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Abstract

In this paper, we give an optimal lower bound for the eigenvalues of the basic Dirac operator on a quaternion-Kähler foliations. The limiting case is characterized by the existence of quaternion-Kähler Killing spinors. We end this paper by giving some examples.

Key words: Basic Dirac operator, quaternion-Kähler foliations, eigenvalues, quaternion-Kähler Killing spinors.

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1 Introduction

On a compact quaternion-Kähler spin manifold (M, g) of dimension $4m \geq 8$, O. Hijazi and J.-L. Milhorat [13] conjectured that any eigenvalue of the Dirac operator satisfies

$$\lambda^2 \geq \frac{m+3}{4(m+2)} S, \tag{1.1}$$

where S denotes the constant scalar curvature (such manifolds are Einstein [1]). They proved that (1.1) is true for $m = 2$ and $m = 3$. For this, they introduced [14] the *twistor operator*, as in the Kähler case, on each eigenbundle associated with the eigenvalues of the fundamental 4-form Ω [12]. Using representation theory, the lower bound (1.1) is established by W. Kramer,

U. Semmelmann and G. Weingart [21]. Their proof is based on the decomposition in two ways of the bundle $TM \otimes TM \otimes \Sigma M$ into parallel subbundles under the action of the group $\mathrm{Sp}_1 \times \mathrm{Sp}_m$.

On a compact Riemannian manifold $(M, g_M, \mathcal{F})$ with a spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M, S. D. Jung [4] gives a Friedrich-type inequality. For Kähler foliations, he also gives a Kirchberg-type inequality for odd complex dimensions [5] where the even case was proved by the author [7]. The main result of this paper is to prove the following theorem:

Theorem 1.1 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a quaternion-Kähler spin foliation $\mathcal{F}$ of codimension $q = 4m$ and a bundle-like metric g_M with a coclosed basic 1-form mean curvature κ. Then the foliation is minimal and any eigenvalue λ of the basic Dirac operator satisfies*

$$\lambda^2 \geq \frac{m+3}{4(m+2)}\sigma^{\nabla}, \tag{1.2}$$

where σ^{∇} denotes the transversal scalar curvature.

Our approach comes from an adaptation of [10] and [20] to the case of Riemannian foliations where the key point is to prove that the mean curvature vanishes since the transversal Ricci curvature is strictly positive. The limiting case is characterized by the existence of quaternion-Kähler Killing spinors (see section 5 for details).

We point out that throughout this paper, we consider a bundle-like metric such that the mean curvature is a basic 1-form and coclosed. The existence of such metric is assured in [3, 16].

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2 Spin Foliations

In this section, we summarize some standard facts about spin foliations. For details, we refer to [4], [6], [7], [17].

Let $(M, g_M, \mathcal{F})$ be a $(p+q)$-dimensional Riemannian manifold with a Riemannian foliation $\mathcal{F}$ of codimension q and let ∇^M be the Levi-civita connection associated with g_M. We denote by L the tangent bundle of TM and $Q = TM/L \simeq L^{\perp}$ the normal bundle and we assume g_M to be a *bundle-like metric* on Q, that means the induced metric g_Q verifies for all $X \in \Gamma(L)$ the holonomy invariance condition that is $\mathcal{L}_X g_Q = 0$, where $\mathcal{L}_X$ is the Lie derivative with respect to X. Let ∇ be the transversal Levi-Civita connection on

Q defined for all $Y \in \Gamma(Q)$ by

$$\nabla_X Y = \begin{cases} \pi[X,Y], & \forall X \in \Gamma(L) \ , \\ \pi(\nabla^M_X Y), & \forall X \in \Gamma(Q) \ , \end{cases}$$

where $\pi : TM \to Q$ denotes the projection. The curvature of ∇ acts on $\Gamma(Q)$ by :

$$R^{\nabla}(X,Y) = -\nabla_X \nabla_Y + \nabla_Y \nabla_X + \nabla_{[X,Y]}, \qquad \forall X, Y \in \chi(M) \, .$$

