Arithmetic Properties of Summands of Partitions

Cécile Dartyge, Andras Sarkozy

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Cécile Dartyge and András Sárközy*

Abstract. Let $d \in \mathbb{N}$, $d \geqslant 2$. We prove that for almost all partitions of an integer the parts are well distributed in residue classes mod d.The limitations of the uniformity of this distribution are also studied.

1. Introduction

In this paper, we will study the distribution in arithmetical progressions of the summands of a general partition of an integer. We expect that this distribution is uniform. When the modulus is a fixed integer, we will show that this is true for almost all partitions. For $k, n, a, d \in \mathbb{N}$ with $1 \leqslant a \leqslant d$ and for a partition λ of n we will use the following notations:

$p(n)$ denotes the number of partitions of n, $p_k(n)$ the number of partitions with at most k parts, $p^{(k)}(n)$ with exactly k parts;

$p(n; a, d)$ is the number of partitions of n all of whose parts are congruent to $a \pmod d$, $p_k(n; a, d)$ the number of such partitions with at most k parts and $p^{(k)}(n; a, d)$ the number of such partitions with exactly k parts;

$\bar{p}(n; a, d)$ is the number of partitions of n none of whose parts is congruent to $a \pmod d$,

$p_{k,a,d}(n)$ is the number of partitions of n such that the number of the parts congruent to $a \pmod d$ is at most k;
$r(n, m)$ denotes the number of partitions of n whose parts are at least m, and $r(n, \{a, b\})$ the number of partitions of n whose parts are not in $\{a, b\}$.

The first result of this paper says that for almost all partitions of n the sum of the parts congruent to a modulo d is close to n/d. For a partition $\lambda = (\lambda_1, \dots, \lambda_s)$ of $n = \lambda_1 + \cdots + \lambda_s$ with $\lambda_1 \geqslant \cdots \geqslant \lambda_s$ we define

$$S_{a,d}(\lambda) := \sum_{\substack{1 \leqslant j \leqslant s \\ \lambda_j \equiv a \,(\mathrm{mod}\, d)}} \lambda_j,$$

and

$$F_{a,d}(\lambda) := \sum_{\substack{1 \leqslant j \leqslant s \\ \lambda_j \equiv a \,(\mathrm{mod}\, d)}} 1.$$

We write:

$$C := \pi \sqrt{\frac{2}{3}}. \tag{1·1}$$

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Theorem 1.1. *Let $1 \leqslant a \leqslant d$ and g a positive and non decreasing function such that $\lim_{n\to+\infty} g(n) = +\infty$ and $g(n) \leqslant n^{1/12}$ for all $n \geqslant 1$. For almost all partitions λ of n we have*

$$S_{a,d}(\lambda) = \frac{n}{d} + O(n^{3/4}g(n)). \tag{1·2}$$

The number of exceptional partitions is $O(p(n)n^2 \exp(\frac{-Cg(n)^2d^2}{8(d-1)}))$.

This result is sharp apart from the g factor as the following Theorem shows:

Theorem 1.2. *For all $d \in \mathbb{N}$, $d \geqslant 2$, $\varepsilon > 0$ there are $n_0 = n_0(d, \varepsilon)$ and $\delta = \delta(d, \varepsilon)$ so that for $n > n_0$, there are more than $(1 - \varepsilon)p(n)$ partitions λ of n with*

$$|S_{1,d}(\lambda) - \frac{n}{d}| > \delta n^{3/4}. \tag{1·3}$$

In 1941, Erdős and Lehner [2] proved that for $k = C^{-1}\sqrt{n}\log n + x\sqrt{n}$, $\lim_{n\to\infty} \frac{p_k(n)}{p(n)} = \exp\left(\frac{-2}{C}e^{-\frac{1}{2}Cx}\right)$. Since this fundamental Theorem, many statistical results have been obtained by Erdős, Szalay, Szekeres, Turán, etc. In particular Szalay and Turán proved ([8] Corollary 1 p. 135) :

for $(\log n)^6 \leqslant j \leqslant \frac{\sqrt{6}}{2\pi}\sqrt{n}\log n - 5\sqrt{n}\log\log n$, the equality

$$\lambda_j = (1 + O((\log n)^{-1}))\frac{\sqrt{6}}{\pi}\sqrt{n}\log\frac{1}{1 - \exp\left(\frac{-\pi j}{\sqrt{6n}}\right)}$$

holds uniformly with the exception of at most $O(p(n)n^{-5/4}\log n)$ partitions of n. In the survey papers of Erdős and Szalay [4], [7] many other results are referred to.

In this paper we will start out from the proof of Erdős and Lehner. We will show that it is possible to adapt it to deduce a result for $p_{k,a,d}(n)$ when a and d are fixed.

Theorem 1.3. *Let $1 \leqslant a \leqslant d$ and $x \in \mathbb{R}$. For $k = C^{-1}\frac{\sqrt{n}}{d}\log n + x\frac{\sqrt{n}}{d}$, we have:*

$$\lim_{n\to\infty} \frac{p_{k,a,d}(n)}{p(n)} = \frac{1}{\Gamma(\frac{a}{d})}\int_{\frac{2}{dC}\exp\frac{-Cx}{2}}^{\infty} e^{-t}t^{\frac{a}{d}-1}dt, \tag{1·4}$$

where Γ is the Gamma function: $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$ for $\Re s > 0$.

It follows that for all $\varepsilon > 0$ there is an $w = w(\varepsilon)$ such that for all but $\varepsilon p(n)$ partitions λ of n we have

$$|F_{a,d}(\lambda) - F_{a',d}(\lambda)| < w\sqrt{n}$$

for all a, a'. This result is sharp as the following Theorem shows:

Theorem 1.4. *Let $d \geqslant 2$. There exists $n_0 = n_0(d)$ such that for any $n > n_0$ and for any $0 < a < b \leqslant d$, there are more than $p(n)/6$ partitions λ of n with*

$$|F_{a,d}(\lambda) - F_{b,d}(\lambda)| > \frac{(a+b)\sqrt{n}}{50ab}. \tag{1·5}$$

Both Theorem 1.3 and the proof of Theorem 1.4 seem to indicate that the uniformity of the distribution of the parts in the residue classes a mod d (where now $0 < a \leqslant d$) is limited by the fact that, probably, the residues classes with "small" a tend to occur slightly more frequently. Indeed, we expect that there is a positive absolute constant $c = c(d)$ so that for $0 < a < a' \leqslant d$, there are more than $(\frac{1}{2} + c)p(n)$ partitions λ satisfying

$$F_{a,d}(\lambda) > F_{a',d}(\lambda). \tag{1·6}$$

Let us call a partition λ "d-regular" if (1·6) holds for all $0 < a < a' \leqslant d$. Probably there are "many" d-regular partitions of n. In a forthcoming paper we will prove that there are more than $c'(d)p(n)$ (where $c'(d) > 0$) d-regular partitions.

