Some regularity results for anisotropic motion of fronts

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SOME REGULARITY RESULTS FOR ANISOTROPIC MOTION OF FRONTS

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Abstract. We study the regularity of propagating fronts whose motion is anisotropic. We prove that there is at most one normal direction at each point of the front; as an application, we prove that convex fronts are $C^{1,1}$. These results are by-products of some necessary conditions for viscosity solutions of quasilinear elliptic equations. These conditions are of independent interest; for instance they imply some regularity for viscosity solutions of nondegenerate quasilinear elliptic equations.

1. Introduction

Following [4, 12], we study propagating fronts whose velocity field $\mathbf{v}_\Phi$ is given by the following geometric law:

$$\mathbf{v}_\Phi = (\kappa_\Phi + g)\mathbf{n}_\Phi,$$

where $\mathbf{n}_\Phi$ and κ_Φ are respectively the inward normal direction and the mean curvature associated with a Finsler metric Φ; g denotes a possible (bounded) driving force.

The main result of this paper states that under appropriate assumptions, there is at most one (outward or inward) "normal direction" at each point of the front.

In order to define the front past singularities, we use the *level-set approach* initiated by Barles [1] and developed by Osher and Sethian [13]. This approach consists in describing the front Γ_t at time t as the zero level-set of a (continuous or discontinuous) function $u : \Gamma_t = \{x : u(x,t) = 0\}$. Choosing

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first a continuous function u_0 such that the initial front Γ_0 coincides with $\{x : u_0(x) = 0\}$ (consider for instance the signed distance function to Γ_0), u turns out to be a solution of the following Hamilton-Jacobi equation:

$$\frac{\partial u}{\partial t} - \Phi^\circ(Du,x)\Big[\mathrm{tr}[D_{\zeta\zeta}\Phi^\circ(Du,x)D^2u] + \Big\langle D_\zeta\Phi^\circ(Du,x), \frac{Du}{|Du|}\Big\rangle$$
$$+\mathrm{tr}[D_{\zeta x}\Phi^\circ(Du,x)] + g(Du,x,t)\Big] = 0, \qquad (1.1)$$

where Du and D^2u denotes the first and second derivative in x of the function u and Φ° denotes the dual metric associated with Φ. This equation is known as the *anisotropic mean curvature equation.* It is solved by using viscosity solutions [7]. The function u depends on the choice of u_0, but not the front Γ_t, even not the two families of sets $O_t = \{x : u(x,t) > 0\}$ and $I_t = \{x : u(x,t) < 0\}$ [8, 5, 11]. The definition of the front is therefore consistent and the notions of "outside"and "inside" become precise.

The study of the normal directions reduces to the study of the semi-jets of discontinuous semisolutions of (1.1). This latter study is persued by using necessary conditions derived for viscosity solutions of degenerate elliptic and parabolic quasilinear equations. Besides, these conditions are of independent interest. For instance, we derive from them regularity of viscosity solutions of nondegenerate quasilinear elliptic and parabolic equations.

The paper is organized as follows. In Section 2, we first give assumptions and recall definitions that are used in the paper. In particular, the Finsler metric and its dual are introduced and the definition of normal directions and semijets are recalled. In Section 3, we state and prove our main results (Theorem 1 and Corollary 1). Eventually, in Section 4, we present the necessary conditions used in the proof of Theorem 1.

2. Assumptions and definitions

In this section, we give assumptions and definitions that are used throughout the paper.

2.1. **Anisotropic motion.** In order to take into account the anisotropy and the inhomogeneity of the environment in which the front propagates, the metric induced by the Euclidian norm is replaced with a so-called *Finsler metric*. In our context, a Finsler metric Φ is the support function of a given compact set denoted by $B_{\Phi^\circ}(x)$:

$$\Phi(\zeta,x) = \max\{\langle\zeta,\zeta^*\rangle : \zeta^* \in B_{\Phi^\circ}(x)\}.$$

The set $B_{\Phi^\circ}(x)$ is referred to as *the Wulff shape.* Here are the assumptions we make concerning Φ and $B_{\Phi^\circ}(x)$.

