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WDM and Directed Star Arboricity

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Abstract: A digraph is *m-labelled* if every arcs is labelled by an integer in $\{1, \dots, m\}$. Motivated by wavelength assignment for multicasts in optical star networks, we study *n-fiber colourings* of labelled digraph which are colourings of the arcs of D such that at each vertex v, for each colour λ, $in(v,\lambda) + out(v,\lambda) \leq n$ with $in(v,\lambda)$ the number of arcs coloured λ entering v and $out(v,\lambda)$ the number of labels l such that there exists an arc leaving v of label l coloured λ. One likes to find the minimum number of colours $\lambda_n(D)$ such that an m-labbelled digraph D has an n-fiber colouring. In the particular case, when D is 1-labelled then $\lambda_n(D)$ is the *directed star arboricty* of D, denoted $dst(D)$. We first show that $dst(D) \leq 2\Delta^-(D) + 1$ and conjecture that if $\Delta^-(D) \geq 2$ then $dst(D) \leq 2\Delta^-(D)$. We also prove that if D is subcubic then $dst(D) \leq 3$ and that if $\Delta^+(D), \Delta^-(D) \leq 2$ then $dst(D) \leq 4$. Finally, we study $\lambda_n(m,k) = \max\{\lambda_n(D) \mid D \text{ is } m\text{-labelled and } \Delta^-(D) \leq k\}$. We show that if $m \geq n$ then $\left\lceil \frac{m}{n} \left\lceil \frac{k}{n} \right\rceil + \frac{k}{n} \right\rceil \leq \lambda_n(m,k) \leq \left\lceil \frac{m}{n} \left\lceil \frac{k}{n} \right\rceil + \frac{k}{n} \right\rceil + C\frac{m^2 \log k}{n}$ for some constant C.

Key-words: WDM optical networks, multicasting, graph colouring, complexity

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WDM et aroboricité étoile des graphes orientés

Résumé :

Un digraphe est appelé m-*étiqueté* si chaque arc possède un label dans l'ensemble $\{1, \ldots, m\}$. Motivé par l'allocation de fréquences dans les réseaux optiques WDM, nous étudions les $n-$fibres colorations d'un digraphe m-étiqueté D. Celles-ci sont les colorations des arcs de D telles que pour chaque sommet v de D et chaque couleur λ, $in(v, \lambda) + out(v, \lambda) \leq n$ où $in(v, \lambda)$ est le nombre d'arcs entrants v colorés avec λ, et $out(v, \lambda)$ est le nombre de labels l tels qu'il existe un arc sortant de v étiqueté l et coloré λ. Le but est de trouver le nombre minimum de couleurs $\lambda_{n,m}(D)$ tel que tout digraphe m-étiqueté admette une n-fibres coloration avec ce nombre de couleurs. Dans le cas particulier d'une seule fibre et lorsque D est $1-$étiqueté, ceci revient à trouver l'arboricité étoile de D, notée $dst(D)$.

Nous démontrons que pour tout digraphe D, on a $dst(D) \leq 2\Delta^-(D) + 1$. Nous étudions ensuite l'arboriciteé étoile des digraphes de degré borné. Nous prouvons que pour les orientations des graphes de degré maximum trois, on a toujours $dst \leq 3$. Pour les orientations régulières des graphes de degré 4, on démontre que $dst \leq 4$. Calculer la valeur exacte de dst est $\mathcal{NP}$-dur même pour la classe de graphes orientés de degré au plus 4. Finalement, nous étudions $\lambda_n(m, k) = \max\{\lambda_n(D) \mid D$ est m-étiquté and $\Delta^-(D) \leq k\}$. Nous prouvons que si $m \geq n$ alors $\left\lceil \frac{m}{n} \left\lceil \frac{k}{n} \right\rceil + \frac{k}{n} \right\rceil \leq \lambda_n(m, k) \leq \left\lceil \frac{m}{n} \left\lceil \frac{k}{n} \right\rceil + \frac{k}{n} \right\rceil + C \frac{m 2 \log k}{n}$ pour une constante C.

Mots-clés : réseaux optiques WDM, multicasting, allocation de fréquences, coloration de graphes, complexité

1 Introduction

The origin of this paper is the study of wavelength assignment for multicasts in star network. Partial results are already obtained by Brandt and Gonzalez [4]. We are given a star network in which a center node is connected by an optical fiber to a set of nodes V. Each node v of V sends a set of multicasts $M_1(v), \ldots, M_{s(v)}(v)$ to the sets of nodes $S_1(v), \ldots, S_{s(v)}(v)$. Using WDM (*wavelength-division multiplexing*), different signals may be sent at the same time through the same fiber but on different wavelengths. The central node is an all-optical transmitter: hence, it may redirect a signal arriving from a node on a particular wavelength to some of the other nodes on the same wavelength. Therefore for each multicast $M_i(v)$, v should send the message to the central node on a set of wavelengths so that the central node redirect it to each node of $S_i(v)$ using one of these wavelengths. The aim is to minimize the total number of used wavelengths.

We first study the very fundamental case when the fiber is unique and each vertex v sends a unique multicast $M(v)$ to the set $S(v)$ of nodes. Let D be the digraph with vertex set V such that the out-neighbourhood of a vertex v is $S(v)$. Note that this is a digraph and not a multidigraph (there is no multiple arcs) as $S(v)$ is a set. Then the problem is to find the smallest k such that there exists a mapping $\phi : V(D) \to \{1, \ldots, k\}$ satisfying the two conditions:

(i) $\phi(uv) \neq \phi(vw)$;

(ii) $\phi(uv) \neq \phi(u'v)$.

Such a mapping is called *directed star k-colouring*. The *directed star arboricity* of a digraph D, denoted by $dst(D)$, is the minimum integer k such that there exists a directed star k-colouring. This notion has been introduced in [6] and is an analog of the *star arboricity* defined in [1]. An *arborescence* is a connected digraph in which every vertex has indegree 1 except one, called *root*, which has indegree 0. A *forest* is the disjoint union of arborescences. A *star* is an arborescence in which the root dominates all the other vertices. A *galaxy* is a forest of stars. Clearly, every colour class of a directed star colouring is a galaxy. Hence, the directed star arboricity of a digraph D is the minimum number of galaxies into which $A(D)$ may be partitioned.

For a vertex v, its indegree $d^-(v)$ corresponds to the number of multicasts it receives. A sensible assumption is that a node receives a bounded number of multicasts. Hence, Brandt and Gonzalez [4] studied the directed star arboricity of a digraph D with maximum indegree Δ^-. They showed that $dst(D) \leq \lceil 5\Delta^-/2 \rceil$. This upper bound is tight if $\Delta^- = 1$ because odd circuits have directed star arboricity 3. However it can be improved for larger value of $\Delta^- = 1$. We conjecture that if $\Delta^- \geq 2$, then $dst(D) \leq 2\Delta^-$.

Conjecture 1 Every digraph D with maximum indegree $k \geq 2$ satisfies $dst(D) \leq 2k$.

This conjecture would be tight as Brandt [3] showed that for every k, there is an acyclic digraph D_k such that $\Delta^-(D_k) = k$ and $dst(D_k) = 2k$. Note that to prove this conjecture, it is sufficient to prove it for $k = 2$ and $k = 3$. Indeed a digraph with maximum indegree $k \geq 2$ has an arc-partition into $k/2$ digraphs with maximum indegree 2 if k is even and into $(k-1)/2$ digraphs with maximum indegree 2 and one with maximum indegree 3. In section 2, we show that $dst(D) \leq 2\Delta^- + 1$ and settle Conjecture 1 for acyclic digraphs.

Remark 2 Note that we restrict ourselves to digraphs, i.e. circuits of length two are permitted, but not multiple arcs. When multiple arcs are allowed, all the bounds above do not hold. Indeed the multidigraph T_k with three vertices u, v and w and k parallel arcs uv, vw and wu satisfies $dst(T_k) = 3k$. Moreover, this example is extremal since every multidigraph satisfies $dst(D) \leq 3\Delta^-$. Indeed let us show it by induction: pick a vertex v with outdegree at most Δ^- in a terminal strong component. A strong component C of a digraph is *terminal* if there is no arc leaving C, i.e. with tail in C and head outside of C. If v has no inneighbour, it is isolated and we remove it. Otherwise, we consider any arc uv. Its colour must be different from the colours of the $d^-(u)$ arcs entering u, the $d^+(v)$ arcs leaving v and the $d^-(v)-1$ other arcs entering v, so at most $3\Delta^- - 1$ arcs in total. Hence, remove the arc uv, apply induction, and extend the colouring to uv. Therefore, for multidigraphs, the bound $dst(D) \leq 3\Delta^-$ is sharp.

We then study the directed star arboricity of a digraph bounded with maximum degree. The *degree* of a vertex v is $d(v) = d^-(v) + d^+(v)$. It corresponds to the degree of the vertex in the underlying multigraph. (We have edges with multiplicity 2 each time there is a circuit of length two in the digraph.) The *maximum degree* of a digraph D, denoted $\Delta(D)$, or simply Δ when D is clearly understood from the context, is $\max\{d(v), v \in V(D)\}$. Let us denote by $\mu(G)$, the maximum multiplicity of an edge in a multigraph. By Vizing's theorem, one can colour the edges of a multigraph with $\Delta(G) + \mu(G)$ colours so that two edges have different colours if they are incident. Since the multigraph underlying a digraph has maximum multiplicity at most two, for any digraph D, $dst(D) \leq \Delta + 2$. We conjecture the following:

Conjecture 3 *Let D be a digraph with maximum degree $\Delta \geq 3$. Then $dst(D) \leq \Delta$.*

This conjecture would be tight since every digraph with $\Delta = \Delta^-$ has directed star arboricity at least Δ. In section 3, we prove Conjecture 3 holds when $\Delta = 3$.