So far we have only studied unrestricted partitions. One might like to study the case of partitions into unequal parts as well. We expect that in the latter case the preponderance of the "small" residue classes disappears, or at least it is less significant.

One may also study other arithmetic properties of the parts in a "random" partition of n. Is it true that the rate of the square-free parts to all the parts used is around $\frac{6}{\pi^2}$ for almost all partitions of n? Is it true that the normal order of the number of prime factors of the parts used is $\log\log n$ for almost all partitions of n? If the answer to the last question is affirmative, then do the parts of a "random" partition satisfy an Erdős-Kac type law? Is it true that the order of magnitude of the frequency of the prime numbers amongst all the parts used is $\frac{1}{\log n}$ for almost all partitions? Perhaps this frequency is $\sim c\frac{1}{\log n}$ for almost all partitions but the value of c can be different for unrestricted partitions, resp. partitions into unequal parts (in the latter case, one would expect $\frac{1}{\log\sqrt{n}} = \frac{2}{\log n}$). We hope to answer some of these questions in subsequent papers.

2. Some asymptotic formulas of Hardy, Ramanujan, Meinardus, Dixmier, Erdős, Nicolas and Sárközy

In this part we will quote some formulas that we will use to prove our results.

First, by a Theorem of Hardy and Ramanjan [5] we have

$$p(n) = \frac{1}{4n\sqrt{3}} e^{C\sqrt{n}}(1 + O(n^{-1/2})). \tag{2·1}$$

We will also need some asymptotic formulas for $p(n; a, d)$ and $\bar{p}(n; a, d)$. Such formula may be derived from Meinardus's Theorem (see Satz 1 and 2 of [6]), when $(a, d) = 1$:

$$p(n; a, d) = A(a, d)n^{-\frac{1}{2}-\frac{a}{2d}} e^{C\sqrt{\frac{n}{d}}}(1 + O(n^{-1/4})) \tag{2·2}$$

with

$$A(a, d) = \Gamma\big(\frac{a}{d}\big)\pi^{\frac{a}{d}-1}2^{-\frac{3}{2}-\frac{a}{2d}}3^{-\frac{a}{2d}}d^{-\frac{1}{2}+\frac{a}{2d}},$$

and

$$\bar{p}(n; a, d) = B(a, d)n^{-\frac{5}{4}+\frac{a}{2d}} e^{C\sqrt{n(1-\frac{1}{d})}}(1 + O(n^{-1/4})) \tag{2·3}$$

with

$$B(a, d) = \frac{d^{\frac{a}{d}-\frac{1}{2}}}{\Gamma(\frac{a}{d})} \frac{1}{2\sqrt{\pi}}\Big((1 - \frac{1}{d})\frac{\pi^2}{6}\Big)^{\frac{1}{4}-\frac{a}{2d}}.$$

The implicit constants in the Landau symbols of (2·2) and (2·3) are uniform in n but depend strongly on a and d. Dixmier and Nicolas [1] proved that uniformly for $m \leqslant n^{1/4}$ we have

$$r(n, m) = p(n)\big(\frac{1}{2}Cn^{-1/2}\big)^{m-1}(m - 1)!(1 + O(m^2n^{-1/2})). \tag{2·4}$$

In the proof of Theorem 1.4 we will need an estimation of $r(n,a,b)$. The following (2·5) is a particular case of the Proposition page 159 of the article of Erdős, Nicolas and Sárközy [3]. For $0<a<b<\sqrt{n}$, we have:

$$r(n,\{a,b\}) = p(n)\frac{abC^2}{4n}(1+O(bn^{-1/2})). \tag{2·5}$$

This Proposition requires in fact that $0<a<b\ll\sqrt{n}$ where the implicit constant is absolute but in our particular case (with only two parts forbidden) their result holds under the condition $0<a<b<\sqrt{n}$.

3. Proof of Theorem 1.1

First we suppose that $(a,d)=1$. We start with the equalities:

(3·1)

$$\begin{aligned} p(n) &= \sum_{m\leqslant n} p(m;a,d)\bar{p}(n-m;a,d)\\ &= \sum_{\substack{m\leqslant n\\ |m-\frac{n}{d}|\leqslant n^{3/4}g(n)}} p(m;a,d)\bar{p}(n-m;a,d) + \sum_{\substack{m\leqslant n\\ |m-\frac{n}{d}|>n^{3/4}g(n)}} p(m;a,d)\bar{p}(n-m;a,d)\\ &= S_1+S_2, \end{aligned}$$

by definition. To prove Theorem 1.1, in view of (2·2) and (2·3) it is sufficient to show that

$$S_2 = O(p(n)n^2\exp(-C\frac{g(n)^2d^2}{8(d-1)})).$$

With the estimations of Meinardus (2·2) and (2·3) we have

$$S_2 \ll_{a,d} \sum_{|m-\frac{n}{d}|>n^{3/4}g(n)} (n-m)^{-\frac{5}{4}+\frac{a}{2d}}m^{-\frac{1}{2}-\frac{a}{2d}}\exp\big(C\sqrt{\frac{m}{d}}+C\sqrt{(n-m)(1-\frac{1}{d}})\big). \tag{3·2}$$

Let $f:[0,n]\to\mathbb{R}, u\mapsto\sqrt{\frac{u}{d}}+\sqrt{(n-u)(1-1/d)}$. This function is positive and concave, its maximum is at $u=n/d$ with $f(n/d)=\sqrt{n}$. Thus we may write:

$$S_2 \ll n\max(\exp(Cf(\frac{n}{d}+n^{3/4}g(n))),\exp(Cf(\frac{n}{d}-n^{3/4}g(n)))). \tag{3·3}$$

By using the formula $\sqrt{1+v}=1+v/2-v^2/8+O(v^3)$ (for $-1<v$), for $-\frac{n}{d}<u<n(1-1/d)$ we have:

$$f(\frac{n}{d}+u) = \sqrt{n}-\frac{u^2}{8n^{3/2}}\frac{d^2}{d-1}+O(\frac{d^2u^3}{n^{5/2}}), \tag{3·4}$$

where the implicite constant is absolute.