A0. (i) The Wulff shape $B_{\Phi^\circ}(x)$ is a compact set that contains the origin in its interior and is symmetric with respect to it;
(ii) $\Phi \in C^2(\mathbb{R}^n\backslash\{0\} \times \mathbb{R}^n)$;
(iii) for all $x \in \mathbb{R}^n$, $\zeta \mapsto [\Phi(\zeta, x)]^2$ is strictly convex.

For a given $x \in \mathbb{R}^n$, the dual metric Φ° is defined as the support function of the set $B_\Phi(x) = \{\zeta \in \mathbb{R}^n : \Phi(\zeta, x) \leqslant 1\}$; this set is known as the *Franck diagram.*

Let us give few examples. If the Finsler metric is simply the Euclidian norm and if there is no driving force, the motion is isotropic and (1.1) reduces to the well-known (*isotropic*) *mean curvature equation*:

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\langle D^2uDu, Du\rangle}{|Du|^2} = 0. \tag{2.1}$$

Equations (1.1) and (2.1) are quasilinear and present a singularity at $Du = 0$. There ellipticity and degeneracy follow from the fact that $\zeta \mapsto \Phi^\circ(\zeta, x)$ is a support function; indeed, a support function is convex and linear along half-lines issued from the origin. Consequently:

$$D_{\zeta\zeta}\Phi^\circ(\zeta, x) \succcurlyeq 0 \text{ and } \zeta \in \mathrm{Ker} D_{\zeta\zeta}\Phi^\circ(\zeta, x),$$

where $\preccurlyeq$ denotes the usual order associated with $\mathbb{S}_n$, the space of $n \times n$ symmetric matrices. A second example of motion is the following: consider a (riemannian) metric $\Phi(\zeta, x) = \Phi(\zeta) = \sqrt{\langle G\zeta, \zeta\rangle}$ where $G \in \mathbb{S}_n$ is definite positive. The associated dual metric turns out to be $\Phi^\circ(\zeta^*) = \sqrt{\langle G^{-1}\zeta^*, \zeta^*\rangle}$. Finally, let us give a third example in which the inhomogeneity of the environment is taken into account: $\Phi(\zeta, x) = a(x)\sqrt{\langle G\zeta, \zeta\rangle}$, where $a \in C^2(\mathbb{R}^n)$ and $a(x) > 0$ for all $x \in \mathbb{R}^n$. The reader can check that in these three examples the kernel of $D_{\zeta\zeta}\Phi^\circ(\zeta, x)$ coincides with $\mathrm{Span}\{\zeta\}$. We next assume that the Finsler metric verifies such a property.

A1. $\forall x \in \mathbb{R}^n, \forall \zeta \in \mathbb{R}^n\backslash\{0\}, \mathrm{Ker} D_{\zeta\zeta}\Phi^\circ(\zeta, x) = \mathrm{Span}\{\zeta\}$.

We also need the following additional assumption.

A2. There exists $L > 0$ such that for all $x, y \in \mathbb{R}^n$ and all $\zeta^* \in \mathbb{R}^n$,

$$|\Phi^\circ(\zeta^*, y) - \Phi^\circ(\zeta^*, x)| \leqslant L\, |\zeta^*|\, |y - x|.$$

