Finding an ordinary conic and an ordinary hyperplane

Olivier Devillers, Asish Mukhopadhyay

published in Nordic Journal of Computing, 6:462-468, 1999

Finding an ordinary conic and an ordinary hyperplane

Olivier Devillers* Asish Mukhopadhyay†

August 24, 2007

Abstract

Given a finite set of non-collinear points in the plane, there exists a line that passes through exactly two points. Such a line is called an *ordinary line*. An efficient algorithm for computing such a line was proposed by Mukhopadhyay et al [10].

In this note we extend this result in two directions. We first show how to use this algorithm to compute an *ordinary conic*, that is, a conic passing through exactly five points, assuming that all the points do not lie on the same conic. Both our proofs of existence and the consequent algorithms are simpler than previous ones. We next show how to compute an ordinary hyperplane in three and higher dimensions.

1 Introduction

Let $\mathcal{S}$ be a set of n points in the plane. A *connecting* line of $\mathcal{S}$ is a line that passes through at least two of its points. A connecting line is said to be *ordinary* if it passes through exactly two points of $\mathcal{S}$.

The problem of establishing the existence of such a line originated with Sylvester [14], who proposed the following problem in 1893:

> If n points in the plane are such that a line passing through any two of them passes through a third point, then are the points collinear?

No solution came forth during the next forty years. In 1943, a positive version of the same problem was proposed by Erds [4], and in the following year a solution by Gallai [5] appeared in print.

Subsequently other proofs also appeared, notable among which were the proofs by Steinberg [13] and Kelly [7]. These results showed that the answer is in the affirmative for real projective geometry in the plane.

*INRIA, BP 93, 06902 Sophia Antipolis, France. Olivier.Devillers@sophia.inria.fr

†Department of Computer Science University of Windsor Windsor, Canada, on leave from IIT Kanpur. asishm@cs.UWindsor.ca, work of this author was done in part while he was visiting INRIA. The support provided by INRIA, financial and otherwise, is gratefully acknowledged.

Therefore if the points of $\mathcal{S}$ are not collinear then there is at least one ordinary line. In fact, Kelly and Moser [7] showed that there are at least $3n/7$ ordinary lines.

A set of points is said to be *co-conic* if all the points lie on one conic. In this paper we address a more general version of the ordinary line problem: given a set of n points in the plane that are not co-conic, find a conic that passes through exactly five points.

Our algorithm provides a constructive proof of the existence of such a conic. Another proof is contained in [15]. Our proof is very simple and allows us to relate a result on the number of ordinary lines to the number of ordinary conics.

The paper is organized as follows. In the next section we discuss some mathematical preliminaries. The algorithm is discussed in the third section. We conclude in the fourth and final section.

2 Preliminaries

2.1 Notations and basic results

Space of conics Let $\mathcal{S}$ be a set of n points in $\mathbb{R}^2$. Let ϕ be the transformation that maps a point $p = (x, y) \in \mathbb{R}^2$ to the point $p^\star = (x^2, y^2, xy, x, y) \in \mathbb{R}^5$.

Under this transformation, $\mathbb{R}^{2\star}$ is the 2 dimensional manifold image of $\mathbb{R}^2$ and $\mathcal{S}^\star$ the map of $\mathcal{S}$in $\mathbb{R}^5$.

If $\mathcal{C}$ is a conic in $\mathbb{R}^2$ with the equation $ax^2+by^2+cxy+dx+ey+f = 0$, then $\phi(\mathcal{C}) = \mathcal{C}^\star$ is the intersection of $\phi(\mathbb{R}^2) = \mathbb{R}^{2\star}$ with the hyperplane $\mathcal{C}^v$: $au + bv + cw + dx + ey + f = 0$ in $\mathbb{R}^5$. We identify the conics of $\mathbb{R}^2$ to hyperplanes of $\mathbb{R}^5$, which can be called *space of conics*.

This idea of mapping points in five dimensions is a natural generalization of the usual space of circles widely used in computational geometry [11, 3, 2] that associates a circle $\mathcal{C}$ in the plane to a point $\mathcal{C}^\bullet$ in three-dimensions and to the polar hyperplane $\mathcal{C}^v$ of $\mathcal{C}^\bullet$ with respect to the unit paraboloid.

