Asymptotic analysis and topological derivatives for shape and topology optimization of elasticity problems in two spatial dimensions

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Abstract. Topological derivatives for elasticity problems are used in shape and topology optimization in structural mechanics. We propose an approach to the asymptotic analysis of singular perturbations of geometrical domains. This approach can be used in order to determine the exact solutions of elasticity boundary value problems in domains with small holes, and determine the explicit asymptotic expansions of solutions with respect to small parameter which describes the radius of internal hole. The elastic potentials of Muskhelishvili gives us an explicite solution in the ring $C(\rho, R) = \{\rho < |x| < R\}$ in the form of complex valued series. The series depends on the small parameter, the radius ρ of the ring, and we are interested in the behavior of the series for the passage $\rho \to 0$. Such analysis leads to the expansion of the elastic energy in the form

$$\mathcal{E}(\rho, R) = \mathcal{E}(0, R) + \rho^2 \mathcal{E}^1(R) + \rho^4 \mathcal{E}^2(R) + \dots ,$$

where $\mathcal{E}^1(R)$ is used to determine the first order topological derivatives of shape functionals, and $\mathcal{E}^2(R)$ can be used to determine the second order topological derivatives of

shape functionals. In the paper the first order term $\mathcal{E}^1(R)$ is given, however the method is general and can be used to determine the subsequent terms of the energy expansion and the topological derivatives of higher order.

Key Words. Shape optimization, topological derivative, optimal design, contact problem, compliance optimization, topology optimization.

1 Introduction

In the engineering literature there are many results concerning the shape optimization of contact problems in elasticity. The boundary variations technique for such problems is described in [36] in the framework of nonsmooth analysis combined with the speed method. Nonsmooth analysis is necessary since the shape differentiability of solutions to contact problems is obtained only in the framework of Hadamard differentiability of metric projections onto polyhedric sets in the appropriate Sobolev spaces. However, to our best knowledge, there is no numerical methods for simultaneous shape and topology optimization [41] of contact problems. The main difficulty in analysis of contact problems is associated with the nonlinearity of the *nonpenetration condition* over the contact zone which makes the boundary value problem nonsmooth. In the paper we propose a method for numerical evaluation of topological derivatives for such problems.

In a series of previous works [37],[38],[39],[40] the authors introduced the notion and developed a method of its efficient computing for the so-called topological derivative. This derivative is applicable to domain functionals defined as integrals of some functions depending on solutions to elliptic boundary value problems e.g. of the Poisson equation or of the elasticity boundary value problems. This derivative approximates the effect of making a small hole in a domain, allowing thus to consider topology changes.

The approach proved to be promising and has been used for shape optimization, the identification of inclusions [14], and extended to other similar problems [20],[19]. Recently it has been generalized to simultaneous topology and shape optimization [41], giving rise to new necessary optimality conditions. We refer also e.g. to [19], [10], [33], for some application of the topological sensitivity analysis in the linear case. Numerical methods are presented in [1], [3], [8], [9], [18]. Asymptotic analysis, in particular in elasticity, is described in [6], [2], [13], [21], [24], [26], [27], [28], [29], [30], [31], [32], [34], [38], [39], [40], and of unilateral problems, including the sensitivity analysis, is considered in [4], [5], [15], [35], [42].

The knowledge of topological derivatives is required for the optimality conditions of simultaneous shape and topology optimization. The topological derivative of a given shape functional can be determined from the variations of the shape functional created by the variations of the topology of geometrical domains. The topology variations are defined by nucleation of small holes or cavities or more generally small defects in geometrical domains. The modern mathematical background for evaluation of such derivatives by the asymptotic analysis techniques of boundary value problems is established in [26]. In [26] the error estimates for asymptotic approximations of solutions to boundary value problems in singularly perturbed geometrical domains are provided in the weighted Hölder spaces. The asymptotic approximations of solutions are used in order to established the explicit formulae for the topological derivatives of shape functionals.

2 Shape optimization and topological derivatives

The shape and topology optimization is one of the most important mathematical problems in structural mechanics. There are numerous application in solid mechanics, we refer to the monographs [7], [17], [22], [23], [25], [36], for the specific examples. In the paper we consider the elasticity boundary values problems, therefore we restrict ourselves to the shape functional in the form of the elastic energy. This corresponds to the minimization of the energy or equivalently maximization of the compliance.

2.1 Contact problems

Our goal is to establish the conical differentiability with respect to the topology variations of solutions for two dimensional contact problem in the elasticity. Let us consider the bounded domain Ω with the boundary $\partial\Omega = \Gamma_0 \cup \Gamma_c$. On Γ_0 the displacement vector of the elastic body is given, on Γ_c the frictionless contact conditions are prescribed. To specify the week formulation we need an expression for the symmetric bilinear form and for the convex set $K \subset H^1(\Omega)^2$.

The method of analysis is the same as in the case of Signorini problem for Laplacian. We start with the formulation of the free boundary problem in unperturbed domain Ω. The form of variational inequality is straightforward.

Contact problem in Ω Find $\mathbf{u} = \mathbf{u}(\Omega) = (u_1, u_2)$ and $\sigma = (\sigma)_{ij}$, $i, j = 1, 2$, such that

$$-\mathbf{div}\, \sigma = \mathbf{f} \qquad \text{in } \Omega \,, \tag{1}$$

$$C\sigma - \epsilon(\mathbf{u}) = 0 \qquad \text{in } \Omega \,, \tag{2}$$

$$\mathbf{u} = 0 \qquad \text{on } \Gamma_0 \,, \tag{3}$$

$$\mathbf{u}\nu \geq 0, \qquad \sigma_\nu \leq 0, \qquad \sigma_\nu \mathbf{u}\nu = 0 \qquad \sigma_\tau = 0 \qquad \text{on } \Gamma_c \,. \tag{4}$$

Here

$$\sigma_\nu = \sigma_{ij}\nu_j\nu_i,\ \sigma_\tau = \sigma\nu - \sigma_\nu = \left\{\sigma_\tau^i\right\}_{i=1}^2,\ \sigma\nu = \left\{\sigma_{ij}\nu_j\right\}_{i=1}^2,$$
$$\epsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}),\ i,j = 1,2,\ \epsilon(\mathbf{u}) = (\epsilon_{ij})_{i,j=1}^2,$$
$$\{C\sigma\}_{ij} = c_{ijk\ell}\sigma_{k\ell},\ c_{ijk\ell} = c_{jik\ell} = c_{k\ell ij},\ c_{ijk\ell} \in L^\infty(\Omega).$$

The Hooke's tensor C satisfies the ellipticity condition

$$c_{ijk\ell}\xi_{ji}\xi_{k\ell} \geq c_0|\xi|^2,\ \forall \xi_{ji} = \xi_{ij},\ c_0 > 0, \tag{5}$$

and we have used the summation convention over repeated indices.

When the topology of Ω is changed, we have the following contact problem in the domain Ω_ρ with the small hole $B(\rho)$.

