EXPLICIT FORMULAS AND MEROMORPHIC EXTENSION OF BESSEL FUNCTIONS ON TANGENT SPACES TO NONCOMPACTLY CAUSAL SYMMETRIC SPACES

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Abstract. Assume that G/H is a noncompactly causal symmetric space with restricted root system of the non-exceptional type and the multiplicity of the short roots is even. Using shift operators we obtain explicit formulas for the Bessel function on the tangent space to G/H at the origin. This enable us to investigate the nature and order of the singularities of the Bessel function, and to formulate a conjecture on this matter.

1. Introduction

Opdam shift operators are multivariable generalizations of the identity

$$\frac{d}{dz}\,{}_2F_1(a,b;c;z) = \frac{ab}{c}\,{}_2F_1(a+1,b+1;c+1;z).$$

The introduction of the concept of shift operators goes back to Koorwinder in the context of orthogonal polynomials in two variables for the root system BC_2 [15]. In his thesis [20, 21], Opdam established the existence of such operators in the setting of hypergeometric functions associated with root systems. Later Heckman gave an effective construction of the shift operators [12]. We should mention the partial contributions made in [23, 24, 3].

Since then, the shift operators found several applications in Harmonic analysis, in the theory of multivariable special functions associated with finite Coxeter groups, and in the integrability of Hamiltonian systems. In a series of papers by Heckman and Opdam, the authors used the shift operators to prove the existence of what is nowadays called the Heckman-Opdam hypergeometric functions. This direction generalizes Harish-Chandra's theory of spherical functions on noncompact Riemannian symmetric spaces. In particular, the shift operators helped to search for explicit formulas for the spherical functions on certain Riemannian symmetric spaces. See e.g. [8, 24, 3, 26].

Another direction has been attempted to use the shift operators to obtain explicit formulas for the spherical functions on noncompact causal symmetric spaces with even multiplicities. This was done by Òlafsson and Pasquale in [18].

Wile the theory of spherical functions has been pursed for a long time, the growing interest in the theory of Bessel functions, either on flat symmetric spaces or associated with arbitrary root systems, is comparably recent. The present paper continue the investigation of the Bessel functions on flat symmetric spaces of the noncompact causal type begun in [6]. Here the list of causal symmetric spaces is larger than the one in [6]. By means of a rational shift operators, we give explicit formulas for the Bessel functions and we investigate their meromorphic extension (in the spectral parameter). These are the main new results of the paper. Further, we formulate a conjecture about the nature and order of the singularities (in the spectral parameter) of the Bessel functions.

To be more precise, let G/H be a noncompact causal symmetric space (NCC for abbreviation). That is G/H is a non-Riemannian symmetric space characterized by the existence of a certain maximal $\mathrm{Ad}(H)$-invariant convex cone $C_{\max}$ in its tangent space $\mathfrak{q}$ at the base point $\{H\}$. For the corresponding Lie algebra, we have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ so the following direct sum $\mathfrak{g} = (\mathfrak{h} \cap \mathfrak{k}) \oplus (\mathfrak{h} \cap \mathfrak{p}) \oplus (\mathfrak{q} \cap \mathfrak{k}) \oplus (\mathfrak{q} \cap \mathfrak{p})$ holds.

Fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p} \cap \mathfrak{q}$. Then $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$ and in $\mathfrak{q}$. Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$ be the restricted root system with respect to $\mathfrak{a}$. It is well known that for every NCC symmetric space Σ is reduced. Set Σ_c and Σ_n to be the set of compact and noncompact roots in Σ. We will denote the Weyl groups for Σ and Σ_c by W and W_0 respectively.

By analyzing a deformation of Faraut-Hilgert-Òlafsson' spherical functions on G/H, it is shown in [6] that for all λ in a specific set $\widetilde{\mathscr{E}}$, the integral representation of the Bessel functions on $\mathfrak{q}$ is given by

$$\Psi_\lambda(m, X) = \int_H e^{-\lambda(\mathscr{P}(\mathrm{Ad}(h)X))} dh, \qquad X \in C^0_{\max},$$

where $\mathscr{P} : \mathfrak{q} \to \mathfrak{a}$ denotes the orthogonal projection. The parameter m refers to the multiplicity function on Σ. See the next section for more details.

Let $\mathscr{O}_s$ be the orbit of the short roots in Σ. Let m_l and m_s be the multiplicities of the long and the short roots in Σ respectively. Henceforth, we will assume that Σ is a non-exceptional root system such that m_s is even. The integer m_l can be either even or odd. See Section 4 for the list of symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ which obey our assumptions. To simplify the presentation of the results, we denote the elements of $\mathfrak{a}$ and $\mathfrak{a}^*_{\mathbb{C}}$ by vectors of length $r := \dim(\mathfrak{a})$. In these circumstances, we prove that for $t \in \mathfrak{a}_-$ and $\lambda \in \widetilde{\mathscr{E}} \cap \{\lambda \in \mathbb{C}^r : \ (\forall \alpha \in \Sigma^+)\ \alpha(\lambda) \neq 0\}$, we have

$$\begin{aligned}\Psi_\lambda((m_l, m_s), t) =& (-1)^{\frac{m_s}{2}|\Sigma_c^+ \cap \mathscr{O}_s|}\, 2^{m_s|\Sigma^+ \cap \mathscr{O}_s|} \prod_{\alpha \in \Sigma^+ \cap \mathscr{O}_s} \langle \lambda, \alpha \rangle^{-m_s} \\ & \mathbb{G}(2 \cdot 1_{\mathscr{O}_s}, (m_l, m_s - 2)) \circ \cdots \circ \mathbb{G}(2 \cdot 1_{\mathscr{O}_s}, (m_l, 0)) \widetilde{\Psi}_\lambda((m_l, 0), t),\end{aligned} \tag{F}$$

where

$$\widetilde{\Psi}_\lambda((m_l, 0), t) = \pi^{-r/2} \sum_{w \in W_0} \prod_{i=1}^r (2t_i \omega(\lambda_i))^{-\frac{m_l}{2} + \frac{1}{2}} K_{\frac{m_l}{2} - \frac{1}{2}}(t_i \omega(\lambda_i)).$$

Here K_ν denotes the Bessel function of the third kind, and $\mathbb{G}(2 \cdot 1_{\mathscr{O}_s}, (m_l, a))$ is the operator that shifts the pair (m_l, a) to $(m_l, a+2)$. In section 3 we recall the construction of the operators $\mathbb{G}$ from [6]. The shift operator $\mathbb{G}$ appeared for the first time in [22] from a different point of view. Formula (F) generalizes [6], since here we do not restrict m_l to be even. In view of (F) we give for several NCC symmetric pairs explicit formulas for the Bessel functions on $\mathfrak{q}$. This study allows to investigate the nature and order of the singularities of $\Psi_\lambda((m_l, m_s), t)$ in the λ-variable. Motivated by this investigation and by a hand made computation for the symmetric pair $(\mathfrak{su}^*(6), \mathfrak{sp}(2, 1))$, we conjecture that for every NCC symmetric pair $(\mathfrak{g}, \mathfrak{h})$ such that Σ is a non-exceptional root system and such that $m_s \in 2\mathbb{N}$, the function $\lambda \mapsto \Psi_\lambda((m_l, m_s), t)$, for $t \in \mathfrak{a}_-$, extends to a meromorphic function in the domain

$$\mathfrak{D} := \{\lambda \in \mathbb{C}^r : \ (\forall 1 \leq i \leq r)\ \lambda_i \mapsto \lambda_i^{-\frac{m_l}{2} + \frac{1}{2}} K_{\frac{m_l}{2} - \frac{1}{2}}(\lambda_i) \text{ is analytic}\}$$

with poles for

$$\begin{cases} \langle\lambda,\alpha\rangle = 0 \quad (\forall\alpha\in\Sigma_c^+), & \text{of order} = m_c - 2,\\ \langle\lambda,\alpha\rangle = 0 \quad (\forall\alpha\in\Sigma_n^+), & \text{of order} = m_n - 1.\end{cases}$$

Here m_c and m_n denotes the multiplicities of the compact and noncompact roots respectively.

Explicit formulas of $\Psi_\lambda((m_l, m_s), t)$ have also significant applications in inverting the "flat" analogue of the Abel transform on $\mathfrak{q}$, and in the proof of the Paley-Wiener theorem for the Bessel Laplace transform defined in terms of integrating against $\Psi_\lambda((m_l, m_s), t)$. See [7] for the symmetric pair $(\mathfrak{su}(r,r), \mathfrak{sl}(r,\mathbb{C})\times\mathbb{R})$. We shall investigate these applications in a forthcoming paper.

In the light of formula (F) and [6, Lemma 4.6], we deduce that if $\Phi_\lambda((m_l, m_s), t)$ denotes the Bessel function on $\mathfrak{p}$ associated with the Riemannian symmetric pair $(\mathfrak{g}, \mathfrak{k})$, then

$$\begin{aligned}\Phi_{-\lambda}((m_l, m_s), t) &= c(m) \prod_{\alpha\in\Sigma^+\cap\mathscr{O}_s} \langle\lambda,\alpha\rangle^{-m_s}\\ &\qquad \mathbb{G}(2\cdot 1_{\mathscr{O}_s}, (m_l, m_s-2))\circ\cdots\circ\mathbb{G}(2\cdot 1_{\mathscr{O}_s}, (m_l, 0))\widetilde{\Phi}_\lambda((m_l, 0), t),\end{aligned}$$

where

$$\widetilde{\Phi}_\lambda((m_l, 0), t) = \pi^{-r/2}\sum_{\omega\in W}\prod_{i=1}^{r}(2t_i\omega(\lambda_i))^{-\frac{m_l}{2}+\frac{1}{2}} K_{\frac{m_l}{2}-\frac{1}{2}}(t_i\omega(\lambda_i)),$$

and $c(m)$ is a constant which depends only on m. Now a similar procedure to the one that we will use in Section 4 for Ψ_λ gives explicit formulas for Φ_λ. This result covers in a unified way the explicit formulas obtained in [2, 17, 26, 5] for certain Riemannian symmetric pairs $(\mathfrak{g}, \mathfrak{k})$ or for certain values of r.

2. **Spherical functions on non-compactly causal symmetric spaces**

This section collects basic notation and some known results that will be needed later on. For a more detailed exposition we refer to [11, 14, 19].

Let G be a connected noncompact semisimple Lie group equipped with a non-trivial involution σ. Denote by H an open subgroup of G satisfying $G_0^\sigma \subset H \subset G^\sigma$, where G^σ is the set of σ-fixed points of G. The quotient G/H is called a symmetric space. Let θ be a Cartan involution on G commuting with σ, and set $K := G^\theta = \{g \in G \;:\; \theta(g) = g\}$. Then K is a maximal compact in G.

Denote by $\mathfrak{g}$ the Lie algebra of G. The involutions of $\mathfrak{g}$ corresponding to σ and θ will be denoted by the same letters. Let $\mathfrak{g} = \mathfrak{k}\oplus_\theta\mathfrak{p} = \mathfrak{h}\oplus_\sigma\mathfrak{q}$ be the decomposition of $\mathfrak{g}$ according to the eigenvalues ±1 of the involutions θ and σ respectively. Note that $\mathfrak{h} = \mathrm{Lie}(H)$ and $\mathfrak{k} = \mathrm{Lie}(K)$. We view $\mathfrak{q}$ as a real euclidean vector space with the inner product

$$(X|Y)_\theta := -B(X, \theta(Y)),$$

where B denotes the Killing form of $\mathfrak{g}$.

In this paper we will always assume that G/H (or equivalently the pair $(\mathfrak{g}, \mathfrak{h})$) is irreducible. That is the only σ-invariant ideals in $\mathfrak{g}$ are the trivial ones. An irreducible symmetric space G/H is said to be causal if there exists a non-empty open $\mathrm{Ad}(H)$-invariant convex cone C in $\mathfrak{q}$, containing no affine lines. By a result due to Vinberg, this is equivalent to the existance of a non-zero $\mathrm{Ad}(K\cap H)$-invariant vector z_0 in $\mathfrak{q}$. If in addition $z_0 \in \mathfrak{p}$, then G/H is called a noncompactly causal symmetric space (NCC for abbreviation). We may normalize the vector z_0 so that the eigenvalues of $\mathrm{ad}(z_0)$ are 0, 1, -1.

Henceforth we will assume that G/H is a NCC symmetric space. We refer to [14, p. 89] for the complete list of NCC symmetric spaces.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}\cap\mathfrak{q}$, and set $r := \dim(\mathfrak{a})$. Note that the NCC structure implies that $\mathfrak{a}$ is also maximal abelian in $\mathfrak{p}$ and in $\mathfrak{q}$. Moreover $z_0 \in \mathfrak{a}$. Let $\Sigma = \Sigma(\mathfrak{g},\mathfrak{a}) \subset \mathfrak{a}^*$ be the restricted root system with respect to $\mathfrak{a}$. It is well known that for every NCC symmetric space, Σ is reduced, i.e. if $\alpha \in \Sigma$ then $2\alpha \notin \Sigma$. Further, Σ can be written as $\Sigma = \Sigma_n^- \cup \Sigma_c \cup \Sigma_n^+$, where

$$\Sigma_n^{\pm} := \{\alpha \in \Sigma : \ \alpha(z_0) = \pm 1\}, \qquad \Sigma_c := \{\alpha \in \Sigma : \ \alpha(z_0) = 0\}.$$

The set Σ_c is the set of compact roots while $\Sigma_n^+ \cup \Sigma_n^-$ is the set of noncompact roots. The Weyl groups associated with Σ and Σ_c will be denoted by W and W_0 respectively.

