Eigenvalues of the transversal Dirac operator on Kahler foliations

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Abstract

In this paper, we prove Kirchberg-type inequalities for any Kähler spin foliation. Their limiting-cases are then characterized as being transversal minimal Einstein foliations. The key point is to introduce the transversal Kählerian twistor operators.

1 Introduction

On a compact Riemannian spin manifold (M^n, g_M), Th. Friedrich [Fri80] showed that any eigenvalue λ of the Dirac operator satisfies

$$\lambda^2 \geq \frac{n}{4(n-1)} S_0, \tag{1.1}$$

where S_0 denotes the infimum of the scalar curvature of M. The limiting case in (1.1) is characterized by the existence of a *Killing spinor*. As a consequence M is Einstein. K.D. Kirchberg [Kir86] established that, on such manifolds any eigenvalue λ satisfies the inequalities

$$\lambda^2 \geq \begin{cases} \frac{m+1}{4m} S_0 & \text{if } m \text{ is odd,} \\ \\ \frac{m}{4(m-1)} S_0 & \text{if } m \text{ is even.} \end{cases}$$

On a compact Riemannian spin foliation $(M, g_M, \mathcal{F})$ of codimension q with a bundle-like metric g_M such that the mean curvature κ is a basic coclosed

1-form, S.D. Jung [Jun01] showed that any eigenvalue λ of the transversal Dirac operator satisfies

$$\lambda^2 \geq \frac{q}{4(q-1)} K_0^{\nabla}, \tag{1.2}$$

where $K_0^{\nabla} = \inf_M (\sigma^{\nabla} + |\kappa|^2)$, here σ^{∇} denotes the transversal scalar curvature with the transversal Levi-Civita connection ∇. The limiting case in (1.2) is characterized by the fact that $\mathcal{F}$ is minimal ($\kappa = 0$) and transversally Einstein (see Theorem 3.1). The main result of this paper is the following:

Theorem 1.1 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a Kähler spin foliation $\mathcal{F}$ of codimension $q = 2m$ and a bundle-like metric g_M. Assume that κ is a basic coclosed 1-form, then any eigenvalue λ of the transversal Dirac operator satisfies:*

$$\lambda^2 \geq \frac{m+1}{4m} K_0^{\nabla} \qquad \text{if } m \text{ is odd}, \tag{1.3}$$

and

$$\lambda^2 \geq \frac{m}{4(m-1)} K_0^{\nabla} \qquad \text{if } m \text{ is even}. \tag{1.4}$$

The limiting case in (1.3) is characterised by the fact that the foliation is minimal and by existence of a transversal Kählerian Killing spinor (see Theorem 4.3). We refer to Theorem 4.4 for the equality case in (1.4).
We point out that Inequality (1.3) was proved by S. D. Jung [JK03] with the additional assumption that κ is *transversally holomorphic*. The author would like to thank Oussama Hijazi for his support.

2 Foliated manifolds

In this section, we summarize some standard facts about foliations. For more details, we refer to [Ton88], [Jun01].
Let (M, g_M) be a $(p+q)$-dimensional Riemannian manifold and a foliation $\mathcal{F}$ of codimension q and let ∇^M be the Levi-civita connection associated with g_M. We consider the exact sequence

$$0 \longrightarrow L \xrightarrow{\iota} TM \xrightarrow{\pi} Q \longrightarrow 0,$$

where L is the tangent bundle of TM and $Q = TM/L \simeq L^{\perp}$ the normal bundle. We assume g_M to be a *bundle-like metric* on Q, that means the induced metric g_Q verifies the holonomy invariance condition,

$$\mathcal{L}_X g_Q = 0, \qquad \forall X \in \Gamma(L),$$

where $\mathcal{L}_X$ is the Lie derivative with respect to X. Let ∇ be the connection on Q defined by:

$$\nabla_X s = \begin{cases} \pi\left[X, Y_s\right], & \forall X \in \Gamma(L)\ , \\ \\ \pi\left(\nabla^M_X Y_s\right), & \forall X \in \Gamma(L^{\perp})\ , \end{cases}$$

where $s \in \Gamma(Q)$ and Y_s is the unique vector of $\Gamma(L^{\perp})$ such that $\pi\left(Y_s\right) = s$. The connection ∇ is metric and torsion-free. The curvature of ∇ acts on $\Gamma(Q)$ by :

$$R^{\nabla}\left(X, Y\right)s = \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s, \qquad \forall X, Y \in \chi\left(M\right).$$