Contact problem in Ω_ρ. Find $\mathbf{u} = \mathbf{u}(\Omega_\rho) = (u_1, u_2)$ and $\sigma = (\sigma)_{ij}$, $i,j = 1,2$, such that

$$-\mathbf{div}\,\sigma = \mathbf{f} \quad \text{in } \Omega_\rho, \tag{6}$$
$$C\sigma - \epsilon(\mathbf{u}) = 0 \quad \text{in } \Omega_\rho, \tag{7}$$
$$\mathbf{u} = 0 \quad \text{on } \Gamma_0, \tag{8}$$
$$\sigma\nu = 0 \quad \text{on } \Gamma_\rho, \tag{9}$$
$$\mathbf{u}\nu \geq 0, \qquad \sigma_\nu \leq 0, \qquad \sigma_\nu \mathbf{u}\nu = 0 \qquad \sigma_\tau = 0 \qquad \text{on } \Gamma_c. \tag{10}$$

We assume for simplicity that the case of isotropic elasticity is considered, thus the symmetric bilinear form associated with the boundary value problem (1)-(4) is given by

$$a(\mathbf{u},\mathbf{u}) = \int_\Omega \left[(\lambda+\mu)(\epsilon_{11}+\epsilon_{22})^2 + \mu(\epsilon_{11}-\epsilon_{22})^2 + \mu\gamma_{12}^2\right] dx, \tag{11}$$

where $\gamma_{12} = 2\epsilon_{12}$ and λ, μ are Lame constants.

The problem (6)-(10) is approximated by the problem with modified bilinear form in the following way.

Approximation of contact problem in Ω_ρ. Let us surround the hole $B(\rho)$ with the circle $\Gamma_R = \partial B(R)$ where R is fixed. Then in the set $\Omega_R = \Omega \setminus \bar{B}(R)$ the solution to the problem (6)-(10) may be approximated by the solution $\mathbf{v}$ of another problem in the whole domain Ω with boundary conditions as in (1)-(4), but a modified bilinear form of the energy functional

$$a(\rho; \mathbf{v}, \mathbf{v}) = a(\mathbf{v},\mathbf{v}) + \rho^2 b(\mathbf{v},\mathbf{v}) + o(\rho^2) \qquad \text{in} \quad H^1(\Omega)^2. \tag{12}$$

The derivative $b(\mathbf{v}, \mathbf{v})$ of the bilinear form $a(\rho; \mathbf{v}, \mathbf{v})$ with respect to ρ^2 at $\rho = 0+$ is given by the expression

$$b(\mathbf{v}, \mathbf{v}) = -2\pi e_{\mathbf{v}}(0) - \frac{2\pi\mu}{\lambda + 3\mu}\big(\sigma_{II}\delta_1 - \sigma_{12}\delta_2\big) \ , \tag{13}$$

where all the quantities are evaluated for the displacement field $\mathbf{v}$ according to formulae given below in terms of the line integrals over Γ_R.
Hence, we can determine the bilinear form $a(\rho; \mathbf{v}, \mathbf{w})$ for all $\mathbf{v}, \mathbf{w}$, from the equality

$$2a(\rho; \mathbf{v}, \mathbf{w}) = a(\rho; \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}) - a(\rho; \mathbf{w}, \mathbf{w}) \ .$$

In the same way the bilinear form $b(\mathbf{v}, \mathbf{w})$ is determined from the formula for $b(\mathbf{v}, \mathbf{v})$.
The convex set is defined in this case by

$$\mathbf{K} = \{\mathbf{v} \in H^1(\Omega)^2 | \mathbf{v}\nu \geq 0 \quad \text{on } \Gamma_c \ , \qquad \mathbf{v} = \mathbf{g} \quad \text{on } \Gamma_0\} \ . \tag{14}$$

Let us consider the following variational inequality which provides a sufficiently precise for our purposes approximation $\mathbf{u}_\rho$ of the solution $\mathbf{u}(\Omega_\rho)$ to contact problem (6)-(10),

$$\mathbf{u}_\rho \in \mathbf{K} \ : \quad a(\rho; \mathbf{u}, \mathbf{v} - \mathbf{u}) \geq L(\rho; \mathbf{v} - \mathbf{u}) \quad \forall v \in \mathbf{K} \ . \tag{15}$$

The result obtained is the following, for simplicity we assume that the linear form $L(\rho; \cdot)$ is independent of ρ.

Theorem 1 *For ρ sufficiently small we have the following expansion of the solution u_ρ with respect to the parameter ρ at $0+$,*

$$\mathbf{u}_\rho = \mathbf{u}(\Omega) + \rho^2 \mathbf{q} + o(\rho^2) \quad \text{in } H^1(\Omega)^2 \ , \tag{16}$$

where the topological derivative $\mathbf{q}$ of the solution $\mathbf{u}(\Omega)$ to the contact problem is given by the unique solution of the following variational inequality

$$\mathbf{q} \in \mathcal{S}_{\mathbf{K}}(\mathbf{u}) = \{\mathbf{v} \in (H^1_{\Gamma_0}(\Omega))^2 | \mathbf{v}\nu \leq 0 \quad \text{on } \Xi(\mathbf{u}) \ , \quad a(0; \mathbf{u}, \mathbf{v}) = 0\} \tag{17}$$

$$a(0; \mathbf{q}, \mathbf{v} - \mathbf{q}) + b(\mathbf{u}, \mathbf{v} - \mathbf{q}) \geq 0 \quad \forall \mathbf{v} \in \mathcal{S}_{\mathbf{K}}(\mathbf{u}) \ . \tag{18}$$

The coincidence set $\Xi(\mathbf{u}) = \{x \in \Gamma_s | \mathbf{u}(x).\nu(x) = 0\}$ is well defined [35] for any function $\mathbf{u} \in H^1(\Omega)^2$, and $\mathbf{u} \in \mathbf{K}$ is the solution of variational inequality (14) *for $\rho = 0$.*

Remark 1 *In the linear case, it can be shown that $\|\mathbf{u}(\Omega_\rho) - \mathbf{u}_\rho\| = o(\rho^2)$ in the norm of appropriate weighted space. We refer the reader to [26] for the related error estimates in the Hölder weighted spaces. In general, we cannot expect that $\mathbf{u}_\rho$ is close to $\mathbf{u}(\Omega_\rho)$ in the vicinity of B_ρ, therefore the weighted spaces should be used for error estimates.*

For the convenience of the reader we provide the explicit formulae for the terms in $b(\mathbf{v},\mathbf{v})$ appearing in (13). They are

$$\begin{aligned} 2\pi e_{\mathbf{v}}(0) = \frac{\lambda+\mu}{\pi^2 R^6}\left(\int_{\Gamma_R}(v_1x_1+v_2x_2)\,ds\right)^2 + \\ +\frac{\mu}{\pi^2R^6}\left(\int_{\Gamma_R}\left((1-9k)(v_1x_1-v_2x_2)+\frac{12k}{R^2}(v_1x_1^3-v_2x_2^3)\right]ds\right)^2 + \\ +\frac{\mu}{\pi^2R^6}\left(\int_{\Gamma_R}\left[(1+9k)(v_1x_2+v_2x_1)-\frac{12k}{R^2}(v_1x_2^3+v_2x_1^3)\right]ds\right)^2, \end{aligned} \tag{19}$$

with

$$\sigma_{II} = \frac{\mu}{\pi R^3}\int_{\Gamma_R}\left[(1-9k)(v_1x_1-v_2x_2)+\frac{12k}{R^2}(v_1x_1^3-v_2x_2^3)\right]ds,$$
$$\sigma_{12} = \frac{\mu}{\pi R^3}\int_{\Gamma_R}\left[(1+9k)(v_1x_2+v_2x_1)-\frac{12k}{R^2}(v_1x_2^3+v_2x_1^3)\right]ds,$$

and

$$\delta_1 = \frac{9k}{\pi R^3}\int_{\Gamma_R}\left[(v_1x_1-v_2x_2)-\frac{4}{3R^2}(v_1x_1^3-v_2x_2^3)\right]ds,$$
$$\delta_2 = \frac{9k}{\pi R^3}\int_{\Gamma_R}\left[(v_1x_2+v_2x_1)-\frac{4}{3R^2}(v_1x_2^3+v_2x_1^3)\right]ds.$$