2.2. **Semi-jets, P-subgradients and P-normals.** We solve (4.1) and (4.2) by using viscosity solutions [7]. In order to ensure the existence of a solution (using for instance results from [10, 12]), we assume throughout the paper that the initial front is bounded. Unboundedness of the domain can be handled with results from [3]. The definition of viscosity solutions is based on the notion of *semi-jets*. Let Ω be a subset of $\mathbb{R}^n$ and u be a numerical function defined on Ω and x be a point in Ω. A couple $(X,p) \in \mathbb{S}_n \times \mathbb{R}^n$ is a so-called *subjet* (resp. a *superjet*) of the function u at x (with respect to Ω) if for all $y \in \Omega$:

$$\frac{1}{2}\langle X(y-x), y-x\rangle + \langle p, y-x\rangle \leqslant u(y) - u(x) + o(|y-x|^2) \quad (2.2)$$

$$\Big(\text{resp. } \frac{1}{2}\langle X(y-x), y-x\rangle + \langle p, y-x\rangle \geqslant u(y) - u(x) + o(|y-x|^2)\Big), \quad (2.3)$$

where $o(.)$ is a function such that $o(h)/h \to 0$ as $h \to 0^+$. The set of all the subjets (resp. superjets) of u at x is denoted by $\mathcal{J}^{2,-}_\Omega u(x)$ (resp. by $\mathcal{J}^{2,+}_\Omega u(x)$). In order to define viscosity solutions for parabolic equations, one must use so-called *parabolic semi-jets* $\mathcal{P}^{2,-}_{\Omega\times[0,T]} u(x,t)$; see [7] for their definition.

A vector p such that there exists $X \in \mathbb{S}_n$ such that $(X,p) \in \mathcal{J}^{2,-}_\Omega u(x)$ is a so-called *P-subgradient* [6] of the function u :

$$\forall y \in \Omega, \langle p, y-x\rangle \leqslant u(y) - u(x) + O(|y-x|^2).$$

The set of all such vectors is referred to as the *proximal subdifferential* of the function u and it is denoted by $\partial_P u(x)$. Analogously, a *proximal superdifferential* (hence *P-supergradients*) can be defined by $\partial^P u(x) = -\partial_P(-u)(x)$. It coincides with the sets of vectors p such that $\exists X \in \mathbb{S}_n : (X,p) \in \mathcal{J}^{2,+}_\Omega u(x)$.

The geometry of a set Ω can be investigated by studying subjets of the function denoted by Zero_Ω defined on Ω and that is identically equal to 0. The proximal subdifferential of this function coincides with the *proximal normal cone* of Ω at x [6]:

$$N_P(\Omega, x) = \{p \in \mathbb{R}^n : \forall y \in \Omega, \langle p, y-x\rangle \leqslant O(|y-x|^2)\}.$$

An element of $N_P(\Omega,x)$ is referred to as a *P-normal*. If p is a P-normal of Ω at x and λ is a nonnegative number, then λp is still a P-normal. From the geometrical viewpoint, one can say that $N_P(\Omega,x)$ is a cone, that is to say it is made of half-lines issued from the origin. Crandall, Ishii and Lions [7] proved that for a set with a C^2 boundary:

$$\mathcal{J}^{2,-}_\Omega \text{Zero}(x) = \{(S(x) - Y, \lambda n(x)) : \lambda \geq 0, Y \succcurlyeq 0\},$$

where $n(x)$ denotes the normal vector and $S(x)$ denotes the second fundamental form extended to $\mathbb{R}^n$ by setting $S = 0$ along $\text{Span}\{n(x)\}$. The proximal normal cone is therefore reduced to $\mathbb{R}^+ n(x) = \{\lambda n(x) : \lambda \geqslant 0\}$. If Ω is a hyperplan and $n \neq 0$ denotes a normal vector from $H^\perp$, then $N_P(\Omega, x)$ is the whole line $\mathrm{Span}\{n\}$.

3. Main results

In this section, we state and prove our main results, namely Theorem 1 and Corollary 1. The proof of Theorem 1 rely on necessary conditions verified by solutions of possibly degenerate elliptic and parabolic quasilinear equations; these conditions are presented in Section 4.