Pinlou and Sopena [9] studied a stronger form of directed star arboricity, called *acircuitic directed star arboricity.* They add the extra condition that any circuit has to have at least three distinct colours. Note that such a notion applies only to *oriented graphs* that are digraphs without circuit of length 2. Indeed such a circuit may not receive 3 colours. They showed that the acircuitic directed star arboricity of a *subcubic* (i.e. each vertex has degree at most 3) oriented graph is at most four. We give a new and very short proof of this result.

A first step towards Conjectures 1 and 3 would be to prove the following statement which is weaker than these two conjectures.

Conjecture 4 *Let $k \geq 2$ and D be a digraph. If $\max(\Delta^-, \Delta^+) \leq k$ then $dst(D) \leq 2k$.*

This conjecture holds and is far from being tight for large values of k. Indeed Guiduli [6] showed that if $\max(\Delta^-, \Delta^+) \leq k$ then $dst(D) \leq k + 20\log k + 84$. Guiduli's proof is based on the fact that, when both out and indegree are bounded, the colour of an arc depends on the colour of few other arcs. This bounded dependency allows the use of the Lovász Local Lemma. This idea was first used by Algor and Alon [1], for the star arboricity of undirected graphs. Note also that Guiduli's result is (almost) tight since there are digraphs D with $\max(\Delta^-, \Delta^+) \leq p$ and $dst(D) \geq p + \Omega(\log p)$ (see [6]). Note also that similarly as for Conjecture 1, it is sufficient to prove Conjecture 4 for $k = 2$ and k=3. In Section 4, we prove that Conjecture 4 holds for $k = 2$. By the above remark, it implies that Conjecture 4 holds for all even k.

In Section 5, we investigate the complexity of finding the directed star arboricity of a digraph. Unsurprisingly, this is an NP-hard problem. More precisely, we show that determining the directed star arboricity of a digraph with out- and indegree at most 2 is NP-complete.

Next, we study the more general (and more realistic) problem in which the center is connected to the onodes of V with n optical fibers. Morover each node may sent several multicasts. We model it as a *labelled* digraph problem: We consider a digraph D on vertex set V. For each multicast $(v, S_i(v))$ we add the set of arcs $A_i(v) = \{vw, w \in S_i(v)\}$ with label i. The label of an arc a is denoted by $l(a)$. Thus for every couple (u, v) of vertices and label i there is at most one arc uv labelled by i. If each vertex sends at most m multicasts, there are at most m labels on the arcs. Such a digraph is said to be *m-labelled.* One wants to find an *n-fiber wavelength assignment* of D, that is a mapping $\Phi : A(D) \to \Lambda \times \{1, \dots, n\} \times \{1, \dots n\}$ in which every arc uv is associated a triple $(\lambda(uv), f^+(uv), f^-(uv))$ such that :

(i) $(\lambda(uv), f^-(uv)) \neq (\lambda(vw), f^+(vw))$;

(ii) $(\lambda(uv), f^-(uv)) \neq (\lambda(u'v), f^-(u'w))$;

(iii) if $l(vw) \neq l(vw')$ then $(\lambda(vw), f^+(vw)) \neq (\lambda(vw'), f^+(vw'))$.

$\lambda(uv)$ corresponds to the wavelength of uv, and $f^+(uv)$ and $f^-(uv)$ the fiber used in u and v respectively. Hence the condition (i) corresponds to the fact that an arc entering v and an arc leaving v have either different wavelengths or different fibers; the condition (ii) corresponds to the fact that two arcs entering v have either different wavelengths or different fibers; the condition (iii) corresponds to the fact that two arcs leaving v with different labels have either different wavelengths or different fibers. The problem is then to find the minimum cardinality $\lambda_n(D)$ of Λ such that there exists an *n-fiber wavelength assignment* of D.

The crucial thing in an n-fiber wavelength assignment is the function λ which assigns colours (wavelengths) to the arcs. It must be an *n-fiber colouring*, that is a function $\phi : A(D) \rightarrow \Lambda$, such that at each vertex v, for each colour $\lambda \in \Lambda$, $in(v, \lambda) + out(v, \lambda) \leq n$ with $in(v, \lambda)$ the number of arcs coloured λ entering v and $out(v, \lambda)$ the number of labels l such that there exists an arc leaving v coloured λ. Once we have an n-fiber colouring, one can easily find a suitable wavelength assignment. For every vertex v and every colour λ, this is done by assigning a different fiber to each arc of colour λ entering v and to each set of arcs of colour λ leaving v and of the same label. Hence $\lambda_n(D)$ is the minimum number of colours such that there exists an n-fiber colouring.

We are particularly intested in $\lambda_n(m, k) = \max\{\lambda_n(D) \mid D \text{ is } m\text{-labelled and } \Delta^-(D) \leq k\}$ that is the maximum number of wavelengths that may be necessary if there are n-fibers and each node sends at most m and receives at most k multicasts. In particular, $\lambda_1(1, k) = \max\{dst(D) \mid \Delta^-(D) \leq k\}$. So our above mentionned results show that $2k \leq \lambda_1(1, k) \leq 2k + 1$. Brandt and Gonzalez showed that for $n \geq 2$ we have $\lambda_n(1, k) \leq \left\lceil \frac{k}{n-1} \right\rceil$. In Section 6, we study the case when $n \geq 2$ and $m \geq 2$. We show that if $m \geq n$ then

$$\left\lceil \frac{m}{n} \left\lceil \frac{k}{n} \right\rceil + \frac{k}{n} \right\rceil \leq \lambda_n(m, k) \leq \left\lceil \frac{m}{n} \left\lceil \frac{k}{n} \right\rceil + \frac{k}{n} \right\rceil + C\frac{m^2 \log k}{n} \qquad \text{for some constant } C.$$

We also show that if $m < n$ then

$$\left\lceil \frac{m}{n} \left\lceil \frac{k}{n} \right\rceil + \frac{k}{n} \right\rceil \leq \lambda_n(m, k) \leq \left\lceil \frac{k}{n-m} \right\rceil.$$

The lower bound generalizes Brandt and Gonzalez [4] results which established this inequality in the particular cases when $k \leq 2$, $m \leq 2$ and $k = m$. The digraphs used to show this lower bound are all acyclic. We show that if $m \geq n$ then this lower bound is tight for acyclic digraphs. Moreover the above mentionned digraphs have large outdegree. Generalizing the result of Guiduli [6], we show that for an m-labelled digraph D with both in- and outdegree bounded by k then few colours are needed:

$$\lambda_n(D) \leq \frac{k}{n} + C'\frac{m^2 \log k}{n} \qquad \text{for some constant } C'.$$

2 Directed star arboricity of digraphs with bounded indegree

Our goal in this section is to approach Conjecture 1. It is easy to see that a forest has directed star arboricity 2. Hence, an idea to prove Conjecture 1 would be to show that every digraph has an arc-partition into Δ^- forests. However this statement is false. Indeed A. Frank [5] (see also [10], p.908) characterized digraphs having an arc-partition into k forests. Let $D = (V, A)$. For any $U \subset V$, the digraph induced by the vertices of U is denoted $D[U]$.

Theorem 5 (A. Frank) *A digraph $D = (V, A)$ has an arc-partition into k forests if and only if $\Delta^-(D) \leq k$ and for every $U \subset V$, the digraph $D[U]$, has at most $k(|U| - 1)$ arcs.*

However, Theorem 5 implies that every digraph D has an arc-partition into $\Delta^- + 1$ forests. Indeed for any $U \subset V$, $\Delta^-(D[U]) \leq \min\{\Delta^-, |U| - 1\}$, so $D[U]$ has at most $\min\{\Delta^-, |U| - 1\} \times |U| \leq (\Delta^- + 1)(|U| - 1)$ arcs. Hence, every digraph has directed star arboricity at most $2\Delta^- + 2$.

Corollary 6 *Every digraph D satisfies $dst(D) \leq 2\Delta^- + 2$.*

We now lessen this upper bound by one.

Theorem 7 *Every digraph D satisfies $dst(D) \leq 2\Delta^- + 1$.*

The idea to prove Theorem 7 is to show that every digraph has an arc-partition into Δ^- forests and a galaxy G. To do so, we prove a stronger result (Lemma 8) by induction.

A *sink* is a vertex with outdegree 0. A *source* is a vertex with indegree 0. A multidigraph is *k-nice* if $\Delta^- \leq k$ and if the tails of parallel arcs, if any, are sources. A *k-decomposition* of a digraph D is an arc-partition into k forests and a galaxy G such that every source of D is isolated in G. Let u be a vertex of D. A k-decomposition of D is *u-suitable* if no arc of G has head u.