We denote by $\rho^{\nabla}, \sigma^{\nabla}$ the transversal Ricci curvature and the scalar curvature respectively associated with ∇. The foliation $\mathcal{F}$ is said to be transversally Einstein if and only if $\rho^{\nabla} = \frac{1}{q}\sigma^{\nabla}\mathrm{Id}$, with constant transversal scalar curvature. The mean curvature of $\mathcal{F}$ is given for all $X \in \Gamma(Q)$ by $\kappa(X) = g_Q(\tau, X)$, where τ is the trace of the second fundamental form II of $\mathcal{F}$ defined by:

$$\begin{array}{rccl} II \, : & \Gamma(L) \times \Gamma(L) & \longrightarrow & \Gamma(Q) \\ & (X,Y) & \longmapsto & II(X,Y) = \pi\left(\nabla^M_X Y\right) . \end{array}$$

We define basic r-forms by :

$$\Omega^r_B(\mathcal{F}) = \{\Phi \in \Lambda^r T^* M | \ X \lrcorner \Phi = 0 \quad \text{and} \quad X \lrcorner d\Phi = 0, \quad \forall X \in \Gamma(L)\} \, ,$$

where d is the exterior derivative and $X\lrcorner$ is the interior product. We denote by $d_B = d|_{\Omega_B(\mathcal{F})}$ where $\Omega_B(\mathcal{F}) = \oplus_{r=0}^{p+q} \Omega^r_B(\mathcal{F})$ and δ_B the adjoint operator of d_B with respect to the induced scalar product. The basic Laplacian is defined as $\Delta_B = d_B\delta_B + \delta_B d_B$. Now we prove the following theorem.

Theorem 2.1 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a Riemannian foliation $\mathcal{F}$ and a bundle-like metric g_M with a coclosed basic* 1-*form κ. Assume that the transversal Ricci curvature is strictly positive, then the mean curvature κ vanishes.*

Proof. In [8, 11], it is proved that the positivity of the transversal Ricci curvature implies the existence of a basic function h such that $\kappa = d_B h$. Then $\Delta_B h = \delta_B d_B h = \delta_B \kappa = 0$. Hence the harmonicity of h implies that the function h is closed, since M is compact. Thus the foliation is minimal. □

Now, we assume that the normal bundle Q carries a spin structure and we denote by $S(\mathcal{F})$ the foliated spinor bundle. The normal bundle acts on the spinor bundle by Clifford multiplication and the transversal Dirac operator [6] is locally given by:

$$D_{tr}\Psi = \sum_{i=1}^{q} e_i \cdot \nabla_{e_i}\Psi - \frac{1}{2}\kappa \cdot \Psi, \tag{2.3}$$

for all $\Psi \in \Gamma(S(\mathcal{F}))$. We can easily prove using Green's theorem [18] that this operator is formally self-adjoint. We define the subspace of basic sections $\Gamma_B(S(\mathcal{F}))$ by

$$\Gamma_B(S(\mathcal{F})) = \{\Psi \in \Gamma(S(\mathcal{F}))|\, \nabla_X\Psi = 0, \quad \forall X \in \Gamma(L)\}.$$

The transversal Dirac operator leaves $\Gamma_B(S(\mathcal{F}))$ invariant if and only if the foliation is isoparametric. Moreover the basic Dirac operator defined by $D_b = D_{tr}|\Gamma_B(S(\mathcal{F}))$, has a discrete spectrum [2] and if the foliation $\mathcal{F}$ is isoparametric with $\delta_B\kappa = 0$, we have the Schrödinger-Lichnerowicz formula for D_b [6]

$$D_b^2\Psi = \nabla^\star\nabla\Psi + \frac{1}{4}K_\sigma^\nabla\Psi,$$

where $K_\sigma^\nabla = \sigma^\nabla + |\kappa|^2$ and

$$\nabla^\star\nabla\Psi = -\sum_{i=1}^{q}\nabla^2_{e_i,e_i}\Psi + \nabla_\kappa\Psi,$$

with $\nabla^2_{X,Y} = \nabla_X\nabla_Y - \nabla_{\nabla_X Y}$, for all $X, Y \in \Gamma(TM)$.