Theorem 1. *Consider a Finsler metric Φ satisfying* **A0, A1** *and* **A2**. *Then the associated propagating front Γ_t, $t > 0$, has at most one "outward normal direction" (resp. "inward normal direction"), that is to say the proximal normal cone at any point of $I_t \cup \Gamma_t$ or at any point of $\overline{I_t}$ (resp. $O_t \cup \Gamma_t$ or $\overline{O_t}$) is at most a line.*

Remarks. 1. Assumptions **A0** and **A2** ensure the existence and uniqueness of the solution u of (1.1). Assumption **A1** can be seen as a regularity assumption on the Franck diagram.

2. Theorem 1 remains valid if the front "fattens" (see [14] for details about the fattening phenomena).

Theorem 1 implies the regularity of convex fronts. See also Theorem 5.5 in [9].

Corollary 1. *Let the metric Φ be independent of the position and such that* **A0, A1** *are satisfied. Assume that the initial front Γ_0 is convex. Then the associated propagating front Γ_t is also convex and is $C^{1,1}$; more precisely, $I_t \cup \Gamma_t$ and $\overline{I_t}$ are convex and their boundary is $C^{1,1}$.*

Let us now prove these two results.

Proof of Theorem 1. Assumptions **A0** and **A2** ensure that the assumptions of Theorem 4.9 in [10] are satisfied. Then, there exists a unique solution of (1.1). In order to prove Theorem 1, we must prove that for a given point of the boundary of $\overline{I_t}$, two P-normals p_1 and p_2 are colinear. Let us choose λ such that $\lambda p_1 + (1-\lambda)p_2 \neq 0$. We know [3] that the function $\mathrm{Zero}_{\overline{I_t}}$ is a supersolution of (1.1). By applying Proposition 1 (see Section 4), we obtain:

$$p_1 - p_2 \in \mathrm{Ker} D_{\zeta\zeta}\Phi^\circ(\lambda p_1 + (1-\lambda)p_2).$$

Using Assumption **A1**, we conclude $p_1 - p_2$ is colinear with $\lambda p_1 + (1-\lambda)p_2$. We conclude that p_1 and p_2 are colinear.

We proceed analogously with the sets $\overline{I_t}$, $O_t \cup \Gamma_t$ and $\overline{O_t}$. $\square$

Proof of Corollary 1. The fact that the front is convex for any time t follows from Theorem 3.1 in [10]. Choosing for u_0 the opposite of the signed distance function to Γ_0, we ensure that the initial datum is Lipschitz and concave. Therefore, Theorem 2.1 in [12] implies that u is Lipschitz; this ensures that u has a sublinear growth. By applying Theorem 3.1 in [10], we know that $x \mapsto u(x,t)$ is concave, hence $I_t \cup \Gamma_t$ and $\overline{I_t}$ are convex sets. The Hahn-Banach theorem ensures the existence of a normal in the sense of convex analysis. Such a normal is also a P-normal [6]. Using the fact that Zero $_{\overline{I_t}}$ and Zero $_{I_t \cup \Gamma_t}$ are supersolutions of (1.1) (see for instance [3]), Theorem 1 implies that there is at most one P-normal. Hence there is exactly one normal in the sense of convex analysis and $C^{1,1}$ regularity follows. □

4. Necessary conditions for elliptic and parabolic quasilinear equations

In the present section, we state necessary conditions that are verified by viscosity sub- and supersolutions (hence by solutions) of quasilinear elliptic equations on a domain $\Omega \subset \mathbb{R}^n$:

$$-\sum_{i,j=1}^{n} a_{i,j}(Du,u,x)\frac{\partial^2 u}{\partial x_i \partial x_j} + f(Du,u,x) = 0, \forall x \in \Omega. \tag{4.1}$$

These equations may be degenerate and/or singular at $Du = 0$. We also study the associated parabolic equations on $\Omega \times [0,T]$:

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{n} a_{i,j}(Du,u,x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + f(Du,u,x,t) = 0, \forall (x,t) \in \Omega \times [0,T]. \tag{4.2}$$

In the following, the $n \times n$ symmetric matrix with entries $(a_{i,j})$ is denoted by A. We assume that (4.1) and (4.2) are degenerate elliptic.