The transversal Ricci curvature is defined by:

$$\begin{array}{rccl} \rho^{\nabla}\ :\ \Gamma(Q) & \longrightarrow & \Gamma(Q) \\ X & \longmapsto & \rho^{\nabla}\left(X\right) = \displaystyle\sum_{j=1}^{q} R^{\nabla}\left(X, e_j\right) e_j. \end{array}$$

Also, we define the transversal scalar curvature :

$$\sigma^{\nabla} = \sum_{i=1}^{q} g_Q\left(\rho^{\nabla}\left(e_i\right), e_i\right) = \sum_{i,j=1}^{q} R^{\nabla}\left(e_i, e_j, e_j, e_i\right),$$

where $\{e_i\}_{i=1,\cdots,q}$ is a local orthonormal frame of Q and $R^{\nabla}(X, Y, Z, W) = g_Q(R^{\nabla}(X, Y)Z, W)$, for all $X, Y, Z, W \in \Gamma(Q)$. The foliation $\mathcal{F}$ is said to be transversally Einstein if and only if

$$\rho^{\nabla} = \frac{1}{q}\sigma^{\nabla}\mathrm{Id},$$

with constant transversal scalar curvature. The mean curvature of Q is given by:

$$\kappa\left(X\right) = g_Q\left(\tau, X\right), \qquad \forall X \in \Gamma(Q),$$

where $\tau = \sum_{l=1}^{p} II\left(e_l, e_l\right)$, with $\{e_l\}_{l=1,\cdots,p}$ is a local orthonormal frame of $\Gamma(L)$ and II is the second fundamental form of $\mathcal{F}$ defined by:

$$\begin{array}{rccl} II\ :\ \Gamma(L) \times \Gamma(L) & \longrightarrow & \Gamma(Q) \\ (X, Y) & \longmapsto & II\left(X, Y\right) = \pi\left(\nabla^M_X Y\right). \end{array}$$

We define basic r-forms by :

$$\Omega^r_B\left(\mathcal{F}\right) = \left\{\Phi \in \Lambda^r T^* M \mid\ X \lrcorner \Phi = 0 \quad \text{and} \quad X \lrcorner d\Phi = 0, \quad \forall X \in \Gamma(L)\right\},$$

where d is the exterior derivative and $X\lrcorner$ is the interior product. Any $\Phi \in \Omega^r_B(\mathcal{F})$ can be locally written as

$$\sum_{1\leq j_1<\cdots<j_r\leq q} \beta_{j_1,\cdots,j_r} dy_{j_1}\wedge\cdots\wedge dy_{j_r},$$

where $\frac{\partial}{\partial x_l}\beta_{j_1,\cdots,j_r} = 0, \quad \forall l = 1,\cdots,p$. With the local expression of basic r-forms, one can verify that κ is closed if $\mathcal{F}$ is isoparametric $(\kappa \in \Omega^1_B(\mathcal{F}))$. For all $r \geq 0$,

$$d\left(\Omega^r_B(\mathcal{F})\right) \subset \Omega^{r+1}_B(\mathcal{F}).$$

We denote by $d_B = d|_{\Omega_B(\mathcal{F})}$ where $\Omega_B(\mathcal{F})$ is the tensor algebra of $\Omega^r_B(\mathcal{F})$. We have the following formulas:

$$d_B = \sum_{i=1}^{q} e_i^\star \wedge \nabla_{e_i} \quad \text{and} \quad \delta_B = -\sum_{i=1}^{q} e_i\lrcorner\nabla_{e_i} + \kappa\,\lrcorner,$$

where δ_B is the adjoint operator of d_B with respect to the induced scalar product and $\{e_i\}_{i=1,\cdots,q}$ is a local orthonormal frame of Q.