Then we apply this formula with $u=\pm n^{3/4}g(n)$:

$$\begin{aligned} S_2 &\ll n\exp(C(\sqrt{n}-\frac{g(n)^2d^2}{8(d-1)}+O(d^2n^{-1/4}g(n)^3)))\\ &\ll n^2p(n)\exp(-C\frac{d^2g^2(n)}{8(d-1)}), \end{aligned} \tag{3·5}$$

when $|g(n)|\leqslant n^{1/12}$. This ends the case $(a,d)=1$.

If $(a,d)>1$ we set $\delta := (a,d)$, $a = a'\delta$ and $d = d'\delta$. It is obvious that

$$p(m;a,d) = \begin{cases} p(\frac{m}{\delta};a',d') & \text{if } m \equiv 0 \,(\mathrm{mod}\,\delta) \\ 0 & \text{in the other cases.} \end{cases}$$

The sum S_2 arising in (3·1) is equal to

$$S_2 = \sum_{\substack{m\leqslant n \\ m\equiv 0\,(\mathrm{mod}\,\delta) \\ |m-\frac{n}{d}|>n^{3/4}g(n)}} p(\frac{m}{\delta};a',d')\bar{p}(n-m;a,d).$$

Then we apply Meinardus's Theorem, and with similar computations as before we obtain an upper bound like (3·5) with an eventually different implicit constant in the Vinogradov symbol $\ll$. This completes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

If a partition λ does not satisfy (1·3) then we have

$$S_{1,d}(\lambda) = m \tag{4·1}$$

with m such that $|m-\frac{n}{d}| \leqslant \delta n^{3/4}$. For fixed m the number of partitions λ satisfying (4·1) is $p(m;1,d)\bar{p}(n-m;1,d)$. Thus it suffices to show that

$$S := \sum_{m:\ |m-\frac{n}{d}|\leqslant \delta n^{3/4}} p(m;1,d)\bar{p}(n-m;1,d) < \varepsilon p(n). \tag{4·2}$$

By (2·2) and (2·3), there are constants $K_1 = K_1(d)$, $K_2 = K_2(d)$ so that for $n\to\infty$ and uniformly for $|m-\frac{n}{d}| \leqslant \delta n^{3/4}$ we have

$$\begin{aligned} p(m;1,d)\bar{p}(n-m;1,d) &= (1+o(1))K_1 m^{-\frac{1}{2}-\frac{1}{2d}}(n-m)^{-\frac{5}{4}+\frac{1}{2d}} \\ &\quad\times \exp\big(C\sqrt{\frac{m}{d}} + C\sqrt{(n-m)(1-\frac{1}{d})}\big) \\ &= (1+o(1))K_2 n^{-\frac{7}{4}} \exp\big(C(\sqrt{\frac{m}{d}} + \sqrt{(n-m)(1-\frac{1}{d})})\big). \end{aligned} \tag{4·3}$$

It follows from (3·4) that uniformly for $|m-\frac{n}{d}| \leqslant \delta n^{3/4}$ we have

$$C(\sqrt{\frac{m}{d}} + \sqrt{(n-m)(1-\frac{1}{d})}) = C\sqrt{n} + O(n^{-3/2}|m-\frac{n}{d}|^2) = C\sqrt{n} + O(\delta^2), \tag{4·4}$$

It follows from (2·1), (4·3), (4·4) that there is a $K_3 = K_3(d)$ so that

$$p(m;1,d)\bar{p}(n-m;1,d) = (1+o(1))K_3 n^{-3/4} p(n)(1+O(\delta^2)) \tag{4·5}$$

uniformly for $|m-\frac{n}{d}| \leqslant \delta n^{3/4}$. If δ is small enough in terms of ε then (4·2) follows from (4·5) and this completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3

As in the proof of Theorem 1.1 we suppose that $(a,d)=1$ and in the last paragraph we will consider the general case.

5.1. Preliminaries

The main idea is to adapt the arguments of Erdős and Lehner [2]. Like in the proof of Theorem 1.1 we start with the splitting

$$\begin{aligned}\frac{p_{k,a,d}(n)}{p(n)} &= \sum_{m\leqslant n}\frac{p_k(m;a,d)\bar{p}(n-m;a,d)}{p(n)}\\ &= \sum_{\substack{m\leqslant n\\ |m-\frac{n}{d}|\leqslant n^{4/5}}}\frac{p_k(m;a,d)\bar{p}(n-m;a,d)}{p(n)}+\sum_{\substack{m\leqslant n\\ |m-\frac{n}{d}|>n^{4/5}}}\frac{p_k(m;a,d)\bar{p}(n-m;a,d)}{p(n)}\\ &= S_3+S_4.\end{aligned}$$

It follows from the proof of Theorem 1.1 (with $g(n)=n^{1/20}$) that we have $S_4=o(1)$.

The main problem is to estimate S_3.

Since $p_k(m;a,d)=p(m;a,d)-\sum_{r\geqslant k+1}p^{(r)}(m;a,d)$, we have the equality:

$$S_3=\sum_{|m-\frac{n}{d}|\leqslant n^{4/5}}\frac{p(m;a,d)\bar{p}(n-m;a,d)}{p(n)}-\sum_{|m-\frac{n}{d}|\leqslant n^{4/5}}\sum_{k+1\leqslant r}\frac{p^{(r)}(m;a,d)\bar{p}(n-m;a,d)}{p(n)}.$$

The first term is $1+o(1)$ by the proof of Theorem 1.1. For the second, to any partition $m=(a+\lambda_1d)+\cdots+(a+\lambda_rd)$ counted in $p^{(r)}(m;a,d)$ we can assign $\lambda_1+\cdots+\lambda_r$ a partition of $\frac{m-ar}{d}$ with at most r summands. This correspondance is one-to-one, so we may write, when $m-ar\equiv 0 \pmod d$, that $p^{(r)}(m;a,d)=p_r(\frac{m-ar}{d})$.