(E) For all $p, u, x(,t)$, $A(p,u,x(,t)) \succcurlyeq 0$.

In Propositions 1 and 2, we prove that the difference of two P-subgradients (resp. P-supergradients) of a supersolution (resp. of a subsolution) of (4.1) or (4.2) is a degenerate direction, that is to say it lies in the kernel of A.

Proposition 1 (The elliptic case)**.** *Consider a supersolution (resp. a subsolution) u of (4.1), a point $x \in \Omega$ and two subjets $(X_i, p_i) \in \mathcal{J}_\Omega^{2,-}u(x)$, $i = 1,2$ (resp. two superjets $(X_i,p_i) \in \mathcal{J}_\Omega^{2,+}u(x)$, $i = 1,2$). Then for any $\lambda \in [0,1]$ such that $\lambda p_1 + (1-\lambda)p_2 \neq 0$, the following holds true:*

$$p_1 - p_2 \in \mathrm{Ker} A(\lambda p_1 + (1-\lambda)p_2, u(x), x).$$

A straightforward consequence of Proposition 1 is the following result dealing with nondegenerate equations.

Corollary 2. *Suppose that the equation* (4.1) *is* nondegenerate, *i.e.,*

$$\langle A(p,u,x)q,q\rangle > 0 \quad \textit{if} \quad q \neq 0.$$

Then a solution $u : \Omega \to \mathbb{R}$ of (4.1) *has "no corners", that is to say the function u has at most one P-subgradient and at most one P-supergradient at any point $x \in \Omega$.*

This corollary applies for instance to the equation associated with the search of minimal surfaces:

$$\operatorname{div}\Big(\frac{Du}{1+|Du|^2}\Big) = 0 \quad \Leftrightarrow -\Delta u + \frac{\langle D^2uDu, Du\rangle}{1+|Du|^2} = 0. \tag{4.3}$$

Before proving Proposition 1, we state its parabolic version.

Proposition 2 (The parabolic case)**.** *Consider a supersolution* (*resp. a subsolution*) *u of* (4.2)*, a point $(x,t) \in \Omega \times [0,T]$ and two parabolic subjets $(X_i, p_i, \alpha_i) \in \mathcal{P}^{2,-}_{\Omega\times[0,T]}u(x,t)$, $i = 1,2$* (*resp. two parabolic superjets $(X_i, p_i, \alpha_i) \in \mathcal{P}^{2,+}_{\Omega\times[0,T]}u(x,t)$, $i = 1,2$*)*. Then for any $\lambda \in [0,1]$ such that $\lambda p_1 + (1-\lambda)p_2 \neq 0$, the following holds true:*

$$p_1 - p_2 \in \operatorname{Ker} A(\lambda p_1 + (1-\lambda)p_2, u(x), x, t).$$

Corollary 3. *Suppose that the equation* (4.2) *is* nondegenerate, *i.e.,*

$$\langle A(p,u,x,t)q,q\rangle > 0 \ \textit{if}\ q \neq 0.$$

Then a solution $u : \Omega \times [0,T] \to \mathbb{R}$ of (4.1) *has "no corners", that is to say the function u has at most one P-subgradient and at most one P-supergradient at any point $x \in \mathbb{R}^n$.*

The Hamilton-Jacobi equation associated with the motion by mean curvature of graphs is an example of nondegenerate quasilinear parabolic equation:

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\langle D^2uDu, Du\rangle}{1+|Du|^2} = 0. \tag{4.4}$$

A class of parabolic equations, including (4.4), is studied by a geometrical approach in [2].

The proof of Proposition 1 relies on the following technical lemma.

Lemma 1. *Consider an arbitrary set Ω and a function $u : \Omega \to \mathbb{R}$. Let x be a point in Ω and (X_i, p_i), $i = 1, 2$, be two subjets of u at x. Then for any matrix $X \in \mathbb{S}_n$ such that $X \preccurlyeq X_i$, $i = 1, 2$, any $\lambda \in [0, 1]$ and any $M > 0$, the following holds true:*

$$(X + M(p_1 - p_2) \otimes (p_1 - p_2), \lambda p_1 + (1 - \lambda)p_2) \in \mathcal{J}^{2,-}_{\Omega} u(x).$$

Let us show how Lemma 1 implies Proposition 1.

