Nilpotent adjacency matrices, random graphs, and quantum random variables

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Abstract

For fixed $n > 0$, the space of finite graphs on n vertices is canonically associated with an abelian, nilpotent-generated subalgebra of the $2n$-particle fermion algebra. Using the generators of the subalgebra, an algebraic probability space of "nilpotent adjacency matrices" associated with finite graphs is defined. Each nilpotent adjacency matrix is a quantum random variable whose m^{th} moment corresponds to the number of m-cycles in the graph G. Each matrix admits a canonical "quantum decomposition" into a sum of three algebraic random variables: $a = a^{\Delta} + a^{\Upsilon} + a^{\Lambda}$, where a^{Δ} is classical while a^{Υ} and a^{Λ} are quantum. Moreover, within the algebraic context, the NP problem of cycle enumeration is reduced to matrix multiplication, requiring no more than n^4 multiplications within the algebra.

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1 Introduction

Links between quantum probability and graph theory have been explored in a number of works. Hashimoto, Hora, and Obata (cf. [6], [13]) obtained limit theorems for increasing sequences of graphs G_n whose adjacency matrices admit a *quantum decomposition* $A_n = {A_n}^+ + {A_n}^-$. Examples include Cayley graphs, Johnson graphs, and distance-regular graphs.

Obata [13] uses this approach to focus on star graphs, which are obtained by gluing together the common origins of a finite number of copies of a given graph. The adjacency matrices of star graphs admit a quantum decomposition of the form $A_n = {A_n}^+ + {A_n}^- + {A_n}^\circ$. Star graphs are of particular interest because they are related to Boolean independence in quantum probability.

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Homogeneous trees are also of interest in quantum probability. These are related to the free independence of Voiculescu [22].

Comb graphs, which provide models of Bose-Einstein condensation, are related to monotone independence discovered by Lu [10] and Muraki [12]. Accardi, Ben Ghorbal, and Obata [2] computed the vacuum spectral distribution of the comb graph by decomposing the adjacency matrix into a sum of monotone independent random variables.

Another work of interest is that of Franz Lehner [8], who investigated the relationships among non-crossing partitions, creation and annihilation operators, and the cycle cover polynomial of a graph. In that work, the cycle indicator polynomials of particular digraphs are used to understand the partitioned moments and cumulants occurring in Fock spaces associated to characters of the infinite symmetric group of Bożejko and Guţă [4].

In contrast to the works cited above, the philosophy of the current work is to begin with an arbitrary finite graph and then to construct an associated algebraic probability space in which the moments of random variables reveal information about the graph. The graph need possess no particular relationship to notions of independence or Fock spaces.

Letting $n = |V|$, the vertices of G can be represented by unit coordinate vectors in $\mathbb{R}^n$. When A is the adjacency matrix of G, a well-known result in graph theory states that $\langle x_0, A^k x_0 \rangle$ corresponds to the number of closed k-walks based at vertex $x_0 \in V(G)$.

We are interested in the related problem of recovering the *k-cycles* based at any vertex x_0. This can be done for any finite graph with the methods described herein.

The algebras used in this paper were originally derived as subalgebras of Clifford algebras. Clifford algebras and quantum logic gates have been discussed in works by W. Li [9] and Vlasov [21]. The role of Clifford algebras in quantum computing has been considered in works by Havel and Doran [7], Matzke [11], and others.

In the $\mathcal{C}\ell$ context, i.e., in terms of the number of multiplications performed in the algebra, the computational complexity of enumerating the Hamiltonian cycles in a graph on n vertices is $O(n^4)$. In this context, some graph problems are moved from complexity class NP into class P.

1.1 Graphs and Adjacency Matrices

A *graph* $G = (V, E)$ is a collection of vertices V and a set E of unordered pairs of vertices called *edges*. A *directed graph* is a graph whose edges are *ordered pairs* of vertices. Two vertices $v_i, v_j \in V$ are *adjacent* if there exists an edge $e = \{v_i, v_j\} \in E$. Adjacency will be denoted $v_i \sim v_j$. A graph is *finite* if V and E are finite sets, that is, if $|V|$ and $|E|$ are finite numbers. A *loop* in a graph is an edge of the form (v, v). A graph is said to be *simple* if 1) it contains no loops, and 2) no unordered pair of vertices appears more than once in E.

A *k-walk* $\{v_0, \ldots, v_k\}$ in a graph G is a sequence of vertices in G with *initial vertex* v_0 and *terminal vertex* v_k such that there exists an edge $(v_j, v_{j+1}) \in E$

for each $0 \leq j \leq k-1$. A k-walk contains k edges. A *self-avoiding walk* is a walk in which no vertex appears more than once. A *closed k-walk* is a k-walk whose initial vertex is also its terminal vertex. A *k-cycle* is a self-avoiding closed k-walk with the exception $v_0 = v_k$. A *Hamiltonian cycle* is an n-cycle in a graph on n vertices; i.e., it contains V. An *Euler circuit* is a closed walk encompassing every edge in E exactly once.

When working with a finite graph G on n vertices, one often utilizes the *adjacency matrix* A associated with G. The adjacency matrix is defined by

$$A_{xy} = \begin{cases} 1 & \text{if } x \sim y, \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

A graph in which multiple edges exist between a given pair of vertices is called a *multigraph.* If a vertex is allowed to be self-adjacent, the edge joining the vertex to itself is referred to as a *loop.* Graphs containing loops are known as *pseudographs.*

1.2 Adjacency matrices as operators on Hilbert space

Let $G = (V, E)$ be a graph on n vertices. Let $\Gamma = \sum_{n=0}^{\infty} \oplus \mathbb{C}\Phi_n$ be the Hilbert space with complete orthonormal basis $\{\Phi_n\}$. Here Φ_0 represents the "vacuum vector." Identifying the vertices of G with the vectors $\{\Phi_k\}$, $(1 \leq k \leq n)$, the adjacency matrix of G is seen to be a bounded linear operator on Γ.

When G is an undirected graph, its adjacency matrix A is symmetric, i.e., Hermitian. Bounded Hermitian operators are random variables in quantum probability theory.