Here

$$k = \frac{\lambda+\mu}{\lambda+3\mu}.$$

2.2 Correction to the energy functional

In this subsection we shall sketch the derivation of formulae for $b(\mathbf{v},\mathbf{v})$ as given in [43]. Let us consider the contribution, in the absence of volume forces, of the energy integral over the circle surrounding the origin (i.e. the potential location of the small hole)

$$e_R(\boldsymbol{u}) = \frac{1}{2}\int_{B(R)}(\boldsymbol{\sigma}:\boldsymbol{\epsilon})\,dx = \frac{1}{2}\int_{\Gamma_R}\boldsymbol{u}^T(\boldsymbol{\sigma}.\boldsymbol{n})\,ds \tag{20}$$

to the global elastic energy.We shall leave the displacement on Γ_R unchanged and consider the distortion to the stress field caused by introducing the small hole, denoted here by $\hat{\boldsymbol{\sigma}}$:

$$\delta e_R = \frac{1}{2}\int_{\Gamma_R}\boldsymbol{u}^T(\hat{\boldsymbol{\sigma}}.\boldsymbol{n})\,ds. \tag{21}$$

This distortion may be explicitely computed using the well known solution of the plane elasticity system in $\mathbf{R}^2$ with small circular void. To this end we need expressions for the values of strains in the center of the circle in terms of the values of displacement on Γ_R. These are calculated as follows (derivation is given in [43]).

Let us define $I_1(k,l)$ and $I_2(k,l)$ as

$$I_1(k,l) = \frac{1}{\alpha(k,l)} \int_{\Gamma_R} u_1 x_1^k x_2^l \, ds \, , \qquad I_2(k,l) = \frac{1}{\beta(k,l)} \int_{\Gamma_R} u_2 x_1^k x_2^l \, ds \, , \tag{22}$$

where

$$\alpha(k,l) = R^{k+l+2} \int_0^{2\pi} \cos^{k+1}\phi \sin^l \phi \, d\phi \, ,$$
$$\beta(k,l) = R^{k+l+2} \int_0^{2\pi} \cos^{k}\phi \sin^{l+1} \phi \, d\phi \, ,$$

whenever these expressions make sense, i.e. if k is odd and l even or vice versa. Observe that $\alpha(k,0) = \beta(0,k)$ and

$$\alpha(1,0) = \pi R^3, \quad \alpha(3,0) = \frac{3}{4}\pi R^5, \quad \alpha(1,2) = \frac{1}{4}\pi R^5, \quad \alpha(5,0) = \frac{5}{8}\pi R^7, \quad \alpha(3,2) = \frac{1}{8}\pi R^7$$

and so on. Furthermore, let

$$\begin{aligned} \delta_1 &= 9k\left([I_1(1,0) - I_2(0,1)] - [I_1(3,0) - I_2(0,3)]\right), \\ \delta_2 &= 9k\left([I_1(0,1) + I_2(1,0)] - [I_1(0,3) + I_2(3,0)]\right). \end{aligned} \tag{23}$$

In terms of these symbols the formulae for the *exact* values of strain components at the point $\boldsymbol{x}_0 = 0$ read as follows:

$$\begin{aligned} \epsilon_{11} + \epsilon_{22} &= I_1(1,0) + I_2(0,1) \, , \\ \epsilon_{11} - \epsilon_{22} &= I_1(1,0) - I_2(0,1) - \delta_1 \, , \\ \gamma_{12} &= I_1(0,1) + I_2(1,0) + \delta_2 \, . \end{aligned} \tag{24}$$

These values are used in (21), which may be rewritten as

$$\begin{aligned} \delta e_R = -\frac{1}{2}\pi\rho^2 \Big[&(\lambda+\mu)(\epsilon_{11}+\epsilon_{22})^2 + \mu(\epsilon_{11}-\epsilon_{22})^2 + \mu\gamma_{12}^2 + \\ &+ \left(1 - \frac{1}{k} + \frac{\rho^2}{R^2}\frac{1}{k}\right)(\sigma_{II}\delta_1 - \sigma_{12}\delta_2)\Big]. \end{aligned} \tag{25}$$

Then due to Hooke's law

$$\delta e_R = -\pi\rho^2 e_u(0) - \frac{1}{2}\pi\rho^2 \Big[\left(1 - \frac{1}{k} + \frac{\rho^2}{R^2}\frac{1}{k}\right)(\sigma_{II}\delta_1 - \sigma_{12}\delta_2)\Big]. \tag{26}$$

which leads immediately to (13).

In order to make clear the origin of (19) we collect below the dependences given by (23),(24) and write down the explicit expression for the terms appearing in (25):

$$
\begin{aligned}
\epsilon_{11} + \epsilon_{22} &= \frac{1}{\pi R^3} \int_{\Gamma_R} (u_1 x_1 + u_2 x_2)\, ds, \\
\epsilon_{11} - \epsilon_{22} &= \frac{1}{\pi R^3} \int_{\Gamma_R} \left[(1 - 9k)(u_1 x_1 - u_2 x_2) + \frac{12k}{R^2}(u_1 x_1^3 - u_2 x_2^3) \right] ds, \\
\gamma_{12} &= \frac{1}{\pi R^3} \int_{\Gamma_R} \left[(1 + 9k)(u_1 x_2 + u_2 x_1) - \frac{12k}{R^2}(u_1 x_2^3 + u_2 x_1^3) \right] ds, \\
\delta_1 &= \frac{9k}{\pi R^3} \int_{\Gamma_R} \left[(u_1 x_1 - u_2 x_2) - \frac{4}{3R^2}(u_1 x_1^3 - u_2 x_2^3) \right] ds, \\
\delta_2 &= \frac{9k}{\pi R^3} \int_{\Gamma_R} \left[(u_1 x_2 + u_2 x_1) - \frac{4}{3R^2}(u_1 x_2^3 + u_2 x_1^3) \right] ds.
\end{aligned}
$$

The above derivation has one drawback, namely due to the use of the particular solution in the infinite medium with a small hole it gives only the first term in the asymptotic expansion of the energy. In the next section we propose the general procedure, so we are able to produce as well the higher order terms in the asymptotic expansion.