Lemma 8 *Let u be a vertex of a k-nice multidigraph D. Then D has a u-suitable k-decomposition.*

Proof. We proceed by induction on $n + k$. We now discuss the connectivity of D:

- If D is not connected, we apply induction on every component.

- If D is strongly connected, every vertex has indegree at least one. Remember also that there is no parallel arcs. Let v be an outneighbour of u. There exists a spanning arborescence T with root v which contains all the arcs with tail v. Let D' be the digraph obtained from D by removing the arcs of T and v. Observe that D' is $(k-1)$-nice. By induction, it has a u-suitable $(k-1)$-decomposition $(F_1, \ldots, F_{k-1}, G)$. Note that F_i, T and G contain all the arcs of D except those with head v. By construction, $G' = G \cup uv$ is a galaxy since no arc of G has head u. Let $u_1, \ldots, u_{l-1}$ be the inneighbours of v distinct from u, where $l \leq k$. Let $F'_i = F_i \cup u_i v$, for all $1 \leq i \leq l - 1$. Then each F'_i is a forest, so $(F_1, \ldots, F_{k-1}, T, G')$ is a u-suitable k-decomposition of D.

- If D is connected but not strongly connected, we consider a strongly connected terminal component D_1. Set $D_2 = D \setminus D_1$. Let u_1 and u_2 be two vertices of D_1 and D_2, respectively, such that u is one of them.

  If D_2 has a unique vertex v (thus $u_2 = v$), since D is connected and D_1 is strong, there exists a spanning arborescence T of D with root v. Now $D' = D \setminus A(T)$ is a $(k-1)$-nice multidigraph, so by induction it has a u-suitable $(k-1)$-decomposition. Adding T to this decomposition, we obtain a u-suitable k-decomposition.

  If D_2 has more than one vertex, by induction, it admits a u_2-suitable k-decomposition $(F_1^2, \ldots, F_k^2, G^2)$. Moreover the digraph D'_1 obtained by contracting D_2 to a single vertex v is k-nice and so has a u_1-suitable k-decomposition $(F_1^1, \ldots, F_k^1, G^1)$. Moreover, since v is a source, it is isolated in G^1. Hence $G = G^1 \cup G^2$ is a galaxy. We now let F_i be the union of F_i^1 and F_i^2 by replacing the arcs of F_i^1 with tail v by the corresponding arcs in D. Then $(F_1, \ldots, F_k, G)$ is a k-decomposition of D which is suitable for both u_1 and u_2.

$\square$

2.1 Acyclic digraphs

It is not hard to show that $dst(D) \leq 2\Delta^-$ when D is acyclic. But we will prove this result in a more constrained way. A *cyclic n-interval* of $\{1, 2, \ldots, p\}$ is a set of n consecutive numbers modulo p. Now for the directed star colouring, we will insist that for every vertex v, the (distinct) colours used to colour the arcs with head v are chosen in a *cyclic k-interval* of $\{1, 2, \ldots, 2k\}$. Thus, the number of possible sets of colours used to colour the entering arcs of a vertex drastically falls from $\binom{2k}{k}$ when every set is *a priori* possible, to $2k$. Note that having consecutives colours on the arcs entering a vertex corresponds to having consecutives wavelengths on the link between the corresponding node and the central one. This is very important for grooming issues. For more details about grooming, we refer to the two comprehensive surveys [7, 8].

We need for this the following result on set of distinct representatives.

Lemma 9 *Let $I_1, \ldots, I_k$ be k non necessary distinct cyclic k-intervals of $\{1, 2, \ldots, 2k\}$. Then $I_1, \ldots, I_k$ admit a set of distinct representatives forming a cyclic k-interval.*

Proof. We consider $I_1, \ldots, I_k$ as a set of p *distinct* cyclic k-intervals $I_1, \ldots, I_p$ with respective multiplicity $m_1, \ldots, m_p$ such that $\sum_{i=1}^{p} m_i = k$. Such a system will be denoted by $((I_1, m_1), \ldots, (I_p, m_p))$. We shall prove the existence of a cyclic k-interval J, such that J can be partitioned into p subsets J_i, $1 \leq i \leq p$, such that $|J_i| = m_i$ and $J_i \subset I_i$. This proves the lemma (by associating distinct elements of J_i to each copy of I_i).

We proceed by induction on p. The result holds trivially for $p = 1$. We have two cases:

- There exists i and j such that $|I_j \setminus I_i| = |I_i \setminus I_j| \leq \max(m_i, m_j)$.

  Suppose without loss of generality that $i < j$ and $m_i \geq m_j$. We apply the induction hypothesis to $((I_1, m_1), \cdots, (I_i, m_i + m_j), \cdots, (I_{j-1}, m_{j-1}), (I_{j+1}, m_{j+1}), \cdots, (I_p, m_p))$, in order to find a cyclic interval J', such that J' admits a partition into subsets J'_r, such that for any $r \neq i, j$, $J'_r \subset I_r$ is a subset of size m_r, and $J'_i \subset I_i$ is of size $m_i + m_j$. We now partition J'_i into two sets J_i and J_j with respective size m_i and m_j, in such a way that $(I_i \setminus I_j) \cap J'_i \subseteq J_i$. Remark that this is possible exactly because of our assumption $|I_j \setminus I_i| = |I_i \setminus I_j| \leq m_i$. Since $J_i \subset I_i$ and $J_j \subset I_j$, this refined partition of J' is the desired one.

- For any i, j we have $|I_j \setminus I_i| = |I_i \setminus I_j| \geq \max(m_i, m_j) + 1$.

  Each I_i intersects exactly $2m_i - 1$ other cyclic k-intervals on less than m_i elements. Since there are $2k$ cyclic k-intervals in total and $\sum_{i=1}^{p}(2m_i - 1) = 2k - p < 2k$, we conclude the existence of a cyclic k-interval J which intersects each I_i in an interval of size at least m_i.

  Let us prove that one can partition J in the desired way. By Hall's matching theorem, it suffices to prove that for every subset $\mathcal{I}$ of $\{1, \ldots, p\}$, $|\bigcup_{i \in \mathcal{I}} I_i \cap J| \geq \sum_{i \in \mathcal{I}} m_i$.

  Suppose for a contradiction that a subset $\mathcal{I}$ of $\{1, \ldots, p\}$ violates this inequality. Such a subset will be called *contracting*. Without loss of generality, we assume that $\mathcal{I}$ is a contracting set with minimum cardinality and that $\mathcal{I} = \{1, \ldots, q\}$. Remark that by the choice of J, we have $q \geq 2$. The set $K := \bigcup_{i \in \mathcal{I}} I_i \cap J$ consists of one or two intervals of J, each containing one extremity of J. By the minimality of $\mathcal{I}$, K must be a single interval (if not, one would take $\mathcal{I}_\infty$ (resp. $\mathcal{I}_\in$), all the elements of $\mathcal{I}$ which contains the first (resp. the second) extremity of J. Then one of I_1 or I_2 would be contracting). Thus, one of the two extremities of J is in every I_i, $i \in \mathcal{I}$. Without loss of generality, we may assume that $(I_1 \cap J) \subset (I_2 \cap J) \subset \cdots \subset (I_q \cap J)$. Now, for every $2 \leq i \leq q$, $|I_i \setminus I_{i-1}| = |(I_i \cap J) \setminus (I_{i-1} \cap J)| \geq \max(m_i, m_{i-1}) + 1 \geq m_i + 1$. But $|\bigcup_{i \in \mathcal{I}} I_i \cap J| = |(I_1 \cap J)| + \sum_{i=2}^{q} |(I_i \cap J) \setminus (I_{i-1} \cap J)|$. So $|\bigcup_{i \in \mathcal{I}} I_i \cap J| \geq \sum_{i=1}^{q} m_i + q - 1$, which is a contradiction.

□

Theorem 10 *Let D be an acyclic digraph with maximum indegree k. D admits a directed star $2k$-colouring such that for every vertex, the colours assigned to its entering arcs are included in a cyclic k-interval of* $\{1, 2, \ldots, 2k\}$.

Proof. By induction on the number of vertices, the result being trivial if D has one vertex. Suppose now that D has at least two vertices. Then D has a sink x. By the induction hypothesis, $D \setminus x$ has a directed star $2k$-colouring c such that for every vertex, the colours assigned to its entering arcs are included in a cyclic k-interval. Let $v_1, v_2, \ldots, v_l$ be the inneighbours of x in D, where $l \leq k$ because $\Delta^-(D) \leq k$. For each $1 \leq i \leq l$, let I'_i be a cyclic k-interval which contains all the colours of the arcs with head v_i. We set $I_i = \{1, \ldots, 2k\} \setminus I'_i$. Clearly, I_i is a cyclic k-interval and the arc $v_i x$ can be coloured by any element of I_i. By Lemma 9, $I_1, \ldots, I_l$ have a set of distinct representatives included in a cyclic n-interval J. Hence assigning J to x, and colouring the arc $v_i x$ by the representative of I_i gives a directed star $2k$-colouring of D. □

Theorem 10 is tight : Brandt [3] showed that for every k, there is an acyclic digraph such that $\Delta^-(D_k) = k$ and $dst(D_k) = 2k$. His construction is the special case of the construction given in Proposition 21 for $n = m = 1$.