Next we use (like Erdős and Lehner) the sieve identity :

$$p_r(\frac{m-ar}{d})=\sum_{j\geqslant 0}(-1)^j\sum_{\substack{1\leqslant r_1<\cdots<r_j\\ 1\leqslant r_1+\cdots+r_j\leqslant \frac{m-ar}{d}-jr}}\frac{p(\frac{m-ar}{d}-jr-\sum_{\ell=1}^j r_\ell)\bar{p}(n-m;a,d)}{p(n)}.$$

Finally

$$S_3=1+o(1)-\sum_{j\geqslant 0}(-1)^jT(j)$$

with

(5·1)

$$T(j)=\sum_{|m-\frac{n}{d}|\leqslant n^{4/5}}\sum_{\substack{k+1\leqslant r\leqslant m/a\\ m\equiv ar \,(\mathrm{mod}\, d)}}\sum_{\substack{1\leqslant r_1<\cdots<r_j\\ 1\leqslant r_1+\cdots+r_j\leqslant \frac{m-ar}{d}-jr}}\frac{p(\frac{m-ar}{d}-jr-\sum_{\ell=1}^j r_\ell)\bar{p}(n-m;a,d)}{p(n)}.$$

More precisely, by the principle of Brun's "simple" sieve we have for all fixed $\nu\geqslant 1$

$$1+o(1)-\sum_{j=0}^{2\nu}(-1)^jT(j)\leqslant S_3\leqslant 1+o(1)-\sum_{j=0}^{2\nu+1}(-1)^jT(j).$$

The integer ν will be fixed large enough, this is the main reason why the error term will be only $o(1)$.

In the next step, we will show that the contribution of the terms $\max(r,r_1,\ldots,r_j)\geqslant n^{3/5}$ is $o(1)$. We will prove the following

Lemma 5.1. *For $j \geqslant 0$, we have*
(5·2)

$$V(j) := \sum_{|m-\frac{n}{d}|\leqslant n^{4/5}} \sum_{\substack{k+1\leqslant r\\ m-ar\equiv 0 \,(\mathrm{mod}\, d)}} \sum_{\substack{1\leqslant r_1<\cdots<r_j\\ 1\leqslant r_1+\cdots+r_j\leqslant \frac{m-ar}{d}-jr\\ \max(r,r_1,\dots,r_j)\geqslant n^{3/5}}} \frac{\bar{p}(n-m;a,d)}{p(n)} p\Big(\frac{m-ar}{d}-jr-\sum_{\ell=1}^{j} r_\ell\Big)$$

$$= o(1).$$

Proof.

If $\max(r,r_1,\dots,r_j)\geqslant n^{3/5}$ then $\frac{m-ar}{d}-jr-\sum_{\ell=1}^{j} r_\ell \leqslant \frac{m}{d}-\frac{n^{3/5}}{d}$. By the estimations (2·1) and (2·3) of Meinardus, Hardy and Ramanujan, we have (the implicit constants may depend on a and d)
(5·3)

$$V(j) \ll \sum_{|m-\frac{n}{d}|\leqslant n^{4/5}} \sum_{\substack{k+1\leqslant r\\ m-ar\equiv 0 \,(\mathrm{mod}\, d)}} \sum_{\substack{1\leqslant r_1<\cdots<r_j\\ 1\leqslant r_1+\cdots+r_j\leqslant \frac{m-ar}{d}-jr\\ \max(r,r_1,\dots,r_j)\geqslant n^{3/5}}} \frac{n(n-m)^{-5/4+\frac{a}{2d}}}{\frac{m-ar}{d}-jr-\sum_{\ell=1}^{j} r_\ell}$$

$$\times \exp\Big(C\sqrt{(n-m)(1-1/d)}-C\sqrt{n}+C\sqrt{\frac{m-ar}{d}-jr-\sum_{\ell=1}^{j} r_\ell}\Big)$$

$$\ll n \sum_{|m-\frac{n}{d}|\leqslant n^{4/5}} \sum_{\substack{k+1\leqslant r\\ m-ar\equiv 0 \,(\mathrm{mod}\, d)}} \sum_{\substack{1\leqslant r_1<\cdots<r_j\\ 1\leqslant r_1+\cdots+r_j\leqslant \frac{m-ar}{d}-jr\\ \max(r,r_1,\dots,r_j)\geqslant n^{3/5}}} \exp\Big(C\sqrt{(n-m)(1-1/d)}\Big)$$

$$\times \exp\Big(-C\sqrt{n}+C\sqrt{\frac{m-n^{3/5}}{d}}\Big).$$

Since $\sqrt{1-t}\leqslant 1-t/2$ for $0\leqslant t\leqslant 1$, we have

$$\sqrt{\frac{m-n^{3/5}}{d}} = \sqrt{\frac{m}{d}}\sqrt{1-\frac{n^{3/5}}{m}} \leqslant \sqrt{\frac{m}{d}}-\frac{n^{1/10}}{2\sqrt{d}}.$$

Thus we have

$$\exp\Big(C\sqrt{\frac{m-n^{3/5}}{d}}\Big) \leqslant \exp\Big(C\Big(\sqrt{\frac{m}{d}}-\frac{n^{1/10}}{2\sqrt{d}}\Big)\Big). \tag{5·4}$$

In the proof of Theorem 1.1 we have remarked that for $0\leqslant m\leqslant n$ we have

$$\exp\Big(C\Big(\sqrt{\frac{m}{d}}+\sqrt{(n-m)(1-\frac{1}{d})}-\sqrt{n}\Big)\Big)\leqslant 1. \tag{5·5}$$

Inserting (5·4) and (5·5) in the upper bound (5·3) we obtain

$$V(j) \ll n^{j+3}\exp\Big(-\frac{n^{1/10}}{2\sqrt{d}}\Big),$$

this completes the proof of Lemma 5.1.

Now we will remove the condition $1\leqslant r_1<\cdots<r_j$. We write

$$T(j) = U(j)+V(j)+E_j, \tag{5·6}$$

with

$$U(j) = \frac{1}{j!}\sum_{|m-\frac{n}{d}|\leqslant n^{4/5}} \sum_{\substack{k+1\leqslant r<n^{3/5}\\ m-ar\equiv 0 \,(\mathrm{mod}\, d)}} \sum_{1\leqslant r_1,\dots,r_j\leqslant n^{3/5}} p\Big(\frac{m-ar}{d}-jr-\sum_{\ell=1}^{j} r_\ell\Big)\frac{\bar{p}(n-m;a,d)}{p(n)}.$$

We will prove that $E_j = o(1)$ $(n\to\infty)$:

Lemma 5.2. *For any fixed j, uniformly in n we have*

$$E_j \ll_{j,a,d} \frac{\exp(-Cx)}{n} U(j-1). \tag{5·7}$$

In the next paragraph, we will prove that $U(j) = O(1)$ uniformly in n thus (5·7) is sufficent to prove that $E_j = o(1)$.