Flats We recall a few basic results from finite-dimensional vector spaces.

A flat F is an affine subspace of $\mathbb{R}^5$ such that for any two points $p, q \in F$, $\alpha p + \beta q \in F$, where $\alpha + \beta = 1$.

A flat is defined by one of its point and its direction $\vec{F} = \{p-q,\ p, q \in F\}$ which is a vectorial subspace of the vectorial space $\mathbb{R}^5$ (a flat is a set of points, its direction a set of vectors).

Two subspaces $\vec{F_1}$ and $\vec{F_2}$ of $\mathbb{R}^5$ are called supplementary if and only if $\vec{F_1} \cap \vec{F_2} = \{0\}$ and $dim(S_1) + dim(S_2) = 5$.

Two flats F_1 and F_2 having supplementary directions have an unique intersection point.

If A and B are two subsets of $\mathbb{R}^5$, we define the affine hull $A \oplus B$ as the smallest flat that contains both A and B.

Point-hyperplane duality Point-hyperplane duality is a common transformation in computational geometry [3, 12]. A point p at distance

r from the origin O is associated with the hyperplane normal to Op at distance $1/r$ from the origin.

This transformation reduces the problem of computing the intersection of a finite set of half-spaces, each containing the origin, to the problem of computing the convex hull of the corresponding points in dual space.

Inversion An inversive transformation maps a point p at distance r from the origin, O, to the point p' at distance $1/r$ from O, lying on the half-line $[Op)$ [12].

This involutary transformation has the interesting properties that the images of spheres and hyperplanes are spheres or hyperplanes. Particularly, spheres passing through O are exchanged with hyperplanes.

2.2 Ordinary line

For completeness, we briefly sketch the algorithm for finding an ordinary line in a finite set of non-collinear and coplanar points.

Let l be a directed line (direction $\vec{v}$) through exactly one point p_0 of $\mathcal{S}$. Let $q_\lambda = p_0 + \lambda\vec{v}$. We find the line passing through at least two points of $\mathcal{S}$ that cuts l in a point q_λ with minimal $\lambda > 0$. Such a line passes through two points consecutive in polar order around p_0 and can thus be found in $O(n \log n)$ time. Either this line is ordinary or a line through p_0 and a point on this line is ordinary. For details see Mukhopadhyay et al. [10].

3 Algorithm

The idea behind the algorithm is to find a hyperplane that passes through exactly five points of $\mathcal{S}^\star$. In the $\mathbb{R}^2$ plane this corresponds to a conic that passes through exactly five points of $\mathcal{S}$.

We first find a conic that passes through exactly three points of $\mathcal{S}$. We do this as follows. We choose $p, q, r \in \mathcal{S}$ and $s, t \notin \mathcal{S}$ such that no three (four) of the five points are collinear. Denote by $\vec{\imath}$ the vector $(1, 0) \in \mathbb{R}^2$ and consider the conic $\mathcal{A}_\theta$ passing through the five points $p, q, r, s, t + \theta\vec{\imath}$.

For any point $\rho \in \mathcal{S}$, there exist at most two values $\theta = \theta_\rho$ or $\theta = \theta'_\rho$ such that $\rho \in \mathcal{A}_\theta$. This is because if

$$a_\theta x^2 + b_\theta y^2 + c_\theta xy + d_\theta x + e_\theta y + f_\theta = 0$$

is the conic that passes through the points, $p, q, r, s, t + \theta\vec{\imath}$ then each of the coefficients is of second degree in θ. So it is easy to determine some θ_0 different from all these values such that $\mathcal{A}_{\theta_0} \cap \mathcal{S} = \{p, q, r\}$.

Now the affine hull $\mathcal{B} = p^\star \oplus q^\star \oplus r^\star$ is a subset of the hyperplane $\mathcal{A}^\star_{\theta_0}$ and so is the affine hull spanned by the points $p^\star$, $s^\star$ and $(t + \theta_0\vec{\imath})^\star$.

Moreover, these two sets intersect in the single point $p^\star$ (see Figure 1).