More generally, one considers the Hilbert space $\ell^2(V)$ of all $\mathbb{C}$-valued functions f on the vertex set of G satisfying

$$|f(x)|^2 < \infty. \tag{1.2}$$

When G is a simple graph, it is well-known [6] that the adjacency matrix of G acts on $\ell^2(V)$ according to

$$Af(x) = \sum_{y \in V} A_{xy} f(y) = \sum_{y \sim x} f(y). \tag{1.3}$$

This notion can also be extended to multigraphs and pseudographs. In an undirected graph with multiple edges or loops, the *edge multiplicity* of a pair (x, y) of adjacent vertices is defined as the number of edges incident to both x and y. In this case, denoting the edge multiplicity of $x \sim y$ by $\varpi(x, y)$, the adjacency matrix of G acts on $\ell^2(V)$ according to

$$Af(x) = \sum_{y \in V} A_{xy} f(y) = \sum_{y \sim x} \varpi(x, y) f(y). \tag{1.4}$$

The *degree* of a vertex $x \in V$, denoted $\kappa(x)$, is defined for any undirected graph by

$$\kappa(x) = \sum_{y \sim x} \varpi(x, y). \tag{1.5}$$

For simple graphs, the degree takes the simpler form

$$\kappa(x) = |\{y \in V : y \sim x\}|. \tag{1.6}$$

Related to the adjacency matrix is another operator called the *combinatorial Laplacian*, which is denoted by Δ. The combinatorial Laplacian associated with G is defined by its action on $\ell^2(V)$ according to

$$\Delta f(x) = \kappa(x) f(x) - \sum_{y \sim x} \omega(x, y) f(y) = \kappa(x) f(x) - A\, f(x). \tag{1.7}$$

The *degree operator* $\hat{\kappa}$ is defined by

$$\hat{\kappa} f(x) = \left(\sum_{y \in V} A_{xy} \right) f(x) = \kappa(x) f(x). \tag{1.8}$$

The combinatorial Laplacian can then be written as $\Delta = \hat{\kappa} - A$.

An *edge-weighted* graph $G = (V, E)$ is a graph together with a real-valued function $\omega : V \times V \to \mathbb{R}$. The adjacency matrix of G is then defined by

$$A_{xy} = \begin{cases} \omega(x, y) & \text{if } x \sim y, \\ 0 & \text{otherwise.} \end{cases} \tag{1.9}$$

In this case, the action of A on $\ell^2(V)$ is defined by replacing ϖ by ω in (1.2).

1.3 Quantum probability: operators as random variables

Let $\mathcal{A}$ be a complex algebra with involution $*$ and unit 1. A positive linear $*$-functional on $\mathcal{A}$ satisfying $\varphi(1) = 1$ is called a *state* on $\mathcal{A}$. The pair $(\mathcal{A}, \varphi)$ is called an *algebraic probability space*.

If $\mathcal{A}$ is commutative, the probability space is *classical*. If $\mathcal{A}$ is non-commutative, the probability space is *quantum*.

Elements of $\mathcal{A}$ are *random variables*. A *stochastic process* is a family (X_t) indexed by an arbitrary set T. Given a stochastic process $X = (X_t)$, the polynomial $*$-algebra $\mathcal{P}(X)$ is a $*$-subalgebra of $\mathcal{A}$. The restriction of φ to this $*$-subalgebra gives a state φ_X on $\mathcal{P}(X)$ called the *distribution* of the process X. If $T = \{1, 2, \ldots, n\}$ is a finite set, then φ_X is called the *joint distribution* of the random variables $\{X_1, \ldots, X_n\}$.

Of particular interest in quantum probability theory are interacting Fock spaces. Let $\lambda_0 = 1$, and let $\{\lambda_i\}_{1 \leq i}$ be a sequence of nonnegative real numbers

Type	Relation	Realized by
Boson	$[B^-, B^+] = 1$	$\lambda_n = n!$.
Fermion	$\{B^-, B^+\} = 1$	$\lambda_0 = \lambda_1 = 1$, $\lambda_n = 0$ for $n \geq 2$.
Free	$B^- B^+ = 1$	$\lambda_n = 1$ for all $n \geq 0$.

Figure 1: Realizations of commutation relations.

such that $\lambda_m = 0 \Rightarrow \lambda_{m+1} = \lambda_{m+2} = \cdots = 0$. In the case that $\lambda_n > 0$ for all n, one defines the linear operators B^+ and B^- by

$$B^+ \Phi_n = \sqrt{\frac{\lambda_{n+1}}{\lambda_n}} \Phi_{n+1}, \ n \geq 0, \tag{1.10}$$

and

$$B^- \Phi_n = \begin{cases} \sqrt{\dfrac{\lambda_n}{\lambda_{n-1}}} \Phi_{n-1}, & n \geq 1 \\ 0 & n = 0. \end{cases} \tag{1.11}$$

On their natural domains, B^+ and B^- are mutually adjoint and closed. Operators B^+ and B^- are called the *creation* and *annihilation operators*, respectively.

The *number operator* is defined by

$$N \Phi_n = n \Phi_n, \ n \geq 0. \tag{1.12}$$

Recalling the definitions of the commutator and anti-commutator, respectively,

$$[a, b] = ab - ba, \tag{1.13}$$

$$\{a, b\} = ab + ba, \tag{1.14}$$

direct calculation shows

$$B^+ B^- \Phi_0 = B^- B^+ \Phi_0 = 0, \tag{1.15}$$

$$\{B^+, B^-\} = \frac{{\lambda_n}^2 + \lambda_{n+1} \lambda_{n-1}}{\lambda_n \lambda_{n-1}}, \ n \geq 1, \tag{1.16}$$

$$\left[B^+, B^-\right] = \frac{{\lambda_n}^2 - \lambda_{n+1} \lambda_{n-1}}{\lambda_n \lambda_{n-1}}, \ n \geq 1, \tag{1.17}$$

$$B^{+n} \Phi_0 = \sqrt{\lambda_n} \Phi_n, \ n \geq 0, \tag{1.18}$$

$$B^{-n} \Phi_n = \sqrt{\lambda_n} \Phi_0, \ n \geq 0. \tag{1.19}$$

Realizations of commutation relations are summarized in Figure 1.