The term E_j contains the $(r_1, \dots, r_j)$ such that there exist $1 \leqslant k < \ell \leqslant j$ with $r_k = r_\ell$ so that, by (2·1),

$$\begin{aligned} E_j &\ll_j \sum_{|m-\frac{n}{d}|\leqslant n^{4/5}} \sum_{\substack{k+1\leqslant r<n^{3/5} \\ m-ar\equiv 0 \,(\mathrm{mod}\, d)}} \sum_{1\leqslant r_1,\dots,r_{j-1}<n^{3/5}} p\Big(\frac{m-ar}{d} - jr - 2r_1 - \sum_{\ell=2}^{j-1} r_\ell\Big) \\ &\quad \times \frac{\bar{p}(n-m;a,d)}{p(n)} \\ &\ll \sum_{|m-\frac{n}{d}|\leqslant n^{4/5}} \sum_{\substack{k+1\leqslant r<n^{3/5} \\ m-ar\equiv 0 \,(\mathrm{mod}\, d)}} \sum_{1\leqslant r_1,\dots,r_{j-1}<n^{3/5}} \frac{\bar{p}(n-m;a,d)}{p(n)} \frac{d^2}{n} \\ &\quad \times \exp\Big(C\Big(\frac{m-ar}{d} - jr - 2r_1 - \sum_{\ell=2}^{j-1} r_\ell\Big)^{1/2}\Big). \end{aligned} \tag{5·8}$$

Then, since $\sqrt{u+h} - \sqrt{u} = h(\sqrt{u+h} + \sqrt{u})^{-1}$, we have

$$\begin{aligned} \sqrt{\frac{m-ar}{d} - jr - 2r_1 - \sum_{\ell=2}^{j-1} r_\ell} &= \sqrt{\frac{m-ar}{d} - (j-1)r - 2r_1 - \sum_{\ell=2}^{j-1} r_\ell} \\ &\quad - \frac{r}{(\frac{m-ar}{d} - jr - 2r_1 - \sum_{\ell=2}^{j-1} r_\ell)^{1/2} + \sqrt{r}}. \end{aligned} \tag{5·9}$$

Finally, since $r > k = \frac{\sqrt{n}}{d}(\frac{\log n}{C} + x)$, we have:

$$\exp\Big(\frac{-rC}{(\frac{m-ar}{d} - jr - 2r_1 - \sum_{\ell=2}^{j-1} r_\ell)^{1/2} + \sqrt{r}}\Big) \ll \exp\Big(-\frac{rC\sqrt{d}}{\sqrt{n}}\Big) = \frac{1}{n}\exp(-Cx). \tag{5·10}$$

If we put (5·9) and (5·10) in (5·8), and using also (2·1) we obtain (5·7).

5.2. The final computations

Now we can begin the computation of the main term $U(j)$. We will use the notation

$$U(j) = \frac{1}{j!} \sum_{|m-\frac{n}{d}|\leqslant n^{4/5}} \frac{\bar{p}(n-m;a,d)}{p(n)} H(m),$$

with

$$\begin{aligned} H(m) &:= \sum_{\substack{k+1\leqslant r<n^{3/5} \\ m-ar\equiv 0 \,(\mathrm{mod}\, d)}} \sum_{1\leqslant r_1,\dots,r_j\leqslant n^{3/5}} p\Big(\frac{m-ar}{d} - jr - \sum_{\ell=1}^{j} r_\ell\Big) \\ &= \sum_{\substack{k+1\leqslant r<n^{3/5} \\ m-ar\equiv 0 \,(\mathrm{mod}\, d)}} \sum_{1\leqslant r_1,\dots,r_j\leqslant n^{3/5}} \frac{d}{m4\sqrt{3}} \exp\Big(C\sqrt{\frac{m-ar}{d} - jr - \sum_{\ell=1}^{j} r_\ell}\Big)\Big(1 + O\Big(\frac{1}{n^{1/2}}\Big)\Big), \end{aligned} \tag{5·11}$$

by (2·1).

When $v \ll n^{3/4}$, we have

$$\sqrt{\frac{m}{d}-v}=\sqrt{\frac{m}{d}}-\frac{v\sqrt{d}}{2\sqrt{m}}+O\Big(\frac{d^{3/2}v^2}{m^{3/2}}\Big).$$

We apply then this formula with $v=\frac{ar}{d}+jr+\sum_{\ell=1}^{j} r_\ell$:

$$H(m)\sim \sum_{\substack{k+1\leqslant r<n^{3/5}\\ m\equiv ar\,(\mathrm{mod}\,d)}}\ \sum_{1\leqslant r_1,\dots,r_j\leqslant n^{3/5}} \frac{d}{m4\sqrt{3}}\exp\Big(C\sqrt{\frac{m}{d}}-\frac{C}{2}\Big(\frac{ar}{d}+jr+\sum_{\ell=1}^{j} r_\ell\Big)\sqrt{\frac{d}{m}}\Big).$$

We recall that $k=\frac{\sqrt{n}}{d}(\frac{\log n}{C}+x)$. The sums over the variables r_ℓ are geometric progressions:

$$\begin{aligned}\sum_{1\leqslant r_1,\dots,r_j\leqslant n^{3/5}} \exp\Big(-\frac{C}{2}\sqrt{\frac{d}{m}}\sum_{\ell=1}^{j} r_\ell\Big) &= \Big(\sum_{1\leqslant r_1\leqslant n^{3/5}} \exp\Big(-\frac{C}{2}r_1\sqrt{\frac{d}{m}}\Big)\Big)^j\\ &=\frac{\exp\big(-\frac{C}{2}j\sqrt{\frac{d}{m}}\big)}{\big(1-\exp(-\frac{C}{2}\sqrt{\frac{d}{m}})\big)^j} \qquad (5·12)\\ &\sim \Big(\frac{2}{C}\sqrt{\frac{m}{d}}\Big)^j.\end{aligned}$$

For any m define $u=u(m)$ by $u\in\{0,\dots,d-1\}$, $m-au\equiv 0\,(\mathrm{mod}\,d)$. Then the sum over $r=u+\lambda d$ becomes:

(5·13)

$$\begin{aligned}\sum_{\substack{k+1\leqslant r<n^{3/5}\\ m\equiv ar\,(\mathrm{mod}\,d)}} \exp\Big(-\frac{C}{2}\Big(\frac{ar}{\sqrt{dm}}+jr\sqrt{\frac{d}{m}}\Big)\Big) &= \sum_{k+1\leqslant u+\lambda d<n^{3/5}} \exp\Big(-\frac{C}{2}\Big(\frac{a(u+\lambda d)}{\sqrt{dm}}+j(u+\lambda d)\sqrt{\frac{d}{m}}\Big)\Big)\\ &\sim \exp\Big(-\frac{kC}{2}\Big(\frac{a}{\sqrt{md}}+j\sqrt{\frac{d}{m}}\Big)\Big)\Big(\frac{C}{2}\sqrt{\frac{d}{m}}(a+jd)\Big)^{-1}.\end{aligned}$$

Inserting (5·12) and (5·13) in the expression of $H(m)$, we obtain

$$H(m)\sim \frac{d}{4m\sqrt{3}}\exp\Big(C\sqrt{\frac{m}{d}}\Big)\Big(\frac{2}{C}\sqrt{\frac{m}{d}}\Big)^j\frac{2}{C}\sqrt{\frac{m}{d}}\frac{1}{a+jd}\exp\Big(-\frac{kC}{2\sqrt{md}}(a+jd)\Big).$$

For $k=\frac{\sqrt{n}}{d}(\frac{\log n}{C}+x)$, and m such that $|m-\frac{n}{d}|\leqslant n^{4/5}$ we have

$$\exp\Big(-\frac{kC}{2\sqrt{md}}(a+jd)\Big)\sim \exp\Big(-\Big(\frac{a}{d}+j\Big)\frac{\log n}{2}-\frac{Cx}{2d}(a+jd)\Big).$$

It follows that

$$\begin{aligned}U(j)\sim{}& \frac{1}{j!}\Big(\frac{2}{dC}\Big)^j\exp\Big(-\frac{Cx}{2}\Big(\frac{a}{d}+j\Big)\Big)\frac{2}{C}\Big(\frac{a}{d}+j\Big)^{-1}\frac{n^{-\frac12-\frac{a}{2d}}}{4\sqrt{3}} \qquad (5·14)\\ &\times \sum_{|m-\frac{n}{d}|\leqslant n^{4/5}} \exp\Big(C\sqrt{\frac{m}{d}}\Big)\frac{\bar{p}(n-m;a,d)}{p(n)}.\end{aligned}$$

Denote the sum over m in (5·14) by Σ_m. To estimate this sum again we use Meinardus 's result (2·2):

$$\Sigma_m \sim \frac{1}{A(a,d)}(\frac{n}{d})^{\frac{1}{2}+\frac{a}{2d}} \sum_{|m-\frac{n}{d}|\leqslant n^{4/5}} \frac{p(m;a,d)\bar{p}(n-m;a,d)}{p(n)}$$
$$\sim \frac{1}{A(a,d)}(\frac{n}{d})^{\frac{1}{2}+\frac{a}{2d}}.$$

Since the general term of the sum over j is convergent we have finally
(5·15)

$$S_1 = 1 - \sum_{j\geqslant 0} \frac{(-1)^j}{j!}\frac{A(a,d)^{-1}}{4\sqrt{3}} d^{-\frac{1}{2}-\frac{a}{2d}} (\frac{2}{dC})^j \frac{2}{C}\frac{1}{j+\frac{a}{d}} \exp(-\frac{Cx}{2}(j+\frac{a}{d})) + o(1)$$
$$= 1 + o(1)$$
$$- \frac{A(a,d)^{-1}}{4\sqrt{3}} d^{-\frac{1}{2}-\frac{a}{2d}} \frac{2}{C}(\frac{2}{dC})^{-\frac{a}{d}} \sum_{j\geqslant 0}\frac{(-1)^j}{j!}(\frac{2}{dC})^{j+\frac{a}{d}} \exp\big(-\frac{Cx}{2}(j+\frac{a}{d})\big)\frac{1}{j+\frac{a}{d}} + o(1).$$

The last sum over j is the series expansion of the truncated gamma function, this sum is equal to $\gamma(\frac{a}{d}, \frac{2}{dC}\exp(-\frac{Cx}{2}))$ where we use the standard notation

$$\gamma(\alpha, u) = \int_0^u e^{-t}t^{\alpha-1}dt = \sum_{j=0}^{\infty} \frac{(-1)^j u^{\alpha+j}}{j!(\alpha+j)}$$

for $\Re\alpha > 0$. Inserting the expression of $A(a,d)$ in equality (5·15) we obtain

$$S_1 = \frac{1}{\Gamma(a/d)}\int_{\frac{2}{dC}\exp(-\frac{Cx}{2})}^{\infty} e^{-t}t^{\frac{a}{d}-1}dt + o(1).$$

The proof of Theorem 1.3 in the case $(a,d) = 1$ is complete.

5.3. The case $(a,d) > 1$

As in the proof of Theorem 1.1, let $\delta := (a,d)$, $a = a'\delta$, $d = d'\delta$. The main difference with the situation $(a,d) = 1$ is that the integer m must satisfy $m \equiv 0 \pmod{\delta}$. The proofs of Lemma 5.1 and Lemma 5.2 are still valid, the first change appears in the computation of $H(m)$ (see (5·11)). Since $m \equiv 0 \pmod{\delta}$, the condition $m - ar \equiv 0 \pmod{\delta}$ is equivalent to $\frac{m}{\delta} - a'r \equiv 0 \pmod{\delta'}$. This has no consequence for the computation (5·12) of the sums over the r_j, $1 \leqslant \ell \leqslant j$, but the sum over r (5·13) is a little different. This time $u = u(m)$ is the element of $\{0,\ldots,d'-1\}$ with $\frac{m}{\delta} - a'u \equiv 0 \pmod{d'}$, and we write $r = u + \lambda d'$:

$$\sum_{\substack{k<r<n^{3/5}\\ \frac{m}{\delta}\equiv a'r \pmod{d'}}} \exp\big(\frac{-C}{2}\sqrt{\frac{d}{m}}(\frac{a'r}{d'}+jr)\big) = \sum_{\frac{k+1-u}{d'}\leqslant\lambda<\frac{n^{3/5}-u}{d'}} \exp\big(\frac{-C}{2}\sqrt{\frac{d}{m}}(\frac{a'}{d'}+j)(u+\lambda d')\big)$$
$$\sim \frac{\exp\big(-\frac{kC}{2}\sqrt{\frac{d}{m}}(\frac{a'}{d'}+j)\big)}{\frac{C}{2}\sqrt{\frac{d}{m}}(a'+jd')}.$$

