time case. To see this, it suffices to write $u(n,\alpha) = x_n^{\{\alpha\}}$ for the α-scale translation, at discrete-time n, of the discrete-time signal $\{x_n\}$. Setting $u_n(\alpha) = u(n,\alpha)$ for fixed n, it becomes clear that the preceding analysis remains valid if the continuous time index t is replaced by the discrete-time index n.

Remark 2.5. In the continuous-time case, if we consider only signals $f(t)$ defined for $t > 0$, then the index t in $g_t(\cdot)$ above becomes superfluous: g_t coincides with g_1 for all t. In the discrete-time context, however, $u(n,\alpha) = x_n^{\{\alpha\}}$ defines a hybrid two variables signal since there is no trivial relation between $u(n,\alpha)$ and $u(m,\alpha)$ unless $n = m$.

3 Discrete-scale invariant systems and signals

From now on, we discretize the scale axis and consider the discrete-scaling of discrete-time signals. For example, one may choose a geometric grid, as is usual with the wavelet transforms: $\alpha_n = \kappa^n, \kappa \neq 1, n \in \mathbb{Z}$ or a combination of, say L, different geometric grids: $\kappa_1^{n_1}\kappa_2^{n_2}\cdots\kappa_L^{n_L}$, $n_i \in \mathbb{Z}$. In the former case, Γ reduces to a hyperbolic cyclic group while in the latter case, the generators κ_i must satisfy: $\kappa_1^{n_1}\cdots\kappa_L^{n_L} = 1 \iff n_1 = \cdots = n_L = 0$.

Before we proceed any farther, let us recall, for the reader's convenient, the basic definitions on Fuchsian groups and automorphic functions that we will use in the sequel. For further details on these topics, see [16], [18], [19].

3.1 Fuchsian groups and automorphic functions

3.1.1 Let G be a group of linear transformations, $T(z) = \frac{az+b}{cz+d}$, $ad-bc = 1$, in the complex plane and let ι denotes the identity transformation. Two points z and z' in $\mathbb{C}$ are said to be *congruent* with respect to G, if $z' = T(z)$ for some $T \in G$ and $T \neq \iota$. Two regions $R, R' \subset \mathbb{C}$ are said to be G-congruent or G-equivalent if there exists a transformation $T \neq \iota$ which sends R to R'. A region R which does not contain any two G-congruent points and such that the neighborhood of any point on the boundary contains G-congruent points of R is called a *fundamental region* for G. A *properly discontinuous* group is a group G having a fundamental region. This amounts to say that the identity transformation is isolated.

Definition 3.1. A *Fuchsian group* is a properly discontinuous group each of whose transformation maps $\mathbb{D}$, $\mathbb{T}$ and $\mathbb{C}\backslash\overline{\mathbb{D}}$ onto themselves. If the cluster points of the centers of the *isometric circles*–$|cz+d| = 1, T \neq \iota$–are nowhere dense on $\mathbb{T}$, then the Fuchsian group is said to be of the *second kind.* Otherwise, the group is of the *first kind.* These cluster points are the *limit points* of the group and they are necessarily located on $\mathbb{T}$.

To simplify the expressions to follow, we will now on write γ, in place of $\gamma_{\{\alpha\}}$, for the elements of our original group Γ (2.1). Likewise, we will write $x_n(\gamma)$ in place of $x_n^{\{\alpha\}}$.

Recall that the scales α are now taken on a discrete grid. So, Γ is clearly a Fuchsian group of the second kind: its limit points are the common fixed points $z_1 = \lambda^2$ and $z_2 = -1$. We denote by

$$\mathcal{F} = \{z \in \mathbb{D} \,:\, |\gamma'(z)| < 1 \quad \text{for all } \gamma \in \Gamma, \gamma \neq \iota\} \tag{3.1}$$

the *normal fundamental domain* of Γ with respect to 0: $\mathcal{F}$ is connected [16].