When there exists some $m \geq 1$ such that $\lambda_m > 0$ but $\lambda_n = 0$ for all $n > m$, one defines the finite dimensional Hilbert space

$$\Gamma = \sum_{n=0}^{m} \oplus \mathbb{C} \Phi_n.$$

The finite-dimensional operators B^+ and B^- are then defined in the obvious way. In this case, $B^+\Phi_m = 0$.

In either the finite- or infinite-dimensional case, the Hilbert space $\Gamma(\mathbb{C}, \{\lambda_n\}) = (\Gamma, \{\lambda_n\}, B^+, B^-)$ is called an *interacting Fock space* associated with $\{\lambda_n\}$.

One can begin with a classical random variable and construct a stochastically equivalent quantum random variable. Given a probability measure μ having finite moments of all orders; that is,

$$\int_{\mathbb{R}} |x|^m \, \mu(dx) < \infty, \ m = 0, 1, 2, \ldots,$$

let $\{P_n\}$ be the associated orthogonal polynomials. The polynomials are assumed to be monic via an appropriate normalization. By orthogonality,

$$P_0(x) = 1, \tag{1.20}$$

$$P_1(x) = x - \alpha_1, \tag{1.21}$$

$$(x - \alpha_{n+1})P_n(x) = P_{n+1}(x) + \omega_n P_{n-1}(x), \ n \geq 1, \tag{1.22}$$

$$\tag{1.23}$$

where the numbers $\{\alpha_n, \omega_n\}_{n=0}^{\infty}$ are called are the *Szegö-Jacobi parameters* of μ.

The book by Szegö [20] provides a detailed treatise on orthogonal polynomials. The work of Asai, Kubo, and Kuo [3] details a generating function method for deriving the orthogonal polynomials and Szegö-Jacobi parameters associated with a probability measure μ.

Letting $\Gamma(\mathbb{C}, \{\lambda_n\})$ be the interacting Fock space associated with $\lambda_0 = 1$, $\lambda_1 = \omega_1$, and $\dfrac{\lambda_{n+1}}{\lambda_n} = \omega_{n+1}$, Accardi and Bożejko [1] proved the existence of an isometry U from $\Gamma(\mathbb{C}, \{\lambda_n\})$ into $L^2(\mathbb{R}, \mu)$ uniquely determined by

$$U\Phi_0 = P_0,$$
$$UB^+U^*P_n = P_{n+1}, \text{ and}$$
$$Q = U(B^+ + B^- + \hat{\alpha}_{n+1})U^*,$$

where Q is the multiplication operator by x densely defined in $L^2(\mathbb{R}, \mu)$ and $\hat{\alpha}_{n+1}$ is the operator

$$\hat{\alpha}_{n+1} = \frac{\alpha_{n+1}}{n} N,$$

so that $\hat{\alpha}_{n+1}\Phi_n = \alpha_{n+1}\Phi_n$. Then,

$$\int_{\mathbb{R}} x^m \mu(dx) = \left\langle \Phi_0, (B^+ + B^- + \hat{\alpha}_{n+1})^m \Phi_0 \right\rangle_{\Gamma}, \ m = 0, 1, 2, \ldots. \tag{1.24}$$

Let $\mathcal{A}$ denote the $*$-algebra generated by Q, and let X be a classical random variable with probability distribution μ having moments of all orders. Then, an algebraic probability space $(\mathcal{A}, \varphi)$ is defined with the state φ determined by

$$\varphi(a) = \langle P_0, aP_0 \rangle_{L^2(\mathbb{R}, \mu)}, \ a \in \mathcal{A}. \tag{1.25}$$

In light of the results of Accardi and Bożejko, the following relation becomes evident:

$$\mathbb{E}(X^m) = \int_{\mathbb{R}} x^m \mu(dx) = \varphi(Q^m), \ m = 1, 2, \ldots. \tag{1.26}$$

In other words, X and Q are stochastically equivalent. The classical random variable X then admits a *quantum decomposition* of the form $B^+ + B^- + \hat{\alpha}_{n+1}$. That is, X admits a decomposition as a linear combination of creation, annihilation, and number operators.

1.4 Operators as adjacency matrices

Graphs can be interpreted as operators on Hilbert spaces. In quantum probability, bounded Hermitian operators on Hilbert spaces are quantum random variables. It is not surprising that quantum random variables can then be interpreted as graphs.

Given an interacting Fock space $(\Gamma, \{\lambda_n\}, B^+, B_-)$ associated with $\{\lambda_n\}$,the operator $A = B^+ + B^- + N$ can also be interpreted as the adjacency matrix associated with an edge-weighted pseudograph. The associated finite graph is constructed with vertex set $\{\Phi_0, \Phi_1, \ldots, \Phi_n\}$, edge weights $\{\sqrt{\frac{\lambda_k}{\lambda_{k-1}}}\}$, and k loops based at vertex Φ_k.

In this context, the adjacency matrix A is the sum of the upper-triangular matrix B^-, the lower triangular-matrix B^+, and the diagonal matrix N. Visualizations of graphs associated with finite-dimensional realizations of the commutation relations from Figure 1 appear in Figures 2, 3, and 4.

A Bernoulli random variable X admits an expression of the form

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{1.27}$$

commonly referred to as "quantum coin-tossing." Labeling the matrices of the right-hand side of (1.27) as f^- and f^+, respectively, X is decomposed into a sum of upper- and lower-triangular nilpotent matrices. It is apparent that by defining $\Phi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\Phi_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, one obtains

$$f^+\Phi_i = \begin{cases} \Phi_1 & \text{if i=0} \\ 0 & \text{otherwise,} \end{cases} \quad f^-\Phi_i = \begin{cases} \Phi_0 & \text{if i=1} \\ 0 & \text{otherwise.} \end{cases} \tag{1.28}$$

The corresponding number operator has the form

$$f_N = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.29}$$

The matrix $A = f^+ + f^- + f_N$ is the adjacency matrix of the pseudograph appearing in Figure 4.