The analogue of (5·14) is

$$U(j) \sim \frac{1}{j!}(\frac{2}{dC})^j n^{-\frac{a'}{d'}} \exp(-\frac{Cx}{2}(\frac{a'}{d'}+j))\frac{2\sqrt{n}}{dC}\frac{1}{a'+jd'}\frac{d^2}{4n\sqrt{3}}$$
$$\times \sum_{\substack{|m-\frac{n}{d}|\leqslant n^{4/5}\\ m\equiv 0 \pmod{\delta}}} \exp(C\sqrt{\frac{m}{d}})\frac{p(n-m;a,d)}{p(n)}. \tag{5·16}$$

When $m \equiv 0 \,(\mathrm{mod}\, \delta)$, by Meinardus's Theorem, we have $p(m; a, d) = p(\frac{m}{\delta}, a', d') \sim A(a', d')\big(\frac{m}{\delta}\big)^{-\frac{1}{2}-\frac{a}{2d}} \exp(C\sqrt{\frac{m}{d}})$. The sum over m in (5·16) is $\sim A(a', d')^{-1}(\frac{n}{d\delta})^{\frac{1}{2}+\frac{a}{2d}}$. Finally we remark that $A(a', d') = A(a, d)\delta^{\frac{1}{2}-\frac{a}{2d}}$ and we can complete the computation as in the end of paragraph 4.2.

The proof of Theorem 1.3 is now complete.

6. Proof of Theorem 1.4

Let $\mathcal{P}$ denote the set of partitions of n. Consider a partition λ of $\mathcal{P}$:

$$x_1 + 2x_2 + 3x_3 + \cdots + nx_n = n \tag{6·1}$$

where $x_1, \ldots, x_n$ are non negative integers. Define $\ell = \ell(\lambda)$ by

$$ax_a + bx_b = \ell; \tag{6·2}$$

then we have

$$\sum_{j \notin \{a,b\}} jx_j = n - \ell. \tag{6·3}$$

Here (6·2) is a partition of ℓ into parts a and b, while (6·3) is a partition of $n - \ell$ into parts not in $\{a, b\}$; denote these partitions by $\Phi = \Phi(\lambda)$ and $\Psi = \Psi(\lambda)$, respectively. For fixed ℓ, write $\mathcal{P}_\ell = \{\lambda : \ \lambda \in \mathcal{P}, \ \ell(\lambda) = \ell\}$.

Let $\delta = (a, b)$. It is obvious that $\mathcal{P}_\ell$ is empty if $\ell \not\equiv 0 \,(\mathrm{mod}\, \delta)$. For $\ell \equiv 0 \,(\mathrm{mod}\, \delta)$ we write $\ell = \delta\ell'$, $a = \delta a'$ and $b = \delta b'$.

Since $(a', b') = 1$ there exists $\beta \in \{0, \ldots, b' - 1\}$ and $\lambda \in \mathbb{Z}$ such that $\ell' = \beta a' + \lambda b'$. When ℓ is large enough it is clear that $\lambda > 0$, in fact we have $\lambda = \frac{\ell'}{b'} + O(1) = \frac{\ell}{b} + O(1)$, where here and in the following, the terms $O(1)$ depend on a and b.

The solutions of (6·2) are $\ell = a(\beta + b'\mu) + b(\lambda - a'\mu)$ with μ such that $\beta + b'\mu$ and $\lambda - a'\mu$ are positive. The number of solutions of (6·2) is $\frac{\ell'}{a'b'} + O(1) = \frac{\ell}{a'b} + O(1)$, while the number of solutions of (6·3) is $r(n - \ell, \{a, b\})$.

Thus we have for $\ell \equiv 0 \,(\mathrm{mod}\, \delta)$

$$|\mathcal{P}_\ell| = \big(\frac{\ell}{a'b} + O(1)\big) r(n - \ell, \{a, b\}). \tag{6·4}$$

Now set

$$\mathcal{P}^* = \{\lambda : \ \lambda \in \mathcal{P}, \ \ell(\lambda) < \sqrt{n}/10\} = \cup_{\ell < \sqrt{n}/10} \mathcal{P}_\ell$$

and

$$\bar{\mathcal{P}} = \mathcal{P} \setminus \mathcal{P}^* = \cup_{\ell \geqslant \sqrt{n}/10} \mathcal{P}_\ell.$$

Then by (2·5) and (6·4) we have

$$\begin{aligned} |\mathcal{P}^*| &= \sum_{\ell < \sqrt{n}/10} |\mathcal{P}_\ell| \\ &= \sum_{\substack{\ell < \sqrt{n}/10 \\ \ell \equiv 0 \,(\mathrm{mod}\, \delta)}} \big(\frac{\ell}{a'b} + O(1)\big) r(n - \ell, \{a, b\}) \\ &= \sum_{\substack{\ell < \sqrt{n}/10 \\ \ell \equiv 0 \,(\mathrm{mod}\, \delta)}} \big(\frac{\ell}{a'b} + O(1)\big) p(n - \ell) \frac{abC^2}{4(n - \ell)} (1 + O(n^{-1/2})). \end{aligned} \tag{6·5}$$

It follows from (2·1) for large n and $\ell < \sqrt{n}/10$ that

(6·6)
$$\begin{aligned}\Big(\frac{\ell}{a'b}+O(1)\Big)\frac{p(n-\ell)}{n-\ell}(1+O(n^{-1/2})) &= \frac{\ell}{a'b}(1+O(\frac{1}{\ell}))\frac{e^{C\sqrt{n-\ell}}}{4(n-\ell)^2\sqrt{3}}(1+O(n^{-1/2}))\\ &\leqslant \frac{\ell}{4a'b\sqrt{3}n^2}e^{C\sqrt{n}}(1+O(\ell^{-1}+n^{-1/2}))\\ &< \frac{e^{C\sqrt{n}}}{40a'b\sqrt{3}n^{3/2}}(1+O(n^{-1/2})).\end{aligned}$$

It follows from (2·1), (6·5) and (6·6) that for large n we have

$$\begin{aligned}|\mathcal{P}^*| &< \frac{C^2}{4}\sum_{\substack{\ell<\sqrt{n}/10\\ \ell\equiv 0 \,(\mathrm{mod}\, \delta)}}\frac{\delta e^{C\sqrt{n}}}{40\sqrt{3}n^{3/2}}(1+O(n^{-1/2}))\\ &< \frac{C^2}{1600\sqrt{3}n}e^{C\sqrt{n}}(1+O(n^{-1/2}))\\ &< \frac{C^2}{400}p(n)(1+O(n^{-1/2})) < \frac{1}{50}p(n)\end{aligned}$$

so that

(6·7)
$$\begin{aligned}|\bar{\mathcal{P}}| &= \sum_{\ell\geqslant\sqrt{n}/10}|\mathcal{P}_\ell| = |\mathcal{P}|-|\mathcal{P}^*|\\ &> p(n)-\frac{1}{50}p(n)=\frac{49}{50}p(n).\end{aligned}$$