3 Subcubic digraphs

Recall that a *subcubic* digraph is a graph with degree at most three. In this section, we first show that the directed star arboricity of a subcubic digraph is at most 3, so proving Conjecture 3 when $\Delta = 3$. We then give a very short proof of a result of Pinlou and Sopena asserting that the acircuitic directed star arboricity of a subcubic digraph is at most 4.

3.1 Directed star arboricity of subcubic digraphs

The aim of this subsection is to prove the following theorem :

Theorem 11 *Every subcubic digraph has directed star arboricity at most* 3.

To do, we need to establish some lemmas to enable us to extend some partial directed star colouring into directed star colouring of the whole digraph. These lemmas need the following definition. Let $D = (V, A)$ be a digraph and S a subset of $V \cup A$. Suppose that each element x of S is assigned a list $L(x)$. A colouring c of S is an *L-colouring* if $c(x) \in L(x)$ for every $x \in S$.

Lemma 12 *Let C be a circuit in which every vertex v receives a list $L(v)$ of two colours among* $\{1, 2, 3\}$ *and each arc a receives the list* $L(a) = \{1, 2, 3\}$. *Then there is no L-colouring c of the arcs and vertices such that $c(x) \neq c(xy)$, $c(y) \neq c(xy)$, and $c(xy) \neq c(yz)$, for all arcs xy and yz if and only if C is odd and all the vertices have the same list.*

Proof. Assume first that every vertex is assigned the same list, say $\{1, 2\}$. If C is odd, it is simple matter to see that we can not find the derired colouring. Indeed, if all vertices have the same colour, then we should have an arc colouring of C with two colours which is impossible. If two adjacente vertices, say x and y, have difefrent colours 1 and 2, then xy should have colour 3, yz (z the other neighbhour of y) should have colour 1 and so z should have colour If C is even, we colour the vertices by 1 and the arcs alternately by 2 and 3.

Now assume that $C = x_1x_2 \dots x_kx_1$ and x_1 and x_2 are assigned different lists. Say $L(x_1) = \{1, 2\}$ and $L(x_2) = \{2, 3\}$. We colour the arc x_1x_2 by 3, the vertex x_2 by 2 and the arc x_2x_3 by 1. Then we colour x_3, x_3x_4, ...,x_k. It remains to colour x_kx_1 and x_1. Two cases may happen: If we can colour x_kx_1 by 1 or 2, we do it and colour x_1 by 2 or 1 respectively. Otherwise the set of colours assigned to x_k and $x_{k-1}x_k$ is $\{1, 2\}$. Hence, we colour x_kx_1 with 3, x_1 by 1, and recolour x_1x_2 by 2 and x_2 by 3. □

Lemma 13 *Let D be a subcubic digraph with no vertex of outdegree two and indegree one. Assume that every arc a has a list of colours $L(a) \subset \{1, 2, 3\}$ such that:*

- *If the head of a is a sink s (a is called a leaving arc), $|L(a)| \geq d^-(s)$.*
- *If a is not a leaving arc and the tail of a is a source (a is called an entering arc), $|L(a)| \geq 2$.*
- *In other cases $|L(a)| = 3$.*
- *If a vertex is the head of at least two entering arcs the union of their lists of colours contains at least three colours.*
- *If all the vertices of an odd circuit are the tails of entering arcs, the union of the lists of colours of these entering arcs contains at least three colours.*

Then D has a directed star L-colouring.

Proof. We colour the graph inductively. Consider a terminal strong component C of D. Since D has no vertex with indegree one and outdegree two, C induces either a singleton or a circuit.

1) Assume that C is a singleton v which is the head of a unique arc $a = uv$. If u has indegree 0, colour a with a colour of its list. If u has indegree 1, and thus total degree 2, colour a by the colour of its list and remove this colour from the list of the arc with head u. If u is the head of e and f, observe that $L(e)$ and $L(f)$ have at least two colours and their union have at least three colours. To conclude, colour a with a colour in its list, remove this colour from $L(e)$ and $L(f)$, remove a, split u into two vertices, one with head e, and the other with head f. Now, choose in their respective lists different colours for the arcs e and f to form the new list $L(e)$ and $L(f)$.
2) Assume that C is a singleton v which is the head of several arcs, including $a = uv$. In this case, we reduce $L(a)$ to a single colour, remove this colour from the other arcs with head v and split v into v_1 which becomes the head of a, and v_2 which becomes the head of the other arcs.
3) Assume that C is a circuit. Every arc entering C has a list of at least two colours. We can apply Lemma 12 to conclude.

□

Proof of Theorem 11. Assume for contradiction that the digraph D has star arboricity more than three and is minimum with respect to the number of arcs for this property. Observe that D has no source, otherwise we simply delete it with its incident arcs, apply induction and extend the colouring since arcs leaving from a source can be coloured arbitrarily. Let D_1 be the subdigraph of D induced by the vertices of indegree at most 1. We denote by D_2 the digraph induced by the other vertices, and by $[D_i, D_j]$ the set of arcs with tail in D_i and head in D_j. We claim that D_1 contains no even circuit. If not, we simply remove the arcs of this even circuit, apply induction and extend the colouring to the arcs of the even circuit since every arc of the circuit has two colours available.

A *critical set* of vertices of D_2 is either a vertex of D_2 with indegree at least two in D_1, or an odd circuit of D_2 having all its inneighbours in D_1. Observe that critical sets are disjoint. For every critical set S, we select two arcs entering S from D_1, called *selected arcs* of S.

Let D' be digraph induced by the arc set $A' = A(D_1) \cup [D_2, D_1]$. Now we define a *conflict graph* on the arcs of D' in the following way:

- Two arcs xy, yv of D' are in conflict, called *normal conflict* at y.
- Two arcs xy, uv of D' are also in conflict if there exists two selected arcs of the same set S with tails y and v. These conflicts are called *selected conflicts* at y and v.

Let us analyse the structure of the conflict graph. Observe first that an arc is in conflict with three arcs : one normal conflict at its tail and at most two (normal or selected) at its head.

We claim that there is no K_4 in the conflict graph. Suppose there is one, then there is 4 arcs which are pairwise in conflict. Since each arc has degree 3, it has a normal conflict at its tail, the digraphs induced by these four arcs contains a circuit. It cannot be a circuit of even length (2 or 4) so it has length 3. It follows that the four arcs a, b, c, d are as in Figure 1 below. Let D^* be the digraph obtained from D by removing the arcs a, b, c, d and their four incident vertices. By minimality of D, D^* admits a directed star 3-colouring which can be extended to D as depicted below depending if the two leaving arcs are coloured the same or differently. This proves the claim.

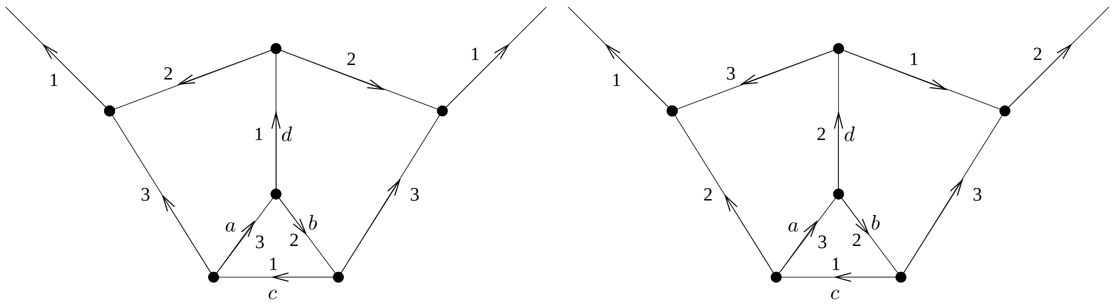

Figure 1: A K_4 in the conflict graph and the two ways of extending the colouring.

Brook's Theorem asserts that every subcubic graph without K_4 is 3-colourable. So the conflict graph admits a 3-colouring c. This gives a colouring of the arcs of D'. Let D'' be the digraph and L be the list-assignment on the arcs of D'' obtained as follow:

- Remove the arcs of D_1 from D,
- Assign to each arc of $[D_2, D_1]$ the singleton list containing the colour it has in D',
- For each arc uv of $[D_1, D_2]$, there is a unique arc tu in $A(D')$, so assign the list $L(uv) = \{1,2,3\} \setminus c(tu)$.
- Assign the list $\{1, 2, 3\}$ to the other arcs.
- If there are vertices with indegree one and outdegree two (they were in D_1), split each of them into one source of degree two and a sink of degree one.

Note that there is a trivial one-to-one correspondence between $A(D'')$ and $A(D) \setminus A(D')$. By the definition of conflict graph and D'', one can easily check that D'' and L satisfies the condition of Lemma 13. Hence D'' admits a directed star L-colouring which union with c is a directed star 3-colouring of D, a contradiction. □

3.2 Acircuitic directed star arboricity

A directed star colouring is *acircuitic* if there is no *bicoloured* circuits, i.e. circuits for which only two colours appears on its arcs. The *acircuitic directed star arboricity* of a digraph D a digraph is the minimum number k of colours such that there exists an acircuitic directed star k-colouring of D. In this subsection, we give a short alternative proof of the following theorem due to Pinlou and Sopena.

Theorem 14 (Pinlou and Sopena [9]) *Every subcubic oriented graph has acircuitic directed star arboricity at most* 4.