3 The transversal Dirac operator on Kähler Foliations

In this section, we start by recalling some facts on Riemannian foliations which could be found in [GK91a], [GK91b], [AG97], [Jun01]. For completeness, we also scketch a straightforward proof of Inequality ((1.2)) established in [Jun01] and end by recalling well-known facts (see [Kir86], [Kir96], [Hij94a], [Hij94b], [JK03]) on Kähler spin foliations.

On a foliated Riemannian manifold $(M, g_M, \mathcal{F})$, a transversal spin structure is a pair $(\mathrm{Spin}Q, \eta)$ where $\mathrm{Spin}Q$ is a Spin_q-principal fibre bundle over M and η a 2-fold cover such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Spin}Q \times \mathrm{Spin}_q \arrow[r] \arrow[d, "\eta\otimes Ad"] & \mathrm{Spin}Q \arrow[r] \arrow[d, "\eta"] & M \\
\mathrm{SO}Q \times \mathrm{SO}_q \arrow[r] & \mathrm{SO}Q \arrow[ur] &
\end{tikzcd}$$

The maps $\mathrm{Spin}Q \times \mathrm{Spin}_q \longrightarrow \mathrm{Spin}Q$, and $\mathrm{SO}Q \times SO_q \longrightarrow \mathrm{SO}Q$, are respectively the actions of Spin_q and SO_q on the principal fibre bundles $\mathrm{Spin}Q$ and

SOQ. In this case, $\mathcal{F}$ is called a transversal spin foliation. We define the foliated spinor bundle by: $S(\mathcal{F}) := \mathrm{Spin}Q \times_\rho \Sigma_q$, where $\rho : \mathrm{Spin}_q \longrightarrow \mathrm{Aut}(\Sigma_q)$, is the complex spin representation and Σ_q is a $\mathbb{C}$ vector space of dimension N with $N = 2^{[\frac{q}{2}]}$, where $[\,]$ stands for the integer part. Recall that the Clifford multiplication $\mathcal{M}$ on $S(\mathcal{F})$ is given by:

$$\begin{array}{rccc} \mathcal{M} : & \Gamma(Q) \times \Gamma(S(\mathcal{F})) & \longrightarrow & \Gamma(S(\mathcal{F})) \\ & (X, \Psi) & \longmapsto & X \cdot \Psi. \end{array}$$

There is a natural Hermitian product on $S(\mathcal{F})$ such that, for all $X, Y \in \Gamma(Q)$, the following relations are true:

$$\begin{array}{rcl} \langle X \cdot \Psi, \Phi \rangle & = & -\langle \Psi, X \cdot \Phi \rangle, \\ X(\langle \Psi, \Phi \rangle) & = & \langle \nabla_X \Psi, \Phi \rangle + \langle \Psi, \nabla_X \Phi \rangle, \\ \nabla_Y (X \cdot \Psi) & = & (\nabla_Y X) \cdot \Psi + X \cdot (\nabla_Y \Psi), \end{array}$$

where ∇ is the Levi-Civita connection on $S(\mathcal{F})$ and $\Psi, \Phi \in \Gamma(S(\mathcal{F}))$.

The transversal Dirac operator [GK91a, GK91b] is locally given by:

$$D_{tr}\Psi = \sum_{i=1}^{q} e_i \cdot \nabla_{e_i}\Psi - \frac{1}{2}\kappa \cdot \Psi, \tag{3.1}$$

for all $\Psi \in \Gamma(S(\mathcal{F}))$. We can easily prove using Green's theorem [YT90] that this operator is formally self adjoint. Furthermore, in [GK91b] it is proved that if $\mathcal{F}$ is isoparametric and $\delta_B \kappa = 0$, then we have the Schrödinger-Lichnerowicz formula:

$$D_{tr}^2 \Psi = \nabla_{tr}^{\star} \nabla_{tr} \Psi + \frac{1}{4} K_\sigma^\nabla \Psi,$$

where $K_\sigma^\nabla = \sigma^\nabla + |\kappa|^2$ and

$$\nabla_{tr}^{\star} \nabla_{tr} \Psi = -\sum_{i=1}^{q} \nabla^2_{e_i, e_i} \Psi + \nabla_\kappa \Psi,$$

with $\nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y}$, for all $X, Y \in \Gamma(TM)$. Denote by $\mathcal{P}$ the transversal twistor operator defined by