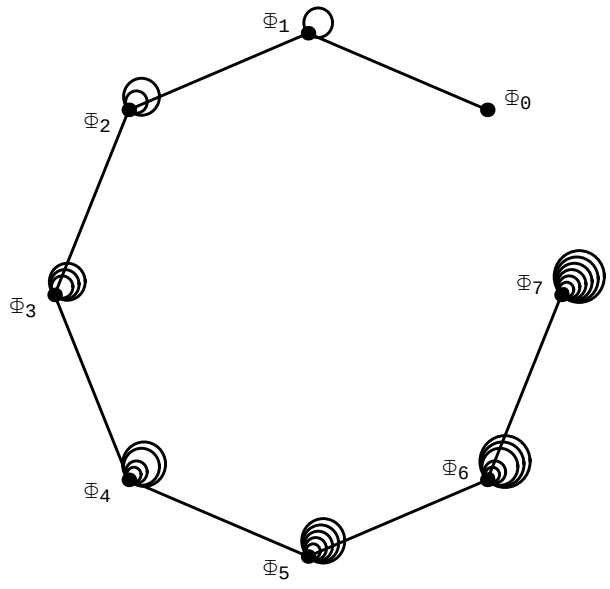


Figure 2: Graph with adjacency matrix $B^+ + B^- + N$ for free commutation relations.

The adjacency matrix $A = B^+ + B^- + N$ corresponding to Figure 2 is

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}, \tag{1.30}$$

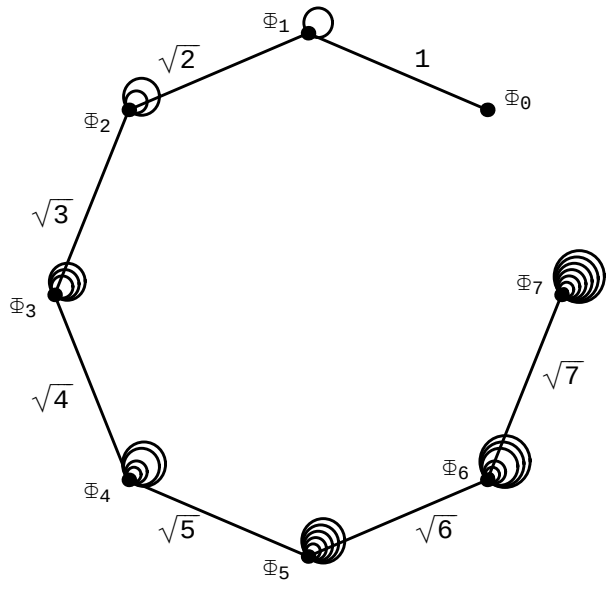


Figure 3: Graph with adjacency matrix $B^+ + B^- + N$ for Boson commutation relations.

where

$$B^+ = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \; B^- = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.31}$$

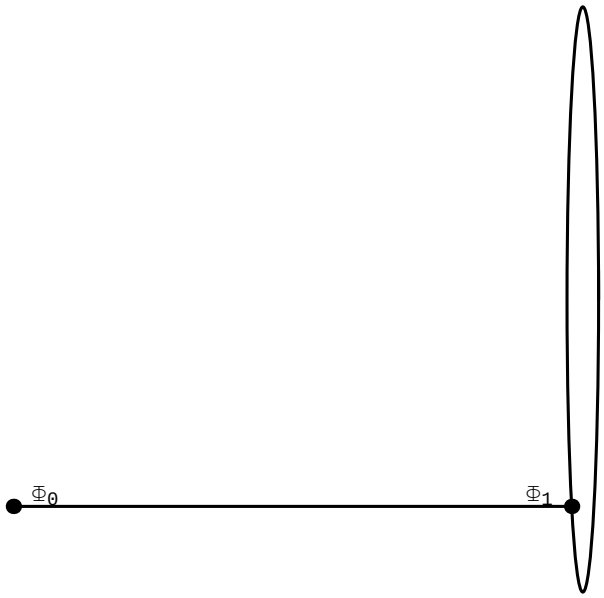


Figure 4: Graph with adjacency matrix $B^+ + B^- + N$ for Fermion commutation relations.

and

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix}. \tag{1.32}$$

Furthermore, with the convention that loops contribute two to the degree of a vertex, the degree mapping is defined by

$$\kappa = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 12 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 14 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 16 \end{pmatrix}. \tag{1.33}$$

2 Nilpotent adjacency matrices

The philosophy of this paper is to begin with an arbitrary graph and use it to construct a quantum random variable whose moments reveal information about the structures contained within the graph.

Let ${\mathcal{C}\ell_n}^{\text{nil}}$ be the associative algebra generated by the unit scalar $\zeta_\emptyset = 1 \in \mathbb{R}$ and the commuting nilpotents $\{\zeta_{\{i\}}\}$, where $1 \leq i \leq n$.

For $n > 0$ and nonnegative integers p, q satisfying $p + q = n$, the Clifford algebra $\mathcal{C}\ell_{p,q}$ is defined as the associative algebra generated by the unit scalar $\mathbf{e}_\emptyset = 1$ and the collection $\{\mathbf{e}_i\}_{1 \leq i \leq n}$ subject to multiplication rules

$$\mathbf{e}_i \, \mathbf{e}_j + \mathbf{e}_j \, \mathbf{e}_i = \begin{cases} 0 & \text{if } i \neq j, \\ 2 & \text{if } i = j \leq p, \\ -2 & \text{if } i = j > p. \end{cases} \tag{2.1}$$

It can be shown that for $n > 0$, the Clifford algebra $\mathcal{C}\ell_{2n,2n}$ is canonically isomorphic to the $2n$-particle fermion algebra. Using this correspondence, the algebra ${\mathcal{C}\ell_n}^{\text{nil}}$ is generated within $\mathcal{C}\ell_{2n,2n}$ by $\zeta_\emptyset = 1$ along with the set $\{\zeta_{\{i\}}\}_{1 \leq i \leq n}$, where $\zeta_{\{i\}} = {f_i}^+ {f_{n+1}}^+$. Here f_i^+ denotes the i^{th} fermion creation operator.

In terms of Pauli matrices, the generators of ${\mathcal{C}\ell_n}^{\text{nil}}$ can be written as

$$\zeta_{\{i\}} = \sigma_0^{\otimes(i-1)} \otimes (\sigma_x + i\sigma_y) \otimes \sigma_0^{\otimes(n-i)}. \tag{2.2}$$

Remark 2.1. Writing the generators $\{\zeta_{\{i\}}\}_{1 \leq i \leq n}$ in terms of fermion annihilation operators is an acceptable alternative.