Let γ be a point of $\mathbb{R}^5$ not in $\mathcal{A}^\star_{\theta_0}$. Translate the affine hull of the points, $p^\star$, $s^\star$ and $(t+\theta\vec{\imath})^\star$ to pass through γ. Let $\mathcal{C}$ denote this translated affine hull.

For any point $\rho \in \mathcal{S} - \{p, q, r\}$, we construct the point $\rho^\dagger = (\mathcal{B} \oplus \rho^\star) \cap \mathcal{C}$. The intersection is exactly one point because the directions of the flats

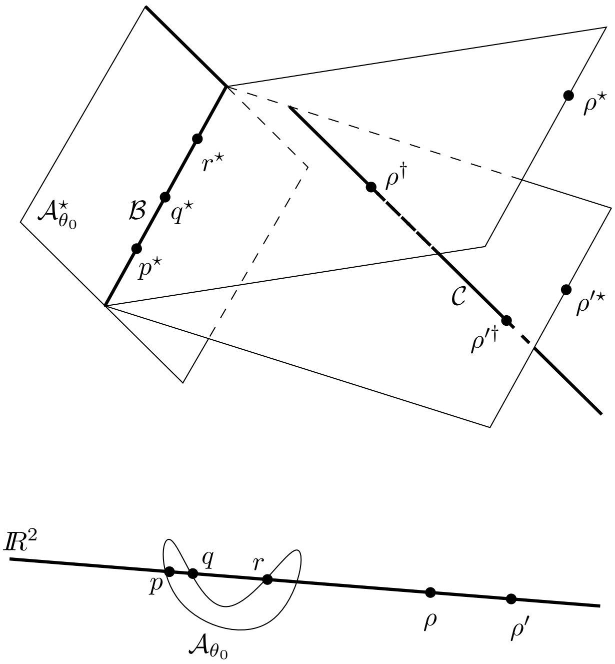

Figure 1: Mapping in 5 dimensions

$\mathcal{B} \oplus \rho^\star$ and $\mathcal{C}$ are supplementary subspaces of the vectorial space $I\!R^5$. Otherwise, $\rho^\star$ would belong to $\mathcal{A}^\star_{\theta_0}$ which is impossible since $\rho \notin \mathcal{A}_{\theta_0}$ by the definition of θ_0.

The set of points $\mathcal{S}^\dagger = \left\{\rho^\dagger; \rho \in \mathcal{S} \ - \{p,q,r\}\right\}$ lies in the two-dimensional plane $\mathcal{C}$. Let l be a line in that plane and $\mathcal{H}$ the hyperplane through $\mathcal{B}$ and l. By construction $\rho^\dagger \in \mathcal{H}$ if and only if $\rho^\star \in \mathcal{H}$; indeed by the definition of $\rho^\dagger$, the line $(\rho^\dagger\rho^\star)$ cuts $\mathcal{B}$ (in $\rho^\ddagger$) and thus an hyperplane $\mathcal{H}$ cannot contains $\mathcal{B}$ ($\Rightarrow \rho^\ddagger \in \mathcal{H}$) and $\rho^\dagger$ without containing $\rho^\star$. Thus at this point we see a complete equivalence between the problem of finding an ordinary line for $\mathcal{S}^\dagger$ and an ordinary conic for $\mathcal{S}$ through the points p, q and r.

If all the points of $\mathcal{S}^\dagger$ are collinear, then all the points of $\mathcal{S}^\star$ are in the same hyperplane and hence all the points of $\mathcal{S}$ are co-conic.

Otherwise, there exists an ordinary line l in $\mathcal{C}$, and the corresponding hyperplane $\mathcal{H}$ contains only five points of $\mathcal{S}^\star$. The corresponding conic is ordinary and passes through p, q, r.