$$\mathcal{P} : \Gamma(S(\mathcal{F})) \xrightarrow{\nabla^{tr}} \Gamma(Q^* \otimes S(\mathcal{F})) \xrightarrow{\pi} \Gamma(\ker \mathcal{M}),$$

where π is the orthogonal projection on the kernel of the Clifford multiplication $\mathcal{M}$. With respect to a local orthonormal frame $\{e_1, \cdots, e_q\}$, for all $\Psi \in \Gamma(S(\mathcal{F}))$, one has

$$\mathcal{P}\Psi = \sum_{i=1}^{q} e_i^* \otimes (\nabla_{e_i}\Psi + \frac{1}{q} e_i \cdot D_{tr}\Psi + \frac{1}{2q} e_i \cdot \kappa \cdot \Psi). \tag{3.2}$$

For any spinor field Ψ, one can easily show that

$$\sum_{i=1}^{q} e_i \cdot \mathcal{P}_{e_i}\Psi = 0. \tag{3.3}$$

Now we give a simple proof of the following theorem:

Theorem 3.1 [Jun01] *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a spin foliation $\mathcal{F}$ of codimension q and a bundle-like metric g_M with $\kappa \in \Omega_B^1(\mathcal{F})$. Assume that $\delta_B \kappa = 0$ and let λ be an eigenvalue of the transversal Dirac operator, then*

$$\lambda^2 \geq \frac{q}{4(q-1)} K_0^{\nabla}. \tag{3.4}$$

Proof. For all $\Psi \in \Gamma(S(\mathcal{F}))$, we have using Identities (3.2), (3.3) (3.1),

$$|\mathcal{P}\Psi|^2 = |\nabla^{tr}\Psi|^2 - \frac{1}{q}|D_{tr}\Psi|^2 - \frac{1}{q}\Re(D_{tr}\Psi, \kappa \cdot \Psi) - \frac{1}{4q}|\kappa|^2|\Psi|^2.$$

For any spinor field Φ, we have that $(\Phi, \kappa \cdot \Phi) = -(\kappa \cdot \Phi, \Phi) = -\overline{(\Phi, \kappa \cdot \Phi)}$, so the scalar product $(\Phi, \kappa \cdot \Phi)$ is a pure imaginary function. Hence for any eigenspinor Ψ of the transversal Dirac operator, we obtain

$$\int_M |\mathcal{P}\Psi|^2 + \frac{1}{4q}\int_M |\kappa|^2|\Psi|^2 = \int_M |\nabla^{tr}\Psi|^2 - \frac{1}{q}\int_M \lambda^2|\Psi|^2,$$

from which we deduce (3.4) with the help of the Schrödinger-Lichnerowicz formula. Finally, we can easily prove in the limiting case that $\mathcal{F}$ is minimal i.e. $\kappa = 0$, and transversally Einstein. $\square$

A foliation $\mathcal{F}$ is called Kähler if there exists a complex parallel orthogonal structure $J : \Gamma(Q) \longrightarrow \Gamma(Q)$ ($\dim Q = q = 2m$). Let Ω be the associated Kähler, i.e., for all $X, Y \in \Gamma(Q)$, $\Omega(X, Y) = g_Q(J(X), Y) = -g_Q(X, J(Y))$. The Kähler form can be locally expressed as

$$\Omega = \frac{1}{2}\sum_{i=1}^{q} e_i \cdot J(e_i) = -\frac{1}{2}\sum_{i=1}^{q} J(e_i) \cdot e_i,$$

and for all $X \in \Gamma(Q)$, we have $[\Omega, X] := \Omega \cdot X - X \cdot \Omega = 2J(X)$. Under the action of the Kähler form, the spinor bundle splits into an orthogonal sum

$$S(\mathcal{F}) = \overset{m}{\underset{r=o}{\oplus}} S_r(\mathcal{F}),$$

where $S_r(\mathcal{F})$ is an eigenbundle associated with the eigenvalue $i\mu_r = i(2r-m)$ of the Kähler form Ω. Moreover, the spinor bundle of a Kähler spin foliation carries a parallel anti-linear map j satisfying the relations:

$$\begin{aligned} j^2 &= (-1)^{\frac{m(m+1)}{2}} Id, \\ [X, j] &= 0, \\ (j\Psi, j\Phi) &= (\Phi, \Psi), \end{aligned}$$