In order to prove this theorem, we need the following lemma.

Lemma 15 *Let D be an acyclic subcubic digraph. Let L be a list-assignment on the arcs of D such that for every arc uv, $|L(uv)| \geq d(v)$. Then D admits a directed star L-colouring.*

Proof. We prove the result by induction on the number of arcs of D, the result holding trivially if D has no arcs.

Since D is acyclic, it has an arc xy with y a sink. Let a be a colour in $L(xy)$. For any arc e distinct from xy, set $L'(e) = L(e) \setminus \{a\}$ if e incident to xy (and thus has head in $\{x, y\}$ since y is a sink), and $L'(e) = L(e)$ otherwise. Then in $D' = D - xy$, we have $|L'(uv)| \geq d(v)$. Hence, by induction hypothesis, D' admits a directed star L'-colouring that can be extended in a directed star L-colouring of D by colouring xy with a. □

Proof of Theorem 14.

Let V_1 be the set of vertices of outdegree at most 1 and $V_2 = V \setminus V_1$. Then every vertex of V_2 has outdegree at least 2 and so indegree at most 1.

Let M be the set of arcs with tail in V_1 and head in V_2. Colour all the arcs of M with 4. Moreover for every circuit C in $D[V_1]$ and $D[V_2]$ choose an arc $e(C)$ and colour it by 4. Note that, by definition of V_1 and V_2, the arc $e(C)$ is not incident to any arc of M and C is the unique cicrcuit containing $e(C)$. Let us denote M_4 the set of arcs coloured 4. Then M_4 is a matching and $D - M_4$ is acyclic.

We shall now find a directed star colouring of $D - M_4$ with $\{1, 2, 3\}$ that creates any bicoloured circuit. If such a circuit exists, 4 would be one of its colour because $D - M_4$ is acyclic and all its arcs coloured 4 would be in M because the arcs of $M_4 \setminus M$ is in a unique circuit which has a unique arc coloured 4. Hence we just have to be careful when colouring arcs in the digraph induced by the endvertices of the arcs of M.

Let us denote the arcs of M by $x_i y_i$, $1 \leq i \leq p$ and set $X = \{x_i, 1 \leq i \leq p\}$ and $Y = \{y_i, 1 \leq i \leq p\}$. Then $x_i \in V_1$ and $y_i \in V_2$. Let E' be the set of arcs with tail in Y and head in X. Let H be the graph with vertex set E' such that an arc $y_i x_j$ is adjacent to an arc $y_k x_l$ if

(a) either $k = l$,

(b) or $j = k$ and $i > j$ and $l > j$.

Since a vertex of X has indegree at most 2 and a vertex of Y has outdegree at most 2, H has maximum degree 3. Moreover H contains no K_4 because two arcs of E' with same tail y_k are not adjacent in H. Hence, by Brooks Theorem, H has a vertex-colouring in $\{1, 2, 3\}$ which is corresponds to a colouring c of the arcs of E'. Since (a) is satisfied c is a directed star colouring. Moreover this colouring creates no bicoloured circuits: indeed a circuit contains a subpath $y_i x_j y_j x_l$ with $i > j$ and $k > j$, whose three arcs are coloured differently by (b).

Let $D' = D - (M_4 \cup E')$. For any arc uv in D', let $L(uv) = \{1, 2, 3\} \setminus \{c(wv) \mid wv \in E'\}$. The set $L(uv)$ is the set of colours in $\{1, 2, 3\}$ that may be assigned to uv without creating any conflict with the already coloured arcs. D' is acyclic and $|L(uv)| \geq d(v)$, so by Lemma 15, it admits a directed star L-colouring and thus D has an acircuitic directed star colouring in $\{1, 2, 3, 4\}$. □

Remark 16 Note that in the acircuitic directed star 4-colouring provided in the proof of Theorem 14 the arcs coloured 4 form a matching.

4 Directed star arboricity of digraphs with maximum in and out-degree two

The goal of this section is to prove that every digraph with outdegree and indegree at most two has directed star arboricity at most four.

Theorem 17 *Let D be a digraph with maximum in and outdegree at most two. Then $dst(D) \leq 4$.*

Thus, conjecture 4 holds for $k = 2$ and hence for all even k. However, the class of digraphs with in and outdegree at most two is certainly not an easy class with respect to directed star arboricity, as we will show in Section 5.

In order to prove Theorem 17, it suffices to show that D contains a galaxy G which spans all the vertices of degree four. Then $D' = D - A(G)$ has maximum degree at most 3. So, by Theorem 11, $dst(D') \leq 3$, so $dst(D) \leq 4$. Hence Theorem 17 is directly implied by the following lemma:

Lemma 18 *Let D be a digraph with maximum indegree and outdegree two. Then D contains a galaxy which spans the set of vertices with degree four.*

In order to prove this lemma, we need some preliminaries:
Let V be a set. An *ordered digraph* on V is a pair $(\leq, D)$ where:

- $\leq$ is a partial order on V.
- D is a digraph with vertex set V.
- D contains the Hasse diagram of $\leq$ (i.e. when $x \leq y \leq z$ implies $x = y$ or $y = z$, then xz is an arc of D).
- If xy is an arc of D, the vertices x, y are $\leq$-comparable.

The arcs xy of D thus belong to two different types: the *forward arcs* when $x \leq y$, and the *backward arcs* when $y \leq x$.

Lemma 19 *Let $(\leq, D)$ be an ordered digraph on V. Assume that every vertex is the tail of at most one backward arc and at most two forward arcs and that the indegree of every vertex of D is at least 2, except possibly one vertex x with indegree 1. Then D contains two arcs ca and bd such that $a \leq b \leq c$, $b \leq d$ and $c \not\leq d$, all four vertices being distinct except possibly $a = b$.*

Proof. Let us consider a counterexample with minimum $|V|$.
An *interval* is a subset I of V which has a minimum m and a maximum M such that $I = \{z : m \leq z \leq M\}$. An interval I is *good* if every arc with tail in I and head outside I has tail M and every backward arc in I has tail M.
Let I be an interval of D. The digraph D/I obtained from D by *contracting* I is the digraph with vertex set $(V \setminus I) \cup \{v_I\}$ such that xy is an arc if and only either $v_I \notin \{x, y\}$ and $xy \in A(D)$, or $x = v_I$ and there exists $x_I \in I$ such that $x_I y \in A(D)$, or $y = v_I$ and there exists $y_I \in I$ such that $xy_I \in A(D)$. Similarly, the partial order $\leq_{/I}$ obtained from $\leq$ by *contracting* I is the partial order on $(V \setminus I) \cup \{v_I\}$ such that $x \leq_{/I} y$ if and only if either $v_I \notin \{x, y\}$ and $x \leq y$, or $x = v_I$ and there exists $x_I \in I$ such

that $x_I \leq y$, or $y = v_I$ and there exists $y_I \in I$ such that $x \leq y_I$. It follows from the definitions that $(\leq_{/I}, D/I)$ is an ordered digraph. Note that if $x \leq_{/I} v_I$ then $x \leq M$ with M the maximum of I.

The crucial point is that if I a good interval of D for which the conclusion of Lemma 19 holds for $(\leq_{/I}, D/I)$, then it holds for $(\leq, D)$. Indeed, suppose there exists two arcs ca and bd of D/I such that $a \leq_{/I} b \leq_{/I} c$, $b \leq_{/I} d$ and $c \not\leq_{/I} d$. Note that since I is good $v_I \neq c$. Let M be the maximum of I.
If $v_I \notin \{a, b, c, d\}$, then ca and bd gives the conclusion for D.
If $v_I = a$ then cM is an arc. Let us show that $M \leq b$. Indeed let x be a maximal vertex in I such that $x \leq b$ and y a minimal vertex such that $x \leq y \leq b$. Since the Hasse diagram of $\leq$ is included in D then xy is an arc so $x = M$ since I is good. Thus cM and bd are the desired arcs.
If $v_I = b$ then Md is an arc and $a \leq M$, so ca and Md are the desired arcs.
If $v_I = d$ then there exists $d_I \in I$ such that bd_I, so ca and bd_I are the desired arcs.

Hence to get a contradiction, it is sufficient to find a good interval I such that $(\leq_{/I}, D/I)$ satisfies the hypotheses of Lemma 19.

Observe that there are at least two backward arcs. Indeed, if there are two minimal elements for $\leq$, there are at least three backward arcs heading to these points (since one of them can be x). And if there is a unique minimum m, by letting m' minimal in $V \setminus m$, at least two arcs are heading to m, m'.

Let M be a vertex which is the tail of a backward arc and which is minimal for $\leq$ for this property. Since two arcs cannot have the same tail, M is not the maximum of $\leq$ (if any). Let Mm be the backward arc with tail M.

We claim that the interval J with minimum m and maximum M is good. Indeed, by the definition of M, no backward arc has its tail in $J \setminus \{M\}$. Moreover, any forward arc bd with its tail in $J \setminus \{M\}$ and its head outside J would give our conclusion (with $a = m$ and $c = M$), a contradiction.

Now consider a good interval I with maximum M which is maximal with respect to inclusion. We claim that if $x \in I$, then there is at least one arc entering I, and if $x \notin I$, there are at least two arcs entering I with different tails.