For each $n > 0$, the algebra ${\mathcal{C}\ell_n}^{\text{nil}}$ has dimension 2^n and is spanned by unit multi-vectors, or *blades*, of the form

$$\zeta_{\underline{i}} = \prod_{\iota \in \underline{i}} \zeta_{\{\iota\}}, \tag{2.3}$$

where $\underline{i} \in 2^{[n]}$ is a canonically ordered multi-index in the power set of $[n] = \{1, 2, \ldots, n\}$.

Let $u \in {\mathcal{C}\ell_n}^{\text{nil}}$, and observe that u has the canonical expansion

$$u = \sum_{\underline{i} \in 2^{[n]}} u_{\underline{i}} \, \zeta_{\underline{i}}, \tag{2.4}$$

where $u_{\underline{i}} \in \mathbb{R}$ for each $\underline{i}$. Define the *dual* $u^\star$ of u by

$$u^\star = \sum_{\underline{i} \in 2^{[n]}} u_{\underline{i}} \, \zeta_{[n] \setminus \underline{i}}. \tag{2.5}$$

The *scalar sum evaluation* of u is defined by

$$\varphi(u) = \sum_{\underline{i} \in 2^{[n]}} u_{\underline{i}}. \tag{2.6}$$

Remark 2.2. The scalar sum evaluation is equivalent to the *Clifford 1-norm* $\|u\|_1 = \sum_{\underline{i} \in 2^{[n]}} |u_{\underline{i}}|$ when all the coefficients in the expansion are nonnegative.

Lemma 2.3. *Let $u, v \in \mathcal{C}\ell_n{}^{\mathrm{nil}}$, and let $\alpha \in \mathbb{R}$. Let $\left\langle \mathcal{C}\ell_n{}^{\mathrm{nil}} \right\rangle_+$ denote those elements of $\mathcal{C}\ell_n{}^{\mathrm{nil}}$ having strictly nonnegative coefficients. Then, the scalar sum evaluation $\varphi : \mathcal{C}\ell_n{}^{\mathrm{nil}} \to \mathbb{R}$ satisfies*

$$\varphi(1) = 1\,, \text{ and} \tag{2.7}$$

$$\varphi(\alpha\, u + v) = \alpha\, \varphi(u) + \varphi(v). \tag{2.8}$$

Moreover,

$$u \in \left\langle \mathcal{C}\ell_n{}^{\mathrm{nil}} \right\rangle_+ \Rightarrow \varphi(u^\star\, u) \geq 0. \tag{2.9}$$

Proof. The result follows immediately from the definitions and is omitted. □

Let $\{\zeta_{\{i\}}\}_{1 \leq i \leq n}$ denote the orthonormal nilpotent generators of $\mathcal{C}\ell_n{}^{\mathrm{nil}}$. Associated with any finite graph $G = (V, E)$ on n vertices is a *nilpotent adjacency matrix* a_n defined by

$$(a_n)_{ij} = \begin{cases} \zeta_{\{j\}} & \text{if } (v_i, v_j) \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

Proposition 2.4. *Let a_n denote the nilpotent adjacency matrix of a graph G on n vertices. Let x_0 represent any fixed vertex in the graph. Let X_m denote the number of m-cycles based at x_0 in G. Then,*

$$\varphi\left(\langle x_0, {a_n}^m x_0 \rangle\right) = \vartheta X_m, \tag{2.11}$$

where

$$\vartheta = \begin{cases} 2 & \textit{if the graph is undirected,} \\ 1 & \textit{otherwise.} \end{cases} \tag{2.12}$$

Proof. Proof is by induction on m. When $m = 2$,

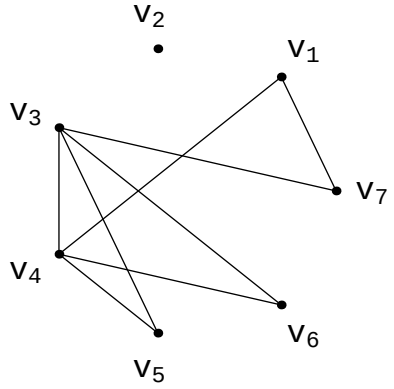
$$\left(a^2\right)_{ii} = (a \times a)_{ii} = \sum_{\ell=1}^{n} a_{i\ell} a_{\ell i}. \tag{2.13}$$

By construction of the nilpotent adjacency matrix,

$$a_{i\ell} \equiv \text{s.a. 1-walks } v_i \to v_\ell, \text{ and} \tag{2.14}$$

$$a_{\ell i} \equiv \text{ s.a. 1-walk } v_\ell \to v_i, \tag{2.15}$$

where s.a. is short for self-avoiding. Hence, the product of these terms corresponds to 2-cycles $v_i \to v_i$.



$$a = \begin{pmatrix} 0 & 0 & 0 & \zeta_{\{4\}} & 0 & 0 & \zeta_{\{7\}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_{\{4\}} & \zeta_{\{5\}} & \zeta_{\{6\}} & \zeta_{\{7\}} \\ \zeta_{\{1\}} & 0 & \zeta_{\{3\}} & 0 & \zeta_{\{5\}} & \zeta_{\{6\}} & 0 \\ 0 & 0 & \zeta_{\{3\}} & \zeta_{\{4\}} & 0 & 0 & 0 \\ 0 & 0 & \zeta_{\{3\}} & \zeta_{\{4\}} & 0 & 0 & 0 \\ \zeta_{\{1\}} & 0 & \zeta_{\{3\}} & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 5: A graph and its nilpotent adjacency matrix.

Now assuming the proposition holds for m and considering the case $m+1$,

$$\left(a^{m+1}\right)_{ii} = \left(a^m \times a\right)_{ii} = \sum_{\ell=1}^{n} \left(a^m\right)_{i\ell} a_{\ell i}. \tag{2.16}$$

Considering a general term of the sum,

$$\left(a^m\right)_{i\ell} = \sum_{\text{s.a. } m\text{-walks } w_m : v_i \to v_\ell} w_m, \text{ and} \tag{2.17}$$

$$a_{\ell i} = \sum_{\text{s.a. 1-walks } w_1 : v_\ell \to v_i} w_1. \tag{2.18}$$

It should then be clear that terms of the product

$$\left(a^m\right)_{i\ell} a_{\ell i} \tag{2.19}$$

are nonzero if and only if they correspond to self-avoiding $m+1$-walks $v_i \to v_\ell \to v_i$. Summing over all vertices v_ℓ gives the total number of $m+1$-cycles based at v_i.