and we have $j\Psi_r = (j\Psi)_{m-r}$. For all $X \in \Gamma(Q)$, we have

$$p_+(X) \cdot S_r(\mathcal{F}) \subset S_{r+1}(\mathcal{F}) \quad \text{and} \quad p_-(X) \cdot S_r(\mathcal{F}) \subset S_{r-1}(\mathcal{F}),$$

where $p_\pm(X) = \frac{X \mp iJ(X)}{2}$. We define the operator $\widetilde{D}_{tr}$ by

$$\widetilde{D}_{tr}\Psi = \sum_{i=1}^{q} J(e_i) \cdot \nabla_{e_i}\Psi - \frac{1}{2} J(\kappa) \cdot \Psi.$$

The local expression of $\widetilde{D}_{tr}$ is independant of the choice of the local frame and by Green's theorem [YT90], we prove that this operator is self-adjoint. On a Kähler spin foliation, the operators D_{tr} and $\widetilde{D}_{tr}$ satisfy:

$$\begin{aligned} [\Omega, D_{tr}] &= 2\widetilde{D}_{tr}, && (3.5) \\ [\Omega, \widetilde{D}_{tr}] &= -2D_{tr}, && (3.6) \\ [\Omega, D_{tr}^2] &= 0, && (3.7) \\ D_{tr}\widetilde{D}_{tr} + \widetilde{D}_{tr}D_{tr} &= 0, && (3.8) \\ \widetilde{D}_{tr}^2 &= D_{tr}^2. && (3.9) \end{aligned}$$

We should point out that Equations (3.7), (3.8) and (3.9) are true under the assumptions that $\mathcal{F}$ is isoparametric and $\delta_B\kappa = 0$. Now we define the two operators D_+ and D_- by

$$D_+ = \frac{1}{2}(D_{tr} - i\widetilde{D}_{tr}) \quad \text{and} \quad D_- = \frac{1}{2}(D_{tr} + i\widetilde{D}_{tr}). \qquad (3.10)$$

Furthermore, D_{tr} splits into D_+ and D_-, and we have the two exact sequences:

$$\Gamma(S_m(\mathcal{F})) \xrightarrow{D_-} \ldots \Gamma(S_r(\mathcal{F})) \xrightarrow{D_-} \Gamma(S_{r-1}(\mathcal{F})) \xrightarrow{D_-} \ldots \Gamma(S_0(\mathcal{F})), \qquad (3.11)$$

$$\Gamma(S_0(\mathcal{F})) \xrightarrow{D_+} \ldots \Gamma(S_r(\mathcal{F})) \xrightarrow{D_+} \Gamma(S_{r+1}(\mathcal{F})) \xrightarrow{D_+} \ldots \Gamma(S_m(\mathcal{F})). \qquad (3.12)$$

4 Eigenvalues of the transversal Dirac operator

In this section, we prove Kirchberg-type inequalities by using the transversal Kählerian twistor operators on Kähler spin foliations. We refer to [Kir90], [Kir92].

Definition 4.1 *On a Kähler spin foliation, we define the transversal Kählerian twistor operators by*

$$\mathcal{P}^{(r)} : \Gamma(S_r(\mathcal{F})) \xrightarrow{\nabla^{tr}} \Gamma(Q^* \otimes S_r(\mathcal{F})) \xrightarrow{\pi_r} \Gamma(\ker \mathcal{M}_r),$$

where $\mathcal{M}_r$ is the transversal Clifford multiplication defined by

$$\begin{array}{rccl} \mathcal{M}_r : & \Gamma(Q^* \otimes S_r(\mathcal{F})) & \longrightarrow & \Gamma(S_{r-1}(\mathcal{F})) \oplus \Gamma(S_{r+1}(\mathcal{F})) \\ & X \otimes \Psi_r & \longmapsto & p_-(X) \cdot \Psi_r \oplus p_+(X) \cdot \Psi_r. \end{array}$$