Any such group is of convergence type [20]:

$$\sum_{\gamma\in\Gamma} \left(1 - |\gamma(z)|^2\right) = \left(1 - |z|^2\right) \sum_{\gamma\in\Gamma} |\gamma'(z)| < \infty \quad z \in \mathbb{D}$$

Then, the Green's function [20] of Γ with respect to a point $\xi \in \mathbb{D}$ is defined as the Blaschke product

$$b_\xi(z) = \prod_{\gamma\in\Gamma} \frac{\gamma(\xi) - z}{1 - \overline{\gamma(\xi)}z} \frac{|\gamma(\xi)|}{\gamma(\xi)} \tag{3.2}$$

It satisfies

$$b_\xi(\varphi(z)) = \mu_\xi(\varphi) b_\xi(z), \quad \forall \varphi \in \Gamma, \tag{3.3}$$

where μ_ξ is the character of Γ associated with b_ξ. A function satisfying the relation (3.3) is said to be *character-automorphic* with respect to Γ while a Γ-periodic function, as for example $|b_\xi(z)| = |b_\xi(\varphi(z))|$, is called *automorphic* with respect to Γ.

3.1.2 We now briefly mention the main properties pertaining to spaces of character-automorphic functions. The materials presented here are essentially borrowed from [20] and [21] (see also [19, 22]). Let $\widehat{\Gamma}$ be the dual group of Γ, *i.e.* the group of (unimodular) characters. For an arbitrary character $\sigma \in \widehat{\Gamma}$, associate the subspaces of the classical space $L_2(\mathbb{T})$

$$L_2^\sigma = \{f \in L_2 \,|\, f \circ \gamma = \sigma(\gamma) f, \, \forall \gamma \in \Gamma\}$$
$$\mathcal{H}_2^\sigma(\mathbb{D}) = L_2^\sigma \bigcap \mathcal{H}_2(\mathbb{D})$$

Theorem 3.2 (Pommerenke [20])**.** *Let Γ be a Fuchsian group without elliptic and parabolic element. We say that Γ is of Widom type if, and only if, the derivative of $b_0(z)$ is of bounded characteristic, i.e. $b_0'(z) = p(z)/q(z)$ with $p, q \in \mathcal{H}_\infty$. Moreover, b_0' has an inner-outer factorization*

$$b_0'(z) = \theta(z) u(z),$$

where the inner factor $\theta(z)$ is character-automorphic

In this case, Widom [23] has shown that the space $\mathcal{H}_\infty^\sigma$ is not trivial for any character $\sigma \in \widehat{\Gamma}$ and we have

Theorem 3.3 (Pommerenke [20])**.** *Let Γ be of Widom type and let $\theta(z)$ be the inner factor of $b_0'(z)$. If σ is any character of Γ and if $h(z)$ is in $\mathcal{H}_p(\mathbb{D}), 1 \leqslant p \leqslant \infty$, then*

$$f(z) = \frac{b_0(z)}{b_0'(z)} \sum_{\gamma \in \Gamma} \overline{\sigma(\gamma)} \theta(\gamma(z)) h(\gamma(z)) \frac{\gamma'(z)}{\gamma(z)} \tag{3.4}$$

is in $\mathcal{H}_p^\sigma(\mathbb{D})$ and

$$\|f\|_p \leqslant \|h\|_p, \quad f(0) = \theta(0) h(0).$$

3.2 Discrete scale-time filtering

We are now ready to present a class of both discrete time and scale stationary signals. The class is modeled by white noise driven linear discrete-time and discrete-scale invariant systems. To proceed, note first that Γ is of Widom-Carleson type since it has only two limit points.

The discrete-scale transforms of a causal discrete-time signal $\{x_n\}_{n \geqslant 0}$ now induce a discrete two-dimensional signal $\{x_n(\gamma)\}_{n \geqslant 0, \gamma \in \Gamma}$.