Now let Q denote the set of those partitions λ of n wich satisfy (1·5). We will show that for all

(6·8)
$$\ell \geqslant \frac{\sqrt{n}}{10} \text{ satisfying } \ell \equiv 0 \,(\mathrm{mod}\, \delta),$$

a positive proportion of the partitions in $\mathcal{P}_\ell$ belongs to Q. In order to prove this, we fix an ℓ satisfying (6·8), and then we introduce an equivalence relation in $\mathcal{P}_\ell$: we say for λ, $\lambda' \in \mathcal{P}_\ell$ that $\lambda \sim \lambda'$ if and only if $\Psi(\lambda) = \Psi(\lambda')$, i.e, in λ and λ' the parts not in $\{a, b\}$ are the same. Consider such an equivalence class $\mathcal{E}$, wich is uniquely determined by the partition Ψ in (6·3). Then $|\mathcal{E}|$ is equal to the number of partitions Φ of form (6·2), so that

(6·9)
$$|\mathcal{E}| = (1+o(1))\frac{\ell}{a'b}$$

(note that ℓ is large by (6·8)). Let Φ_0 denote the partition (6·2) with $x_b = \lambda - a'\Big[\frac{\ell}{2a'b}\Big]$:

(6·10)
$$a\Big(\beta+b'\Big[\frac{\ell}{2a'b}\Big]\Big)+b\Big(\lambda-a'\Big[\frac{\ell}{2a'b}\Big]\Big)=\ell,$$

and define the partition $\lambda_0 \in \mathcal{E}$ by $\Phi(\lambda_0) = \Phi_0$ (while $\Psi(\lambda_0) = \Psi$ is the same for every element of $\mathcal{E}$). Next define the subset $\mathcal{E}^+$ of $\mathcal{E}$ in the following way:

(i) if $F_{a,d}(\lambda_0) \geqslant F_{b,d}(\lambda_0)$ then let $\mathcal{E}^+$ denote the set of the partitions $\lambda \in \mathcal{E}$ such that $x_b \leqslant \ell/4b$ in (6·2),

(ii) if $F_{a,d}(\lambda_0) < F_{b,d}(\lambda_0)$ then let $\mathcal{E}^+$ denote the set of the partitions $\lambda \in \mathcal{E}$ such that $\frac{3\ell}{4b} \leqslant x_b \leqslant \frac{\ell}{b}$ in $\Phi(\lambda)$ in (6·2). Since $x_b = \lambda - a'\mu$, with $\lambda = \frac{\ell}{b} + O(1)$, in the first case we have $\lambda - \frac{\ell}{4b} + O(1) \leqslant a'\mu \leqslant \lambda$ and in the second we have $O(1) \leqslant a'\mu \leqslant \lambda - \frac{3\ell}{4b} + O(1)$. Then, by (6·9), in both cases we have

$$|\mathcal{E}^+| = \frac{\ell}{4a'b} + O(1) > \frac{1}{5}|\mathcal{E}|. \tag{6·11}$$

Moreover, for all $\lambda \in \mathcal{E}^+$ we have

$$F_{a,d}(\lambda) = x_a + (F_{a,d}(\lambda_0) - (\beta + b'\Big[\frac{\ell}{2a'b}\Big])) = -\frac{bx_b}{a} + F_{a,d}(\lambda_0) + \frac{\ell}{2a} + O(1)$$

and

$$F_{b,d}(\lambda) = x_b + (F_{b,d}(\lambda_0) - \lambda + a'\Big[\frac{\ell}{2a'b}\Big]) = x_b + F_{b,d}(\lambda_0) - \frac{\ell}{2b} + O(1)$$

whence

$$F_{a,d}(\lambda) - F_{b,d}(\lambda) = F_{a,d}(\lambda_0) - F_{b,d}(\lambda_0) + \ell\frac{(a+b)}{2ab} - x_b\frac{(a+b)}{a} + O(1).$$

It follows that in case (i) we have

$$F_{a,d}(\lambda) - F_{b,d}(\lambda) \geqslant (a+b)\big(\frac{\ell}{2ab} - \frac{\ell}{4ab}\big) + O(1) = (1+o(1))\frac{(a+b)\ell}{4ab} > \frac{(a+b)\ell}{5ab}$$

while in case (ii),

$$F_{a,d}(\lambda) - F_{b,d}(\lambda) < (a+b)\big(\frac{\ell}{2ab} - \frac{3\ell}{4ab}\big) + O(1) = -(1+o(1))\frac{(a+b)\ell}{4ab} < -\frac{(a+b)\ell}{5ab}$$

(recall that ℓ is large by (6·8)). Thus by (6·8) in both cases we have

$$|F_{a,d}(\lambda) - F_{b,d}(\lambda)| > \frac{(a+b)\ell}{5ab} \geqslant \frac{(a+b)\sqrt{n}}{50ab}$$

whence

$$\mathcal{E}^+ \subset Q. \tag{6·12}$$

By (6·11) and (6·12) we have

$$\begin{aligned} |\mathcal{P}_\ell \cap Q| &= \sum_{\mathcal{E}} |\mathcal{E} \cap Q| \\ &\geqslant \sum_{\mathcal{E}} |\mathcal{E}^+| \\ &> \frac{1}{5}\sum_{\mathcal{E}} |\mathcal{E}| = \frac{1}{5}|\mathcal{P}_\ell| \end{aligned} \tag{6·13}$$

(for all $\ell \geqslant \sqrt{n}/10$). It follows from (6·7) and (6·13) that

$$\begin{aligned} |Q| \geqslant |\mathcal{P} \cap Q| &= \sum_{\ell \geqslant \sqrt{n}/10} |\mathcal{P}_\ell \cap Q| \\ &> \frac{1}{5}\sum_{\ell \geqslant \sqrt{n}/10} |\mathcal{P}_\ell| > \frac{49}{250}p(n) > \frac{p(n)}{6} \end{aligned}$$

wich completes the proof of Theorem 1.4.

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