Call m_1 the minimum of I and m_2 any minimal element of $I \setminus m_1$. First assume that x is in I. There are at least three arcs with heads m_1 or m_2. One of them is m_1m_2, one of them can be with tail M, but there is still one left with tail not in I. Now assume that x is not in I. There are at least two arcs with heads m_1 or m_2 and tails not in I. If the tails are different, we are done. If the tails are the same, say v, observe that vm_1 and vm_2 are both backward of both forward (otherwise v would be in I). Since both can not be backward vm_1 and vm_2 are forward. Hence the interval with minimum v and maximum M is a good interval, contradicting the maximality of I. This proves the claim.

This claim implies that $(\leq_{/I}, D/I)$ satisfies the hypotheses of Lemma 19, yielding a contradiction. □

Proof of Theorem 18. Let G be a galaxy of D which spans a maximum number of vertices of degree four. Suppose for contradiction that some vertex x with degree four is not spanned.

An *alternating path* is an oriented path ending at x, starting by an arc of G, and alternating with arcs of G and arcs of $A(D) \setminus A(G)$. We denote by $\mathcal{A}$ the set of arcs of G which belong to an alternating path.

Claim 1 *Every arc of $\mathcal{A}$ is a component of G.*

Proof. Indeed, if uv belongs to $\mathcal{A}$, it starts some alternating path P. Thus, if u has outdegree more than one in G, the digraph with set of arcs $A(G) \triangle A(P)$ is a galaxy and spans $V(G) \cup x$. □

Claim 2 *There is no circuits alternating arcs of $\mathcal{A}$ and arcs of $A(D) \setminus \mathcal{A}$.*

Proof. Assume that there is such a circuit C. Consider a shortest alternating path P starting with some arc of $\mathcal{A}$ in C. Now the digraph with arcs $A(G) \triangle (A(P) \cup A(C))$ is a galaxy which spans $V(G) \cup x$, contradicting the maximality of G. □

We now endow $\mathcal{A} \cup x$ with a partial order structure by letting $a \leq b$ if there exists an alternating path starting at a and ending at b. The fact that this relation is a partial order relies on Claim 2. Observe that x is the maximum of this order.

We also construct a digraph $\mathcal{D}$ on vertex set $\mathcal{A} \cup x$ and all arcs $uv \to st$ such that us or vs is an arc of D (and $uv \to x$ such that ux or vx is an arc of D).

Claim 3 *The pair $(\mathcal{D}, \leq)$ is an ordered digraph. Moreover an arc of $\mathcal{A}$ is the tail of at most one backward arc and two forward arcs and x is the tail of at most two backward arcs.*

Proof. The fact that the Hasse diagram of $\leq$ is contained in $\mathcal{D}$ follows from the fact that if $uv \leq st$ belongs to the Hasse diagram of $\leq$, there is an alternating path starting by $uvst$, in particular, the arc vs belongs to D, and thus $uv \to st$ in $\mathcal{D}$.

Suppose that $uv \to st$ and then vs or us is an arc of D. If vs is an arc, then because there is no alternating circuit, st follows uv on some alternating path so $uv \leq st$. In this case, $uv \to st$ is forward. If us is an arc of D, we claim that $st \leq uv$. Indeed, if an alternating path P starting at st does not contain uv, the galaxy with arcs $(A(G) \triangle A(P)) \cup \{us\}$ spans $V(G) \cup x$ contradicting the maximality of G. In this case, $uv \to st$ is backward.

It follows that an arc uv of $\mathcal{A}$ is the tail of at most one backward arc since this arc and uv are the two arcs leaving u in D and the tail of at most two forward arcs since v has outdegree at most 2. Furthermore, since x has outdegree at most two, it follows that x is the tail of at most two backward arcs. □

Claim 4 *The indegree of every vertex of $\mathcal{D}$ is two.*

Proof. Let uv be a vertex of $\mathcal{D}$ which starts an alternating path P. If u has indegree less than two, and thus does not belong to the set of vertices of degree four, the galaxy with arcs $A(G) \triangle A(P)$ spans more vertices of degree four than G, a contradiction. Let s and t be the two inneighbours of u in D. An element of $\mathcal{A} \cup x$ contains s otherwise the galaxy with arcs $(A(G) \triangle A(P)) \cup \{su\}$ spans $V(G) \cup x$ and contradicts the maximality of G. Similarly an element of $\mathcal{A} \cup x$ contains t.

Observe that the same element of $\mathcal{A} \cup x$ cannot contain both s and t (either the arc st or the arc ts), otherwise the arcs su and tu would be both backward or forward, which is impossible. □

At this stage, in order to apply Lemma 19, we just need to insure that the backward outdegree of every vertex is at most one. Since the only element of $\mathcal{D}$ which is the tail of two backward arcs is x, we simply delete any of these two backward arcs. The indegree of a vertex of $\mathcal{D}$ decreases by one but we are still fulfilling the hypothesis of Lemma 19.

Hence according to this lemma, $\mathcal{D}$ contains two arcs ca and bd such that $a \leq b \leq c$, $b \leq d$ and $c \not\leq d$. Keep in mind that a, b, c, d are elements of $\mathcal{A} \cup x$. In particular, there is an alternating path P containing a, b, d (in this order) which does not contain c. Setting $a = a_1a_2$ and $c = c_1c_2$, note that the backward arc ca corresponds to the arc c_1a_1 in D. We reach a contradiction by considering the galaxy with arcs $(A(G) \triangle A(P)) \cup \{c_1a_1\}$ which spans $V(D') \cup x$. □

5 Complexity

The digraphs with directed star arboricity 1 are the galaxies. So one can polynomially decide if $dst(D) = 1$. Deciding whether $dst(D) = 2$ or not is also easy since we just have to check that the conflict graph (with vertex set the arcs of D, two distinct arcs xy, uv being in conflict when $y = u$ or $y = v$) is bipartite. However for larger value, as expected, it is NP-complete to decide if a digraph has directed star arboricity at most k. This is illustrated by the next result:

Theorem 20 *The following problem is NP-complete:*
INSTANCE: A digraph D with $\Delta^+(D) \leq 2$ and $\Delta^-(D) \leq 2$.
QUESTION: Is $dst(D)$ at most 3?

Proof. The proof is a reduction to 3-edge-colouring of 3-regular graphs. To see this, consider a 3-regular graph G. It admits an orientation D such that every vertex has in and outdegree at least 1. Let D' be the digraph obtained from D by replacing every vertex with indegree 1 and outdegree 2 by the subgraph H depicted in Figure 2 which has also one entering arc (namely a) and two leaving arcs (b and c). It is easy to check that in any directed star 3-colouring of H, the three arcs a, b and c get different

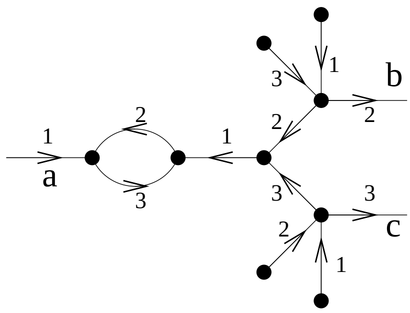

Figure 2: The graph H and one of its directed star 3-colouring

colours. Moreover if these three arcs are precoloured with three different colours, we can extend this to a directed star 3-colouring of H. Such a colouring with a coloured 1, b coloured 2 and c coloured 3 is given in Figure 2. Furthermore, a vertex with indegree 2 and outdegree 1 must have its three incident arcs coloured differently in a directed star 3-colouring. So $dst(D') = 3$ if and only if G is 3-edge colourable. □

6 Multiple fibers

In this section we consider the problem with $n \geq 2$ fibers. More precisely, we give some bounds on $\lambda_n(m,k)$. We first give a lower bound on $\lambda_n(m,k)$.

Proposition 21 $$\lambda_n(m,k) \geq \left\lceil \frac{m}{n} \left\lceil \frac{k}{n} \right\rceil + \frac{k}{n} \right\rceil$$

Proof. Consider the following m-labelled digraph $G_{n,m,k}$ with vertex set $X \cup Y \cup Z$ such that :

- $|X| = k$, $|Y| = 2^{(m+1)k^2}$ and $|Z| = m\binom{|Y|}{k}$.
- For any $x \in X$ and $y \in Y$ there is an arc xy (of whatever label).
- For every set S of k vertices of Y and integer $1 \leq i \leq m$, there is a vertex z_S^i in Z which is dominated by all the vertices of S via arcs labelled i.