The following theorem is a consequence of the above discussion and the fact that there are at least $\frac{3n}{7}$ ordinary lines:

> **Theorem:** *Given a set $\mathcal{S}$ of n points in the plane that are not co-conic, then for any three non collinear points in $\mathcal{S}$, there exist at least $\frac{3(n-3)}{7}$ ordinary conics of $\mathcal{S}$ that pass though these three points. Furthermore, such an ordinary conic can be found in $O(n \log n)$ time.*

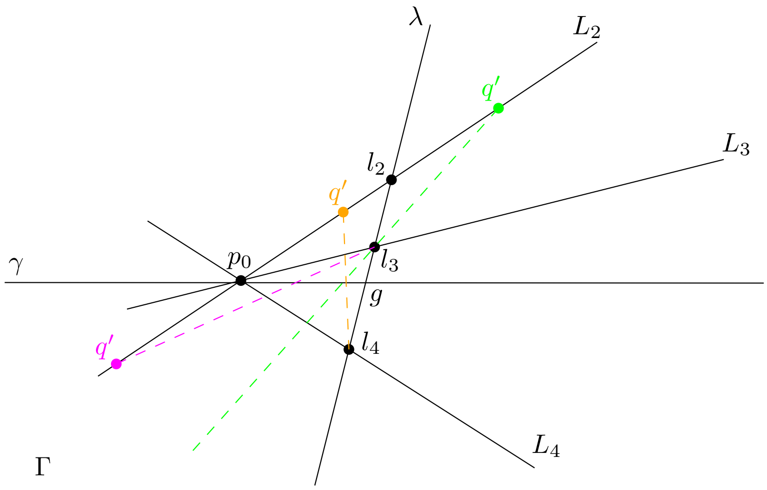

Figure 2: Ordinary plane in three dimensions

4 Ordinary plane in three dimensions

Given a set $\mathcal{S}$ of $n(\geq 3)$ points in three space, a connecting plane (that is, a plane through some three points of $\mathcal{S}$) is defined to be ordinary if all but one of the points of $\mathcal{S}$ that lie on it are collinear. Such a plane always exists, unless all the points of $\mathcal{S}$ are collinear.

A plane that passes through exactly three points is certainly ordinary in the sense of this definition; however, such a plane need not exist.

As an example, place three or more points on each of two skew lines in three space. This configuration of points has no connecting plane that is defined by exactly three points, all the ordinary planes contain one of the two lines, (see Motzkin [9]).

We show that the ideas sketched in Section 2.2 can be generalized to three and higher dimensions to compute a plane that is ordinary in the sense of the above definition.

Let p_0 be a point of $\mathcal{S}$ and γ a line through p_0. Let Λ be a plane through three points p_1, p_2, p_3 of $\mathcal{S}$ such that its distance to p_0, measured along γ, is minimum among all possible connecting planes of $\mathcal{S}$ that intersect γ.

If Λ is ordinary, we are done; otherwise, set $g = \gamma \cap \Lambda$ and let Γ be an arbitrary plane containing γ; finally, set $\lambda = \Lambda \cap \Gamma$.

Let p_1, p_2, p_3 and p_4 be points of $\mathcal{S}$ in Λ such that no 3-tuple of the form $p_1 p_i p_j$, $i, j \in \{2, 3, 4\}$, are collinear (such points exist in Λ since Λ is not ordinary). We consider the planes through $p_0 p_1$ and p_2, p_3, p_4 respectively. Let L_2, L_3 and L_4 be their respective intersections with Γ and l_2, l_3 and l_4 their respective intersections with λ.

Assume, without loss of generality, that l_2 is separated from g along λ by l_3 or l_4. Then we claim that the plane determined by the points p_0, p_1, p_2 is ordinary. That is all the points, barring p_0, are on the line determined by p_1, p_2.

If not, let q be a point, distinct from p_0, lying outside this line. In fact, if $q \notin \Lambda \cap L_2$, one of the two planes qp_1p_3 or qp_1p_4 must pass between g and p_0, contradicting the definition of Λ. The dashed lines on Figure 2 show the different cases for the intersection of the plane with Γ depending of the position of $q' = (p_1q) \cap L_2$.

It remains to find Λ efficiently. It is clear that if we consider the cell in the arrangement of the $O(n^3)$ planes defined by points of $\mathcal{S} - \{p_0\}$, that contains p_0, Λ is incident on the facet that is hit by γ.