Choose a positive system $\Sigma_c^+ \subset \Sigma_c$ and set $\Sigma^+ := \Sigma_n^+ \cup \Sigma_c^+$. Then Σ^+ is a system of positive roots in Σ. Define the convex cone in $\mathfrak{a}$ by

$$c_{\max} = \{X \in \mathfrak{a} : \ (\forall \alpha \in \Sigma_n^+)\ \alpha(X) \le 0\},$$

and let $\mathfrak{n} := \sum_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$ be the sum of the root spaces corresponding to the positive roots. The analytic subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$ will be denoted by the capital letters.

Let $S = H\exp(c_{\max}^{\circ})H$ be the H-bi-invariant semigroup in G. By [11], $S \subset NAH$. For $x \in NAH$, we will write $x = n\exp(A(x))h$ with $n \in N$, $A(x) \in \mathfrak{a}$, and $h \in H$.

For $\alpha \in \Sigma$, let $m_\alpha := \dim(\mathfrak{g}_\alpha)$, and denote by $m : \Sigma \to \mathbb{N}$, $\alpha \mapsto m_\alpha$ the W-invartiant multiplicity function on Σ. Define $\mathscr{E}$ to be the set of $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ for which the function $h \mapsto e^{\langle \rho(m)-\lambda, A(hx)\rangle}$ is integrable over H. Here $\rho(m) := 1/2\sum_{\alpha\in\Sigma^+} m_\alpha \alpha$. In [16] the authors show that

$$\mathscr{E} = \{\lambda \in \mathfrak{a}_{\mathbb{C}}^* : \ (\forall \alpha \in \Sigma_n^+)\ \operatorname{Re}\lambda(H_\alpha) < 2 - m_\alpha\},$$

where $H_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] \subset \mathfrak{a}$ so that $\alpha(H_\alpha) = 2$.

For $\lambda \in \mathscr{E}$, let $\varphi_\lambda(m,\cdot)$ be the function defined on S by

$$\varphi_\lambda(m,s) = \int_H e^{\langle \rho(m)-\lambda, A(hs)\rangle}\, dh. \tag{2.1}$$

By [11], the convergence of the integral defining $\varphi_\lambda(m,s)$ depends on λ and not on s. Clearly $\varphi_\lambda(m,\cdot)$ is an $\mathrm{Ad}(H)$-invariant function, and it is well known that $\varphi_\lambda(m,\cdot)$ is an eigenfunction of all G-invariant differential operators on G/H. The function $\varphi_\lambda(m,\cdot) : S \to \mathbb{C}$ is the so-called spherical function on G/H.

Define the open Weyl chamber $\mathfrak{a}_- \subset \mathfrak{a}$ by

$$\mathfrak{a}_- = \{X \in \mathfrak{a} : \ (\forall \alpha \in \Sigma^+)\ \alpha(X) < 0\},$$

and denote by A_- the analytic subgroup of G with Lie algebra $\mathfrak{a}_-$. Notice that A_- is a subset of $S \cap A = \mathrm{Int}(\overline{W_0 A_-})$. In [19] the author proves that $\varphi_\lambda(m,\cdot)$ admits on A_- a Harish-Chandra type expansion. More precisely, for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, define the following c-functions

$$c_0(m,\lambda) := \frac{\tilde{c}_0(m,\lambda)}{\tilde{c}_0(m,\rho(m))}, \qquad c_1(m,\lambda) := \frac{\tilde{c}_1(m,\lambda)}{\tilde{c}_1(m,\rho(m))},$$

where

$$\tilde{c}_0(m,\lambda) = \prod_{\alpha\in\Sigma_c^+} \frac{\Gamma\left(\frac{\lambda(H_\alpha)}{2}\right)}{\Gamma\left(\frac{\lambda(H_\alpha)}{2}+\frac{m_\alpha}{2}\right)},$$

and

$$\tilde{c}_1(m,\lambda) = \prod_{\alpha\in\Sigma_n^+} \frac{\Gamma\left(-\frac{\lambda(H_\alpha)}{2}-\frac{m_\alpha}{2}+1\right)}{\Gamma\left(-\frac{\lambda(H_\alpha)}{2}+1\right)}.$$

For $\lambda \in \mathscr{E}$ generic and $a \in A_-$, we have the following expansion formula

$$\varphi_\lambda(m,a) = c_1(m,\lambda)\sum_{w\in W_0} c_0(m,w\lambda)\Phi_{w\lambda}(m,a), \tag{2.2}$$

where

$$\Phi_\lambda(m,a) = a^{\rho(m)-\lambda}\sum_{\mu\in\mathbb{N}[\Sigma^+]}\Gamma_\mu(m,\lambda)a^\mu, \qquad a\in A_-,$$

with $\Gamma_0(m,\lambda) = 1$ and $\Gamma_\mu(m,\lambda) \in \mathbb{C}$ are obtained by means of a recurrence formula. In the light of (2.8), the function $\lambda \mapsto \varphi_\lambda(m,a)$, for $a \in A_-$, extends to a meromorphic function in $\mathfrak{a}^*_\mathbb{C}$. For more details on the theory of spherical functions for NCC symmetric spaces we refer to [11, 19].

3. **Opdam's shift operators**

For fixed $H \in \mathfrak{a}$, the trigonometric Dunkl-Heckman differential-difference operator $T(H,m)$ is defined by

$$T(H,m) = \partial(H) + \sum_{\alpha\in\Sigma^+}\frac{m_\alpha}{2}\alpha(H)\coth\alpha\otimes(1-r_\alpha),$$

where $\partial(H)$ denotes the directional derivative in the direction of H, and

$$r_\alpha(\lambda) := \lambda - \lambda(H_\alpha)\alpha, \qquad \lambda\in\mathfrak{a}^*.$$

Let $\mathscr{O}\subset\Sigma$ be an orbit of the Weyl group W in Σ. The orbit $\mathscr{O}$ defines the signature $\varepsilon_\mathscr{O} : W \to \{\pm 1\}$ given by $\varepsilon_\mathscr{O}(r_\alpha) = -1$ for $\alpha\in\mathscr{O}$ and $\varepsilon_\mathscr{O}(r_\alpha) = 1$ otherwise.

Set $\mathscr{O}^+ := \mathscr{O}\cap\Sigma^+$, and write

$$\pi_{\mathscr{O}^+}(X) = \prod_{\alpha\in\mathscr{O}^+}\operatorname{sh}\alpha(X), \qquad X\in\mathfrak{a}.$$

For any $H \in \mathfrak{a}$, Opdam's elementary shift operators associated with an orbit $\mathscr{O}$ are defined by

$$G(2\cdot 1_\mathscr{O}, m) = \operatorname{Res}\Big[\pi^{-1}_{\mathscr{O}^+}\sum_{w\in W}\varepsilon_\mathscr{O}(w)T(wH,m)^{|\mathscr{O}^+|}\Big],$$

where $1_\mathscr{O} : \Sigma \to \{0,1\}$ denotes the characteristic function of $\mathscr{O}$, and Res is the operation that consists of placing all the reflection terms to the right and then restrict the operator onto the space of W-invariant functions. The operator $G(2\cdot 1_\mathscr{O}, m)$ satisfies

$$G(2\cdot 1_\mathscr{O}, m)\Phi_\lambda(m,a) = \frac{\tilde{c}(m,-\lambda)}{\tilde{c}(m+2\cdot 1_\mathscr{O},-\lambda)}\Phi_\lambda(m+2\cdot 1_\mathscr{O},a), \tag{3.1}$$

where

$$\tilde{c}(m,\lambda)=\prod_{\alpha\in\Sigma^+}\frac{\Gamma\left(\frac{\lambda(H_\alpha)}{2}\right)}{\Gamma\left(\frac{\lambda(H_\alpha)}{2}+\frac{m_\alpha}{2}\right)},\tag{3.2}$$

and $m+2\cdot 1_{\mathscr{O}}:\Sigma\to\mathbb{N}$ denotes the multiplicity function defined by $m+2\cdot 1_{\mathscr{O}}(\alpha)=m_\alpha+2$ for $\alpha\in\mathscr{O}$ and $m+2\cdot 1_{\mathscr{O}}(\alpha)=m_\alpha$ otherwise. In the light of (3.3) and the Harish-Chandra expansion (2.8), one can check that for $\lambda\in\mathscr{E}$ generic and $a\in A_-$

$$G(2\cdot 1_{\mathscr{O}},m)\varphi_\lambda(m,a)=\frac{c_{G/H}(m,\lambda)}{c_{G/H}(m+2\cdot 1_{\mathscr{O}},\lambda)}\frac{\tilde{c}(m,-\lambda)}{\tilde{c}(m+2\cdot 1_{\mathscr{O}},-\lambda)}\varphi_\lambda(m+2\cdot 1_{\mathscr{O}},a),\tag{3.3}$$

where

$$c_{G/H}(m,\lambda):=c_0(m,\lambda)c_1(m,\lambda).\tag{3.4}$$

For all $\zeta>0$ and $\mathscr{O}\subset\Sigma$, let $G^{(\zeta)}(2\cdot 1_{\mathscr{O}},m)$ be the deformed elementary shift operator

$$\mathrm{Res}\Big[\pi_{\zeta,\mathscr{O}^+}^{-1}\sum_{w\in W}\varepsilon_{\mathscr{O}}(w)T^{(\zeta)}(wH,m)^{|\mathscr{O}^+|}\Big],$$

where

$$\pi_{\zeta,\mathscr{O}^+}(X)=\prod_{\alpha\in\mathscr{O}^+}\operatorname{sh}\zeta\alpha(X),\qquad X\in\mathfrak{a},$$

and

$$T^{(\zeta)}(H,m)=\frac{1}{\zeta}\partial(H)+\sum_{\alpha\in\Sigma^+}\frac{m_\alpha}{2}\alpha(H)\sum_{\kappa=0}^{\infty}B_{2\kappa}\frac{2^{2\kappa}\zeta^{2\kappa-1}\alpha^{2\kappa-1}}{(2\kappa)!}\otimes(1-r_\alpha),$$

with B_ℓ is the Bernoulli number. We may think of $T^{(\zeta)}(H,m)$ as the Dunkl-Heckman differential-difference operator associated with the triple $(\Sigma^{(\zeta)},m^{(\zeta)},\mathfrak{a})$, where $\Sigma^{(\zeta)}$ is the root system given by

$$\Sigma^{(\zeta)}=\{\zeta\alpha:\ \alpha\in\Sigma\},\tag{3.5}$$

and $m^{(\zeta)}$ is the multiplicity function defined by

$$m^{(\zeta)}:\Sigma^{(\zeta)}\to\mathbb{N},\ \zeta\alpha\mapsto m_\alpha.\tag{3.6}$$

It is shown in [6] that, under the weak topology, the following limit

$$\mathbb{G}(2\cdot 1_{\mathscr{O}},m):=\lim_{\zeta\to 0}\zeta^{2|\mathscr{O}^+|}G^{(\zeta)}(2\cdot 1_{\mathscr{O}},m)$$

exists, and

$$\mathbb{G}(2\cdot 1_{\mathscr{O}},m)=\mathrm{Res}\Big[\widetilde{\pi}_{\mathscr{O}^+}^{-1}\sum_{w\in W}\varepsilon_{\mathscr{O}}(w)T^{\circ}(wH,m)^{|\mathscr{O}^+|}\Big],\tag{3.7}$$

where

$$\widetilde{\pi}_{\mathscr{O}^+}(X)=\prod_{\alpha\in\mathscr{O}^+}\alpha(X),$$

and

$$T^{\circ}(H,m)=\partial(H)+\sum_{\alpha\in\Sigma^+}\frac{m_\alpha}{2}\alpha(H)\alpha^{-1}\otimes(1-r_\alpha)$$

denotes the so-called rational Dunkl operator. The operator $\mathbb{G}(2\cdot 1_{\mathscr{O}},m)$ satisfies

$$\mathbb{G}(2\cdot 1_{\mathscr{O}},m)\circ\triangle(m)=\triangle(m+2\cdot 1_{\mathscr{O}})\circ\mathbb{G}(2\cdot 1_{\mathscr{O}},m),\tag{3.8}$$

where

$$\begin{aligned}\triangle(m) &:= \operatorname{Res}\Big[\sum_{j=1}^{r} T^{\circ}(H_j,m)^2\Big],\\ &= \sum_{j=1}^{r}\partial(H_j)^2+\sum_{\alpha\in\Sigma+} m_\alpha\frac{\partial_\alpha}{\alpha},\end{aligned}$$

with $\{H_j\}_{j=1}^r$ is an orthonormal basis of $\mathfrak{a}$. The relation (3.11) means that $\mathbb{G}(2\cdot 1_{\mathscr{O}}, m)$ transforms the eigenfunctions of $\triangle(m)$ to the eigenfunctions of $\triangle(m+2\cdot 1_{\mathscr{O}})$ and shifts in that sense from m to $m+2\cdot 1_{\mathscr{O}}$. Further, for two orbits $\mathscr{O}$ and $\mathscr{O}'$ in Σ, we have

$$\mathbb{G}(2\cdot(1_{\mathscr{O}}+1_{\mathscr{O}'}), m)=\mathbb{G}(2\cdot 1_{\mathscr{O}}, m+2\cdot 1_{\mathscr{O}'})\circ\mathbb{G}(2\cdot 1_{\mathscr{O}'}, m).$$

The operator $\mathbb{G}(2\cdot 1_{\mathscr{O}}, m)$ appeared for the first time in [22] from a different point of view.