For all $r \in \{0, \ldots, m\}$ and $\Psi_r \in \Gamma(S_r(\mathcal{F}))$, we have

$$\mathcal{P}^{(r)}\Psi_r = \sum_{i=1}^{q} e_i^* \otimes (\nabla_{e_i}\Psi_r + a_r p_-(e_i) \cdot \mathcal{D}_+\Psi_r + b_r p_+(e_i) \cdot \mathcal{D}_-\Psi_r), \qquad (4.1)$$

where $\mathcal{D}_\pm = D_\pm + \frac{1}{2}p_\pm(\kappa)$ with $a_r = \frac{1}{2(r+1)}$ and $b_r = \frac{1}{2(m-r+1)}$. For any spinor field $\Psi_r \in \Gamma(S_r(\mathcal{F}))$, we can easily prove

$$\sum_{i=1}^{q} e_i \cdot \mathcal{P}^{(r)}_{e_i}\Psi_r = 0. \qquad (4.2)$$

Remark 4.2 *For any non zero eigenvalue λ of D_{tr}, there exists a spinor field $\Psi \in \Gamma(S(\mathcal{F}))$ called of type $(r, r+1)$, such that $D_{tr}\Psi = \lambda\Psi$ and $\Psi = \Psi_r + \Psi_{r+1}$, with $r \in \{0, \cdots, m-1\}$. By using* (3.10), (3.11) *and* (3.12) *it follows that $D_-\Psi_r = D_+\Psi_{r+1} = 0$, $D_-\Psi_{r+1} = \lambda\Psi_r, D_+\Psi_r = \lambda\Psi_{r+1}$ and $\|\Psi_r\|_{L^2} = \|\Psi_{r+1}\|_{L^2}$.*

Proof. Let φ be an eigenspinor of D_{tr}. There exists an r such that φ_r does not vanish. Let $\Psi = \frac{1}{\lambda}D_-D_+\varphi_r + D_+\varphi_r$, one can easily get that $D_{tr}\Psi = \lambda\Psi$.

Theorem 4.3 *Let $(M, g_M, \mathcal{F})$ be a compact Riemannian manifold with a Kähler spin foliation $\mathcal{F}$ of codimension $q = 2m$ and a bundle-like metric g_M*

with $\kappa \in \Omega^1_B(\mathcal{F})$ and $\delta_B \kappa = 0$. Then any eigenvalue λ of the transversal Dirac operator, satisfies

$$\lambda^2 \geq \frac{m+1}{4m} K_0^\nabla. \tag{4.3}$$

If Ψ is an eigenspinor of type $(r, r+1)$ associated with an eigenvalue λ satisfying equality in (4.3), then $r = \frac{m-1}{2}$, the foliation $\mathcal{F}$ is minimal and for all $X \in \Gamma(Q)$, the spinor Ψ satisfies

$$\nabla_X \Psi + \frac{\lambda}{2(m+1)} (X \cdot \Psi - i\varepsilon J(X) \cdot \bar{\Psi}) = 0, \tag{4.4}$$

where $\varepsilon = (-1)^{\frac{m-1}{2}}$, and $\bar{\Psi} := (-1)^r (\Psi_r - \Psi_{r+1})$. As a consequence m is odd and $\mathcal{F}$ is transversally Einstein with non negative constant transversal curvature σ^∇.

Proof. For all $\Psi_r \in \Gamma(S_r(\mathcal{F}))$, using Identities (4.1) and (4.2), we have

$$\begin{aligned} |\mathcal{P}^{(r)} \Psi_r|^2 &= \sum_{i=1}^q |\mathcal{P}^{(r)}_{e_i} \Psi_r|^2 = \sum_{i=1}^q (\mathcal{P}^{(r)}_{e_i} \Psi_r, \nabla_{e_i} \Psi_r) \\ &= \sum_{i=1}^q (\nabla_{e_i} \Psi_r + a_r p_-(e_i) \cdot \mathcal{D}_+ \Psi_r \\ &\quad + b_r p_+(e_i) \cdot \mathcal{D}_- \Psi_r, \nabla_{e_i} \Psi_r). \end{aligned}$$

Finally we obtain,

$$|\mathcal{P}^{(r)} \Psi_r|^2 = |\nabla^{tr} \Psi_r|^2 - a_r |\mathcal{D}_+ \Psi_r|^2 - b_r |\mathcal{D}_- \Psi_r|^2. \tag{4.5}$$