The correction factor $\vartheta = 2$ in the undirected case accounts for the two possible orientations of each cycle. An orientation correction is not needed when the graph is directed. □

Additional properties of nilpotent adjacency matrices over ${\mathcal{C}\ell_n}^{\mathrm{nil}}$ can be found in [16] and [19].

Example 2.5. The two 5-cycles contained in the 7-vertex graph of Figure 5 are recovered by examining the trace of a^5. Computations performed using Mathematica reveal

$$\frac{\mathrm{tr}(a^5)}{10} = \zeta_{\{1,3,4,5,7\}} + \zeta_{\{1,3,4,6,7\}}. \tag{2.20}$$

3 Nilpotent Adjacency Matrices as Quantum Random Variables

Let $\mathcal{M}$ denote the *-algebra generated by $n \times n$ nilpotent adjacency matrices with involution defined by $a^* = (a^\star)^\dagger$. Here, * denotes the dual defined by linear extension of ${\zeta_{\underline{i}}}^\star = \zeta_{[n]\setminus\underline{i}}$, and † denotes the matrix transpose.

The *norm* of $a \in \mathcal{M}$ is defined by

$$\|a\|^2 = \int \mathrm{tr}(a^* a)\, d\zeta_n \cdots d\zeta_1. \tag{3.1}$$

Remark 3.1. This is the *Clifford-Frobenius norm* constructed in [16].

Let $\varphi(u)$ denote the scalar sum evaluation of $u \in {\mathcal{C}\ell_n}^{\mathrm{nil}}$, and define $\psi : \mathcal{M} \to \mathbb{R}$ by

$$\psi(a) = \frac{\varphi\left(\mathrm{tr}(a)\right)}{n}. \tag{3.2}$$

Direct computation shows that ψ is a linear mapping and that $\psi(1_{\mathcal{M}}) = 1$. The positivity requirement for states, $\psi(a^*\, a) \geq 0$, does not generally hold in the *-algebra, however.

The collection of nilpotent adjacency matrices $\mathcal{A} \subset \mathcal{M}$ under matrix multiplication defined in terms of the algebra product constitutes a semigroup with involution $*$. On this semigroup, the map ψ satisfies the positivity requirement, and $(\mathcal{A}, \psi)$ is a quantum probability space.

The proofs of the following propositions are virtually identical and follow naturally from Proposition 2.4.

Proposition 3.2 (Cycles in finite graphs)**.** *Let $a \in \mathcal{A}$. Letting X_m denote the number of m-cycles in the graph associated with a,*

$$\psi(a^m) = \frac{\vartheta\, m}{n} X_m, \tag{3.3}$$

where ϑ is defined as in (2.12)*. Thus, the nilpotent adjacency matrix $a \in \mathcal{A}$ is a quantum random variable whose m^{th} moment in the state ψ corresponds to the number of m-cycles in the graph associated with a.*

Proposition 3.3 (Time-homogeneous random walks on finite graphs)**.** *Let M denote a stochastic matrix corresponding to an n-state Markov chain, and let τ denote the nilpotent stochastic matrix defined by*

$$\tau_{ij} = M_{ij}\,\zeta_{\{j\}}\,, \tag{3.4}$$

where $\zeta_{\{j\}}$ is a nilpotent generator of the abelian algebra ${\mathcal{C}\ell_n}^{\mathrm{nil}}$.

Let the state ψ be defined as in (3.2)*, and let ϑ be defined as in* (2.12)*. Define the random variable w_m on the space of m-step walks by*

$$w_m = \begin{cases} 1 & \text{if the } m\text{-walk forms a cycle,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

Then,

$$\psi(\tau^m) = \frac{\vartheta\, m}{n}\mathbb{E}(w_m). \tag{3.6}$$

In other words, τ is a quantum random variable whose m^{th} moment corresponds to the expected number of m-walks forming cycles in the n-state Markov chain associated with matrix M.

Proposition 3.4 (Cycles in random graphs)**.** *Consider a random directed graph on n vertices, corresponding to edge-existence matrix A, defined by*

$$A_{ij} = \mathbb{P}\left((v_i, v_j) \in E\right). \tag{3.7}$$

It is assumed that the probabilities are pairwise-independent. Let ξ denote the nilpotent adjacency matrix defined by

$$\xi_{ij} = A_{ij}\,\zeta_{\{j\}}\,, \tag{3.8}$$

where $\zeta_{\{j\}}$ is a nilpotent generator of the abelian algebra ${\mathcal{C}\ell_n}^{\mathrm{nil}}$.

Let the state ψ be defined as in (3.2)*, let ϑ be defined as in* (2.12)*, and define the random variable z_m as the number of m-cycles occurring in the graph. Then,*

$$\psi(\xi^m) = \frac{\vartheta\, m}{n}\mathbb{E}(z_m). \tag{3.9}$$

That is, ξ is a quantum random variable whose m^{th} moment corresponds to the expected number of m-cycles occurring in the graph.

4 Decomposition of Nilpotent Adjacency Matrices

In the work of Hashimoto, Hora, and Obata (cf. [6], [13]), fixing a vertex v_0 in a finite graph induces a stratification of all the vertices by associating each vertex with the length of the shortest path linking it with v_0. This stratification is then used to define a quantum decomposition of the graph's adjacency matrix.

The nilpotent adjacency matrix of a graph also admits a quantum decomposition as the sum of three algebraic random variables: one classical, and two quantum. The decomposition considered here differs from that of Hashimoto, et al.

The nilpotent adjacency matrix a of any finite graph has the canonical quantum decomposition

$$a = a^{\Delta} + a^{\Lambda} + a^{\Upsilon}, \tag{4.1}$$

where

$$a^{\Delta}{}_{ij} = \delta_{ij} a_{ij}, \tag{4.2}$$

$$a^{\Upsilon}{}_{ij} = \theta_{ij} a_{ij}, \tag{4.3}$$

$$a^{\Lambda}{}_{ij} = (1 - \theta_{ij}) a_{ij}. \tag{4.4}$$

Here δ_{ij} is the Kronecker delta function, and θ_{ij} is the *ordering symbol* defined by

$$\theta_{ij} = \begin{cases} 1 & \text{if } i < j \\ 0 & \text{otherwise.} \end{cases} \tag{4.5}$$

The matrix a^{Δ} is an element of the multiplicative semigroup $\mathcal{D}$ of diagonal nilpotent adjacency matrices, which is commutative. Hence, a^{Δ} is a classical algebraic random variable in the space $(\mathcal{D}, \psi)$.