4. **Bessel functions associated with NCC symmetric spaces**

For $\zeta>0$ and $\lambda\in\mathfrak{a}_{\mathbb{C}}^*$, write

$$\Psi_\lambda(m,X):=\lim_{\zeta\to 0}\varphi_{\lambda/\zeta}(m,\exp(\zeta X)),\qquad X\in c_{\max}^0,\tag{4.1}$$

when ever the limit exists. We may think of $\varphi_{\lambda/\zeta}(m,\exp(\zeta\,\cdot))$ as the spherical function associated with the triple $(\Sigma^{(\zeta)}, m^{(\zeta)}, \mathfrak{a})$ defined by (2.9) and (2.10). Using the integral representation (2.4) of the spherical functions, it was shown in [6] that for all λ belongs to the set

$$\widetilde{\mathscr{E}}:=\{\lambda\in\mathfrak{a}_{\mathbb{C}}^*:\ (\forall\alpha\in\Sigma_n^+)\ \operatorname{Re}\lambda(H_\alpha)<0\},$$

the limit Ψ and its derivatives exist. Moreover, its integral representation is given by

$$\Psi_\lambda(m,X)=\int_H e^{-\langle\lambda,\mathscr{P}(\operatorname{Ad}(h)X)\rangle}dh,$$

where $\mathscr{P}:\mathfrak{q}\to\mathfrak{a}$ denotes the orthogonal projection. Note that for all $h\in H$ and $X\in c_{\max}^0$, $\mathscr{P}(\operatorname{Ad}(h)X)\in c_{\max}^0$. The function $\Psi_\lambda(m,\cdot)$ is called the Bessel function on the flat symmetric space $H\ltimes\mathfrak{q}/H\simeq\mathfrak{q}$. Further, if $S(\mathfrak{a}_{\mathbb{C}})$ denotes the symmetric algebra over $\mathfrak{a}_{\mathbb{C}}$ considered as the space of polynomial functions on $\mathfrak{a}_{\mathbb{C}}^*$, then $\Psi_\lambda(m,\cdot)$ satisfies

$$p(T^{\circ}(m))\Psi_\lambda(m,X)=p(\lambda)\Psi_\lambda(m,X),\qquad \forall p\in S(\mathfrak{a}_{\mathbb{C}})^W,$$

where $T^{\circ}(m)=(T^{\circ}(H_1,m),\ldots,T^{\circ}(H_r,m))$. On the other hand, by means of (3.5), we have

$$\begin{aligned}\varphi_{\lambda/\zeta}(m+2\cdot 1_{\mathscr{O}},\exp(\zeta X)) &= \frac{c_{G/H}(m+2\cdot 1_{\mathscr{O}},\lambda/\zeta)}{c_{G/H}(m,\lambda/\zeta)}\frac{\tilde{c}(m+2\cdot 1_{\mathscr{O}},-\lambda/\zeta)}{\tilde{c}(m,-\lambda/\zeta)}\\ &\qquad G^{(\zeta)}(2\cdot 1_{\mathscr{O}},m)\varphi_{\lambda/\zeta}(m,\exp(\zeta X)),\end{aligned}\tag{4.2}$$

where $c_{G/H}$ and $\widetilde{c}$ are given respectively by (3.4) and (3.6). Since $\Gamma(z+a)/\Gamma(z+b)\sim z^{a-b}$ as $z\to\infty$, equation (??) and (??) implies that for all $X\in\mathfrak{a}_-$ and $\lambda\in\widetilde{\mathscr{E}}\cap\{\lambda\in\mathfrak{a}_{\mathbb{C}}^*:$ $(\forall\alpha\in\Sigma^+)\ \lambda(H_\alpha)\neq 0\}$

$$\Psi_\lambda(m+2\cdot 1_{\mathscr{O}},X)=(-1)^{|\Sigma_0^+\cap\mathscr{O}|}\prod_{\alpha\in\mathscr{O}^+}\left(\frac{\lambda(H_\alpha)}{2}\right)^{-2}\mathbb{G}(2\cdot 1_{\mathscr{O}},m)\Psi_\lambda(m,X).\tag{4.3}$$

Recall that for every NCC symmetric space, the root system Σ is reduced. Thus, if we write the multiplicity function m as $(m_{\Sigma\setminus\mathscr{O}}, m_{\mathscr{O}})$, where $m_{\Sigma\setminus\mathscr{O}} : \Sigma\setminus\mathscr{O} \to \mathbb{N}$ and $m_{\mathscr{O}} : \mathscr{O} \to \mathbb{N}$, then each component is a constant function. By abuse of notation, we will denote $m_{\Sigma\setminus\mathscr{O}}(\alpha)$, for all $\alpha \in \Sigma\setminus\mathscr{O}$, by $m_{\Sigma\setminus\mathscr{O}}$, and $m_{\mathscr{O}}(\alpha)$, for all $\alpha\in\mathscr{O}$, by $m_{\mathscr{O}}$.

Let us assume that $m_{\mathscr{O}} \in 2\mathbb{N}$. By virtue of (4.2)

$$\Psi_\lambda((m_{\Sigma\setminus\mathscr{O}}, m_{\mathscr{O}}), X) = (-1)^{|\Sigma_0^+\cap\mathscr{O}|\frac{m_{\mathscr{O}}}{2}} \prod_{\alpha\in\mathscr{O}^+}\left(\frac{\lambda(H_\alpha)}{2}\right)^{-m_{\mathscr{O}}} \tag{4.4}$$
$$\mathbb{G}(2\cdot 1_{\mathscr{O}}, (m_{\Sigma\setminus\mathscr{O}}, m_{\mathscr{O}}-2))\circ\cdots\circ\mathbb{G}(2\cdot 1_{\mathscr{O}}, (m_{\Sigma\setminus\mathscr{O}}, 0))\Psi_\lambda((m_{\Sigma\setminus\mathscr{O}}, 0), X).$$

From now on we will work with NCC symmetric spaces such that Σ is a non-exceptional root system. That is, if $r = \dim(\mathfrak{a})$, then Σ is either of type A_{r-1}, B_r, C_r, or D_r. Thus Σ consists of short roots with multiplicity m_s, and, possibly, long roots with multiplicity m_l. Furthermore, we will assume that Σ obeys the following condition:

(**H**) *The multiplicity m_s of the short roots is even.*

Let $\{\gamma_1,\dots,\gamma_r\}\subset\Sigma_n$ be a maximal strongly orthogonal subset such that γ_j is the highest element of Σ_n strongly orthogonal to $\{\gamma_{j+1},\dots,\gamma_r\}$ for $j = r,\dots,1$. We will use the realization of the reduced root system Σ in terms of the vectors γ_i as described in [4].

1. Assume that Σ is of type A_{r-1}, i.e.

$$\Sigma = \{\pm(\gamma_j-\gamma_i) :\ 1\le i<j\le r\}.$$

Let the symmetric group S_r acts on the subscripts of the γ_i, then the Weyl group W associated with Σ is isomorphic to S_r. In the present case, we say that there exists in Σ only one W-orbit $\mathscr{O}_s = \{\pm(\gamma_j-\gamma_i)\}_{i<j}$ consisting of short roots. The irreducible NCC symmetric pairs of type A_{r-1} which satisfy the condition (**H**) are:

$(\mathfrak{sl}(p+q,\mathbb{C}), \mathfrak{su}(p,q))$,	with $m_l = 0$ and $m_s = 2$,
$(\mathfrak{su}^*(2(p+q)), \mathfrak{sp}(p,q))$,	with $m_l = 0$ and $m_s = 4$,
$(\mathfrak{e}_{6(-26)}, \mathfrak{f}_{4(-20)})$,	with $m_l = 0$ and $m_s = 8$.

Here $p+q=r$ for the first two pairs, while $r=3$ for the third one. The set of compact and non-compact roots in Σ^+ are given respectively by

$$\Sigma_c^+ = \{(\gamma_j-\gamma_i) :\ 1\le i<j\le p\}\cup\{(\gamma_{p+j}-\gamma_{p+i}) :\ 1\le i<j\le q\} = A_{p-1}\times A_{q-1},$$

and

$$\Sigma_n^+ = \{(\gamma_{p+j}-\gamma_i) :\ 1\le i\le p,\ 1\le j\le q\},$$

with $p=1$ and $q=2$ for the symmetric pair $(\mathfrak{e}_{6(-26)}, \mathfrak{f}_{4(-20)})$. The Weyl group W_0 associated with Σ_c is isomorphic to $S_p\times S_q$.

2. Assume that the root system Σ is of type B_r, i.e.

$$\Sigma = \{\pm\gamma_i\ (1\le i\le r),\ \ \pm(\gamma_j\pm\gamma_i)\ (1\le i<j\le r)\}.$$

The Weyl group W is isomorphic to the semidirect product of $(\mathbb{Z}/2\mathbb{Z})^r$ and S_r. The two possible W-orbits in Σ are the orbit $\mathscr{O}_l = \{\pm\gamma_i\}_i$ of the long roots and the orbit

$\mathscr{O}_s = \{\pm(\gamma_j \pm \gamma_i)\}_{i<j}$ of the short roots. The irreducible NCC symmetric pairs of type B_r which satisfy the hypothesis **(H)** are:

$$\begin{array}{ll} (\mathfrak{so}(2r+1,\mathbb{C}), \mathfrak{so}(2r-1,2)), & \text{with } m_l = 2 \text{ and } m_s = 2, \\ (\mathfrak{so}(r+2k,r), \mathfrak{so}(r+2k-1,1) \times \mathfrak{so}(r-1,1)), & \text{with } m_l = 1 \text{ and } m_s = 2k. \end{array}$$

The set of compact and non-compact roots in Σ^+ are given respectively by

$$\Sigma_c^+ = \{\gamma_i \ (2 \le i \le r),\ (\gamma_j \pm \gamma_i)\ (2 \le i < j \le r)\} = B_{r-1},$$

and

$$\Sigma_n^+ = \{\gamma_1,\ (\gamma_j \pm \gamma_1)\ (2 \le j \le r)\}.$$

The little Weyl group W_0 is isomorphic to the semidirect product of $(\mathbb{Z}/2\mathbb{Z})^{r-1}$ and S_{r-1}.

3. Assume that Σ is of type C_r, i.e.

$$\Sigma = \{\pm 2\gamma_i \ (1 \le i \le r),\ \pm(\gamma_j \pm \gamma_i)\ (1 \le i < j \le r)\}.$$

The Weyl group W is isomorphic to that of B_r. The two W-orbits in Σ are the orbit $\mathscr{O}_l = \{\pm 2\gamma_i\}_i$ of the long roots, and the orbit $\mathscr{O}_s = \{\pm(\gamma_j \pm \gamma_i)\}_{i<j}$ of the short roots. The irreducible NCC symmetric pairs of type C_r which satisfy the hypothesis **(H)** are:

$$\begin{array}{ll} (\mathfrak{su}(r,r), \mathfrak{sl}(r,\mathbb{C}) \times \mathbb{R}), & \text{with } m_l = 1 \text{ and } m_s = 2, \\ (\mathfrak{so}^*(4r), \mathfrak{su}^*(2r) \times \mathbb{R}), & \text{with } m_l = 1 \text{ and } m_s = 4, \\ (\mathfrak{sp}(r,\mathbb{C}), \mathfrak{sp}(r,\mathbb{R})), & \text{with } m_l = 2 \text{ and } m_s = 2, \\ (\mathfrak{sp}(r,r), \mathfrak{sp}(r,\mathbb{C})), & \text{with } m_l = 3 \text{ and } m_s = 4, \\ (\mathfrak{e}_{7(-25)}, \mathfrak{e}_{6(-26)} \times \mathbb{R}), & \text{with } m_l = 1 \text{ and } m_s = 8. \end{array}$$

The set of compact and non-compact roots in Σ^+ are given respectively by

$$\Sigma_c^+ = \{(\gamma_j - \gamma_i) :\ 1 \le i < j \le r\} = A_{r-1},$$

and

$$\Sigma_n^+ = \{2\gamma_i \ (1 \le i \le r),\ (\gamma_j + \gamma_i)\ (1 \le i < j \le r)\}.$$

For the exceptional symmetric pair, $r = 3$. The Weyl group W_0 is isomorphic to the symmetric group S_r.

4. Assume that Σ is a root system of type D_r, i.e.

$$\Sigma = \{\pm(\gamma_j \pm \gamma_i) :\ 1 \le i < j \le r\}.$$

The Weyl group W is isomorphic to the semidirect product of S_r and $(\mathbb{Z}/2\mathbb{Z})^r_{\text{even}} := \{(\varepsilon_1, \ldots, \varepsilon_r) :\ \varepsilon_i = \pm 1,\ \prod_{i=1}^r \varepsilon_i = 1\}$. The only orbit of W in Σ is $\mathscr{O}_s = \{\pm(\gamma_j \pm \gamma_i)\}_{i<j}$. The NCC symmetric pairs of type D_r which obey the hypothesis **(H)** are:

$$\begin{array}{ll} (\mathfrak{so}(2r,\mathbb{C}), \mathfrak{so}^*(2r)), & \text{with } m_l = 0 \text{ and } m_s = 2, \\ (\mathfrak{so}(2r,\mathbb{C}), \mathfrak{so}(2r-2,2)), & \text{with } m_l = 0 \text{ and } m_s = 2, \end{array}$$

For the first symmetric pair, the set of compact and non-compact roots in Σ^+ are given respectively by

$$\Sigma_c^+ = \{(\gamma_j - \gamma_i) :\ 1 \le i < j \le r\} = A_{r-1},$$

and

$$\Sigma_n^+ = \{(\gamma_j + \gamma_i) :\ 1 \le i < j \le r\}.$$

The Weyl group W_0 is isomorphic to S_r.
For the second symmetric pair, the set of compact and non-compact roots in Σ^+ are given respectively by

$$\Sigma_c^+ = \{(\gamma_j \pm \gamma_i) :\ 2 \leq i < j \leq r\} = D_{r-1},$$

and

$$\Sigma_n^+ = \{(\gamma_j \pm \gamma_1) :\ 2 \leq j \leq r\}.$$

The Weyl group W_0 is isomorphic to the semidirect product of S_{r-1} and $(\mathbb{Z}/2\mathbb{Z})^{r-1}_{\text{even}}$.