Proof of Proposition 1. Let $X \in \mathbb{S}_n$ be such that $X \preccurlyeq X_i$ for $i = 1, 2$ and consider any $\lambda \in [0, 1]$ and any $M > 0$. By applying Lemma 1 to the supersolution u of (4.1) and by denoting p the vector $\lambda p_1 + (1 - \lambda)p_2$ and q the vector $p_1 - p_2$, we conclude that: $(X + Mq \otimes q, p) \in \mathcal{J}^{2,-}_{\Omega} u(x)$. As u is a supersolution of (4.1) and $p \neq 0$, the following holds true:

$$-\mathrm{tr}\,[A(p, u(x), x)(X + Mq \otimes q)] + f(p, u(x), x) \geqslant 0.$$

Dividing by M and letting $M \to +\infty$ yields:

$$0 \leqslant \langle A(p, u(x), x)q, q\rangle = \mathrm{tr}\,[A(p, u(x), x)q \otimes q] \leqslant 0.$$

The first inequality follows from the ellipticity of (4.1). We conclude that $q \in \mathrm{Ker} A(p, u(x), x)$.

If the function u is a subsolution, apply the lemma to the function $-u$ and use it analogously. $\square$

One can easily give a parabolic version of this lemma and use it to prove Proposition 2. We omit these details and we turn to the proof of Lemma 1.

Proof of Lemma 1. By considering $v(y) = u(x+y)-u(x)$, we may assume that $x = 0$ and $u(x) = 0$. Let us denote $p = \lambda p_1 + (1 - \lambda)p_2$ and $q = p_1 - p_2$. A straightforward calculus shows us that for any real number r such that $|r| \leqslant \min(\frac{2\lambda}{M}, \frac{2(1-\lambda)}{M})$:

$$\frac{1}{2} Mr^2 \leqslant \max\{(1 - \lambda)r, -\lambda r\}.$$

Therefore, for any y such that $|\langle q, y\rangle| \leqslant \min(\frac{2\lambda}{M}, \frac{2(1-\lambda)}{M})$, we get:

$$\frac{1}{2} M\langle q, y\rangle^2 \leqslant \max\{(1 - \lambda)\langle q, y\rangle, -\lambda\langle q, y\rangle\}.$$

Finally, for any y in a neighbourhood of the origin such that $x + y \in \Omega$, we get:

$$\frac{1}{2}\langle (X + Mq \otimes q)y, y\rangle + \langle p, y\rangle = \frac{1}{2}\langle Xy, y\rangle + \frac{1}{2} M\langle q, y\rangle^2 + \langle p, y\rangle$$

$$\leqslant \max\left\{\frac{1}{2}\langle X_1 y, y\rangle + (1-\lambda)\langle q, y\rangle + \langle p, y\rangle, \frac{1}{2}\langle X_2 y, y\rangle - \lambda\langle q, y\rangle + \langle p, y\rangle\right\}$$
$$= \max\left\{\frac{1}{2}\langle X_1 y, y\rangle + \langle p_1, y\rangle, \frac{1}{2}\langle X_2 y, y\rangle + \langle p_2, y\rangle\right\} \leqslant v(y) + o(|y|^2).$$

We have therefore proved that $(X + Mq\otimes q, p) \in \mathcal{J}^{2,-}_{\Omega-x} v(0) = \mathcal{J}^{2,-}_{\Omega} u(x)$. $\square$

Remark. Using Lemma 1, necessary conditions can be derived for any general nonlinear elliptic equation $F(D^2u, Du, u, x) = 0$ if (E) is satisfied and if $X \mapsto F(X, p, u, x)$ is positively homogenous.

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