By a point-plane duality transformation, with p_0 as center, the cell containing p_0 is mapped into the convex hull of the $O(n^3)$ vertices of an arrangement of $n-1$ planes (see paragraph 2.1).

In two dimensions we have to compute the convex hull of the vertices of an arrangement of $n-1$ lines and it is not difficult to see that a vertex can be on the convex hull only if the two lines have consecutive slopes (in the set of all slopes). In three dimensions, the phenomenon is similar if we consider the Gaussian diagram of the normals to the $n-1$ planes (the convex hull of the unit normal vectors to $n-1$ planes) [1, 6, 8]. Three planes define a vertex of the convex hull only if their normal vectors define a face of the Gaussian diagram. Since the Gaussian diagram can be computed in $O(n \log n)$ time, we get the following result:

> **Theorem:** *If $\mathcal{S}$ a set of n non-coplanar points in 3-space, then an ordinary plane can be found in $O(n \log n)$ time.*

The same ideas extend to higher dimensions. We can find an ordinary hyperplane with the help of a Gaussian diagram. The complexity is identical to the complexity of computing the convex hull in that dimension.

5 Conclusions

In this note we have shown that an algorithm for finding a line through exactly two points of a given set of non-collinear points can be used to find a conic through exactly five points, if we assume that all the points are not co-conic.

It is particularly easy to find an ordinary circle passing through a chosen point, p, of the given set of points, if we allow for a degenerate circle. We simply apply an inversion transformation with p as the center of inversion (see paragraph 2.1). Solve the ordinary line problem for the remaining $n-1$ transformed points. We have a degenerate circle if the ordinary line found passes through p, else its image is an ordinary circle passing through p. We also conclude that at least $3(n-1)/7$ ordinary circles pass through a chosen point.

By applying a stereographic projection we note that if n points on a real sphere do not lie in the same plane then there is plane containing exactly three of them.

References

[1] Francis Avnaim and Jean-Daniel Boissonnat. Simultaneous containment of several polygons. In *Proc. 3rd Annu. ACM Sympos. Comput. Geom.*, pages 242–250, 1987.

[2] Olivier Devillers, Stefan Meiser, and Monique Teillaud. The space of spheres, a geometric tool to unify duality results on Voronoi diagrams. In *Proc. 4th Canad. Conf. Comput. Geom.*, pages 263–268, 1992.

[3] H. Edelsbrunner. *Algorithms in Combinatorial Geometry*, volume 10 of *EATCS Monographs on Theoretical Computer Science*. Springer-Verlag, Heidelberg, West Germany, 1987.

[4] P. Erdős. Problem no 4065. *Amer. Math. Monthly*, 51:169, 1944.

[5] T. Gallai. Solution to problem no 4065. *Amer. Math. Monthly*, 51:169–171, 1944.

[6] Leonidas J. Guibas and R. Seidel. Computing convolutions by reciprocal search. *Discrete Comput. Geom.*, 2:175–193, 1987.

[7] L. Kelly and W. Moser. On the number of ordinary lines determined by n points. *Canad. J. Math.*, 10:210–219, 1958.

[8] T. Lozano-Pérez. Spatial planning: A configuration space approach. *IEEE Trans. Comput.*, C-32:108–120, 1983.

[9] T. Motzkin. The lines and planes connecting the points of a finite set. *Trans. Amer. Math. Soc.*, 70:451–464, 1951.

[10] A. Mukhopadhyay, A. Agrawal, and R. M. Hosabettu. On the ordinary line problem in computational geometry. *Nordic J. Comput.*, 4:330–341, 1997.

[11] D. Pedoe. *Geometry, a comprehensive course*. Dover Publications, New York, 1970.

[12] F. P. Preparata and M. I. Shamos. *Computational Geometry: An Introduction*. Springer-Verlag, 3rd edition, October 1990.

[13] R. Steinberg. Solution to problem no 4065. *Amer. Math. Monthly*, 51:169–171, 1944.

[14] J. J. Sylvester. Mathematical question 11851. *Educational Times*, 59:98, 1893.

[15] J. A. Wiseman and P. R. Wilson. A Sylvester theorem for conic sections. *Discrete Comput. Geom.*, 3:295–305, 1988.