Consider the Euclidean space $\mathbb{R}^r$ with the usual orthonormal basis $\{e_1, \ldots, e_r\}$ and inner product $\langle \cdot, \cdot \rangle$. Henceforth we will use the following identifications:

(i) When Σ is of type B_r, C_r, or D_r: Let $E = \mathbb{R}^r$. Identify the complex dual $\mathfrak{a}_{\mathbb{C}}^*$ with the complexification $E_{\mathbb{C}}$ of E via the map

$$\lambda = (\lambda_1, \ldots, \lambda_r) \mapsto -\sum_{j=1}^{r} \lambda_j \gamma_j.$$

Thus, if Σ of type B_r or C_r, then

$$\mathfrak{a}_- \equiv \{t = (t_1, \ldots, t_r) \in E :\ 0 < t_1 < \cdots < t_r\},$$

and if Σ of type D_r, then

$$\mathfrak{a}_- \equiv \{t = (t_1, \ldots, t_r) \in E :\ t_1 < \cdots < t_r\},$$

(ii) When Σ is of type A_{r-1}: Let E be the r-dimensional subspace of $\mathbb{R}^r$ orthogonal to $e_1 + \cdots + e_r$. Identify $\mathfrak{a}_{\mathbb{C}}^*$ with $E_{\mathbb{C}}$ as above. Thus

$$\mathfrak{a}_- \equiv \{t = (t_1, \ldots, t_r) \in E :\ t_1 < \cdots < t_r\}.$$

Now let us go back to the identity (4.3) where we will fix from now on the orbit $\mathscr{O}$ to be $\mathscr{O}_s$. In view of the hypothesis **(H)**, we have

$$\begin{aligned}\Psi_\lambda((m_l, m_s), t) =& (-1)^{\frac{m_s}{2}|\Sigma_c^+ \cap \mathscr{O}_s|}\, 2^{m_s|\Sigma^+\cap \mathscr{O}_s|} \prod_{\alpha \in \Sigma^+ \cap \mathscr{O}_s} \langle \lambda, \alpha \rangle^{-m_s} \\ & \mathbb{G}(2\cdot 1_{\mathscr{O}_s}, (m_l, m_s - 2)) \circ \cdots \circ \mathbb{G}(2 \cdot 1_{\mathscr{O}_s}, (m_l, 0)) \widetilde{\Psi}_\lambda((m_l, 0), t), \quad (4.5)\end{aligned}$$

where

$$\widetilde{\Psi}_\lambda((m_l, 0), t) = \pi^{-r/2} \sum_{w \in W_0} \prod_{i=1}^{r} (2t_i \omega(\lambda_i))^{-\frac{m_l}{2}+\frac{1}{2}} K_{\frac{m_l}{2}-\frac{1}{2}}(t_i \omega(\lambda_i)).$$

Here K_ν denotes the Bessel function of the third kind. We remind the reader that the function $z \mapsto K_\nu(z)$ is analytic on the complex plane cut along the negative real axis.

In the sequel we will need the following obvious lemma.

Lemma 4.1. *Suppose $f_1, f_2, \ldots, f_r$ are smooth functions on a real interval I. Let*

$$F(x_1, \ldots, x_r) := \frac{\det\limits_{1\leq i,j \leq r} (f_i(x_j))}{\prod\limits_{1 \leq i < j \leq r} (x_i - x_j)}.$$

Then for all x_{i_0} and x_{j_0} in I

$$\lim_{x_{i_0}\to x_{j_0}} F(x_1,\dots,x_{i_0},\dots,x_{j_0},\dots,x_r) = \prod_{\substack{1\le i<j\le r\\ (i,j)\neq(i_0,j_0)}} (x_i-x_j)^{-1}\times \begin{vmatrix} f_1(x_1) & \cdots & f_r(x_1)\\ \vdots & \vdots & \vdots\\ f_1(x_{i_0-1}) & \cdots & f_r(x_{i_0-1})\\ f_1'(x_{i_0}) & \cdots & f_r'(x_{i_0})\\ f_1(x_{i_0+1}) & \cdots & f_r(x_{i_0+1})\\ \vdots & \vdots & \vdots\\ f_1(x_r) & \cdots & f_r(x_r)\end{vmatrix}.$$

Depending on the type of the restricted root system Σ we are now going to investigate the nature and order of the singularities of the Bessel functions.

4.1. **The cases with restricted root system of type A_{r-1}.** When Σ is of type A_{r-1}, formula (4.4) reads

$$\begin{aligned}\Psi_\lambda((0,m_s),t) &= (-1)^{\frac{m_s}{2}(p(p-1)/2+q(q-1)/2)}2^{m_s r(r-1)/2}\prod_{1\le i<j\le r}(\lambda_i-\lambda_j)^{-m_s} \quad (4.6)\\ &\qquad \mathbb{G}(2\cdot 1_{\mathscr{O}_s},(0,m_s-2))\circ\cdots\circ\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(0,0))\widetilde{\Psi}_\lambda((0,0),t).\end{aligned}$$

Due an observation made by Berezin in [1], the elementary shift operator $\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(0,0))$ is given by

$$\prod_{1\le i<j\le r}\frac{\partial_i-\partial_j}{t_i-t_j},$$

where $\partial_i=\partial/\partial t_i$ for $1\le i\le r$. The function $\widetilde{\Psi}_\lambda((0,0),t)$ is given by

$$\sum_{\omega\in W_0}\prod_{1\le i\le r} e^{-t_i\omega(\lambda_i)}. \qquad (4.7)$$

Here we used the fact that $K_{-1/2}(z)=(\pi/2z)^{1/2}e^{-z}$.

4.1.1. **The symmetric pair $(\mathfrak{sl}(p+q,\mathbb{C}),\mathfrak{su}(p,q))$.** For the present pair, formula (??) becomes

$$\widetilde{\Psi}_\lambda((0,0),t)=\sum_{(\sigma,\tau)\in S_p\times S_q}\prod_{1\le i\le p} e^{-t_i\lambda_{\sigma(i)}}\prod_{1\le i\le q} e^{-t_{p+i}\lambda_{p+\tau(i)}}.$$

It is easy to check that

$$\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(0,0))\widetilde{\Psi}_\lambda((0,0),t)=\prod_{1\le i<j\le r}\left(\frac{\lambda_i-\lambda_j}{t_i-t_j}\right)\det_{1\le i,j\le p}(e^{-t_i\lambda_j})\det_{1\le i,j\le q}(e^{-t_{p+i}\lambda_{p+j}}).$$

Thus the Bessel function associated with the symmetric pair $(\mathfrak{sl}(p+q,\mathbb{C}),\mathfrak{su}(p,q))$ is given by

$$\Psi_\lambda((0,2),t) = \frac{(-1)^{p(p-1)/2+q(q-1)/2}2^{r(r-1)}}{\prod_{1\le i\le p}\prod_{1\le j\le q}(\lambda_i-\lambda_{p+j})\prod_{1\le i<j\le r}(t_i-t_j)}\Big\{\frac{\det_{1\le i,j\le p}(e^{-t_i\lambda_j})}{\prod_{1\le i<j\le p}(\lambda_i-\lambda_j)}\Big\}\Big\{\frac{\det_{1\le i,j\le q}(e^{-t_{p+i}\lambda_{p+j}})}{\prod_{1\le i<j\le q}(\lambda_{p+i}-\lambda_{p+j})}\Big\}.$$

By Lemma 3.1, the function $\lambda\mapsto\Psi_\lambda((0,2),t)$ extends to a meromorphic function in $\mathbb{C}^r$ with simple poles for $\lambda_i-\lambda_{p+j}=0$ $(1\le i\le p,\ 1\le j\le q)$. Thus we have proved:

Proposition 4.2. *For the symmetric pair* $(\mathfrak{sl}(p+q,\mathbb{C}),\mathfrak{su}(p,q))$, *we have*

$$\Psi_\lambda((0,2),t) = (-1)^{p(p-1)/2+q(q-1)/2}2^{r(r-1)}\frac{\det_{1\le i,j\le p}(e^{-t_i\lambda_j})\det_{1\le i,j\le q}(e^{-t_{p+i}\lambda_{p+j}})}{\prod_{1\le i<j\le r}(\lambda_i-\lambda_j)\prod_{1\le i<j\le r}(t_i-t_j)}.$$

For fixed $t\in\mathfrak{a}_-$, *the map* $\lambda\mapsto\Psi_\lambda((0,2),t)$ *extends to a meromorphic function in* $\mathbb{C}^r$ *with simple poles for* $\langle\lambda,\alpha\rangle=0$ $(\forall\alpha\in\Sigma_n^+)$.

Remark 4.3. For the complex symmetric pair $(\mathfrak{sl}(p+q,\mathbb{C}),\mathfrak{su}(p,q))$, and indeed for every complex NCC symmetric pair, an explicit formula for the spherical function φ_λ is well known [11] (see also [10]). Thus one may recover the above formula for $\Psi_\lambda((0,2),t)$ in a direct fashion by applying the limit formula (4.1).

4.1.2. **The symmetric pair** $(\mathfrak{su}^*(2(p+q)),\mathfrak{sp}(p,q))$**.** In view (??), a similar calculation to the one above implies that the Bessel function associated with $(\mathfrak{su}^*(2(p+q)),\mathfrak{sp}(p,q))$ is given by

$$\Psi_\lambda((0,4),t) = \frac{2^{2r(r-1)}}{\prod_{1\le i<j\le r}(\lambda_i-\lambda_j)^3}\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(0,2))\Big\{\frac{\det_{1\le i,j\le p}(e^{-\lambda_i t_j})\det_{1\le i,j\le q}(e^{-\lambda_{p+i}t_{p+j}})}{\prod_{1\le i<j\le r}(t_i-t_j)}\Big\}.$$

By means of [8], we have

$$\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(0,2))\circ\frac{1}{\prod_{1\le i<j\le r}(t_i-t_j)} = \frac{1}{\prod_{1\le i<j\le r}(t_i-t_j)^2}\mathscr{D}_r\circ\cdots\circ\mathscr{D}_2,$$

where $\mathscr{D}_\ell$, for $2\le\ell\le r$, denotes the differential operators obtained after replacing z_i by $\partial_i-\partial_\ell$ in the following polynomial on z

$$\widetilde{\mathscr{D}}_\ell = \exp\Big(-\sum_{i=1}^{\ell-1}\frac{2}{(t_i-t_\ell)}\frac{\partial}{\partial z_i}+\sum_{1\le i<j\le \ell-1}\frac{2}{(t_i-t_j)^2}\frac{\partial^2}{\partial z_i\partial z_j}\Big)\prod_{1\le i\le\ell-1}z_i.$$

It follows that the Bessel function associated with $(\mathfrak{su}^*(2(p+q)),\mathfrak{sp}(p,q))$ extends to a meromorphic function of λ in $\mathbb{C}^r$ with the following possible poles:

$$\begin{array}{lll} \lambda_i-\lambda_j=0 & (1\le i<j\le p), & \text{of order}\le 2,\\ \lambda_{p+i}-\lambda_{p+j}=0 & (1\le i<j\le q), & \text{of order}\le 2,\\ \lambda_i-\lambda_{p+j}=0 & (1\le i\le p,\ 1\le j\le q), & \text{of order}\le 3.\end{array}$$

Thus we have proved:

Proposition 4.4. *For the symmetric pair* $(\mathfrak{su}^*(2(p+q)), \mathfrak{sp}(p,q))$*, we have*

$$\Psi_\lambda((0,4),t) = \frac{2^{2r(r-1)}}{\prod_{1\le i<j\le r}(\lambda_i-\lambda_j)^3 \prod_{1\le i<j\le r}(t_i-t_j)^2} \mathscr{D}_r \circ \cdots \circ \mathscr{D}_2 \Big\{ \det_{1\le i,j\le p}(e^{-\lambda_i t_j}) \det_{1\le i,j\le q}(e^{-\lambda_{p+i} t_{p+j}}) \Big\}, \tag{4.8}$$

where $\mathscr{D}_\ell$*, for* $2 \le \ell \le r$*, are the differential operators as defined above. For fixed* $t \in \mathfrak{a}_-$*, the map* $\lambda \mapsto \Psi_\lambda((0,4),t)$ *extends to a meromorphic function in* $\mathbb{C}^r$ *with the possible poles for* $\langle\lambda,\alpha\rangle = 0$ $(\forall\alpha \in \Sigma_c^+)$ *and* $\langle\lambda,\alpha\rangle = 0$ $(\forall\alpha\in\Sigma_n^+)$*, of orders less or equal than* 2 *and* 3 *respectively.*

Example 4.5. Assume that $p = 1$ and $q = 1$. Thus, the differential operator $\mathscr{D}_2$ is given by

$$(\partial_1 - \partial_2) - \frac{2}{(t_1-t_2)}, \tag{4.9}$$

and formula (4.7) becomes

$$\Psi_\lambda((0,4),t) = \frac{2^4}{(\lambda_1-\lambda_2)^3(t_1-t_2)^2}\Big\{(\lambda_2-\lambda_1) - \frac{2}{(t_1-t_2)}\Big\} e^{-t_1\lambda_1} e^{-t_2\lambda_2}.$$

Clearly the function $(\lambda_1,\lambda_2) = \lambda \mapsto \Psi_\lambda((0,4),t)$ has poles for $\lambda_1 - \lambda_2 = 0$ with order 3.