3 Quaternion-Kähler Foliations

In this section, we review some basic relations on quaternion-Kähler spin foliations [13] also we give basic ingredients for the estimate which could be found in [10] .

A foliation $\mathcal{F}$ of codimension $q = 4m$ is said to be quaternion-Kähler if its principal bundle of oriented orthonormal frames SOQ admits a reduction P to the subgroup $\mathrm{Sp}_1\cdot\mathrm{Sp}_m := \mathrm{Sp}_1\times_{\mathbb{Z}_2}\mathrm{Sp}_m \subset \mathrm{SO}_{4m}$. This is equivalent to the existence of a subbundle E of $\mathrm{End}(Q)$ of rank 3 which admits a local frame $\{J_\alpha\}_{\alpha=1,2,3}$ such that the metric g_Q is hermitian for $J_\alpha, \alpha = 1, 2, 3$ and verifies

$$\begin{cases} J_\alpha \circ J_\beta = -\delta_{\alpha\beta}\mathrm{Id} + \varepsilon^{123}_{\alpha\beta\gamma}J_\gamma, \\ \nabla J_\alpha = \sum_{\beta=1}^3 \omega_\alpha^\beta J_\beta, \end{cases} \tag{3.1}$$

where ω_α^β are the local 1-forms on M and $\varepsilon^{123}_{\alpha\beta\gamma} = \pm 1$ if (α, β, γ) is even or odd permutation of $(1, 2, 3)$. We note that a quaternion-Kähler foliation is transversally Einstein [1], hence it admits a constant scalar curvature which is supposed to be positive throughout this paper. A consequence of the definition is the existence of a parallel 4-form Ω defined by $\Omega = \sum_{\alpha=1}^3 \Omega_\alpha\wedge\Omega_\alpha$, where the Ω_α are the local Kähler 2-forms associated with J_α. The 4-form Ω can be written as

$$\Omega = \sum_{\alpha=1}^{3}\Omega_\alpha\cdot\Omega_\alpha + 6m\mathrm{Id}. \tag{3.2}$$

Under the action of Ω, the foliated spinor bundle $S(\mathcal{F})$ splits into an orthogonal sum

$$S(\mathcal{F}) = \overset{m}{\underset{r=0}{\oplus}} S_r(\mathcal{F}),$$

where $S_r(\mathcal{F})$ is the eigenbundle associated with the eigenvalue $\mu_r = 6m - 4r(r+2)$ of Ω. Moreover, the action of the group $\mathrm{Sp}_1 \times \mathrm{Sp}_m$ splits the bundle $Q^{\mathbb{C}} \otimes S_r(\mathcal{F})$ into [21]

$$\begin{aligned} Q^{\mathbb{C}} \otimes S_r(\mathcal{F}) \quad = \quad & W_{r+1,\bar{r}}(\mathcal{F}) \oplus W_{r-1,\bar{r}}(\mathcal{F}) \oplus W_{r+1,r-1}(\mathcal{F}) \oplus W_{r-1,r+1}(\mathcal{F}) \\ & \oplus W_{r-1,r-1}(\mathcal{F}) \oplus W_{r+1,r+1}(\mathcal{F}), \end{aligned} \tag{3.3}$$

where $W_{r,s}(\mathcal{F})$ denotes the space of the irreducible representation of the group $\mathrm{Sp}_1 \times \mathrm{Sp}_m$ with dominant weight

$$(r, 1, \cdots, 1, \underbrace{0, \cdots, 0}_{s}),$$

and $W_{r,\bar{s}}(\mathcal{F})$ is the space of the irreducible representation of the group $\mathrm{Sp}_1 \times \mathrm{Sp}_m$ with dominant weight