3 Explicit elasticity solution in the ring

Let us consider the plane elasticity problem in the ring centered around zero with external radius R and internal radius ρ. We use polar coordinates (r, θ) with $\mathbf{e}_r$ pointing outwards and $\mathbf{e}_\theta$ perpendicularly in the counter-clockwise direction. Assume that the displacement on the outer boundary is given, while the inner boundary is free. We want to compare solution to such a problem to the one defined in the full circle, with the same displacement data. To this end we shall construct the exact representation of both solutions, using the complex variable method of [23]. It has been shown there that

$$
\begin{aligned}
\sigma_{rr} - i\sigma_{r\theta} &= 2\mathrm{Re}\,\phi' - e^{2i\theta}(\bar{z}\phi'' + \psi') \\
\sigma_{rr} + i\sigma_{\theta\theta} &= 4\mathrm{Re}\,\phi' \\
2\mu(u_r + iu_\theta) &= e^{-i\theta}(\kappa\phi - z\bar{\phi}' - \bar{\psi})
\end{aligned}
\tag{27}
$$

where $\phi,\ \psi$ are given by complex series

$$\begin{aligned} \phi &= A\log(z) + \sum_{k=-\infty}^{k=+\infty} a_k z^k \\ \psi &= -\kappa\bar{A}\log(z) + \sum_{k=-\infty}^{k=+\infty} b_k z^k \end{aligned} \tag{28}$$

Here μ – Lame constant, ν – Poisson ratio, $\kappa = 3 - 4\nu$ in the plain strain case, and $\kappa = (3-\nu)/(1+\nu)$ for plane stress.

The displacement data are given in the form of Fourier series,

$$2\mu(u_r + iu_\theta) = \sum_{k=-\infty}^{k=+\infty} A_k e^{ik\theta} \tag{29}$$

The traction-free condition on some circle means $\sigma_{rr} = \sigma_{r\theta} = 0$. After taking into account simple relations

$$\bar{z} = r^2\frac{1}{z}, \qquad z = re^{i\theta}, \qquad e^{-i\theta} = r\frac{1}{z}$$

we get for displacements the formula

$$\begin{aligned} 2\mu(u_r + iu_\theta) = {} & 2\kappa A r\log(r)\frac{1}{z} - \bar{A}\frac{1}{r}z + \\ & + \sum_{p=-\infty}^{p=+\infty} [\kappa r a_{p+1} - (1-p)\bar{a}_{1-p} r^{-2p+1} - \bar{b}_{-(p+1)} r^{-2p-1}] z^p \end{aligned} \tag{30}$$

Hence, comparing (29) and (30) gives, assuming substitution $r := R$,

$$\begin{aligned} &p = -1: && 2\kappa A r\log(r) + (\kappa a_0 - \bar{b}_0) - 2\bar{a}_2 r^2 = A_{-1} \\ &p = 0: && \kappa r a_1 - r\bar{a}_1 - \frac{1}{r}\bar{b}_{-1} = A_0 \\ &p = 1: && -\bar{A} + \kappa r^2 a_2 - \bar{b}_{-2}\frac{1}{r^2} = A_1 \\ &p \notin \{-1, 0, +1\}: && \kappa r^{p+1} a_{p+1} - (1-p)\bar{a}_{1-p} r^{-p+1} - \bar{b}_{-(p+1)} r^{-(p+1)} = A_p \end{aligned} \tag{31}$$

Similarly we obtain representation of tractions on some circle

$$\begin{aligned} \sigma_{rr} - i\sigma_{r\theta} = {} & 2A\frac{1}{z} + (\kappa+1)\frac{1}{r^2}\bar{A}z + \\ & + \sum_{p=-\infty}^{p=+\infty} [(1+p)(1-p)a_{p+1} + (1-p)\bar{a}_{1-p} r^{-2p} - (p-1)\frac{1}{r^2} b_{p-1}] z^p \end{aligned} \tag{32}$$

and the resulting traction-free condition, assuming substitution $r := \rho$,

$$\begin{aligned}
&p = -1: && 2A + 2\bar{a}_2 r^2 + 2\frac{1}{r^2} b_{-2} = 0\\
&p = 0: && a_1 + \bar{a}_1 + \frac{1}{r^2} b_{-1} = 0\\
&p = 1: && (\kappa+1)\frac{1}{r^2}\bar{A} = 0\\
&p \notin \{-1,0,+1\}: && (1+p)(1-p)a_{p+1} + (1-p)\bar{a}_{1-p} r^{-2p} - (p-1)\frac{1}{r^2} b_{p-1} = 0
\end{aligned} \tag{33}$$

Denote $d_0 = \kappa a_0 - \bar{b}_0$. For the full circle we must eliminate singularities, i.e. $b_{-k} = a_{-k} = A = 0$ for $k = 1, 2, \ldots$ and then using (31) obtain

$$\begin{aligned}
d_0^0 &= A_{-1} + \frac{2}{\kappa}\bar{A}_1\\
\operatorname{Re} a_1^0 &= \frac{1}{(\kappa-1)R}\operatorname{Re} A_0, \qquad \operatorname{Im} a_1^0 = \frac{1}{(\kappa+1)R}\operatorname{Im} A_0\\
a_k^0 &= \frac{1}{\kappa R^k} A_{k-1}, \qquad k = 2, 3, \ldots\\
b_k^0 &= -\frac{1}{R^k}[(k+2)\frac{1}{\kappa}A_{k+1} + \bar{A}_{-(k+1)}], \qquad k = 1, 2, \ldots
\end{aligned} \tag{34}$$

Let us check the formulas for uniformly stretched (compressed) circle. Then $A_0 = 2\mu u_r(R)$ and other A_k vanish. Hence only

$$a_1^0 = \frac{2\mu}{(\kappa-1)R} u_r(R)$$

is different from 0 and $\psi = 0$, $\phi = a_1^0 z$. According to e.g. [16] the solution should be $u_r(r) = u_r(R)r/R$. Indeed, using (27) we get

$$2\mu(u_r + iu_\theta) = r\frac{1}{z}(\kappa a_1^0 z - \bar{a}_1^0 z] = 2\mu\frac{r}{R}u_r(R).$$

The other test of (31) is applying it to the ring with different displacement conditions at $r = \rho$ and $r = R$. Then these formulas allow, after longer calculations, to obtain the same solution as in [11].

In further analysis we consider the ring and assume $\rho < 0.5R$. Then from (31) for $r = R$ and (33) for $r = \rho$, by considering first $p \in \{-1, 0, 1\}$, we get $A = 0$ and

$$d_0 = A_{-1} + \frac{2R^4}{\kappa R^4 + \rho^4}\bar{A}_1$$

$$a_2 = \frac{R^2}{\kappa R^4 + \rho^4} A_1$$

$$\mathrm{Re}\, a_1 = \frac{R}{(\kappa - 1)R^2 + 2\rho^2} \mathrm{Re}\, A_0, \qquad \mathrm{Im}\, a_1 = \frac{1}{\kappa + 1} \mathrm{Im}\, A_0$$

$$b_{-1} = -2\rho^2 \mathrm{Re}\, a_1 = -\frac{2\rho^2 R}{(\kappa - 1)R^2 + 2\rho^2} \mathrm{Re}\, A_0$$

$$b_{-2} = -\rho^4 \bar{a}_2 = -\frac{\rho^4 R^2}{\kappa R^4 + \rho^4} \bar{A}_1$$

Here we may again test the correctness of formulas. From [16] we get the the solution for the uniformly stretched ring with unloaded inner boundary: It reads, in the plain strain case,

$$u_r(r) = \left[\frac{\rho^2}{\rho^2 + (1 - 2\nu)} \frac{1}{r} + \frac{1 - 2\nu}{\rho^2 + (1 - 2\nu)} r \right] u_R(1)$$

Again, $A_0 = 2\mu u_r(R = 1)$, and from (27)

$$a_1 = \frac{1}{(\kappa - 1) + 2\rho^2} 2\mu u_R(1), \qquad b_{-1} = -\frac{2\rho^2}{(\kappa - 1) + 2\rho^2} 2\mu u_R(1).$$