Suppose there exists an n-fiber colouring of $G_{n,m,k}$ with $c < \lceil \frac{m}{n} \lceil \frac{k}{n} \rceil + \frac{k}{n} \rceil$ colours. For $y \in Y$ and $1 \leq i \leq m$, let $C_i(y)$ be the set of colours assigned to the arcs labelled i leaving y. For $0 \leq j \leq n$, let P_j the set of colours used on j arcs entering y (and necessarily with two differents fibers). Then $\sum_{j=0}^{n} j|P_j| = k$

as k arcs enter y. Moreover $(P_0, P_1, \dots, P_n)$ is a partition of the set of colours so $\sum_{j=0}^{n} |P_j| = c$. Now each colour of P_j may appear in at most $n-j$ of the $C_i(y)$, so

$$\sum_{i=1}^{m} |C_i(y)| \leq \sum_{j=0}^{n} (n-j)|P_j| = n\sum_{j=0}^{n} |P_j| - \sum_{j=0}^{n} j|P_j| = cn - k.$$

Because $|Y| = 2^{(m+1)k^2}$, there is a set S of k vertices y of Y having the same m-uple $(C_1(y), \dots, C_m(y)) = (C_1, \dots, C_m)$. Without loss of generality, we may assume $|C_1| = min\{|C_i| \mid 1 \leq i \leq m\}$. Hence $|C_1| \leq \frac{cn-k}{m}$. But the vertex z_S^1 has indegree k so $|C_1| \geq k/n$. Since $|C_1|$ is an integer, we have $\lfloor \frac{cn-k}{m} \rfloor \geq |C_1| \geq \lceil k/n \rceil$. So $c \geq \frac{m}{n} \lceil \frac{k}{n} \rceil + \frac{k}{n}$. Since c is an integer, we get $c \geq \lceil \frac{m}{n} \lceil \frac{k}{n} \rceil + \frac{k}{n} \rceil$, a contradiction. □

Note that the graph $G_{n,m,k}$ is acyclic. The following lemma shows that, if $m \geq n$, one cannot expect better lower bounds by considering acyclic digraphs. Indeed $G_{n,m,k}$ is the m-labelled acyclic digraph with indegree at most k for which an n-fiber colouring requires the more colours.

Lemma 22 *Let D be an acyclic m-labelled digraph with $\Delta^- \leq k$. If $m \geq n$ then $\lambda_n(D) \leq \lceil \frac{m}{n} \lceil \frac{k}{n} \rceil + \frac{k}{n} \rceil$.*

Proof. Since D is acyclic, its vertex set admits an ordering $(v_1, v_2, \dots, v_p)$ such that if $v_j v_{j'}$ is an arc then $j < j'$.

By induction on q, we shall find an n-fiber colouring of $D[\{v_1, \dots, v_q\}]$ together with sets $C_i(v_r)$, $1 \leq i \leq m$ and $1 \leq r \leq q$, of $\lceil k/n \rceil$ colours such that, in the future, assigning a colour in $C_i(v_r)$ to an arc labelled i leaving v_r will fullfill the condition of n-fiber colouring at v_r.

Starting the process is easy. We may take as $C_i(v_1)$ any $\lceil k/n \rceil$-sets such that a colour appears in at most n of them.

Suppose now that we have an n-fiber colouring of $D[\{v_1, \dots, v_{q-1}\}]$ and that, for $1 \leq i \leq m$ and $1 \leq r \leq q-1$, the set $C_i(v_r)$ is determined. Let us colour the arcs entering v_q. Each of these arcs $v_r v_q$ may be assigned one of the $\lceil k/n \rceil$ colours of $C_{l(v_r v_q)}(v_r)$. Since a colour may be assigned to n arcs (using different fibers) entering v_q, one can assign a colour and fiber to each such arc. It remains to determine the $C_i(v_q)$, $1 \leq i \leq m$.

For $0 \leq j \leq n$, let P_j be the set of colours assigned to j arcs entering v_q. Let $N = \sum_{i=0}^{n} (n-j)|P_j|$ and $(c_1, c_2, \dots, c_N)$ be a sequence of colours such that each colour of P_j appears exactly $n-j$ times and consecutively. For $1 \leq i \leq m$, set $C_i(v_q) = \{c_a \mid a \equiv i \mod m\}$. As $n \leq m$, a colour appears at most once in each $C_i(v_q)$. Moreover, $N = n \lceil \frac{m}{n} \lceil \frac{k}{n} \rceil + \frac{k}{n} \rceil - k \geq m \lceil \frac{k}{n} \rceil$. So for $1 \leq i \leq m$, $|C_i(v_q)| \geq \lceil \frac{k}{n} \rceil$. □

Lemma 22 gives a tight upper bound on $\lambda_n(D)$ for acyclic digraphs. We shall prove an upper bound for general digraphs. To do so, we fisrt give an upper bound on $\lambda_n(D)$ for m-labelled digraphs with bounded in- and outdegre In this case, oone can derive from the following theorem of Guiduli that "few" colours are needed. Note that the graphs $G_{n,m,k}$ requires lots of colours but have very large outdegree.

Theorem 23 (Guiduli [6]) *If $\Delta^-, \Delta^+ \leq k$ then $dst(D) \leq k + 20\log k + 84$. Moreover D admits a directed star colouring with $k + 20\log k + 84$ colours such that for each vertex v there are at most $10\log k + 42$ colours assigned to its leaving arcs.*

The proof of Guiduli's Theorem can be modified to obtain the following statement for m-labelled digraphs.

Theorem 24 *Let $f(n,m,k) = \left\lceil \frac{k + (10m^2+5)\log k + 80m^2 + m + 21}{n} \right\rceil$ and D be an m-labelled digraph with $\Delta^-, \Delta^+ \leq k$. Then $\lambda_n(D) \leq f(n,m,k)$. Moreover D admits a n-fiber colouring with $f(n,m,k)$ such that for each vertex v and each label l, there are at most $g(m,k) = \lceil (10m+5)\log k + 40m + 21 \rceil$ colours assigned to the arcs labelled l leaving v.*

Note that Theorem 24 in the case $n = m = 1$ is a bit better than Theorem 23. Indeed it shows that if $\Delta^-, \Delta^+ \leq k$ then $dst(D) \leq k + 15\log k + 102$. It is due to Lemma 7 which is a bit better than Guiduli's one because it uses Theorem 7 ($dst \leq 2\Delta^- + 1$) whereas Guiduli uses $dst \leq 3\Delta^-$. However the method is identical.

Definition 25 Given a family of sets $\mathcal{F} = (A_i, i \in I)$ A *transversal* of $\mathcal{F}$ is a family of distinct elements $(t_i, i \in I)$ with $\forall i, t_i \in A_i$.

Lemma 26 *Let D be a m-labelled digraph with $\Delta^- \leq k$. Suppose that for each vertex v, there are m disjoint lists $L_v^1, ..., L_v^m$ of c colours each being a subset of $\{1, ..., k+c\}$. If for each vertex v, the family $\{L_y^i \mid yx \in E(D)$ and yx is labelled $i\}$ has a transversal, then there is a 1-fiber colouring of D with $k + (2m^2 + 1)c + m$ colours such that for each vertex v and label l, at most $(2m + 1)c + 1$ colours are assigned to arcs labelled l leaving v.*

Proof. Using the transversal to colour the entering arcs at each vertex, we obtain a colouring with few conflicts. Indeed there is no conflict between arcs entering a same vertex. So the only possible conflict are between an arc entering a vertex v and an arc leaving v. Since arcs leaving a vertex v use at most $m.c$ colours (those of $L_v^1 \cup ... \cup L_v^m$), there are at most $m.c$ arcs entering v having the same colour as an arc leaving c. Removing such entering arcs for every vertex v, we obtain a digraph D' for which the colouring with the $k+c$ colours is a 1-fiber colouring. We now want to colour the arcs of $D - D'$ with few extra colours. Consider a label $1 \leq l \leq m$, let D'_l be the digraph induced by the arcs of $D - D'$ labelled l. Then D'_l has indegree at most $m.c$. So by Theorem 7, we can partition D'_l in $2m.c+1$ star forests. Thus D can be 1-fiber coloured with $k + c + m(2m.c + 1)$ colours. Moreover, in the above described colouring, arcs labelled l leaving a vertex v have a colour in L_v^l or corresponding to one of the $2m.c + 1$ star forests of D'_l. So at most $(2m + 1)c + 1$ colours are assigned to arcs labelled l leaving v. □

Theorem 27 (N. Alon, C. McDiarmid, B. Reed, 1992 [2]) *Let k and c be positive integers with $k \geq c \geq 5\log k + 20$. Choose independent random subsets $S_1, \ldots, S_k$ of $X = \{1, \ldots, k + c\}$ as follows. For each i choose S_i by performing c independent uniform samplings from X. Then the probability that $S_1, \ldots, S_k$ do not have a transversal is at most $k^{3-\frac{c}{2}}$*

Proof of Theorem 24. It suffices to prove the result for $n = 1$. Indeed a 1-fiber colouring satisfying the conditions of the theorem to an n-fiber colouring satisfying the conditions by replacing the colour $qn + r$ with $1 \leq r \leq n$ by the colour $q + 1$ on fiber r.

Let $c = \lceil 5\log k + 20 \rceil$ We can assume $k \geq m.c$. For all vertices x, select $m.c$ different ordered elements $e_1, e_2, \cdots, e_{m.c}$ independently and uniformly. For all $1 \leq i \leq m$, let $L_x^i = \{e_{ci+1}, \cdots, e_{c(i+1)}\}$. Each set has the same distribution as the c elements where chosen uniformly and independently.