$$(r, 2, 1, \cdots, 1, \underbrace{0, \cdots, 0}_{s}).$$

The last two bundles in (3.3) are respectively isomorphic to $S_{r-1}(\mathcal{F})$ and $S_{r+1}(\mathcal{F})$. We denote by m_r the restriction of the Clifford multiplication to $Q^{\mathbb{C}} \otimes S_r(\mathcal{F})$. The kernel of m_r splits into an orthogonal sum

$$\mathrm{Ker}\, m_r = W_{r+1,\bar{r}}(\mathcal{F}) \oplus W_{r-1,\bar{r}}(\mathcal{F}) \oplus W_{r+1,r-1}(\mathcal{F}) \oplus W_{r-1,r+1}(\mathcal{F}).$$

This comes from the computation of the image of m_r of the maximal vector of each component of (3.3). Thus the restriction of m_r to $W_{r-1,r-1}(\mathcal{F})$ (resp. $W_{r+1,r+1}(\mathcal{F})$) is an isomorphism onto $S_{r-1}(\mathcal{F})$ (resp. $S_{r+1}(\mathcal{F})$). Let $(.\,,.)$ be the usual hermitian product on $Q^{\mathbb{C}} \otimes S(\mathcal{F})$. Since $(m_r(.), m_r(.))$ and $(.\,,.)$ are $(\mathrm{Sp}_1 \times \mathrm{Sp}_m)$-invariant scalar products on both $W_{r-1,r-1}(\mathcal{F})$ and $W_{r+1,r+1}(\mathcal{F})$, one gets from Schur lemma

$$\forall w \in W_{r-1,r-1}(\mathcal{F}), \;\; |m_r(w)|^2 = \frac{2(r+1)(m-r+1)}{r}|w|^2, \tag{3.4}$$

and,

$$\forall w \in W_{r+1,r+1}(\mathcal{F}), \;\; |m_r(w)|^2 = \frac{2(r+1)(m+r+3)}{r+2}|w|^2. \tag{3.5}$$

In order to obtain a similar result for the other terms in (3.3), we locally define the operator $\widetilde{m} : \Gamma(Q^{\mathbb{C}} \otimes S(\mathcal{F})) \longrightarrow \Gamma(E^{\mathbb{C}} \otimes S(\mathcal{F}))$ by

$$\widetilde{m}(X \otimes \Psi) = \sum_{\alpha=1}^{3} J_\alpha \otimes (J_\alpha(X) \cdot \Psi), \tag{3.6}$$

for all $X \in \Gamma(Q)$ and $\Psi \in \Gamma(S(\mathcal{F}))$. We denote by $\widetilde{m}_r$ the restriction of $\widetilde{m}$ to $Q^{\mathbb{C}} \otimes S_r(\mathcal{F})$. As above, computing the image of $\widetilde{m}_r$ of maximal vector of each component of (3.3), the kernel of $\widetilde{m}_r$ splits into

$$\operatorname{Ker} \widetilde{m}_r = W_{r+1,\bar{r}}(\mathcal{F}) \oplus W_{r-1,\bar{r}}(\mathcal{F}).$$

Using the same argument as in (3.4) and (3.5), one gets from Schur lemma

$$\begin{aligned}
\forall w \in W_{r+1,r-1}(\mathcal{F}),\ |\widetilde{m}_r(w)|^2 &= 4(m-r+1)|w|^2, &(3.7)\\
\forall w \in W_{r-1,r+1}(\mathcal{F}),\ |\widetilde{m}_r(w)|^2 &= 4(m+r+3)|w|^2, &(3.8)\\
\forall w \in W_{r-1,r-1}(\mathcal{F}),\ |\widetilde{m}_r(w)|^2 &= \frac{2(r-1)(m-r+1)}{r}|w|^2, &(3.9)\\
\forall w \in W_{r+1,r+1}(\mathcal{F}),\ |\widetilde{m}_r(w)|^2 &= \frac{2(r+3)(m+r+3)}{r+2}|w|^2. &(3.10)
\end{aligned}$$

4 The main Result

In this section, we show (1.2) by using the decomposition of the bundle $Q^{\mathbb{C}} \otimes S_r(\mathcal{F})$ given in the above section . We refer to [10], [19], [21].