Hence $\phi = a_1 z, \ \psi = b_{-1} \frac{1}{z}$ and

$$\begin{aligned} 2\mu u_r(r) &= r \frac{1}{z} \left[\kappa a_1 z - z a_1 - b_{-1} \frac{1}{r^2} z \right] = \\ &= \left[(\kappa - 1) \frac{1}{(\kappa - 1) + 2\rho^2} r + \frac{2\rho^2}{(\kappa - 1) + 2\rho^2} \frac{1}{r} \right] 2\mu u_R(1). \end{aligned}$$

Taking into account that $\kappa = 3 - 4\nu$ we get the same result.
Observe, that

$$\begin{aligned} d_0 - d_0^0 &= -\rho^4 \frac{2}{\kappa(\kappa R^4 + \rho^4)} \bar{A}_1 \\ a_1 - a_1^0 &= -\rho^2 \frac{2}{(\kappa - 1)R((\kappa - 1)R^2 + 2\rho^2)} \mathrm{Re}\, A_0 \\ a_2 - a_2^0 &= -\rho^4 \frac{1}{\kappa R^2(\kappa R^4 + \rho^4)} A_1 \end{aligned} \tag{35}$$

There remains to compute the rest of terms. Taking $p = +k, \ \ k = 2, 3, \ldots$ in (31) for $r = R$ and (33) for $r = \rho$ gives

$$\begin{aligned} \kappa a_{k+1} R^{k+1} + (k - 1)\bar{a}_{-(k-1)} R^{-(k-1)} - \bar{b}_{-(k+1)} R^{-(k+1)} &= A_k \\ (k + 1) a_{k+1} \rho^{2(k+1)} + \bar{a}_{-(k-1)} \rho^2 + b_{k-1} \rho^{2k} &= 0 \end{aligned} \tag{36}$$

and for $p = -k, \quad k = 2, 3, \ldots$

$$\begin{aligned} \kappa a_{-(k-1)} R^{-(k-1)} - (k+1)\bar{a}_{k+1} R^{k+1} - \bar{b}_{k-1} R^{k-1} &= A_{-k} \\ -(k-1) a_{-(k-1)} \rho^2 + \bar{a}_{k+1} \rho^{2(k+1)} + b_{-(k+1)} &= 0 \end{aligned} \tag{37}$$

We eliminate $a_{-(k-1)}$ and $b_{-(k+1)}$ using

$$\begin{bmatrix} a_{-(k-1)} \\ b_{-(k+1)} \end{bmatrix} = \begin{bmatrix} -(k+1)\rho^{2k} & , & -\rho^{2(k-1)} \\ -k^2 \rho^{2(k+1)} & , & -(k-1)\rho^{2k} \end{bmatrix} \cdot \begin{bmatrix} \bar{a}_{k+1} \\ \bar{b}_{k-1} \end{bmatrix} = T_k(\rho) \cdot \begin{bmatrix} \bar{a}_{k+1} \\ \bar{b}_{k-1} \end{bmatrix} \tag{38}$$

and get the system which may be rewritten as

$$S_k(\rho) \cdot \begin{bmatrix} a_{k+1} \\ b_{k-1} \end{bmatrix} = \begin{bmatrix} A_k \\ \bar{A}_{-k} \end{bmatrix} \tag{39}$$

with entries

$$\begin{aligned} S_k(\rho)_{11} &= \kappa R^{k+1} - (k^2-1) R^{-(k-1)} \rho^{2k} + k^2 R^{-(k+1)} \rho^{2(k+1)} \\ S_k(\rho)_{12} &= -(k-1)(R^{-(k-1)} \rho^{2(k-1)} - R^{-(k+1)} \rho^{2k}) \\ S_k(\rho)_{21} &= -(k+1)(R^{k+1} + \kappa R^{-(k-1)} \rho^{2k}) \\ S_k(\rho)_{22} &= -R^{k-1} - \kappa R^{-(k-1)} \rho^{2(k-1)} \end{aligned} \tag{40}$$

In fact the formulas (38), (40) are correct also for $k = 0, 1$ and $\rho > 0$.

The matrix $S_k(\rho)$ is a perturbation of $S_k(0)$, which would produce the solution for the full circle, namely $a^0_{k+1}, \ b^0_{k-1}$. Observe that $T_k(0) = 0$. Direct computations lead to estimates

$$|a_3 - a^0_3| \le \Lambda \left(|A_2| \rho^4 + |A_{-2}| \rho^2 \right) \tag{41}$$

and for $k = 4, 5, \ldots$

$$|a_k - a^0_k| \le \Lambda \left(|A_{k-1}| \rho^{3(k-1)/2} + |A_{-(k-1)}| \rho^{3(k-2)/2} \right) \tag{42}$$

where the exponent $k/2$ has been used to counteract the growth of k^2 in terms like $k^2 \rho^{k/2}$. Similarly

$$|b_1 - b^0_1| \le \Lambda \left(|A_2| \rho^4 + |A_{-2}| \rho^2 \right) \tag{43}$$

and for $k = 2, 3, \ldots$

$$|b_k - b^0_k| \le \Lambda \left(|A_{k+1}| \rho^{3(k+1)/2} + |A_{-(k+1)}| \rho^{3k/2} \right) \tag{44}$$

From relation (38) we get another estimate

$$|a_{-k}| \le \Lambda \rho^{2k} \left(|A_{k+1}| + |A_{-(k+1)}| \right), \quad k = 1, 2, \ldots \tag{45}$$

end

$$|b_{-k}| \leq \Lambda\rho^{2(k-1)}\left(|A_{k-1}| + |A_{-(k-1)}|\right), \quad k = 3, 4, \ldots \tag{46}$$

Here Λ is a constant independent from ρ and A_i. Observe that the corrections proportional to ρ^2 are present only in a_1, a_3, b_{-1}, a_{-1}. The rest is of the order at least $O(\rho^3)$ (in fact $O(\rho^4)$).

The condition for A_i, namely

$$\sum_{k=-\infty}^{k=+\infty} \sqrt{1+k^2}\, |A_k|^2 \leq \Lambda_0 \tag{47}$$

together with bounds (35),(41),(42),(43), (44),(45),(46) ensure, that the expression for elastic energy concentrated in the ring splits into the one corresponding to the full circle, correction proportional to ρ^2 and the rest, which is uniformly of the order $O(\rho^3)$ for all $\mathbf{u}$ satisfying

$$\|\mathbf{u}\|^2_{\mathbf{H}^{1/2}(\Gamma_R)} \leq \Lambda_0$$

4 Numerical illustration

Rugby ball. Let us take $u_r = s_0 \cos^2\theta = \frac{1}{2}s_0 + \frac{1}{2}s_0 \cos 2\theta$. Hence

$$A_k = [\ldots, \frac{1}{2}\mu s_0, 0,\ A_0 = \mu s_0\ , 0, \frac{1}{2}\mu s_0, \ldots]$$

The resulting distortion for $\rho = 0.2$ and $r = 0.3$ are shown in Fig.1 (solid line - undeformed, dashed - deformed ring, dotted - deformed ball):

Bubble. Now we take $u_r = s_0 \sin 4\theta$. Hence

$$A_k = [\ldots, \mu s_0 i, 0, 0, 0,\ A_0 = 0\ , 0, 0, 0, -\mu s_0 i, \ldots]$$

The resulting distortions for $\rho = 0.2$ and $r = 0.3$ are shown in Fig.2 using the same types of lines.