Example 4.6. Assume that $p = 2$ and $q = 1$. The operator $\mathscr{D}_2$ is given by (4.8) while $\mathscr{D}_3$ is defined by

$$(\partial_1-\partial_3)(\partial_2-\partial_3) - 2\frac{(\partial_2-\partial_3)}{(t_1-t_3)} - 2\frac{(\partial_1-\partial_3)}{(t_2-t_3)} + \frac{2}{(t_1-t_2)^2} + \frac{4}{(t_1-t_3)(t_2-t_3)}.$$

A straitforward calculation shows that

$$\begin{aligned}
&(\partial_1-\partial_3)(\partial_2-\partial_3) - 2\frac{(\partial_2-\partial_3)}{(t_1-t_3)} - 2\frac{(\partial_1-\partial_3)}{(t_2-t_3)} \left[\mathscr{D}_2\left(\det_{1\le i,j\le 2}(e^{-t_i\lambda_j}) e^{-t_3\lambda_3}\right)\right] \\
&= (\lambda_2-\lambda_1) e^{-t_1\lambda_1} e^{-t_2\lambda_2} e^{-t_3\lambda_3}\left(1 - \frac{2}{(t_1-t_2)(\lambda_1-\lambda_2)}\right) \\
&\quad \Big\{-2\frac{(\lambda_3-\lambda_1)}{(t_2-t_3)} - 2\frac{(\lambda_3-\lambda_2)}{(t_1-t_3)} + (\lambda_3-\lambda_1)(\lambda_3-\lambda_2)\Big\} \\
&\quad + (\lambda_2-\lambda_1) e^{-\lambda_2 t_1} e^{-\lambda_1 t_2} e^{-t_3\lambda_3}\left(1 + \frac{2}{(t_1-t_2)(\lambda_1-\lambda_2)}\right) \\
&\quad \Big\{-2\frac{(\lambda_3-\lambda_2)}{(t_2-t_3)} - 2\frac{(\lambda_3-\lambda_1)}{(t_1-t_3)} + (\lambda_3-\lambda_1)(\lambda_3-\lambda_2)\Big\} \\
&\quad + 4\frac{\det_{1\le i,j\le 2}(e^{-t_i\lambda_j}) e^{-t_3\lambda_3}}{(t_1-t_2)^2}\Big\{\frac{1}{(t_2-t_3)} - \frac{1}{(t_1-t_3)} + \frac{1}{(t_1-t_2)}\Big\}.
\end{aligned}$$

Thus the spherical function associated with $(\mathfrak{su}^*(6), \mathfrak{sp}(2,1))$ is given by

$$\begin{aligned}
\Psi_\lambda((0,4),t) &= \frac{2^{13}}{\prod\limits_{1\le i\le 2}(\lambda_i-\lambda_3)^3}\frac{2t_1-t_2-t_3}{(t_1-t_2)^2(t_1-t_3)^2(t_2-t_3)^3}\Psi_{\rm c}(\lambda_1,\lambda_2;t_1,t_2)e^{-t_3\lambda_3}\\
&+\frac{2^{12}}{\prod\limits_{1\le i\le 2}(\lambda_i-\lambda_3)^3(\lambda_1-\lambda_2)^2}\frac{1}{\prod\limits_{1\le i\le 2}(t_i-t_3)^2(t_1-t_2)^2}\\
&\Big[e^{-t_1\lambda_1}e^{-t_2\lambda_2}e^{-t_3\lambda_3}\left(1-\frac{2}{(t_1-t_2)(\lambda_1-\lambda_2)}\right)\\
&\Big\{-2\frac{(\lambda_3-\lambda_1)}{(t_2-t_3)}-2\frac{(\lambda_3-\lambda_2)}{(t_1-t_3)}+(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)\Big\}\\
&+e^{-\lambda_2t_1}e^{-\lambda_1t_2}e^{-t_3\lambda_3}\left(1+\frac{2}{(t_1-t_2)(\lambda_1-\lambda_2)}\right)\\
&\Big\{-2\frac{(\lambda_3-\lambda_2)}{(t_2-t_3)}-2\frac{(\lambda_3-\lambda_1)}{(t_1-t_3)}+(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)\Big\}\Big]\\
&+\frac{2^{14}}{\prod\limits_{1\le i<j\le 3}(t_i-t_j)^2(t_1-t_2)^2}\Big\{\frac{1}{(t_2-t_3)}-\frac{1}{(t_1-t_3)}+\frac{1}{(t_1-t_2)}\Big\}\\
&\frac{\det\limits_{1\le i,j\le 2}(e^{-t_i\lambda_j})e^{-t_3\lambda_3}}{\prod\limits_{1\le i<j\le 3}(\lambda_i-\lambda_j)^3}.
\end{aligned}$$

Here

$$\begin{aligned}
\Psi_{\rm c}(\lambda_1,\lambda_2;t_1,t_2) &:= \frac{1}{(t_1-t_2)^2(\lambda_1-\lambda_2)^2}\Big\{\left(1-\frac{2}{(t_1-t_2)(\lambda_1-\lambda_2)}\right)e^{-(\lambda_1t_1+\lambda_2t_2)}\\
&+\left(1+\frac{2}{(t_2-t_1)(\lambda_1-\lambda_2)}\right)e^{-(\lambda_1t_2+\lambda_2t_1)}\Big\}
\end{aligned}$$

denotes the Bessel function associated with the Riemannian symmetric pair $(\mathfrak{su}(4), \mathfrak{sp}(2))$ which is holomorphic in $\mathbb{C}^2$. It follows that for $r=3$, the poles of the function $\lambda \mapsto \Psi_\lambda((0,4),t)$ are $\lambda_i-\lambda_3=0$ $(i=1,2)$ of order 3, and $\lambda_1-\lambda_2=0$ of order 2.

4.1.3. **The symmetric pair** $(\mathfrak{e}_{6(-26)}, \mathfrak{f}_{4(-20)})$**.** For the present pair

$$\widetilde{\Psi}_\lambda((0,0),t) = e^{-t_1\lambda_1}\sum_{\sigma\in S_2} e^{-t_2\lambda_{\sigma(2)}}e^{-t_3\lambda_{\sigma(3)}}.$$

Using [3, 24], we get

$$\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(0,6))\circ\cdots\circ\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(0,0)) = \frac{1}{(t_1-t_2)^4(t_2-t_3)^4(t_3-t_1)^4}\,\mathscr{D}_3\circ\mathscr{D}_2\circ\mathscr{D}_1\circ\mathscr{D}_0,$$

where

$$\begin{aligned}\mathscr{D}_\ell &= (\partial_1-\partial_2)\circ(\partial_2-\partial_3)\circ(\partial_3-\partial_1)-\frac{2\ell}{(t_1-t_2)}(\partial_2-\partial_3)\circ(\partial_3-\partial_1)\\
&\quad -\frac{2\ell}{(t_2-t_3)}(\partial_3-\partial_1)\circ(\partial_1-\partial_2)-\frac{2\ell}{(t_3-t_1)}(\partial_1-\partial_2)\circ(\partial_2-\partial_3)\\
&\quad +\left(\frac{4\ell^2}{(t_1-t_2)(t_2-t_3)}-\frac{\ell(\ell-1)}{(t_3-t_1)^2}\right)(\partial_3-\partial_1)\\
&\quad +\left(\frac{4\ell^2}{(t_2-t_3)(t_3-t_1)}-\frac{\ell(\ell-1)}{(t_1-t_2)^2}\right)(\partial_1-\partial_2)\\
&\quad +\left(\frac{4\ell^2}{(t_3-t_1)(t_1-t_2)}-\frac{\ell(\ell-1)}{(t_2-t_3)^2}\right)(\partial_2-\partial_3)\\
&\quad -2\ell(\ell-1)(\ell+2)\left(\frac{1}{(t_1-t_2)^3}+\frac{1}{(t_2-t_3)^3}+\frac{1}{(t_3-t_1)^3}\right)\\
&\quad -\frac{6\ell^2(\ell+1)}{(t_1-t_2)(t_2-t_3)(t_3-t_1)}. \qquad (4.10)\end{aligned}$$

It follows that the Bessel function associated with $(\mathfrak{e}_{6(-26)},\mathfrak{f}_{4(-20)})$ is given by

$$\Psi_\lambda((0,8),t) = \frac{2^{24}}{\prod_{2\le j\le 3}(\lambda_1-\lambda_j)^7(\lambda_2-\lambda_3)^6\prod_{1\le i<j\le 3}(t_i-t_j)^4}\mathscr{D}_3\circ\mathscr{D}_2\circ\mathscr{D}_1\Big\{e^{-t_1\lambda_1}\frac{\det_{2\le i,j\le 3}(e^{-t_i\lambda_j})}{(\lambda_2-\lambda_3)}\Big\}.$$

It follows that $\lambda\mapsto\Psi_\lambda((0,8),t)$ extends to a meromorphic function in $\mathbb{C}^r$ with the following possible poles:

$$\begin{array}{lll}\lambda_1-\lambda_i=0 & (i=2,3), & \text{of order}\le 7,\\ \lambda_2-\lambda_3=0, & & \text{of order}\le 6.\end{array}$$

Thus we have proved:

Proposition 4.7. *For the symmetric pair $(\mathfrak{e}_{6(-26)},\mathfrak{f}_{4(-20)})$, we have*

$$\Psi_\lambda((0,8),t) = \frac{2^{24}}{\prod_{1\le i<j\le 3}(\lambda_i-\lambda_j)^7\prod_{1\le i<j\le 3}(t_i-t_j)^4}\mathscr{D}_3\circ\mathscr{D}_2\circ\mathscr{D}_1\Big\{e^{-t_1\lambda_1}\det_{2\le i,j\le 3}(e^{-t_i\lambda_j})\Big\},$$

wher $\mathscr{D}_\ell$, for $1\le\ell\le 3$, are the same as in (4.9). For fixed $t\in\mathfrak{a}_-$, the map $\lambda\mapsto\Psi_\lambda((0,8),t)$ extends to a meromorphic function in $\mathbb{C}^r$ with the possibles poles for $\langle\lambda,\alpha\rangle=0$ ($\forall\alpha\in\Sigma_c^+$) and $\langle\lambda,\alpha\rangle=0$ ($\forall\alpha\in\Sigma_n^+$), of orders less or equal than 6 and 7 respectively.

4.2. **The cases with restricted root system of type B_r.** When Σ is a root system of type B_r, formula (??) becomes

$$\begin{aligned}\Psi_\lambda((m_l,m_s),t) &= 2^{m_s r(r-1)}\prod_{1\le i<j\le r}(\lambda_i^2-\lambda_j^2)^{-m_s}\\
&\quad \mathbb{G}(2\cdot 1_{\mathscr{O}_s},(m_l,m_s))\circ\cdots\circ\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(m_l,0))\widetilde{\Psi}_\lambda((m_l,0),t).\end{aligned}$$

Using [9], it follows that the shift operator $\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(m_l,0))$ is given by

$$\prod_{1\le i<j\le r}(t_i^2-t_j^2)^{-1}\prod_{1\le i<j\le r}\Big(\mathscr{L}_{m_l}(t_i,\partial_i)-\mathscr{L}_{m_l}(t_j,\partial_j)\Big), \tag{4.11}$$

where

$$\mathscr{L}_{m_l}:=\frac{d^2}{dt^2}+\frac{m_l}{t}\frac{d}{dt}.$$

4.2.1. **The symmetric pair $(\mathfrak{so}(2r+1,\mathbb{C}),\mathfrak{so}(2r-1,2))$.** For the present pair, $m_l=2$ and

$$\widetilde{\Psi}_\lambda((2,0),t)=\frac{e^{-\lambda_1 t_1}}{2\lambda_1 t_1}\sum_{\omega\in\mathbb{Z}_2^{r-1}\times S_{r-1}}\prod_{2\le i\le r}\frac{e^{-t_i\omega(\lambda_i)}}{2t_i\omega(\lambda_i)}.$$

An easy calculation gives

$$\begin{aligned}
&\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(2,0))\widetilde{\Psi}((2,0),\lambda,t)\\
&=\frac{e^{-\lambda_1 t_1}}{2\lambda_1 t_1}\frac{\prod_{2\le j\le r}(\lambda_1^2-\lambda_j^2)}{\prod_{1\le i<j\le r}(t_i^2-t_j^2)\prod_{2\le j\le r}2t_i\lambda_i}\\
&\quad\sum_{\sigma\in S_{r-1}}\prod_{2\le i<j\le r}(\lambda_{\sigma(i)}^2-\lambda_{\sigma(j)}^2)\sum_{\varepsilon\in\mathbb{Z}_2^{r-1}}\prod_{2\le i\le r}\varepsilon_i\prod_{2\le i\le r}e^{-\varepsilon_i t_i\lambda_{\sigma(i)}}\\
&=\frac{e^{-\lambda_1 t_1}}{2\lambda_1 t_1}\frac{\prod_{2\le j\le r}(\lambda_1^2-\lambda_j^2)}{\prod_{1\le i<j\le r}(t_i^2-t_j^2)\prod_{2\le j\le r}t_i\lambda_i}\sum_{\sigma\in S_{r-1}}\prod_{2\le i<j\le r}(\lambda_{\sigma(i)}^2-\lambda_{\sigma(j)}^2)\prod_{2\le i\le r}\mathrm{sh}(t_i\lambda_{\sigma(i)})\\
&=\frac{e^{-t_1\lambda_1}}{2\lambda_1 t_1}\frac{\prod_{1\le i<j\le r}(\lambda_i^2-\lambda_j^2)}{\prod_{1\le i<j\le r}(t_i^2-t_j^2)\prod_{2\le j\le r}t_j\lambda_j}\det_{2\le i,j\le r}(\mathrm{sh}(t_i\lambda_j)).
\end{aligned}$$