Let λ be an eigenvalue of D_{tr} and let Ψ an eigenspinor of type $(r, r+1)$. Applying Equality (4.5) to Ψ_r, one gets

$$\begin{aligned} |\mathcal{P}^{(r)} \Psi_r|^2 &= |\nabla^{tr} \Psi_r|^2 - a_r \lambda^2 |\Psi_{r+1}|^2 - a_r \lambda \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r) \\ &\quad - \frac{a_r}{4} |p_+(\kappa) \cdot \Psi_r|^2 - \frac{b_r}{4} |p_-(\kappa) \cdot \Psi_r|^2. \end{aligned}$$

By the Schrödinger-Lichnerowicz formula and by the fact that Ψ_r and Ψ_{r+1} have the same L^2-norms, we get

$$\begin{aligned} \int_M |\mathcal{P}^{(r)} \Psi_r|^2 + \frac{a_r}{4} \int_M |p_+(\kappa) \cdot \Psi_r|^2 + \frac{b_r}{4} \int_M |p_-(\kappa) \cdot \Psi_r|^2 &= \\ \int_M ((1 - a_r)\lambda^2 - \frac{1}{4} K_\sigma^\nabla) |\Psi_r|^2 - a_r \lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa) \cdot \Psi_r). \end{aligned} \tag{4.6}$$

Similarly applying (4.5) to Ψ_{r+1}, we obtain

$$\int_M |\mathcal{P}^{(r+1)}\Psi_{r+1}|^2 + \frac{a_{r+1}}{4}\int_M |p_+(\kappa)\cdot\Psi_{r+1}|^2 + \frac{b_{r+1}}{4}\int_M |p_-(\kappa)\cdot\Psi_{r+1}|^2 = \int_M ((1-b_{r+1})\lambda^2 - \frac{1}{4}K_\sigma^\nabla)|\Psi_{r+1}|^2 + b_{r+1}\lambda\int_M \Re(\Psi_{r+1}, p_+(\kappa)\cdot\Psi_r), \quad (4.7)$$

where $K_\sigma^\nabla = \sigma^\nabla + |\kappa|^2$. In order to get rid the term $\lambda\int_M \Re(\Psi_{r+1}, p_+(\kappa)\cdot\Psi_r)$, since the l.h.s. of (4.6) and (4.7) are non negative, dividing (4.6) by a_r and (4.7) by b_{r+1} then summing up, we find by substituting the values of a_r and b_{r+1},

$$\lambda^2 \geq \frac{m+1}{4m}K_0^\nabla.$$

Now, we discuss the limiting case of Inequality (4.3). Dividing (4.6) by a_r and (4.7) by b_{r+1} then summing up as before, and substituting a_r, b_{r+1} and λ^2 by their values, we easily deduce that $\kappa = 0$, $\mathcal{P}^{(r)}\Psi_r = 0$ and $\mathcal{P}^{(r+1)}\Psi_{r+1} = 0$. Hence by (4.6), we find that $\lambda^2 = \frac{1}{4(1-a_r)}\sigma_0 = \frac{m+1}{4m}\sigma_0$ where $\sigma_0 = \inf\limits_M \sigma^\nabla$, then $r = \frac{m-1}{2}$ and m is odd. It remains to prove that Ψ satisfies (4.4). For $r = \frac{m-1}{2}$, by definition of the Kählerian twistor operators, for all $j \in \{1, \cdots, q\}$, we obtain

$$\nabla_{e_j}\Psi_r + \frac{\lambda}{m+1}p_-(e_j)\cdot\Psi_{r+1} = 0,$$

and

$$\nabla_{e_j}\Psi_{r+1} + \frac{\lambda}{m+1}p_+(e_j)\cdot\Psi_r = 0.$$

Summing up the two equations, we get (4.4) for $X = e_j$. Using Ricci identity in (4.4), one easily proves that $\mathcal{F}$ is transversally Einstein. □