3.2.1 Discrete-scale filtering

a. The scale transform of $\{ax_n + by_n\}$ is obviously $\{ax_n(\gamma) + by_n(\gamma)\}$.

b. For n_0 fixed, let $\{x_{n_0}(\gamma)\}_{\gamma\in\Gamma}$ and $\{h_{n_0}(\gamma)\}_{\gamma\in\Gamma}$ be two discrete-scale signals. Their discrete convolution product is given by:

$$y_{n_0}(\gamma) = (x_{n_0} \star h_{n_0})(\gamma) = \sum_{\varphi\in\Gamma} x_{n_0}(\varphi) h_{n_0}(\gamma \circ \varphi^{-1}) = (h_{n_0} \star x_{n_0})(\gamma) \quad (3.5)$$

c. Unit scale-pulse: To define the discrete-scale unit pulse, let

$$\Delta(z) = \frac{b_0(z)}{z \prod_{\gamma\neq\iota} |\gamma(0)|} = \sum_{k\geqslant 0} \Delta_k z^k,$$

and consider the scale transform $\{\Delta_k(\gamma)\}_{\gamma\in\Gamma}$ of the signal $\{\Delta_k\}_{k\geqslant 0}$. Then

Lemma 3.4. *The scale signal* $\{\tilde{\delta}(\gamma)\}_{\gamma\in\Gamma}$ *defined by* $\tilde{\delta}(\gamma) = \Delta_0(\gamma)$ *satisfies*

$$\tilde{\delta}(\gamma) = \begin{cases} 1 & \textit{if } \gamma = \iota \\ 0 & \textit{else} \end{cases} \quad (3.6)$$

Proof. The signal $\{\Delta_k(\gamma)\}_{\gamma\in\Gamma}$ is obtained by:

$$\Delta_{\gamma^{-1}}(z) = \Phi_0^{\{\gamma\}}(z)\Delta(\gamma(z)) = \sigma(\gamma)\Phi_0^{\{\gamma\}}(z)\frac{b_0(z)}{\gamma(z)\prod_{\gamma\neq\iota}|\gamma(0)|} = \sum_{k\geqslant 0} \Delta_k(\gamma) z^k$$

Now $b_0(z)/\gamma(z)$ vanishes at zero if, and only if, $\gamma \neq \iota$ □

The scale unit-pulse $\tilde{\delta}(\cdot)$ is the neutral element of the scale convolution operation.

d. Henceforth, we interpret (3.5) as the input-output relation of the linear discrete scale-invariant filter with impulse response $h_{n_0}(\cdot)$. Indeed, we may readily check that a scale translation of γ_0 on the input map $\gamma \mapsto x_{n_0}(\gamma)$ yields the corresponding output map $\gamma \mapsto y_{n_0}(\gamma)$, with the same scale translation:

$$\{\gamma \mapsto x_{n_0}(\gamma \circ \gamma_0^{-1})\} \mapsto \{\gamma \mapsto y_{n_0}(\gamma \circ \gamma_0^{-1})\} \quad (3.7)$$

e. The scale filter (3.5) is stable or BIBO-stable if the impulse response satisfies $\sum_{\gamma\in\Gamma} |h_{n_0}(\gamma)|^2 < \infty$ or $\sum_{\gamma\in\Gamma} |h_{n_0}(\gamma)| < \infty$, respectively.

f. Causality may be defined in a straightforward manner. It suffices to take the "positive" scales as those γ corresponding to $m_\alpha = \alpha \geqslant 1$.

g. The Fourier transform of the discrete-scale signal $x_{n_0}(\gamma)$ is obtained by discretizing (2.9). It is given by the function defined on $\widehat{\Gamma}$,

$$\hat{x}_{n_0}(\sigma) = \sum_{\gamma\in\Gamma} x_{n_0}(\gamma)\overline{\sigma(\gamma)} \tag{3.8}$$

and the inverse Fourier transform reads as:

$$x_{n_0}(\gamma) = \int_{\widehat{\Gamma}} \hat{x}_{n_0}(\sigma)\sigma(\gamma)d\mu(\sigma), \tag{3.9}$$

where $\mu(\cdot)$ denotes, as before, the Haar measure on $\widehat{\Gamma}$[2]. The scale frequency response of the filter is thus corresponds to the Fourier transform of its impulse response.