As a consequence

$$\Psi_\lambda((2,2),t)=\frac{2^{2r(r-1)-1}e^{-t_1\lambda_1}}{\lambda_1\prod_{1\le i\le r}t_i\prod_{1\le i<j\le r}(t_i^2-t_j^2)\prod_{2\le j\le r}(\lambda_1^2-\lambda_j^2)}\left\{\frac{\det_{2\le i,j\le N}(\mathrm{sh}(t_i\lambda_j))}{\prod_{2\le j\le r}\lambda_j\prod_{2\le i<j\le r}(\lambda_i^2-\lambda_j^2)}\right\}.$$

By Lemma 3.1, the Bessel function associated with $(\mathfrak{so}(2r+1,\mathbb{C}),\mathfrak{so}(2r-1,2))$ extends to a meromorphic function of λ in $\mathbb{C}^r$ with simple poles for $\lambda_1=0$ and $\lambda_1\pm\lambda_j=0$ $(2\le j\le r)$. Thus we have proved:

Proposition 4.8. *For the symmetric pair $(\mathfrak{so}(2r+1,\mathbb{C}),\mathfrak{so}(2r-1,2))$, we have*

$$\Psi_\lambda((2,2),t)=2^{2r(r-1)-1}\frac{e^{-t_1\lambda_1}\det_{2\le i,j\le r}(\mathrm{sh}(t_i\lambda_j))}{\prod_{1\le i\le r}\lambda_i\prod_{1\le i\le r}t_i\prod_{1\le i<j\le r}(t_i^2-t_j^2)\prod_{1\le i<j\le r}(\lambda_i^2-\lambda_j^2)}.$$

For fixed $t \in \mathfrak{a}_-$, the map $\lambda \mapsto \Psi_\lambda((2,2),t)$ extends to a meromorphic function in $\mathbb{C}^r$ with simple poles for $\langle \lambda, \alpha \rangle = 0$ $(\forall \alpha \in \Sigma_n^+)$.

4.2.2. **The symmetric pair** $(\mathfrak{so}(r+2k,r), \mathfrak{so}(r+2k-1,1) \times \mathfrak{so}(r-1,1))$**.** Here $m_l = 1$ and

$$\widetilde{\Psi}_\lambda((1,0),t) = K_0(t_1\lambda_1) \sum_{\omega \in S_{r-1} \times \mathbb{Z}_2^{r-1}} \prod_{2 \le i \le r} K_0(t_i \omega(\lambda_i)).$$

Using the fact that $K_0(z)$ is a solution to

$$u'' + \frac{1}{z} u' = u,$$

we deduce that[1]

$$\begin{aligned}
&\mathbb{G}(2 \cdot 1_{\mathscr{O}_s}, (1,0)) \widetilde{\Psi}_\lambda((1,0),t) \\
&\quad = \frac{K_0(\lambda_1 t_1) \prod\limits_{2 \le i \le r} (\lambda_1^2 - \lambda_i^2)}{\prod\limits_{1 \le i < j \le r} (t_i^2 - t_j^2)} \sum_{\sigma \in S_{r-1}} \prod_{2 \le i < j \le r} (\lambda_{\sigma(i)}^2 - \lambda_{\sigma(j)}^2) \sum_{\varepsilon \in \mathbb{Z}_2^{r-1}} \prod_{2 \le i \le r} K_0(t_i \varepsilon_i \lambda_{\sigma(i)}) \\
&\quad = \frac{K_0(\lambda_1 t_1) \prod\limits_{1 \le i < j \le r} (\lambda_i^2 - \lambda_j^2)}{\prod\limits_{1 \le i < j \le r} (t_i^2 - t_j^2)} \sum_{\sigma \in S_{r-1}} (-1)^\sigma \prod_{2 \le i \le r} K_0(t_i \lambda_{\sigma(i)}) + K_0(-t_i \lambda_{\sigma(i)}) \\
&\quad = \frac{K_0(\lambda_1 t_1) \prod\limits_{1 \le i < j \le r} (\lambda_i^2 - \lambda_j^2)}{\prod\limits_{1 \le i < j \le r} (t_i^2 - t_j^2)} \det_{2 \le i,j \le r} \Big(K_0(t_i \lambda_j) + K_0(-t_i \lambda_j) \Big)
\end{aligned}$$

This leads to

$$\begin{aligned}
&\Psi_\lambda((1,2k),t) \\
&\quad = \frac{2^{2kr(r-1)}}{\prod\limits_{2 \le i < j \le r} (\lambda_i^2 - \lambda_j^2)^{2k-2} \prod\limits_{2 \le i \le r} (\lambda_1^2 - \lambda_i^2)^{2k-1}} \mathbb{G}(2 \cdot 1_{\mathscr{O}_s}, (1,2k-2)) \circ \cdots \circ \mathbb{G}(2 \cdot 1_{\mathscr{O}_s}, (1,2)) \\
&\qquad \Bigg\{ \frac{K_0(\lambda_1 t_1) \det\limits_{2 \le i,j \le r} \Big(K_0(t_i \lambda_j) + K_0(-t_i \lambda_j) \Big)}{\prod\limits_{1 \le i < j \le r} (t_i^2 - t_j^2) \prod\limits_{2 \le i < j \le r} (\lambda_i - \lambda_j) \prod\limits_{2 \le i < j \le r} (\lambda_i + \lambda_j)} \Bigg\}.
\end{aligned}$$

By Lemma 3.1, the function $\lambda \mapsto \Psi_\lambda((1,2k),t)$ extends to a meromorphic function in

$$\mathfrak{D} = \{\lambda \in \mathbb{C}^r : \ \lambda_i \in \mathbb{C} \backslash]\infty, 0]\},$$

with the following possible poles:

$$\begin{array}{lll}
\lambda_1 \pm \lambda_j = 0 & (2 \le j \le r), & \text{of order} \le 2k-1, \\
\lambda_i \pm \lambda_j = 0 & (2 \le i < j \le r), & \text{of order} \le 2k-2.
\end{array}$$

Thus we have proved:

[1] $K_0(-z) = K_0(z) - i\pi I_0(z)$ (cf. [25, p. 428])

Proposition 4.9. *For the symmetric pair* $(\mathfrak{so}(r+2k,r),\mathfrak{so}(r+2k-1,1)\times\mathfrak{so}(r-1,1))$, *we have*

$$\begin{aligned}\Psi_\lambda((1,2k),t) &= \frac{2^{2kr(r-1)}}{\prod\limits_{1\le i<j\le r}(\lambda_i^2-\lambda_j^2)^{2k-1}}\,\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(1,2k-2))\circ\\ &\circ\cdots\circ\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(1,2))\Big\{\frac{K_0(\lambda_1t_1)\det\limits_{2\le i,j\le r}\Big(K_0(t_i\lambda_j)+K_0(-t_i\lambda_j)\Big)}{\prod\limits_{1\le i<j\le r}(t_i^2-t_j^2)}\Big\},\end{aligned}$$

where $\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(1,2\ell))$, *for* $1\le\ell\le k-1$, *denotes the differential operator (??). For fixed* $t\in\mathfrak{a}_-$, *the map* $\lambda\mapsto\Psi_\lambda((1,2k),t)$ *extends to a meromorphic function in the set* $\mathfrak{D}=\{\lambda\in\mathbb{C}^r\;:\;\lambda_i\in\mathbb{C}\backslash]\infty,0]\}$ *with the possible poles for* $\langle\lambda,\alpha\rangle=0$ $(\forall\alpha\in\Sigma_c^+)$ *and* $\langle\lambda,\alpha\rangle=0$ $(\forall\alpha\in\Sigma_n^+)$, *of orders less or equal than* $2k-2$ *and* $2k-1$ *respectively.*

Example 4.10. The Bessel function associated with $(\mathfrak{so}(r+2,r),\mathfrak{so}(r+1,1)\times\mathfrak{so}(r-1,1))$ is given by

$$\Psi_\lambda((1,2),t)=2^{2r(r-1)}\frac{K_0(\lambda_1t_1)\det\limits_{2\le i,j\le r}\Big(K_0(t_i\lambda_j)+K_0(-t_i\lambda_j)\Big)}{\prod\limits_{1\le i<j\le r}(\lambda_i^2-\lambda_j^2)\prod\limits_{1\le i<j\le r}(t_i^2-t_j^2)}.$$

The function $\lambda\mapsto\Psi_\lambda((1,2),t)$ extends to a meromorphic function in $\mathfrak{D}$ with simple poles for $\lambda_1\pm\lambda_i=0$, with $2\le i\le r$.

4.3. **The cases with restricted root system of type C_r.** When Σ is a root system of type C_r, we have

$$\begin{aligned}\Psi_\lambda((m_l,m_s),t) &= (-1)^{\frac{sr(r-1)}{4}}2^{m_sr(r-1)}\prod_{1\le i<j\le r}(\lambda_i^2-\lambda_j^2)^{-m_s} \\ &\qquad \mathbb{G}(2\cdot 1_{\mathscr{O}_s},(m_l,m_s-2))\circ\cdots\circ\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(m_l,0))\widetilde{\Psi}_\lambda((m_l,0),t).\end{aligned}\tag{4.12}$$

The shift operator $\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(m_l,0))$ is the same as in (??).

4.3.1. **The symmetric pair $(\mathfrak{su}(r,r),\mathfrak{sl}(r,\mathbb{C})\times\mathbb{R})$.** Considerations similar to the case 4.2.2 yield

$$\begin{aligned}\Psi_\lambda((1,2),t) &= \frac{(-1)^{\frac{r(r-1)}{2}}2^{2r(r-1)}}{\prod\limits_{1\le i<j\le r}(\lambda_i^2-\lambda_j^2)^2\prod\limits_{1\le i<j\le r}(t_i^2-t_j^2)}\sum_{\sigma\in S_r}\prod_{1\le i<j\le r}(\lambda_{\sigma(i)}^2-\lambda_{\sigma(j)}^2)\prod_{i=1}^rK_0(t_i\lambda_{\sigma(i)})\\ &= \frac{(-1)^{\frac{r(r-1)}{2}}2^{2r(r-1)}}{\prod\limits_{1\le i<j\le r}(\lambda_i+\lambda_j)\prod\limits_{1\le i<j\le r}(t_i^2-t_j^2)}\Big\{\frac{\det\limits_{1\le i,j\le r}(K_0(t_i\lambda_j))}{\prod\limits_{1\le i<j\le r}(\lambda_i-\lambda_j)}\Big\}.\end{aligned}$$

By virtue of Lemma 3.1, the Bessel function associated with $(\mathfrak{su}(r,r),\mathfrak{sl}(r,\mathbb{C})\times\mathbb{R})$ extends to a meromorphic function of λ in

$$\mathfrak{D}=\{\lambda\in\mathbb{C}^r:\;\lambda_i\in\mathbb{C}\backslash]\infty,0]\}$$

with simple poles for $\lambda_i+\lambda_j=0$ $(1\le i<j\le r)$.

Proposition 4.11. *For the symmetric pair $(\mathfrak{su}(r,r), \mathfrak{sl}(r,\mathbb{C})\times\mathbb{R})$, we have*

$$\Psi_\lambda((1,2),t) = (-1)^{\frac{r(r-1)}{2}} 2^{2r(r-1)} \frac{\det\limits_{1\le i,j\le r}(K_0(t_i\lambda_j))}{\prod\limits_{1\le i<j\le r}(\lambda_i^2-\lambda_j^2)\prod\limits_{1\le i<j\le r}(t_i^2-t_j^2)}.$$

For fixed $t\in\mathfrak{a}_-$, the map $\lambda\mapsto\Psi_\lambda((1,2),t)$ extends to a meromorphic function in the set $\mathfrak{D}=\{\lambda\in\mathbb{C}^r:\ \lambda_i\in\mathbb{C}\backslash]\infty,0]\}$ with simple poles for $\langle\lambda,\alpha\rangle=0$ $(\forall\alpha\in\Sigma_n^+)$.