Elements a^{Λ} and a^{Υ}, respectively, reside in the semigroups $\mathcal{L}$ and $\mathcal{U}$ of lower- and upper-triangular nilpotent adjacency matrices with matrix multiplication, which are noncommutative. Hence, a^{Λ} and a^{Υ} are quantum algebraic random variables in the spaces $(\mathcal{L}, \psi)$ and $(\mathcal{U}, \psi)$, respectively.

In particular, let γ represent the diagonal matrix defined by

$$\gamma = \begin{pmatrix} \zeta_{\{1\}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \zeta_{\{n\}} \end{pmatrix}. \tag{4.6}$$

Then, recalling the (finite) adjacency matrix realizations of the commutation relations of Table 1, the following correspondence is seen:

$$B^{+}\gamma = a^{\Lambda} \tag{4.7}$$

$$B^{-}\gamma = a^{\Upsilon} \tag{4.8}$$

$$N\gamma = a^{\Delta}. \tag{4.9}$$

5 Extending to Infinite Dimensions

Denote by $\left(\mathcal{C}\ell_n{}^{\text{nil}}\right)^n$ the n-fold Cartesian product of $\mathcal{C}\ell_n{}^{\text{nil}}$, i.e. the space of n-tuples from the algebra. Let $\{x_i\}_{1 \le i \le n}$ denote the collection of basis

coordinate vectors of the form $x_i = (0, \ldots, 0, \underbrace{1}_{i^{\text{th}} \text{ pos.}}, 0, \ldots, 0)$ such that for $\vec{u} = (u_1, u_2, \ldots, u_n) \in \left(\mathcal{C}\ell_n{}^{\text{nil}}\right)^n$, one has

$$\langle \vec{u}, x_i \rangle = u_i. \tag{5.1}$$

The following finite-dimensional result follows from the previous results and definitions.

Proposition 5.1 (Generating function method)**.** *Fix $n > 0$, and let G be a finite graph on n vertices with associated nilpotent adjacency matrix a. Coordinate basis vectors $x_1, \ldots, x_n$ are also associated with the vertices of G by construction of a. Let X_k denote the number of k-cycles based at an arbitrary vertex x_0 in G. Then,*

$$\vartheta X_k = \varphi\left(\frac{\partial^k}{\partial t^k} \langle x_0, \exp(ta)\, x_0 \rangle\right)\Big|_{t=0}. \tag{5.2}$$

Define the infinite-dimensional nilpotent-generated algebra $\mathcal{C}\ell^{\text{nil}}$ by

$$\mathcal{C}\ell^{\text{nil}} = \bigoplus_{n=1}^{\infty} \mathcal{C}\ell_n{}^{\text{nil}}. \tag{5.3}$$

Define the space $\mathcal{V}\left(\mathcal{C}\ell^{\text{nil}}\right)$ by

$$\mathcal{V}\left(\mathcal{C}\ell^{\text{nil}}\right) = \bigoplus_{n=1}^{\infty} \left(\mathcal{C}\ell_n{}^{\text{nil}}\right)^n. \tag{5.4}$$

The algebra $\mathcal{M}_n$ of $n \times n$ matrices with entries in $\mathcal{C}\ell_n{}^{\text{nil}}$ is a $*$-algebra of operators on $\left(\mathcal{C}\ell_n{}^{\text{nil}}\right)^n$. Define the $*$-algebra $\mathcal{M}$ of operators on $\mathcal{V}\left(\mathcal{C}\ell^{\text{nil}}\right)$ by

$$\mathcal{M} = \bigoplus_{i=1}^{\infty} \mathcal{M}_n. \tag{5.5}$$

Remark 5.2. Note that with inner product defined by

$$(a, b) = \int \operatorname{tr}(a^* b)\, d\zeta_{\{n\}}, \ldots, d\zeta_{\{1\}}, \tag{5.6}$$

$\mathcal{M}$ is a Hilbert space.

Denote by $\mathcal{A}_n$ the semigroup of nilpotent adjacency matrices associated with finite graphs on n vertices under matrix multiplication. Denote by $\mathcal{A}$ the collection of operators in $\mathcal{M}$ that represent nilpotent adjacency matrices. In other words,

$$\mathcal{A} = \{a \in \mathcal{M} \,:\, (\exists \xi \in \mathcal{A}_n)(\forall i \in \{1, \ldots, n\})\, [\langle x_i, a x_i \rangle = \langle x_i, \xi x_i \rangle]\}. \tag{5.7}$$

Observe that φ is a state on the semigroup $\mathcal{A}$.

Define a sequence of operators $\{a_n\}$, $n \geq 1$, in $\mathcal{A}$ such that for each n, a_n is an operator on $\left(\mathcal{C}\ell_n{}^{\text{nil}}\right)^n$. The sequence $\{a_n\}$ will be said to converge to the operator a if for each $k \geq 0$ and any coordinate basis vector x_0, the following equation holds:

$$\lim_{n\to\infty} \varphi\left(\left\langle x_0, {a_n}^k x_0\right\rangle\right) = \varphi\left(\left\langle x_0, a^k x_0\right\rangle\right). \tag{5.8}$$

Denote this convergence by $a_n \xrightarrow{w} a$, and notice that this implies

$$\lim_{n\to\infty} \varphi\left(\left\langle x_0, \exp\left(a_n\right) x_0\right\rangle\right) = \varphi\left(\left\langle x_0, \exp\left(a\right) x_0\right\rangle\right). \tag{5.9}$$

An immediate consequence of this convergence is found in the following proposition.