Theorem 4.1 *Under the same conditions as in Theorem* 2.1 *with the assumption that the foliation $\mathcal{F}$ has a quaternion-Kähler spin structure of codimension $q = 4m$, then the mean curvature κ vanishes and any eigenvalue λ of the basic Dirac operator satisfies*

$$\lambda^2 \geq \frac{m+3}{4(m+2)}\sigma^\nabla,$$

where σ^∇ denotes the transversal scalar curvature.

Proof. The fact that $\mathcal{F}$ is minimal comes from Theorem 2.1 since the transversal scalar curvature is supposed to be positive. For the second part, according to the decomposition (3.3), for any $\Psi \in \Gamma_B(S_r(\mathcal{F}))$, the covariant derivative $\nabla\Psi$ splits into

$$\begin{aligned}
\nabla\Psi = \ & (\nabla\Psi)_{r+1,\bar{r}} + (\nabla\Psi)_{r-1,\bar{r}} + (\nabla\Psi)_{r+1,r-1} + (\nabla\Psi)_{r-1,r+1}\\
& + (\nabla\Psi)_{r-1,r-1} + (\nabla\Psi)_{r+1,r+1}.
\end{aligned} \tag{4.1}$$

In order to compute the norm of $\nabla\Psi$, since the last two terms in the above equation are sections in the subbundles $S_{r-1}(\mathcal{F})$ and $S_{r+1}(\mathcal{F})$ respectively, we get from (3.4) and (3.5),

$$|(\nabla\Psi)_{r-1,r-1}|^2 = \frac{r}{2(r+1)(m-r+1)}|D_-\Psi|^2, \tag{4.2}$$

and,

$$|(\nabla\Psi)_{r+1,r+1}|^2 = \frac{r+2}{2(r+1)(m+r+3)}|D_+\Psi|^2, \tag{4.3}$$

where $D_-\Psi = (D_b\Psi)_{r-1}$ and $D_+\Psi = (D_b\Psi)_{r+1}$. Similar results could be obtained for the other terms in (4.1) by using the definition of the operator $\widetilde{m}$ in (3.6). For this, we consider for any spinor Ψ the operator $D_\alpha\Psi$ locally defined by $\sum_{i=1}^{4m} J_\alpha(e_i)\cdot\nabla_{e_i}\Psi$. Hence we have $\widetilde{m}(\nabla\Psi) = \sum_{\alpha=1}^{3} J_\alpha \otimes D_\alpha\Psi$ and we get that

$$|\widetilde{m}(\nabla\Psi)|^2 = \sum_{\alpha=1}^{3}|D_\alpha\Psi|^2.$$

On the other hand, Equations (3.7), (3.8), (3.9), (3.10) imply that

$$\begin{aligned}
|\widetilde{m}((\nabla\Psi)_{r+1,r-1})|^2 &= 4(m-r+1)|(\nabla\Psi)_{r+1,r-1}|^2,\\
|\widetilde{m}((\nabla\Psi)_{r-1,r+1})|^2 &= 4(m+r+3)|(\nabla\Psi)_{r-1,r+1}|^2,\\
|\widetilde{m}((\nabla\Psi)_{r-1,r-1})|^2 &= \frac{2(r-1)(m-r+1)}{r}|(\nabla\Psi)_{r-1,r-1}|^2,\\
|\widetilde{m}((\nabla\Psi)_{r+1,r+1})|^2 &= \frac{2(r+3)(m+r+3)}{r+2}|(\nabla\Psi)_{r+1,r+1}|^2.
\end{aligned}$$