Let A_x be the event that the family $\{L_y^i \mid yx \in E(D)$ and yx is labelled $i\}$ fails to have a transversal. By Theorem 27, $P(A_x) \leq k^{3-c/2}$. Furthermore, the event A_x is independent of all A_y for which there is no vertex z such that both zx and zy are in $E(D)$. The dependency graph for the events has degree at most k^2 so we can apply Lovász Local Lemma. We obtain that there exists a family of lists satisfying conditions of Lemma 26. This lemma gives the desired colouring. □

Any digraph D may be decomposed into an acyclic digraph D_a and an *eulerian* digraph D_e, that is such that for every vertex v, $d^-_{D_e}(V) = d^+_{D_e}(v)$. Indeed consider an eulerian subdigraph D_e of D which has a maximum number of arcs. Then the digraph $D_a = D - D_e$ is necessarily acyclic. Hence by Theorems 24 and 22, if $m \geq n$ then $\lambda_n(D) \leq \lceil \frac{m}{n} \lceil \frac{k}{n} \rceil + \frac{k}{n} \rceil + f(n, m, k)$. But we will now lessen this bound by roughly $\frac{k}{n}$.

Theorem 28 *If $n \leq m$, then*

$$\lambda_n(m,k) \leq \left\lceil \frac{m}{n}\left\lceil \frac{k}{n}\right\rceil + \frac{k}{n}\right\rceil + 2m\frac{\lceil (10m+5)\log k + 40m + 21\rceil}{n}.$$

Proof. Let D be an m-labelled digraph with $\Delta^-(D) \leq k$. Consider a decomposition of D into an eulerian digraph D_e and an acyclic digraph D_a. We first apply Theorem 24 and n-fiber colour the arcs of D_e with $f(n,m,k)$ colours such that at most $g(m,k)$ colours are assigned to the arcs leaving each vertex.

We shall extend the n-fiber colouring of D_e to the arcs of D_a in a way similar to the proof of Lemma 22, i.e. we will assign to each vertex v sets $C_i(v)$, $1 \leq i \leq m$ of $\lceil k/n + m.g(m,k)\rceil$ colours such that an arc labelled i leaving v will be labelled using a colour in $C_i(v)$.

Let $(v_1,\ldots,v_n)$ be an ordering of the vertices of A such that if $v_jv_{j'}$ is an arc then $j < j'$.

We start to build the $C_i(v_1)$ with the colours assigned to the leaving arcs of v_1 labelled i. The vertex v_1 has at most k entering arcs. Each of them forbid one type (colour, fiber). In the colouring of D_e induced by Theorem 24, there are at most $m.g(m,k)$ types assigned to the arcs leaving v_1. So there are at least $\lceil \frac{mk}{n^2}\rceil + \lceil \frac{k}{n}\rceil + 2m\frac{g(m,k)}{n} - k - m.g(m,k) \geq m\lceil \frac{k}{n}\rceil + m.g(m,k)$ types unused at vertex v_1. Since $n \leq m$, we can partition these types into m sets of size at least $\frac{k}{n}$ such that no two types having the same colour are in the same set. These sets are the $C_i(v_1)$.

Suppose that the sets have been defined for v_1 up to v_{q-1} and that all the arcs v_iv_j for $i < q$ and $j < q$ have a colour. We now give a colour to the arcs of type v_iv_q for $i < q$.

There are k_e arcs entering v_q in D_e which are already coloured. So it remains to give a colour to $k_a \leq k - k_e$ arcs. Each uncoloured arc may be assigned a colour in a list of size at least $\lceil \frac{k}{n} + m.g(m,k)\rceil$. This gives a choice between $n.\lceil \frac{k}{n} + m.g(m,k)\rceil$ different types. k_e types are forbidden by the entering arcs in D_e while at most $m.g(m,k)$ types are forbidden by the leaving arcs in D_e. Then it remains at least $n.\lceil \frac{k}{n} + m.g(m,k)\rceil - k_e - m.g(m,k) \geq k_a$ types for each entering arcs of D_a. So one can assign distinct available colours to each of the k_a arcs entreing v_q.

We then build the $C_i(v_q)$ similarly as for v_1.

This process finished, we obtain an n-fiber colouring of D using $\lceil \frac{mk}{n^2}\rceil + \lceil \frac{k}{n}\rceil + 2m\frac{g(m,k)}{n}$ colours. □

Theorem 28 gives an upper bound on $\lambda_n(m,k)$ when $m \geq n$. We now give an upper bound when $m < n$.

Proposition 29 *If $m < n$ then $\lambda_n(m,k) \leq \left\lceil \frac{k}{n-m}\right\rceil$.*

Proof. Let D be an m-labelled digraph with $\Delta^- \leq k$. For each vertex v, we give to its entering arcs a colour such that none of them is used more than $n-m$ times. This is possible as there are at most $k \leq (n-m)\left\lceil \frac{k}{n-m}\right\rceil$ arcs entering v. Then we have $in(v,\lambda) \leq n-m$. Moreover each arc vw is given a colour by w. Since D is m-labelled, a colour λ can be used to colour an arc of at most m different labels, i.e. $out(v,\lambda) \leq m$. Consequently $in(v,\lambda) + out(v,\lambda) \leq n$. This give a proper n-fiber colouring. □

References

[1] I. Algor and N. Alon, The star arboricity of graphs, *Discrete Mathematics* **75** (1989), 11–22.

[2] N. Alon, C. McDiarmid, B. Reed, Star arboricity, *Combinatorica* **12** (1992), 375–380.

[3] R. Brandt, Multicasting using WDM in Multifiber Optical Star Networks, Thesis, UCSB, September 2003.

[4] R. Brandt and T. F. Gonzalez, Wavelength assignment in multifiber optical star networks under the multicasting communication mode, *Journal of Interconnection Networks* **6** (2005), 383–405.

[5] A. Frank, Covering branchings, *Acta Sci. Math. (Szeged)* **41** (1979), 77–81.

[6] B. Guiduli, On incidence coloring and star arboricity of graphs, *Discrete Mathematics* **163** (1997), 275–278.

[7] A. Lardies, R. Gupta and R. Patterson. Traffic grooming in a multi-layer network. *Opt. Netw. Mag.*, **2** (2001), 91–99.

[8] E. Modiano and P.J. Lin. Traffic grooming in WDM networks. *IEEE Communications Magazine*, **39**(7), (2001), 124–129.

[9] A. Pinlou and E. Sopena, The acircuitic directed star arboricity of subcubic graph is at most four, *Discrete Mathematics*, to appear.

[10] A. Schrijver, Combinatorial optimization. Polyhedra and efficiency. *Algorithms and Combinatorics* 24, Springer-Verlag, Berlin, 2003.

7 Appendix

7.1 Alternative proof of Theorem 14

In this subsection, we give an alternative proof of Theorem 14 using Theorem 11.

Proof. As every subcubic digraph is the subgraph of a cubic digraph, its suffices to prove the result for cubic digraph. So let D be a cubic digraph. By Theorem 11, it admits a directed star 3-colouring c of D.

We will now select one arc e_C per bicoloured circuit C and recolour it with 4. If C is dominating (no arcs enters C) or is dominated (no arcs leaves C) let e_C be any arc of C, otherwise let e_C be an arc whose tail has indegree 2 and head outdegree 2.

It is easy to see that the set of arcs now coloured 4 is a matching. Note moreover that the arcs incident to an arc uv coloured 4 have there colours in the set of two colours, the one of $\{1,2,3\}\setminus\{c\}$ where c is the colour initially assigned to uv.

The way we have selected the recoloured arcs assures us we have a 4-directed star colouring. Moreover, there is no circuits bicoloured with two colours in $\{1,2,3\}$. However, we may have bicoloured circuits with arcs coloured 4, so we need to do an extra recolouring. Note that the arcs coloured 4 of such circuits were not selected for dominating or dominated circuits.

An arc uv coloured 4 may be in at most 2 bicoloured circuit and that if it is in two such cicuits one is bicoloured 4 and c_1 and the other 4 and c_2 with $c_1 \neq c_2$. In particular, the two circuits enter u and leave v with different arcs.

Now for each bicoloured circuit C, we *choose* an arc f_C coloured 4 in order to maximizes the number of arcs chosen by two circuits.

Recolour each arc uv that has been chosen by two circuits, say C_1 and C_2, bicoloured 4 and c_1 and 4 and c_2 respectively, with the colour $c_3 \in \{1,2,3\}\setminus\{c_1,c_2\}$. Doing so we still have a directed star colouring and the cicuit C_1 and C_2 are no more bicoloured. Moreover, we do not create any new bicoloured circuit : indeed, a circuit C containing the arc uv must go either through the one of the two arcs vw_i of C_i, for $i \in \{1,2\}$, which is coloured c_i. But then w_i has outdegree one and the arc leaving w_i is in both C and C_i and so colour 4. Hence the three colours c_3, c_i and 4 appears on C.

We will now recolour an arc of each remaining bicoloured circuit C one after another. Such a recolouring will make C no more bicoloured and create no new bicoloured circuits. Hence at the end, we wil have a 4-acircuitic directed star colouring of D.

Let us consider such a circuit C. Its chosen arc uv has been chosen only once. Note that uv does not belong to any other such circuit since the number of arc chosen twice is maximized. In particular, all the arcs incident to uv have not been recoloured. Let tu be the arc preceding uv in C. Recolour tu with the colour c_3 in $\{1,2,3\} \setminus \{c_1, c_2\}$. This is valid since u is the head of an arc coloured 4 and thus has outdegree 2. Moreover, this does not creates any bicoloured circuit since all the circuits containing tu must contain one of the arcs leaving v which are both coloured in $\{c_1 c_2\}$.

□

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