3.2.2 Discrete scale-time stationary signals

Let $\{w_n\}_{n\geqslant 0}$ be a zero-mean random *i.i.d* signal, with unit variance. We have

Lemma 3.5. *If $\{w_n(\gamma)\}_{n\geqslant 0}$ is the scale transform of $\{w_n\}_{n\geqslant 0}$ at some fixed scale γ, then $\{w_n(\gamma)\}_{n\geqslant 0}$ is also a zero-mean random i.i.d signal, with unit variance.*

Proof. That $\{w_n(\gamma)\}_{n\geqslant 0}$ is zero-mean is clear since the scale transform is a linear operator. Collect now the samples w_n and $w_n(\gamma)$ in the infinite vectors $W = [w_0\ w_1\ \cdots]^t$ and $W_\gamma = [w_0(\gamma)\ w_1(\gamma)\ \cdots]^t$, respectively. Then the Proposition 2.3 shows that these two vectors are related by:
$W_\gamma = \left[\mathbf{\Phi}^{\{\gamma\}}\right]^* W$ and we have:

$$\mathsf{E}(W_\gamma W_\gamma^*) = \left[\mathbf{\Phi}^{\{\gamma\}}\right]^* \mathsf{E}(WW^*)\mathbf{\Phi}^{\{\gamma\}} = \left[\mathbf{\Phi}^{\{\gamma\}}\right]^* \mathbf{\Phi}^{\{\gamma\}} = I.$$

□

Proposition 3.6. For a given character σ, let $H^\sigma(z) \in \mathcal{H}_2^\sigma(\mathbb{D}) \subset \mathcal{H}_2(\mathbb{D})$ be the transfer function of a discrete-time linear invariant filter. Let the filter be driven by a zero-mean random *i.i.d* signal $\{w_n\}_{n\geqslant 0}$. Denote by $\{x_n\}_{n\geqslant 0}$ the corresponding ouput and by $(R)_{i,j} = r_{i-j}$, its correlation matrix. Then the discrete time-scale signal $\{x_n(\gamma)\}_{n\geqslant 0,\gamma\in\Gamma}$ satisfy the following

[2]Recall that by the Pontryagin duality [17], $\widehat{\Gamma}$ is compact since Γ is discrete.

1. For each $\gamma \in \Gamma$, $\{x_n(\gamma)\}_{n\geqslant 0}$ is a wide-sense stationary signal,
2. $\mathsf{E}[x_n(\gamma)\overline{x_m(\gamma)}] = \mathsf{E}[x_n\overline{x_m}] = r_{n-m}$, $\forall\, \gamma \in \Gamma$,
3. For each fixed n, $\mathsf{E}[x_n(\gamma)\overline{x_n(\varphi)}] = \mathsf{E}[x_n\overline{x_n(\varphi\circ\gamma^{-1})}] = \rho_n^{(n)}(\varphi\circ\gamma^{-1})$, $\forall\,\gamma,\varphi \in \Gamma$, where the samples of the signal $\{\rho_k^{(n)}\}_{k\geqslant 0}$ are taken from the $(n+1)^{th}$ column of R.

Proof. In terms of Z transform, the input-output relation of the filter reads as: $X(z) = H^\sigma(z)W(z)$ where $W(z)$ and $X(z)$ stand for the Z transforms of the input and output, respectively. Replacing z by $\gamma(z)$ and multiplying both sides of this equality by $\phi_0^{\{\gamma\}}(z)$ yields:

$$X_{\gamma^{-1}}(z) = \sigma(\gamma)H^\sigma(z)W_{\gamma^{-1}}(z).$$

This shows that $\{w_n(\gamma)\}_{n\geqslant 0}$ and $\{x_n(\gamma)\}_{n\geqslant 0}$ are, respectively, the input and output of the causal and stable filter with transfer function $\sigma(\gamma)H^\sigma(z)$. Item 1 is thus obvious in view of the above lemma.

Owing that $|\sigma(\gamma)| = 1$, $\forall\,\gamma \in \Gamma$, this lemma shows again that the power density spectrum of $\{x_n(\gamma)\}_{n\geqslant 0}$ does not depend on γ. This establishes item 2.