4.3.2. **The symmetric pair $(\mathfrak{so}^*(4r), \mathfrak{su}^*(2r)\times\mathbb{R})$.** A similar calculation to one illustrated above gives

$$\begin{aligned}\Psi_\lambda((4,1),t) &= \frac{2^{4r(r-1)}}{\prod\limits_{1\le i<j\le r}(\lambda_i-\lambda_j)^2\prod\limits_{1\le i<j\le r}(\lambda_i+\lambda_j)^3}\\ &\qquad\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(1,2))\Big\{\frac{\det\limits_{1\le i,j\le r}(K_0(\lambda_i t_j))}{\prod\limits_{1\le i<j\le r}(t_i^2-t_j^2)\prod\limits_{1\le i<j\le r}(\lambda_i-\lambda_j)}\Big\}.\end{aligned}$$

Thus, the function $\lambda\mapsto\Psi_\lambda((4,1),t)$ extends meromorphically in

$$\mathfrak{D}=\{\lambda\in\mathbb{C}^r:\ \lambda_i\in\mathbb{C}\backslash]\infty,0]\}$$

with the following possible poles

$$\begin{array}{lll}\lambda_i-\lambda_j=0 & (1\le i<j\le r), & \text{of order}\le 2,\\ \lambda_i+\lambda_j=0 & (1\le i<j\le r), & \text{of order}\le 3.\end{array}$$

Proposition 4.12. *For the symmetric pair $(\mathfrak{so}^*(4r), \mathfrak{su}^*(2r)\times\mathbb{R})$, we have*

$$\Psi_\lambda((0,2),t) = \frac{2^{4r(r-1)}}{\prod\limits_{1\le i<j\le r}(\lambda_i^2-\lambda_j^2)^3}\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(1,2))\Big\{\frac{\det\limits_{1\le i,j\le r}(K_0(\lambda_i t_j))}{\prod\limits_{1\le i<j\le r}(t_i^2-t_j^2)}\Big\},$$

where $\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(1,2))$ denotes the differential operator (??). For fixed $t\in\mathfrak{a}_-$, the map $\lambda\mapsto\Psi_\lambda((0,2),t)$ extends to a meromorphic function in $\mathfrak{D}=\{\lambda\in\mathbb{C}^r:\ \lambda_i\in\mathbb{C}\backslash]\infty,0]\}$ with the possible poles for $\langle\lambda,\alpha\rangle=0$ $(\forall\alpha\in\Sigma_c^+)$ and $\langle\lambda,\alpha\rangle=0$ $(\forall\alpha\in\Sigma_n^+)$, of orders less or equal than 2 and 3 respectively.

4.3.3. **The symmetric pair $(\mathfrak{sp}(r,\mathbb{C}), \mathfrak{sp}(r,\mathbb{R}))$.** For the present case

$$\widetilde{\Psi}_\lambda((2,0),t) = \sum_{\sigma\in S_r}\prod_{i=1}^r(2t_i\lambda_{\sigma(i)})^{-1}e^{-t_i\lambda_{\sigma(i)}},$$

and

$$\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(2,0))\widetilde{\Psi}_\lambda((2,0),t) = \frac{\prod\limits_{1\le i<j\le r}(\lambda_i^2-\lambda_j^2)}{\prod\limits_{1\le i<j\le r}(t_i^2-t_j^2)}\sum_{\sigma\in S_r}\varepsilon(\sigma)\prod_{i=1}^r\frac{e^{-t_i\lambda_{\sigma(i)}}}{2t_i\lambda_{\sigma(i)}}.$$

Now formula (??) reads

$$\Psi_\lambda((2,2),t)=\frac{(-1)^{\frac{r(r-1)}{2}}2^{r(r-1)-r}}{\prod\limits_{1\le i\le r}\lambda_i\prod\limits_{1\le i<j\le r}(\lambda_i+\lambda_j)\prod\limits_{1\le i\le r}t_i\prod\limits_{1\le i<j\le r}(t_i^2-t_j^2)}\Big\{\frac{\det\limits_{1\le i,j\le r}(e^{-\lambda_i t_j})}{\prod\limits_{1\le i<j\le r}(\lambda_i-\lambda_j)}\Big\}.$$

Hence, the map $\lambda\mapsto\Psi_\lambda((2,2),t)$ extends to a meromorphic function in $\mathbb{C}^r$ with simple poles for $\lambda_i=0$ $(1\le i\le r)$ and $\lambda_i+\lambda_j=0$ $(1\le i<j\le r)$.

Proposition 4.13. *For the symmetric pair* $(\mathfrak{sp}(r,\mathbb{C}),\mathfrak{sp}(r,\mathbb{R}))$, *we have*

$$\Psi_\lambda((2,2),t)=(-1)^{\frac{r(r-1)}{2}}2^{r(r-1)-r}\frac{\det\limits_{1\le i,j\le r}(e^{-\lambda_i t_j})}{\prod\limits_{1\le i\le r}t_i\prod\limits_{1\le i\le r}\lambda_i\prod\limits_{1\le i<j\le r}(t_i^2-t_j^2)\prod\limits_{1\le i<j\le r}(\lambda_i^2-\lambda_j^2)}.$$

For fixed $t\in\mathfrak{a}_-$, *the map* $\lambda\mapsto\Psi_\lambda((2,2),t)$ *extends to a meromorphic function in* $\mathbb{C}^r$ *with simple poles for* $\langle\lambda,\alpha\rangle=0$ $(\forall\alpha\in\Sigma_n^+)$.

4.3.4. **The symmetric pair** $(\mathfrak{sp}(r,r),\mathfrak{sp}(r,\mathbb{C}))$**.** Here we have $m_l=3$, and therefore

$$\widetilde{\Psi}_\lambda((3,0),t)=\pi^{-r/2}\sum_{\sigma\in S_r}\prod_{i=1}^r(2t_i\lambda_{\sigma(i)})^{-1}K_1(t_i\lambda_{\sigma(i)}).$$

Using the fact that

$$\mathscr{L}_3(t,dt)\left[\frac{K_1(\lambda t)}{\lambda t}\right]=\frac{\lambda^2}{2\lambda t}K_1(\lambda t),$$

we deduce that

$$\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(3,0))\widetilde{\Psi}_\lambda((3,0),t)=2^{-r}\pi^{-r/2}\prod_{1\le i<j\le r}\left(\frac{\lambda_i^2-\lambda_j^2}{t_i^2-t_j^2}\right)\det_{1\le i,j\le r}\left(\frac{K_1(t_i\lambda_j)}{t_i\lambda_j}\right).$$

Hence the Bessel function can be written as

$$\begin{aligned}\Psi_\lambda((3,4),t)\quad=&\quad\frac{2^{r(4r-1)}\pi^{-r/2}}{\prod\limits_{1\le i<j\le r}(\lambda_i+\lambda_j)^3\prod\limits_{1\le i<j\le r}(\lambda_i-\lambda_j)^2\prod\limits_{1\le i\le r}\lambda_i^2}\\ &\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(3,2))\Big\{\frac{\det\limits_{1\le i,j\le r}(\lambda_iK_1(t_i\lambda_j))}{\prod\limits_{1\le i\le r}t_i\prod\limits_{1\le i<j\le r}(t_i^2-t_j^2)\prod\limits_{1\le i<j\le r}(\lambda_i-\lambda_j)}\Big\}.\end{aligned}$$

It follows that the function $\lambda\mapsto\Psi_\lambda((3,4),t)$ extends meromorphically in

$$\mathfrak{D}=\{\lambda\in\mathbb{C}^r:\ \lambda_i\in\mathbb{C}\backslash]\infty,0[\}$$

with the following possible poles:

$$\begin{array}{lll}\lambda_i=0 & (1\le i\le r) & \text{of order}\le 2\\ \lambda_i-\lambda_j=0 & (1\le i<j\le r) & \text{of order}\le 2\\ \lambda_i+\lambda_j=0 & (1\le i<j\le r) & \text{of order}\le 3.\end{array}$$

Proposition 4.14. *For the symmetric pair* $(\mathfrak{sp}(r,r), \mathfrak{sp}(r,\mathbb{C}))$, *we have*

$$\Psi_\lambda((3,4),t) = \frac{2^{r(4r-1)}\pi^{-r/2}}{\prod_{1\le i<j\le r}(\lambda_i^2-\lambda_j^2)^3 \prod_{1\le i\le r}\lambda_i}\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(3,2))\Big\{\frac{\det_{1\le i,j\le r}(K_1(\lambda_i t_j))}{\prod_{1\le i\le r} t_i \prod_{1\le i<j\le r}(t_i^2-t_j^2)}\Big\},$$

where $\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(3,2))$ *denotes the differential operator (??). For fixed* $t\in\mathfrak{a}_-$, *the map* $\lambda\mapsto\Psi_\lambda((3,4),t)$ *extends to a meromoprhic function on* $\mathfrak{D}=\{\lambda\in\mathbb{C}^r:\ \lambda_i\in\mathbb{C}\backslash]\infty,0[\}$ *with the possible poles for* $\langle\lambda,\alpha\rangle=0$ $(\forall\alpha\in\Sigma_c^+)$ *and* $\langle\lambda,\alpha\rangle=0$ $(\forall\alpha\in\Sigma_n^+)$, *of orders less or equal than* 2 *and* 3 *respectively.*

4.3.5. **The symmetric pair** $(\mathfrak{e}_{7(-25)},\mathfrak{e}_{6(-26)}\times\mathbb{R})$**.** For this pair, recall that $r=3$. Clearly we have

$$\Psi_\lambda((1,8),t) = \frac{2^{72}\pi^{-3/2}}{\prod_{1\le i<j\le 3}(\lambda_i+\lambda_j)^7\prod_{1\le i<j\le 3}(\lambda_i-\lambda_j)^6}$$
$$\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(1,6))\circ\cdots\circ\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(1,2))\Big\{\frac{\det_{1\le i,j\le 3}(K_0(t_i\lambda_j))}{\prod_{1\le i<j\le 3}(t_i^2-t_j^2)\prod_{1\le i<j\le 3}(\lambda_i-\lambda_j)}\Big\}.$$

Thus, the function $\lambda\mapsto\Psi_\lambda((1,8),t)$ extends to a meromorphic function in

$$\mathfrak{D}=\{\lambda\in\mathbb{C}^3:\ \lambda_i\in\mathbb{C}\backslash]\infty,0]\}$$

with the following possible poles

$$\begin{array}{lll}\lambda_i-\lambda_j=0 & (1\le i<j\le 3), & \text{of order}\le 6\\ \lambda_i+\lambda_j=0 & (1\le i<j\le 3), & \text{of order}\le 7.\end{array}$$

Proposition 4.15. *For the symmetric pair* $(\mathfrak{e}_{7(-25)},\mathfrak{e}_{6(-26)}\times\mathbb{R})$, *we have*

$$\Psi_\lambda((1,8),t) = \frac{2^{72}\pi^{-3/2}}{\prod_{1\le i<j\le 3}(\lambda_i^2-\lambda_j^2)^7}$$
$$\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(1,6))\circ\cdots\circ\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(1,2))\Big\{\frac{\det_{1\le i,j\le 3}(K_0(t_i\lambda_j))}{\prod_{1\le i<j\le 3}(t_i^2-t_j^2)}\Big\},$$

where $\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(1,2\ell))$, *for* $1\le\ell\le 3$, *denotes the differential operator (??). For fixed* $t\in\mathfrak{a}_-$, *the map* $\lambda\mapsto\Psi_\lambda((1,8),t)$ *extends to a meromorphic function in* $\mathfrak{D}=\{\lambda\in\mathbb{C}^3:\ \lambda_i\in\mathbb{C}\backslash]\infty,0]\}$ *with the possible poles for* $\langle\lambda,\alpha\rangle=0$ $(\forall\alpha\in\Sigma_c^+)$ *and* $\langle\lambda,\alpha\rangle=0$ $(\forall\alpha\in\Sigma_n^+)$, *of orders less or equal than* 2 *and* 3 *respectively.*

4.4. **The cases with restricted root system of type** D_r**.** For both pairs $(\mathfrak{so}(2r,\mathbb{C}),\mathfrak{so}^*(2r))$ and $(\mathfrak{so}(2r,\mathbb{C}),\mathfrak{so}(2r-2,2))$, we have $m_l=0$ and $m_s=2$. It follows that

$$\Psi_\lambda((0,2),t) = \frac{(-1)^{|\Sigma_c^+\cap\mathscr{O}_s|}2^{2r(r-1)}}{\prod_{1\le i<j\le r}(\lambda_i^2-\lambda_j^2)^2}\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(0,0))\Big(\sum_{\omega\in W_0}\prod_{i=1}^r e^{-t_i\omega(\lambda_i)}\Big), \tag{4.13}$$

where

$$\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(0,0))=\prod_{1\le i<j\le r}\left(\frac{\partial_i^2-\partial_j^2}{t_i^2-t_j^2}\right). \tag{4.14}$$

4.4.1. **The symmetric pair $(\mathfrak{so}(2r,\mathbb{C}),\mathfrak{so}^*(2r))$.** It is easy to check that

$$\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(0,0))\widetilde{\Psi}_\lambda((0,0),t)=\prod_{1\le i<j\le r}\left(\frac{\lambda_i^2-\lambda_j^2}{t_i^2-t_j^2}\right)\det_{1\le i,j\le r}(e^{-t_i\lambda_j}).$$

Now formula (3.3) implies that

$$\Psi_\lambda((0,2),t)=\frac{(-1)^{\frac{r(r-1)}{2}}2^{2r(r-1)}}{\prod_{1\le i<j\le r}(t_i^2-t_j^2)\prod_{1\le i<j\le r}(\lambda_i+\lambda_j)}\left\{\frac{\det_{1\le i,j\le r}(e^{-t_i\lambda_j})}{\prod_{1\le i<j\le r}(\lambda_i-\lambda_j)}\right\},$$

and the function $\lambda\mapsto\Psi_\lambda((0,2),t)$ extends to a meromorphic function in $\mathbb{C}^r$ with simple poles for $\lambda_i+\lambda_j=0$ $(1\le i<j\le r)$.