Hence by the above equations and (4.2), (4.3), we conclude for any $\Psi \in \Gamma_B(S_r(\mathcal{F}))$ that

$$\begin{aligned}
\sum_{\alpha=1}^{3}|D_\alpha\Psi|^2 &= 4(m-r+1)|(\nabla\Psi)_{r+1,r-1}|^2 + 4(m+r+3)|(\nabla\Psi)_{r-1,r+1}|^2\\
&\quad + \frac{r+3}{r+2}|D_+\Psi|^2 + \frac{r-1}{r+1}|D_-\Psi|^2.
\end{aligned} \tag{4.4}$$

Then using equations (4.2), (4.3), (4.4) and by (4.1), we write the norm of $\nabla\Psi$ as

$$\begin{aligned}
|\nabla\Psi|^2 &= |(\nabla\Psi)_{r+1,\bar{r}}|^2 + |(\nabla\Psi)_{r-1,\bar{r}}|^2 + \frac{2(r+1)}{m+r+3}|(\nabla\Psi)_{r+1,r-1}|^2\\
&\quad + \frac{1}{4(m+r+3)}\sum_{\alpha=1}^{3}|D_\alpha\Psi|^2 + \frac{1}{4(m+r+3)}|D_+\Psi|^2\\
&\quad + \frac{m+3r+1}{4(m-r+1)(m+r+3)}|D_-\Psi|^2.
\end{aligned} \tag{4.5}$$

Now let λ be any eigenvalue of the basic Dirac operator, then there exists an eigenspinor Ψ, called of type $(r, r+1)$, such that

$$D_b\Psi = \lambda\Psi \quad \text{and} \quad \Psi = \Psi_r + \Psi_{r+1},$$

with $r \in \{0, \cdots, m-1\}$. In [13], it is showed that for any spinor $\Psi \in \Gamma_B(S(\mathcal{F}))$, we have

$$\int_M \sum_{\alpha=1}^{3} |D_\alpha \Psi|^2 = 3\int_M (D_b^2\Psi, \Psi) + \frac{\sigma^\nabla}{4m(m+2)}\int_M ((\Omega - 6m)\cdot\Psi, \Psi).$$

Therefore, applying Equation (4.5) to Ψ_{r+1} and integrating over M, one gets since $D_-\Psi_{r+1} = \lambda\Psi_r$ and $D_+\Psi_{r+1} = 0$

$$0 \ \leq \ ||\nabla\Psi_{r+1}||_{L^2}^2 - a_r\lambda^2||\Psi_{r+1}||_{L^2}^2 + b_r\sigma^\nabla||\Psi_{r+1}||_{L^2}^2 - c_r\lambda^2||\Psi_r||_{L^2}^2,$$

where,

$$\begin{cases} a_r = \frac{3}{4(m+r+4)}, \\ \\ b_r = \frac{(r+1)(r+3)}{4m(m+2)(m+r+4)}, \\ \\ c_r = \frac{m+3r+4}{4(m-r)(m+r+4)}. \end{cases}$$

Finally with the help of the Schrödinger-Lichnerowicz formula and the fact that Ψ_r and Ψ_{r+1} have the same L^2-norms, we get (1.2). $\square$

5 The Limiting case

Let λ be the first eigenvalue satisfying equality in (1.2) and Ψ an eigenspinor of type $(r, r+1)$. From the proof of Theorem 2.1, one gets necessarily that $r = 0$ and the following equations [10]

$$\begin{cases} |\nabla\Psi_0|^2 = \frac{1}{m+3}|D_b\Psi_0|^2, \\ \\ |\nabla\Psi_1|^2 = \frac{1}{4m}|D_b\Psi_1|^2 + \frac{1}{4(m+4)}\sum_{\alpha=1}^{3}|D_\alpha\Psi_1|^2. \end{cases} \tag{5.1}$$

Furthermore, the spinor Ψ_1 satisfies

$$\begin{aligned} &\textstyle\sum_{\alpha=1}^{3}\Omega_\alpha \cdot D_\alpha\Psi_1 = 0, \\ &\textstyle\sum_{\beta,\gamma}\varepsilon_{\alpha\beta\gamma}^{123}\Omega_\beta \cdot D_\gamma\Psi_1 = 8D_\alpha\Psi_1, \quad \forall\alpha = 1,2,3. \end{aligned} \tag{5.2}$$