To prove item 3, observe that for each fixed n,

$$\begin{aligned}\mathsf{E}[x_n\overline{x_n(\gamma)}] &= \sum_{k\geqslant 0}\mathsf{E}[x_n\overline{x}_k]\overline{\phi_k^{\{\gamma\}}}(n)\\ &= \sum_{k\geqslant 0} r_{n-k}\overline{\phi_k^{\{\gamma\}}}(n) = e_n^t R\left[\mathbf{\Phi}^{\{\gamma\}}\right]^* e_n\end{aligned}$$

where e_n is the infinite column vector with 1 in the $(n+1)^{th}$ position and 0 elsewhere. On the other hand, we have

$$\mathsf{E}[x_n(\varphi)\overline{x_n(\gamma\circ\varphi)}] = \sum_{k,m\geqslant 0} r_{k-m}\phi_k^{\{\varphi\}}(n)\overline{\phi_m^{\{\gamma\circ\varphi\}}}(n) = e_n^t\mathbf{\Phi}^{\{\gamma\}}R\left[\mathbf{\Phi}^{\{\gamma\circ\varphi\}}\right]^* e_n.$$

Now, the Γ-invariance of the power density spectrum implies that $R = \mathbf{\Phi}^{\{\gamma\}}R\left[\mathbf{\Phi}^{\{\gamma\}}\right]^*$ for all γ and, as we already know, $\mathbf{\Phi}^{\{\gamma^{-1}\}} = \left[\mathbf{\Phi}^{\{\gamma\}}\right]^*$ and $\mathbf{\Phi}^{\{\gamma\circ\varphi\}} = \mathbf{\Phi}^{\{\gamma\}}\mathbf{\Phi}^{\{\varphi\}}$. Hence,

$$\mathsf{E}[x_n(\varphi)\overline{x_n(\gamma\circ\varphi)}] = e_n^t R\left[\mathbf{\Phi}^{\{\gamma\}}\right]^* e_n = e_n^t\left[\mathbf{\Phi}^{\{\gamma\}}\right]^* Re_n = \mathsf{E}[x_n\overline{x_n(\gamma)}].$$

Finally, note that $e_n^t\left[\mathbf{\Phi}^{\{\gamma\}}\right]^* Re_n$ is simply the n^{th} sample (counting from 0) of the scale transform of the signal whose samples form the vector $\rho^{(n)} = Re_n$. This completes the proof. □

4 Discrete-time self-similarity

Deducing a class of self-similar systems from the preceding analysis is now easy. The following lemma is easy to show.

Lemma 4.1. *Let ν be a given real number. Consider the function ϑ defined by*

$$\vartheta(z) = \frac{(\lambda - \overline{\lambda} z)^{\nu}}{(1+z)^{1+\nu}} \tag{4.1}$$

Then, we have

$$\vartheta_{\gamma}(z) = \alpha^{1/2+\nu}\vartheta(z)$$

Proof. Direct inspection □

Proposition 4.2. Any element of the space

$$\mathcal{H}^{\sigma}_{2,\nu}(\mathbb{D}) = \vartheta(z)\mathcal{H}^{\sigma}_{2}(\mathbb{D}),$$

with $\nu > -1$, is the transfer function of a discrete-time self-similar system with parameter ν.

Proof. An element $H(z)$ of $\mathcal{H}^{\sigma}_{2,\nu}(\mathbb{D})$ is of the form $H(z) = \vartheta(z)G(z)$ with $G(z) \in \mathcal{H}^{\sigma}_{2}(\mathbb{D})$. Thus,

$$H_{\gamma}(z) = \vartheta_{\gamma}(z)G(\gamma^{-1}(z)) = \overline{\sigma}(\gamma)\alpha^{1/2+\nu}\vartheta(z)G(z) = \overline{\sigma}(\gamma)\alpha^{1/2+\nu}H(z).$$

□

Since $\mathcal{H}^{\sigma}_{2}(\mathbb{D}) \in \mathcal{H}_{2}(\mathbb{D})$, it comes that $\mathcal{H}^{\sigma}_{2,\nu}(\mathbb{D})$ is a subspace of the fractional Hardy space $\mathcal{H}_{2,\nu}(\mathbb{D})$ studied in our previous work [15]. More on the properties of this space and espacially on the character-automorphic extension of the Schur algorithm are in progress.

References

[1] B. B. Mandelbrot and J. W. Van Ness, "Fractional Brownian motions, fractional noises and applications," *SIAM Review*, vol. 10, no. 4, pp. 422–437, 1968.

[2] J. A. Barnes and D. W. Allan, "A statistical model of flicker noise," *Proceedings of the IEEE*, vol. 54, pp. 176–178, 1966.