Proposition 4.16. *For the symmetric pair $(\mathfrak{so}(2r,\mathbb{C}),\mathfrak{so}^*(2r))$, we have*

$$\Psi_\lambda((0,2),t)=(-1)^{\frac{r(r-1)}{2}}2^{2r(r-1)}\frac{\det_{1\le i,j\le r}(e^{-t_i\lambda_j})}{\prod_{1\le i<j\le r}(\lambda_i^2-\lambda_j^2)\prod_{1\le i<j\le r}(t_i^2-t_j^2)}.$$

For fixed $t\in\mathfrak{a}_-$, the map $\lambda\mapsto\Psi_\lambda((0,2),t)$ extends to a meromorphic function in $\mathbb{C}^r$ with simple poles for $\langle\lambda,\alpha\rangle=0$ $(\forall\alpha\in\Sigma_n^+)$.

4.4.2. **The symmetric pair $(\mathfrak{so}(2r,\mathbb{C}),\mathfrak{so}(2r-2,2))$.** Applying the shift operator (??) to the function

$$\widetilde{\Psi}_\lambda((0,0),t)=e^{-t_1\lambda_1}\sum_{w\in S_{r-1}\times\mathbb{Z}_{2,\mathrm{even}}^{r-1}}\prod_{i=2}^{r}e^{-t_i w(\lambda_i)}$$

gives

$$\begin{aligned}
&\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(0,0))\widetilde{\Psi}_\lambda((0,0),t)\\
&\quad=2^{r-1}e^{-t_1\lambda_1}\prod_{1\le i<j\le r}\left(\frac{\lambda_i^2-\lambda_j^2}{t_i^2-t_j^2}\right)\sum_{\epsilon\in\mathbb{Z}_{2,\mathrm{even}}^{r-1}}\sum_{\sigma\in S_{r-1}}(-1)^\sigma\prod_{i=2}^{r}e^{-t_i\epsilon_{\sigma(i)}\lambda_{\sigma(i)}}\\
&\quad=2^{r-1}e^{-t_1\lambda_1}\prod_{1\le i<j\le r}\left(\frac{\lambda_i^2-\lambda_j^2}{t_i^2-t_j^2}\right)\Big(\det_{2\le i,j\le r}(\mathrm{ch}(t_i\lambda_j))+(-1)^r\det_{2\le i,j\le r}(\mathrm{sh}(t_i\lambda_j))\Big).
\end{aligned}$$

Hence formula (3.3) reads

$$\Psi_\lambda((0,2),t)=\frac{2^{(2r+1)(r-1)}(-1)^{\frac{r(r-1)}{2}}}{\prod_{1\le i<j\le r}(t_i^2-t_j^2)}\frac{e^{-t_1\lambda_1}}{\prod_{2\le j\le r}(\lambda_1^2-\lambda_j^2)}\left\{\frac{\det_{2\le i,j\le r}(\mathrm{ch}(t_i\lambda_j))+(-1)^r\det_{2\le i,j\le r}(\mathrm{sh}(t_i\lambda_j))}{\prod_{2\le i<j\le r}(\lambda_i^2-\lambda_j^2)}\right\}.$$

It is now clear that the function $\lambda\mapsto\Psi_\lambda((0,2),t)$ extends to a meromorphic function in $\mathbb{C}^r$ with simple poles for $\lambda_1\pm\lambda_j=0$ $(2\le j\le r)$.

Proposition 4.17. *For the symmetric pair* $(\mathfrak{so}(2r,\mathbb{C}),\mathfrak{so}(2r-2,2))$*, we have*

$$\Psi_\lambda((0,2),t)=2^{(2r+1)(r-1)}(-1)^{\frac{r(r-1)}{2}}\frac{e^{-t_1\lambda_1}\Big(\det\limits_{2\le i,j\le r}(\mathrm{ch}(t_i\lambda_j))+(-1)^r\det\limits_{2\le i,j\le r}(\mathrm{sh}(t_i\lambda_j))\Big)}{\prod\limits_{1\le i<j\le r}(t_i^2-t_j^2)\prod\limits_{1\le i<j\le r}(\lambda_i^2-\lambda_j^2)}.$$

For $t\in\mathfrak{a}_-$*, the map* $\lambda\mapsto\Psi_\lambda((0,2),t)$ *extends to a meromorphic functions in* $\mathbb{C}^r$ *with simple poles for* $\langle\lambda,\alpha\rangle=0$ $(\forall\alpha\in\Sigma_n^+)$.

5. The main theorem

For $m\in\mathbb{N}$, denote by $\mathfrak{D}_m$ the complex plane for which the function $z\mapsto z^{-\frac{m}{2}+\frac{1}{2}}K_{\frac{m}{2}-\frac{1}{2}}(z)$ is analytic.

Taking all our results from the previous discussions together, the following theorem holds.

Theorem 5.1. (i) *Assume that* $(\mathfrak{g},\mathfrak{h})$ *is one of the following NCC symmetric pairs:*

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{sl}(p+q,\mathbb{C})$	$\mathfrak{su}(p,q)$
$\mathfrak{so}(2r+1,\mathbb{C})$	$\mathfrak{so}(2r-1,2)$
$\mathfrak{su}(r,r)$	$\mathfrak{sl}(r,\mathbb{C})\times\mathbb{R}$
$\mathfrak{sp}(r,\mathbb{C})$	$\mathfrak{sp}(r,\mathbb{R})$
$\mathfrak{so}(2r,\mathbb{C})$	$\mathfrak{so}^*(2r)$
$\mathfrak{so}(2r,\mathbb{C})$	$\mathfrak{so}(2r-2,2)$

Then the function $\lambda\mapsto\Psi_\lambda((m_l,m_s),t)$ *extends meromorphically in the product domain* $\mathfrak{D}_{m_l}^r$ *with poles for*

$$\langle\lambda,\alpha\rangle=0\qquad(\forall\alpha\in\Sigma_n^+)$$

of order m_n-1.

(ii) *Assume that* $(\mathfrak{g},\mathfrak{h})$ *is one of the following NCC symmetric pairs:*

$\mathfrak{g}$	$\mathfrak{h}$
$\mathfrak{su}^*(2(p+q))$	$\mathfrak{sp}(p,q)$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_{4(-20)}$
$\mathfrak{so}(r+2k,r)$	$\mathfrak{so}(r+2k-1,1)\times\mathfrak{so}(r-1,1)$
$\mathfrak{so}^*(4r)$	$\mathfrak{su}^*(2r)\times\mathbb{R}$
$\mathfrak{sp}(r,r)$	$\mathfrak{sp}(r,\mathbb{C})$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)}\times\mathbb{R}$

Then the function $\lambda\mapsto\Psi_\lambda((m_l,m_s),t)$ *extends meromorphically in the product domain* $\mathfrak{D}_{m_l}^r$ *with the following possible poles*

$$\begin{cases}\langle\lambda,\alpha\rangle=0 & (\forall\alpha\in\Sigma_c^+), \quad \text{of order}\le m_c-2,\\ \langle\lambda,\alpha\rangle=0 & (\forall\alpha\in\Sigma_n^+), \quad \text{of order}\le m_n-1.\end{cases}$$

Here m_c *and* m_n *denote the multiplicity of the compact and non-compact roots respectively.*

Recall that for the symmetric pairs listed in part (i) of Theorem 5.1, the multiplicity of the compact roots is equal to 2. Motivated by the main result above and Example 4.5 we conjecture that:

Conjecture 5.2. *Let* $(\mathfrak{g},\mathfrak{h})$ *be a NCC symmetric pair with restricted root system of the non-exceptional type such that* **(H)** *holds. Then the function* $\lambda \mapsto \Psi_\lambda((m_l,m_s),t)$*, for fixed* $t\in\mathfrak{a}_-$*, extends to a meromorphic function in the product* $\mathfrak{D}^r_{m_l}$ *with poles for*

$$\begin{cases} \langle\lambda,\alpha\rangle=0 \quad (\forall\alpha\in\Sigma_c^+), & \textit{of order} = m_c-2,\\ \langle\lambda,\alpha\rangle=0 \quad (\forall\alpha\in\Sigma_n^+), & \textit{of order} = m_n-1.\end{cases}$$

6. Concluding remark

Let $\mathfrak{k}^d := (\mathfrak{k}\cap\mathfrak{h})\oplus i(\mathfrak{p}\cap\mathfrak{h})$ and $\mathfrak{p}^d := i(\mathfrak{k}\cap\mathfrak{q})\oplus(\mathfrak{p}\cap\mathfrak{q})$. Then $\mathfrak{g}^d := \mathfrak{k}^d\oplus\mathfrak{p}^d$ is a semisimple Lie algebra with Cartan involution $\theta_{|\mathfrak{g}^d}$, and $\mathfrak{k}^d$ is the corresponding maximal compactly embedded subalgebra. Denote by G^d and K^d the analytic Lie subgroups of $G_\mathbb{C}$ with Lie algebras $\mathfrak{g}^d$ and $\mathfrak{k}^d$ respectively. The symmetric space G^d/K^d is the so-called Riemannian dual of G/H. The NCC structure of G/H implies that $G^d/K^d\simeq G/K$.

Recall that the maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}\cap\mathfrak{q}$ is also maximal in $\mathfrak{p}$ and $\mathfrak{q}$, i.e. a Cartan subspace for both G/H and $G^d/K^d\simeq G/K$. For $\lambda\in\mathfrak{a}^*_\mathbb{C}$ and $X\in\mathfrak{p}$, let

$$\Phi_\lambda(m,X)=\int_K e^{\lambda(\mathrm{Ad}(k)X)}dk.$$

The function $\Phi_\lambda(m,\cdot)$ is the so-called Bessel function on the flat symmetric space $K\ltimes\mathfrak{p}/K\simeq\mathfrak{p}$ with spectral parameter λ. It is well known that the function $\lambda\mapsto\Phi_\lambda(m,X)$ is holomorphic on $\mathfrak{a}^*_\mathbb{C}$. We refer to [13] for more details on the theory of Bessel functions on $\mathfrak{p}$. See also [22].

Set $\rho_n(m) := \sum_{\alpha\in\Sigma_n^+} m_\alpha/2$, and recall the identifications of $\mathfrak{a}$ and $\mathfrak{a}^*_\mathbb{C}$ with $\mathbb{R}^r$ and $\mathbb{C}^r$. In [6], it is shown that for $t\in\mathfrak{a}_-$, the relation

$$\Phi_{-\lambda}(m,t)=\frac{(-1)^{\rho_n(m)}}{\widetilde{c}_0(m,\rho(m))}\sum_{\omega\in W\backslash W_0}\Psi_{\omega(\lambda)}(m,t) \tag{6.1}$$

holds as equality of meromorphic functions in $\mathbb{C}^r$.

Henceforth we will assume that $\Sigma=\Sigma(\mathfrak{g},\mathfrak{a})$ is a restricted root system of the non-exceptional type such that the hypothesis **(H)** holds. In view of formula (3.3) and the fact that m_s is even, it follows from (??) that

$$\begin{aligned}\Phi_{-\lambda}((m_l,m_s),t) \;=\;& \frac{(-1)^{\frac{m_s}{2}|\Sigma_c^+\cap\mathscr{O}_s|+\rho_n(m)}\,2^{m_s|\Sigma^+\cap\mathscr{O}_s|}}{\widetilde{c}_0(m,\rho(m))}\prod_{\alpha\in\Sigma^+\cap\mathscr{O}_s}\langle\lambda,\alpha\rangle^{-m_s}\\ & \mathbb{G}(2\cdot 1_{\mathscr{O}_s},(m_l,m_s-2))\circ\cdots\circ\mathbb{G}(2\cdot 1_{\mathscr{O}_s},(m_l,0))\widetilde{\Phi}_\lambda((m_l,0),t),\end{aligned}$$

where

$$\widetilde{\Phi}_\lambda((m_l,0),t)=\pi^{-r/2}\sum_{\omega\in W}\prod_{i=1}^r(2t_i\omega(\lambda_i))^{-\frac{m_l}{2}+\frac12}K_{\frac{m_l}{2}-\frac12}(t_i\omega(\lambda_i)).$$

Now one may continue the discussion as in Section 4 to obtain explicit formulas for the Bessel functions $\Phi_\lambda((m_l,m_s),t)$ associated with the following Riemannian symmetric pairs:

Σ	$\mathfrak{g}$	$\mathfrak{k}$
A_{r-1}	$\mathfrak{sl}(r,\mathbb{C})$	$\mathfrak{su}(r)$
	$\mathfrak{su}^*(2r)$	$\mathfrak{sp}(r)$
	$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_4$
B_r	$\mathfrak{so}(2r+1,\mathbb{C})$	$\mathfrak{so}(2r+1)$
	$\mathfrak{so}(r+2k,r)$	$\mathfrak{so}(r+2k)\times\mathfrak{so}(r)$
C_r	$\mathfrak{su}(r,r)$	$\mathfrak{s}(\mathfrak{u}(r)\oplus\mathfrak{u}(r))$
	$\mathfrak{so}^*(4r)$	$\mathfrak{su}(2r)$
	$\mathfrak{sp}(r,\mathbb{C})$	$\mathfrak{sp}(r)$
	$\mathfrak{sp}(r,r)$	$\mathfrak{sp}(r)\times\mathfrak{sp}(r)$
	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_6\oplus\mathbb{R}$
D_r	$\mathfrak{so}(2r,\mathbb{C})$	$\mathfrak{so}(2r)$

This result generalizes [17, 27, 2, 5], where the authors give explicit formulas for $\Phi_\lambda((m_l,m_s),t)$ only for some Riemannian symmetric pairs or for particular values of r.

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