Moreover for all $X \in \Gamma(Q)$, we have the *quaternion-Kähler Killing* equations [10], [19], [20]

$$\nabla_X \Psi_0 = -\frac{\lambda}{m+3} p_1(X) \cdot \Psi_1, \tag{5.3}$$

and,

$$\nabla_X \Psi_1 = -\frac{\lambda}{4m} X \cdot \Psi_0 - \frac{1}{4(m+4)} \sum_{\alpha=1}^{3} J_\alpha(X) \cdot D_\alpha \Psi_1, \tag{5.4}$$

where for all $X \in \Gamma(Q)$, the operator p_1 is defined by (see [14])

$$\begin{cases} p_1(X) = \frac{1}{8}(5X + \mathcal{J}(X)), \\ \\ \mathcal{J}(X) = \frac{1}{4}[\Omega, X]. \end{cases}$$

In order to prove (5.3), we define the transversal quaternion-Kähler twistor operator, denoted by $\mathcal{P}^0$, on the bundle $S_0(\mathcal{F})$ whose the image lies in the bundle $Q^* \otimes S_0(\mathcal{F})$ (see [14] for the details). For any spinor field $\psi_0 \in \Gamma_B(S_0(\mathcal{F}))$, we write

$$\mathcal{P}^0 \psi_0 = \sum_{i=1}^{4m} e_i \otimes (\nabla_{e_i} \psi_0 + \frac{1}{m+3} p_1(e_i) \cdot D_b \psi_0),$$

where $\{e_i\}_{i=1,\cdots,4m}$ is a local orthonormal frame of $\Gamma(Q)$. By a straightforward computation and with the definition of p_1, we easily verify that $\sum_{i=1}^{4m} e_i \cdot \mathcal{P}^0_{e_i} \psi_0 = 0$. Hence the image of $\mathcal{P}^0$ lies in the kernel of Clifford multiplication m_0. Since $\mathcal{P}^0_{e_i} \psi_0$ is a section on $S_0(\mathcal{F})$, we deduce with the definition of the operator $\mathcal{J}$, that $\sum_{i=1}^{4m} \mathcal{J}(e_i) \cdot \mathcal{P}^0_{e_i} \psi_0 = 0$. Then

$$\begin{aligned} |\mathcal{P}^0 \psi_0|^2 &= \sum_{i=1}^{4m} (\mathcal{P}^0_{e_i} \psi_0, \mathcal{P}^0_{e_i} \psi_0) \\ &= \sum_{i=1}^{4m} (\mathcal{P}^0_{e_i} \psi_0, \nabla_{e_i} \psi_0) \\ &= |\nabla \psi_0|^2 + \frac{1}{m+3} \sum_{i=1}^{4m} (p_1(e_i) \cdot D_b \psi_0, \nabla_{e_i} \psi_0). \end{aligned} \tag{5.5}$$

Since Clifford multiplication by $\mathcal{J}$ is symmetric, one can easily verify that $(\nabla_{e_i} \psi_0, p_1(e_i) \cdot D_b \psi_0) = -(e_i \cdot \nabla_{e_i} \psi_0, D_b \psi_0)$. Then for any spinor $\psi_0 \in \Gamma(S_0(\mathcal{F}))$, Equation (5.5) reduces to

$$|\mathcal{P}^0 \psi_0|^2 = |\nabla \psi_0|^2 - \frac{1}{m+3} |D_b \psi_0|^2,$$

which vanishes by (5.1) for the spinor field Ψ_0. Thus Equation (5.3) is satisfied for $X = e_i$. $\square$