[3] C. W. J. Granger and R. Joyeux, "An introduction to long memory time series models and fractional differencing," *Journ. of times series analysis*, vol. 1, no. 1, pp. 15–29, 1980.

[4] J. R. M. Hosking, "Fractional differencing," *Biometrika*, vol. 68, pp. 165–176, 1981.

[5] O. Magré and M. Guglielmi, "Modeling and analysis of fractional Brownian motions," *Chaos, Solitons and Fractals*, vol. 8, no. 3, pp. 377–388, 1997.

[6] P. Carmona, L. Coutin, and G. Montseny, "A diffusive Markovian representation of fractional Brownian motion with hurst parameter less than 1/2," Prepublication, 1998.

[7] B. Yazıcı and R. L. Kashyap, "A class of second-order stationary self-similar processes for $1/f$ phenomena," *IEEE Trans. Signal Processing*, vol. 45, no. 2, pp. 396–410, Feb 1997.

[8] C. J. Nuzman and H. Vincent Poor, "Reproducing kernel Hilbert space methods for wide-sense self-similar processes," *Ann. Appl. Probab.*, vol. 11, no. 4, pp. 1199–1219, 2001.

[9] J. Lamperti, "Semi-stable stochastic processes," *Trans. Amer. Math. Soc.*, vol. 104, pp. 62–78, 1962.

[10] P. Amblard P. Borgnat and P. Flandrin, "Scale invariances and Lamperti transformations for stochastic processes," *Journ. Phys. A: Math. Gen.*, vol. 38, no. 10, pp. 2081–2101, 2005.

[11] O. Cappé, E. Moulines, J-C. Pesquet, A. Petropulu, and X. Yang, "Long-range dependance and heavy-tail modeling for teletraffic data," *IEEE Signal Processing Mag.*, vol. 19, no. 3, pp. 14–27, May 2002.

[12] Y. Meyer, F. Sellan, and M.S. Taqqu, "Wavelets, generalized white noise and fractional integration: the synthesis of fractional Brownian motion," *J. Fourier Anal. Appl.*, vol. 5, no. 5, pp. 465–494, 1999.

[13] P. Abry, P. Gonçalvès, and P. Flandrin, "Wavelets, spectrum analysis and $1/f$ processes," in *Wavelets and statistics*. Lecture notes in Statistics, 1995.

[14] R. Narasimha S. Lee, W. Zhao and R.M. Rao, "Discrete-time models for statistically self-similar signals," *IEEE Trans. Signal Processing*, vol. 51, no. 5, pp. 1221–1230, 2003.

[15] M. Mboup, "On the structure of self-similar systems: A Hilbert space approach," in *Operator Theory: Advances and applications*, vol. OT-143, pp. 273–302. Birkhäuser-Verlag, 2003.

[16] L. R. Ford, *Automorphic functions*, Chelsea, New-York, 2nd edition, 1951.

[17] E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis*, vol. I/II, Springer, Berlin, Göttingen Heidelberg, 1963/1970.

[18] C. L. Siegel, *Topics in Complex Function Theory*, vol. 2: Automorphic Functions and Abelian Integrals, Wiley, New York, 1988.

[19] M. Hasumi, *Hardy classes on infinitely connected Riemann surfaces*, vol. 1027 of *Lecture Notes in Math.*, Springer-Verlag, Berlin, 1983.

[20] CH. Pommerenke, "On the Green's function of Fuchsian groups," *Ann. Acad. Sci. Fenn.*, vol. 2, pp. 409–427, 1976.

[21] M. Sodin and P. Yuditskii, "Almost periodic Jacobi matrices with homogeneous spectrum, infinite dimensional Jacobi inversion and Hardy spaces of character-automorphic functions," *The Journ. of Geometric Analysis*, vol. 7, no. 3, pp. 387–435, 1997.

[22] M. V. Samokhin, "Some classical problems in the theory of analytic functions in domains of Parreau-Widom type," *Math USSR Sbornik*, vol. 73, pp. 273–288, 1992.

[23] H. Widom, "H^p sections of vector bundles over Riemann surfaces," *Ann. of Math.*, vol. 94, pp. 304–323, 1971.