Asymptotics. In the second numerical experiment - bubble - only A_{-4} and A_4 were nonzero, what means, that the difference between positions of the contour $r = 0.3$ for full circle and the ring should behave like ρ^6. In the first experiment it should be ρ^2. In Fig.3 we show the slopes of this difference for $\rho = [0.05, 0.10, 0.15, 0.20, 0.25]$ in log–log graph (solid - rugby, dashed - bubble):

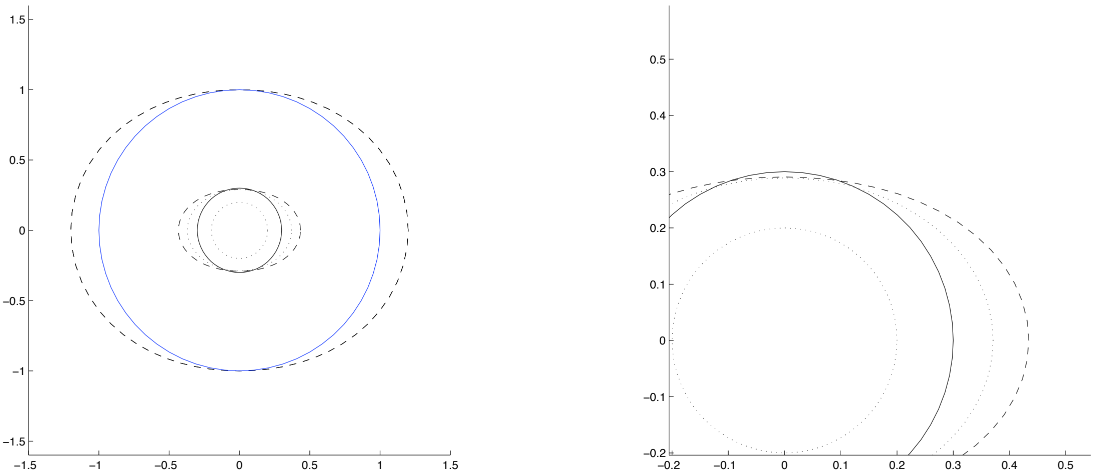

Figure 1: Rugby-like distortion and the detail near the central hole (exaggerated).

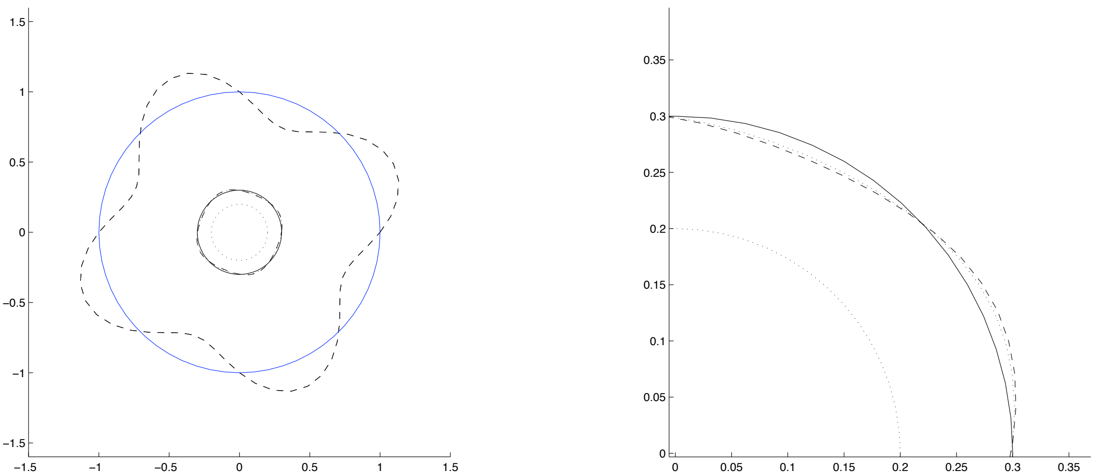

Figure 2: Bubble-like distortion and the detail near the central hole (exaggerated).

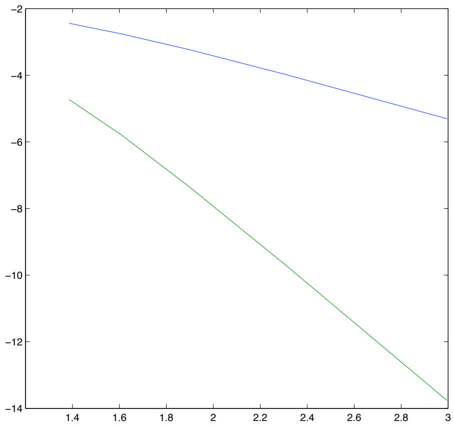

Figure 3: The slope of diminishing influence of the hole for rugby and bubble (below) type of distortions.

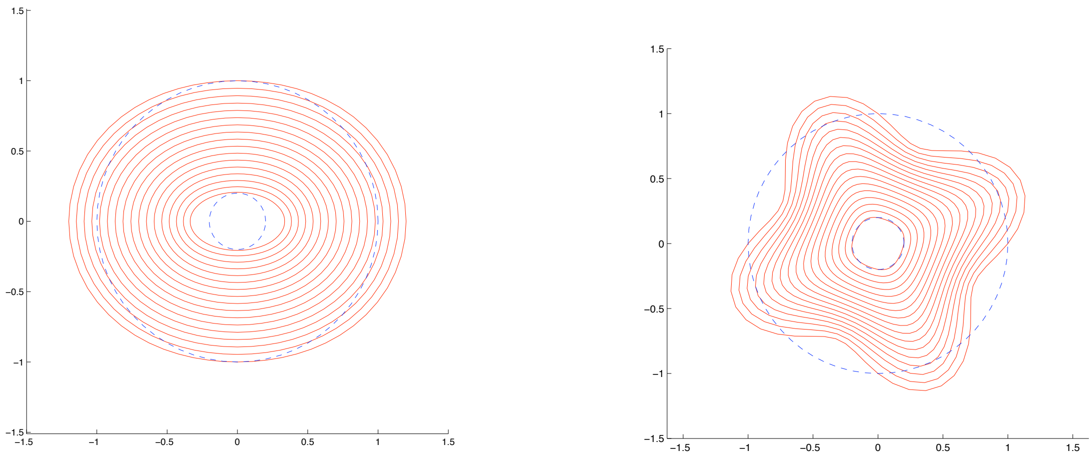

Figure 4: The pattern of distortions for both experiments.

Comparison of distortions. The deformations for $\rho = 0.2$ and several intermidiate radii (dashed - undeformed, solid - deformed contours) are shown in Fig.4. It may again be seen that the influence of the outer boundary distortion vanishes quicker for the bubble case, where A_k with smaller indices are nonzero.