Proposition 5.3 (Ascending chains)**.** *Let $\{a_n\} \subset \mathcal{A}$ be a sequence of operators such that for each $n > 0$, a_n represents the nilpotent adjacency matrix associated with a finite graph on n vertices. Let t be a real-valued parameter.*

Let $X_k(n)$ denote the number of k-cycles based at vertex x_0 in the n^{th} graph of the sequence. Then,

$$\vartheta X_k(n) = \varphi\left(\frac{\partial^k}{\partial t^k}\left\langle x_0, \exp\left(t a_n\right) x_0\right\rangle\right)\Bigg|_{t=0}. \tag{5.10}$$

Assuming $a_n \xrightarrow{w} a$ as $n \to \infty$ and fixing $k \geq 1$, one finds

$$\vartheta \lim_{n\to\infty} X_k(n) = \varphi\left(\frac{\partial^k}{\partial t^k}\left\langle x_0, \exp\left(t a\right) x_0\right\rangle\right)\Bigg|_{t=0}. \tag{5.11}$$

6 Applications

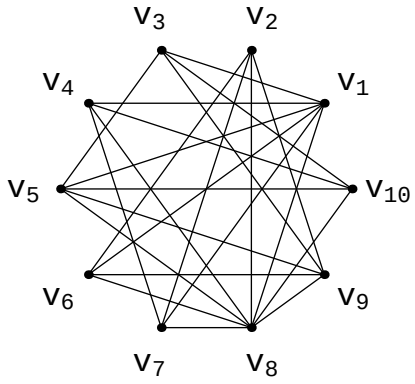
Letting X_k denote the number of k-cycles contained in a simple random graph on n vertices, the nilpotent adjacency matrix method has been used to recover higher moments of X_k [19].

The nilpotent adjacency matrix method has been used to give a graph-theoretic construction of iterated stochastic integrals [18]. In particular, stochastic integrals of $L^2 \otimes \mathcal{C}\ell_{p,q}$-valued processes are defined and recovered with this approach.

A graph-free algebraic construction of iterated stochastic integrals of classical processes has also been developed using sequences within a growing chain of fermion algebras [17].

Algebraic generating functions for Stirling numbers of the second kind, Bell numbers, and Bessel numbers have been defined [14].

Example 6.1. In Figure 6, the number of Hamiltonian cycles contained in a randomly generated, 10-vertex graph are recovered by examining the trace of a^{10}.



$$a = \begin{pmatrix}
0 & 0 & \zeta_{\{3\}} & \zeta_{\{4\}} & \zeta_{\{5\}} & \zeta_{\{6\}} & \zeta_{\{7\}} & \zeta_{\{8\}} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \zeta_{\{6\}} & \zeta_{\{7\}} & \zeta_{\{8\}} & \zeta_{\{9\}} & 0 \\
\zeta_{\{1\}} & 0 & 0 & 0 & \zeta_{\{5\}} & 0 & 0 & 0 & \zeta_{\{9\}} & \zeta_{\{10\}} \\
\zeta_{\{1\}} & 0 & 0 & 0 & 0 & 0 & \zeta_{\{7\}} & \zeta_{\{8\}} & 0 & \zeta_{\{10\}} \\
\zeta_{\{1\}} & 0 & \zeta_{\{3\}} & 0 & 0 & 0 & 0 & \zeta_{\{8\}} & \zeta_{\{9\}} & \zeta_{\{10\}} \\
\zeta_{\{1\}} & \zeta_{\{2\}} & 0 & 0 & 0 & 0 & 0 & \zeta_{\{8\}} & \zeta_{\{9\}} & 0 \\
\zeta_{\{1\}} & \zeta_{\{2\}} & 0 & \zeta_{\{4\}} & 0 & 0 & 0 & \zeta_{\{8\}} & 0 & 0 \\
\zeta_{\{1\}} & \zeta_{\{2\}} & 0 & \zeta_{\{4\}} & \zeta_{\{5\}} & \zeta_{\{6\}} & \zeta_{\{7\}} & 0 & \zeta_{\{9\}} & \zeta_{\{10\}} \\
0 & \zeta_{\{2\}} & \zeta_{\{3\}} & 0 & \zeta_{\{5\}} & \zeta_{\{6\}} & 0 & \zeta_{\{8\}} & 0 & 0 \\
0 & 0 & \zeta_{\{3\}} & \zeta_{\{4\}} & \zeta_{\{5\}} & 0 & 0 & \zeta_{\{8\}} & 0 & 0
\end{pmatrix}$$

```
In[56]:= Simplify[Tr[MatrixPower[a, 10]]] / 10 / 2
```

$$\text{Out[56]= } 176\,\zeta_{\{1,2,3,4,5,6,7,8,9,10\}}$$

Figure 6: A 10-vertex graph containing 176 Hamiltonian cycles.

6.1 Complexity

Within the algebraic context, the problem of cycle enumeration in finite graphs is reduced to matrix multiplication. Hence, enumerating the k-cycles in a graph on n vertices requires kn^3 *geometric products*. Consequently, fixing k and allowing n to vary, enumerating the k-cycles in a finite graph on n vertices has complexity (in the $\mathcal{C}\ell$ context) equal to $O(n^3)$.

Let $Clops(n)$ denote the number of algebra multiplications performed by an algorithm processing a data set of size n. An algorithm will be said to have $\mathcal{C}\ell$ *complexity* $O(f(n))$ if for every $k \in \mathbb{N}$, $\exists c \in \mathbb{R}$ such that

$$n \geq k \Rightarrow |Clops(n)| \leq c|f(n)|. \tag{6.1}$$

The following lemmas reflect the time complexity of computing powers of matrices.

Lemma 6.2. *Fixing k and allowing n to vary, enumerating the k-cycles in a finite graph on n vertices has $\mathcal{C}\ell$ complexity $O(n^3)$.*

Lemma 6.3. *Enumerating the Hamiltonian cycles in a finite graph on n vertices has $\mathcal{C}\ell$ complexity $O(n^4)$.*

These lemmas emphasize the potential power of a computer architecture based on geometric algebra. Given a device in which the product of any pair $u, v \in \mathcal{C}\ell^{\text{nil}}$ could be computed in polynomial time, the NP class problem of cycle enumeration would be reduced to class P.

Remark 6.4. Using the Coppersmith-Winograd algorithm, multiplying two $n \times n$ matrices can be done in $O(n^{2.376})$ time [5]. Applying this result to nilpotent adjacency matrices further reduces the $\mathcal{C}\ell$ time complexity [15].

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