Now, we will prove Equation (5.4). The proof consists in computing the sum

$$\sum_{i=1}^{4m} |\nabla_{e_i}\Psi_1 + \tfrac{1}{4m} e_i \cdot D_b\Psi_1 + \tfrac{1}{4(m+4)} \sum_{\alpha=1}^{3} J_\alpha e_i \cdot D_\alpha \Psi_1|^2 =$$

$$|\nabla\Psi_1|^2 + \tfrac{1}{4m}|D_b\Psi_1|^2 + \tfrac{1}{16(m+4)^2} \sum_{i=1}^{4m} |\sum_{\alpha=1}^{3} J_\alpha e_i \cdot D_\alpha\Psi_1|^2 \tag{5.6}$$

$$+\tfrac{1}{2m} \sum_{i=1}^{4m} (\nabla_{e_i}\Psi_1, e_i \cdot D_b\Psi_1) + \tfrac{1}{2(m+4)} \sum_{i=1}^{4m} (\nabla_{e_i}\Psi_1, J_\alpha e_i \cdot D_\alpha\Psi_1).$$

The fact that Clifford multiplication by e_i and $J_\alpha(e_i)$ is skew-symmetric, the last terms are easily computed and it remains to compute the third term in the r.h.s. of (5.6) . For this, using a local orthonormal frame $\{J_\alpha e_i\}_{i=1,\cdots,4m}$ and (3.1), it follows

$$\begin{aligned}
\sum_{i=1}^{4m} |\sum_{\alpha=1}^{3} J_\alpha e_i \cdot D_\alpha \Psi_1|^2 &= \sum_{i,\alpha,\beta} (J_\alpha e_i \cdot D_\alpha\Psi_1, J_\beta e_i \cdot D_\beta \Psi_1) \\
&= \sum_{i,\alpha,\beta} (D_\alpha\Psi_1, e_i \cdot J_\beta J_\alpha e_i \cdot D_\beta\Psi_1) \\
&= 4(m+4) \sum_{\alpha=1}^{3} |D_\alpha \Psi_1|^2.
\end{aligned} \tag{5.7}$$

The last identity in (5.7) comes from (3.1) and (5.2). Finally substituting (5.7) and using (2.3), Equation (5.6) reduces to

$$\sum_{i=1}^{4m} |\nabla_{e_i}\Psi_1 + \tfrac{1}{4m} e_i \cdot D_b\Psi_1 + \tfrac{1}{4(m+4)} \sum_{\alpha=1}^{3} J_\alpha e_i \cdot D_\alpha \Psi_1|^2 =$$

$$|\nabla\Psi_1|^2 - \tfrac{1}{4m}|D_b\Psi_1|^2 - \tfrac{1}{4(m+4)} \sum_{\alpha=1}^{3} |D_\alpha\Psi_1|^2,$$

which vanishes by (5.1). $\square$

Example 1 We consider the compact manifold $N = M \times \mathbb{HP}^m$, where M is a compact Riemannian manifold of dimension p and $\mathbb{HP}^m$ is the quaternionic projective space with its standard metric. Let g_N be the product metric on N. We define a foliation $\mathcal{F}$ on N by its leaves of the form $M \times \{y\}$ where $y \in \mathbb{HP}^m$. This is a Riemannian foliation on N and g_N is a bundle-like metric with totally geodesic fibers. Since the fibers of the normal bundle are the tangent space of $\mathbb{HP}^m$, then it carries a quaternion-Kähler spin structure and the basic Dirac operator coincides with the one on $\mathbb{H}P^m$ where the eigenvalues are computed in [9]. Hence the limiting case in (1.2) is achieved.

Example 2 Let M be a compact 3-Sasakian manifold and consider the foliation on M defined by its Killing vector fields. This is a Riemannian foliation with a bundle-like metric and totally geodesic fibers diffeomorphic to $\Gamma \setminus S^3$ where Γ is a finite subgroup of Sp_1 [15]. It induces a quaternion-Kähler spin structure on the normal bundle with positive transversal scalar curvature. If M is either S^{4q+3} or $\mathbb{RP}^{4q+3}$, then it projects onto $\mathbb{H}P^m$ (Hopf fibration). Since the fibers of the normal bundle are isomorphic to the tangent space of $\mathbb{H}P^m$, then equality in (1.2) is achieved.

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