5 Explicit expansion of elastic energy

The elastic energy contained in the ring has the form

$$2\mathcal{E}(\rho,R) = \int_{C(\rho,R)} \boldsymbol{\sigma}(\boldsymbol{u}_\rho) : \boldsymbol{\epsilon}(\boldsymbol{u}_\rho)\,dx = \int_{\Gamma_R} \boldsymbol{u}_\rho \boldsymbol{\sigma}(\boldsymbol{u}_\rho).\boldsymbol{n}\,ds. \tag{48}$$

Since $\boldsymbol{u}_\rho = \boldsymbol{u}$ on Γ_ρ,

$$2\mathcal{E}(\rho,R) = \int_{\Gamma_R} \boldsymbol{u}\boldsymbol{\sigma}(\boldsymbol{u}_\rho).\boldsymbol{n}\,ds. \tag{49}$$

Now $\boldsymbol{\sigma}(\boldsymbol{u}_\rho)$ is in fact of the form $\boldsymbol{\sigma}(\boldsymbol{u}_\rho) = \boldsymbol{\sigma}_\rho(\boldsymbol{u})$, because $\boldsymbol{u}_\rho = \boldsymbol{u}$ on Γ_R, which means that $\boldsymbol{u}_\rho = \boldsymbol{u}_\rho(\boldsymbol{u})$. If we split $\boldsymbol{\sigma}_\rho$ into

$$\boldsymbol{\sigma}_\rho(\boldsymbol{u}) = \boldsymbol{\sigma}^0 + \rho^2\boldsymbol{\sigma}^1(\boldsymbol{u}) + O(\rho^4) \tag{50}$$

then

$$2\mathcal{E}(\rho,R) = 2\mathcal{E}(0,R) + 2\rho^2\int_{\Gamma_R} \boldsymbol{u}\boldsymbol{\sigma}^1(\boldsymbol{u}).\boldsymbol{n}\,ds + +O(\rho^4). \tag{51}$$

Thus the problem reduces to finding $\boldsymbol{\sigma}^1(\boldsymbol{u})$.

From (27), (28) we know that $\boldsymbol{\sigma}_\rho(\boldsymbol{u})$ is a linear function of infinite vectors $\boldsymbol{a},\boldsymbol{b}$, while $\boldsymbol{\sigma}^0(\boldsymbol{u})$ is the same function of $\boldsymbol{a}^0,\boldsymbol{b}^0$. Here $\boldsymbol{a}^0,\boldsymbol{b}^0$ are computed for $B(R)$, while $\boldsymbol{a},\boldsymbol{b}$ correspond to $C(\rho,R)$. In order obtain $\boldsymbol{\sigma}^1(\boldsymbol{u})$ it is enough to express $\boldsymbol{a},\boldsymbol{b}$ as

$$\begin{aligned}\boldsymbol{a} &= \boldsymbol{a}^0 + \rho^2\boldsymbol{a}^1 + O(\rho^4)\\ \boldsymbol{b} &= \boldsymbol{b}^0 + \rho^2\boldsymbol{b}^1 + O(\rho^4)\end{aligned} \tag{52}$$

because then

$$\boldsymbol{\sigma}^1(\boldsymbol{u}) = \boldsymbol{\sigma}^1(\boldsymbol{a}^1,\boldsymbol{b}^1;\boldsymbol{u}).$$

Let us observe as well that

$$\int_{\Gamma_R} \boldsymbol{u}\boldsymbol{\sigma}^1(\boldsymbol{u}).\boldsymbol{n}\,ds = R\int_0^{2\pi} (\sigma^1_{rr}u_r + \sigma^1_{r\theta}u_\theta)\,d\theta = R\int_0^{2\pi} \Re[(\sigma^1_{rr} - i\sigma^1_{r\theta})(u_r + iu_\theta)]\,d\theta \tag{53}$$

The analysis of formulae (34) for $\boldsymbol{a}^0,\boldsymbol{b}^0$ and their counterparts $\boldsymbol{a},\boldsymbol{b}$ leads to the conclusion, that the only nonzero terms in $\boldsymbol{a}^1,\boldsymbol{b}^1$ will be $a^1_3, a^1_1, a^1_{-1}, b^1_{-1}, b^1_1$.

Taking into account that $A = 0$ in (28) for our problem,

$$\begin{aligned}\phi &= \phi^0 + \rho^2\phi^1 + O(\rho^4)\\ \psi &= \psi^0 + \rho^2\psi^1 + O(\rho^4)\end{aligned} \tag{54}$$

where

$$\phi^1 = a^1_{-1}\frac{1}{z} + a^1_1 z + a^1_3 z^3, \qquad \psi^1 = b^1_{-1}\frac{1}{z} + b^1_1 z. \tag{55}$$

As a result

$$\sigma^1_{rr} - i\sigma^1_{r\theta} = 2\Re[-a^1_{-1}\frac{1}{z^2} + a^1_1 + 3a^1_3 z^2] - e^{2i\theta}(6a^1_3|\bar{z}|^2 + 2a^1_{-1}\frac{\bar{z}}{z^3} + b^1_1 - b^1_{-1}\frac{1}{z^2}). \tag{56}$$

Computing corrections to coefficients. The matrix $S_k(\rho)$ may be for $k = 2$ expressed as

$$S_2(\rho) = S_2(0) + \rho^2 S^1_2 + O(\rho^4).$$

Using (40)

$$S_2(0)\begin{bmatrix} a^0_3 \\ b^0_1 \end{bmatrix} = \begin{bmatrix} A_2 \\ \bar{A}_{-2} \end{bmatrix}$$

we may get

$$\begin{bmatrix} a^1_3 \\ b^1_1 \end{bmatrix} = -S^{-1}_2(0)S^1_2\begin{bmatrix} a^0_3 \\ b^0_1 \end{bmatrix} \tag{57}$$

where

$$S_2(0) = \begin{bmatrix} \kappa R^3 & , & 0 \\ -3R^3 & , & -R \end{bmatrix} \qquad S^1_2 = \begin{bmatrix} 0 & , & -1/R \\ 0 & , & -\kappa/R \end{bmatrix}.$$

Similarly

$$T_2(\rho) = 0 + \rho^2 T^1_2 + O(\rho^4), \qquad T^1_2 = \begin{bmatrix} 0 & , & -1 \\ 0 & , & 0 \end{bmatrix}.$$

Hence, using (38),(57) we get finally

$$\begin{aligned} a^1_{-1} &= -\bar{b}^0_1 \quad , \quad a^1_3 = \frac{1}{\kappa R^4}b^0_1 \\ b^1_1 &= \frac{3+\kappa^2}{\kappa R^2}b^0_1 \quad , \quad a^1_1 = -\frac{2}{(\kappa-1)R^2}\Re a^0_1 \\ b^1_{-1} &= -2\Re a^0_1 \end{aligned} \tag{58}$$

Since, as is obvious, we need only A_0, A_2, A_{-2} and

$$A_k = \frac{\mu}{\pi}\int_0^{2\pi}(u_r + iu_\theta)e^{-ik\theta}\,d\theta$$

as well as

$$u_r + iu_\theta = (u_1 + iu_2)e^{-i\theta}$$

we obtain

$$
\begin{aligned}
A_0 &= \frac{\mu}{\pi}\int_0^{2\pi}(u_1+iu_2)e^{-i\theta}\,d\theta \\
A_2 &= \frac{\mu}{\pi}\int_0^{2\pi}(u_1+iu_2)e^{-3i\theta}\,d\theta \\
A_0 &= \frac{\mu}{\pi}\int_0^{2\pi}(u_1+iu_2)e^{+i\theta}\,d\theta
\end{aligned} \tag{59}
$$