Theorem 4.4 *Under the same conditions as in Theorem 4.3 for m even, any eigenvalue λ of the transversal Dirac operator satisfies*

$$\lambda^2 \geq \frac{m}{4(m-1)}K_0^\nabla. \quad (4.8)$$

If Ψ is an eigenspinor of type $(r, r+1)$ associated with an eigenvalue satisfying equality in (4.8), then $r = \frac{m}{2}$, the foliation $\mathcal{F}$ is minimal and Ψ satisfies for all $X \in \Gamma(Q)$,

$$\nabla_X\Psi_{r+1} = -\frac{\lambda}{q}(X - iJX)\cdot\Psi_r. \quad (4.9)$$

Proof. Let Ψ an eigenspinor of type $(r, r+1)$ associated with any eigenvalue λ of the transversal Dirac operator D_{tr}. Recalling Equalities (4.6) and (4.7), we have

$$0 \leq \int_M ((1-a_r)\lambda^2 - \frac{1}{4}K_\sigma^\nabla)|\Psi_r|^2 - a_r\lambda\int_M \Re(\Psi_{r+1}, p_+(\kappa)\cdot\Psi_r), \quad (4.10)$$

and

$$0 \leq \int_M ((1-b_{r+1})\lambda^2 - \frac{1}{4}K_\sigma^\nabla)|\Psi_{r+1}|^2 + b_{r+1}\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa)\cdot\Psi_r). \quad (4.11)$$

Hence if $\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa)\cdot\Psi_r) \leq 0$, then by (4.11)

$$\lambda^2 \geq \frac{1}{4(1-b_{r+1})}K_0^\nabla,$$

The antilinear isomorphism j sends $S_r(\mathcal{F})$ to $S_{m-r}(\mathcal{F})$. This allows the choice of μ_r to be non negative (i.e. $r \geq \frac{m}{2}$) where μ_r is the eigenvalue associated with Ψ_r. Then a careful study of the graph of the function $\frac{1}{1-b_{r+1}}$, yields (4.8).

On the other hand if $\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa)\cdot\Psi_r) > 0$. Applying Equation (4.5) to the spinor $j\Psi$, which is a spinor of type $(m-(r+1), m-r)$, we find the same inequalities as (4.10) and (4.11), then

$$\lambda^2 > \frac{1}{1-a_r}\frac{K_0^\nabla}{4}.$$

As before we can choose $\mu_{m-(r+1)} \geq 0$ (i.e. $r \leq \frac{m}{2}-1$). A careful study of the graph of the function $\frac{1}{1-a_r}$ gives Inequality (4.8).
Now we discuss the limiting case of (4.8). As we have seen, it could not be achieved if $\lambda \int_M \Re(\Psi_{r+1}, p_+(\kappa)\cdot\Psi_r) > 0$, so only the other case should be considered. By (4.7), one has

$$\begin{aligned}
&\textstyle\int_M |\mathcal{P}^{(r+1)}\Psi_{r+1}|^2 + \frac{a_{r+1}}{4}\int_M |p_+(\kappa)\cdot\Psi_{r+1}|^2 \\
&\textstyle+\frac{b_{r+1}}{4}\int_M |p_-(\kappa)\cdot\Psi_{r+1}|^2 - b_{r+1}\lambda\int_M \Re(\Psi_{r+1}, p_+(\kappa)\cdot\Psi_r) = \\
&\textstyle(1-b_{r+1})\int_M (\frac{m}{4(m-1)}K_0^\nabla - \frac{1}{4(1-b_{r+1})}K_\sigma^\nabla)|\Psi_{r+1}|^2.
\end{aligned}$$

Since $\frac{m}{m-1} = \inf_{r\geq\frac{m}{2}} \frac{1}{1-b_{r+1}}$, and the l.h.s. of (4) is non negative, we deduce that $\kappa = 0, \mathcal{P}^{r+1}\Psi_{r+1} = 0$ and $\frac{m}{m-1} = \frac{1}{1-b_{r+1}}$ so $r = \frac{m}{2}$. It remains to show that Equation (4.9) holds. For this, take $X = e_j$ where $\{e_j\}_{j=1,\cdots,q}$ is a local orthonormal frame. For $r = \frac{m}{2}$, and by definition of the Kählerian twistor operators, for all $j \in \{1,\cdots,q\}$, we obtain

$$\nabla_{e_j}\Psi_{r+1} + \frac{\lambda}{q}(e_j - iJe_j)\cdot\Psi_r = 0.$$

$\square$

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