Now we shall return to the integral (53). Using (56) we get

$$
\sigma^1_{rr} - i\sigma^1_{r\theta} = B^1_0 + B^1_2 e^{2i\theta} + B^1_{-2}e^{-2i\theta}, \tag{60}
$$

where

$$
B^1_0 = 2a^1_1 - \frac{1}{r^2}b^1_{-1}, \qquad B^1_2 = -\frac{1}{r^2}\bar{a}^1_{-1} - 3a^1_3R^2 + b^1_1
$$

$$
B^1_{-2} = 3\bar{a}^1_3R^2 - \frac{3}{R^2}a^1_{-1}
$$

From (53) is obvious that only a part of displacement will contribute to the integral, namely

$$
u_r + iu_\theta = \frac{1}{2\mu}[A_{-2}e^{-2i\theta} + A_0 + A_2e^{2i\theta}].
$$

Then

$$
\begin{aligned}
&\frac{1}{2\mu R}\int_0^{2\pi}(B^1_0 + B^1_2e^{2i\theta} + B^1_{-2}e^{-2i\theta})(A_{-2}e^{-2i\theta} + A_0 + A_2e^{2i\theta})\,d\theta = \\
&= \frac{\pi}{\mu R}(B^1_0A_0 + B^1_2A_{-2} + B^1_{-2}A_2)
\end{aligned} \tag{61}
$$

Now we are able to collect all formulae and obtain the final expression, using (34),(60),(61)

$$
\begin{aligned}
&\int_{\Gamma_R} \boldsymbol{u}\boldsymbol{\sigma}^1(\boldsymbol{u}).\boldsymbol{n}\,ds = \\
&= \frac{1}{R^2}\Big[\frac{2(\kappa-2)}{(\kappa-1)^2}(\Re A_0)^2 - (\kappa+1)|A_{-2}|^2 - \frac{9(\kappa+1)}{\kappa^2}|A_2|^2 - \frac{6(\kappa+1)}{\kappa}\Re(A_2A_{-2})\Big].
\end{aligned} \tag{62}
$$

Numerical scheme. From (59) it follows that

$$
\begin{aligned}
\Re A_0 &= \frac{\mu}{\pi}\int_0^{2\pi}(u_1\cos\theta + u_2\sin\theta)\,d\theta \\
A_2 &= \frac{\mu}{\pi}\int_0^{2\pi}(u_1\cos 3\theta + u_2\sin 3\theta)\,d\theta + i\frac{\mu}{\pi}\int_0^{2\pi}(u_2\cos 3\theta - u_1\sin 3\theta)\,d\theta \\
A_{-2} &= \frac{\mu}{\pi}\int_0^{2\pi}(u_1\cos\theta - u_2\sin\theta)\,d\theta + i\frac{\mu}{\pi}\int_0^{2\pi}(u_2\cos\theta + u_1\sin\theta)\,d\theta
\end{aligned} \tag{63}
$$

Here values of displacements are taken as $u_i(R\cos\theta, R\sin\theta)$. After discretization these integrals constitute weighted sums of values of u_i at certain points on Γ_R. If we assume piecewise linear approximation over triangles, then it is well known that

$$u_i^h(\boldsymbol{x}) = \boldsymbol{x}^T \begin{bmatrix} x_1^1 & x_2^1 & 1 \\ x_1^2 & x_2^2 & 1 \\ x_1^3 & x_2^3 & 1 \end{bmatrix}^{-1} U_i = \boldsymbol{x}^T M^{-1} U_i = (M^{-T}\boldsymbol{x})^T U_i = \boldsymbol{c}^T U_i$$

where $u_i^h(\boldsymbol{x})$ is a value of the approximation of u_i at a point $\boldsymbol{x}$ inside the triangle defined by vertices $\boldsymbol{x}^1, \boldsymbol{x}^2, \boldsymbol{x}^3$ and $U_i = [U_i^1, U_i^2, U_i^3]^T$ is a vector of the values of u_i at these vertices. Observe that $\boldsymbol{c}$ is a vector of weights with which nodal values enter into the expression for $u_i^h(\boldsymbol{x})$.

Let now $\boldsymbol{U} = [u_1^1, u_2^1, \ldots, u_1^K, u_2^K]^T$ be a vector of nodal values of $\boldsymbol{u}$ for the global triangulation. Then we may write down the following formulae

$$\frac{\mu}{\pi}\int_0^{2\pi} u_1 \cos\theta \, d\theta = \boldsymbol{c}_{11}^T \boldsymbol{U} \qquad \frac{\mu}{\pi}\int_0^{2\pi} u_2 \sin\theta \, d\theta = \boldsymbol{s}_{21}^T \boldsymbol{U}$$
$$\frac{\mu}{\pi}\int_0^{2\pi} u_1 \cos 3\theta \, d\theta = \boldsymbol{c}_{13}^T \boldsymbol{U} \qquad \frac{\mu}{\pi}\int_0^{2\pi} u_2 \sin 3\theta \, d\theta = \boldsymbol{s}_{23}^T \boldsymbol{U}$$
$$\frac{\mu}{\pi}\int_0^{2\pi} u_1 \sin\theta \, d\theta = \boldsymbol{s}_{11}^T \boldsymbol{U} \qquad \frac{\mu}{\pi}\int_0^{2\pi} u_2 \cos\theta \, d\theta = \boldsymbol{c}_{21}^T \boldsymbol{U}$$
$$\frac{\mu}{\pi}\int_0^{2\pi} u_1 \sin 3\theta \, d\theta = \boldsymbol{s}_{13}^T \boldsymbol{U} \qquad \frac{\mu}{\pi}\int_0^{2\pi} u_2 \cos 3\theta \, d\theta = \boldsymbol{c}_{23}^T \boldsymbol{U}$$

Here $\boldsymbol{s}_{ij}, \boldsymbol{c}_{ij}$ are sparse vectors of weights with which nodal values of $\boldsymbol{u}$ enter into appropriate integrals. In this notation

$$\begin{aligned} (\Re A_0)^2 &= \|(\boldsymbol{c}_{11} + \boldsymbol{s}_{21})^T \boldsymbol{U}\|^2 \\ |A_2|^2 &= \|(\boldsymbol{c}_{13} + \boldsymbol{s}_{23})^T \boldsymbol{U}\|^2 + \|(\boldsymbol{c}_{23} - \boldsymbol{s}_{13})^T \boldsymbol{U}\|^2 \\ |A_{-2}|^2 &= \|(\boldsymbol{c}_{11} - \boldsymbol{s}_{21})^T \boldsymbol{U}\|^2 + \|(\boldsymbol{c}_{21} + \boldsymbol{s}_{11})^T \boldsymbol{U}\|^2 \\ \Re(A_2 A_{-2}) &= \boldsymbol{U}^T(\boldsymbol{c}_{13} + \boldsymbol{s}_{23})(\boldsymbol{c}_{11} - \boldsymbol{s}_{21})\boldsymbol{U} - \boldsymbol{U}^T(\boldsymbol{c}_{23} - \boldsymbol{s}_{13})(\boldsymbol{c}_{21} + \boldsymbol{s}_{11})\boldsymbol{U} \end{aligned} \tag{64}$$

Taking into account (62) we may conclude that the first term in the correction of energy is a well defined quadratic form. Similar, only more complicated expressions may be obtained for further terms corresponding to ρ^4 and higher.

The derivation described above is new and while it gives the same result as in Section 2 for the first term, it is much more general.

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