Hardness of Approximating the Traffic Grooming Problem

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Le groupage est un problème central dans l'étude des réseaux optiques. Dans cet article, on propose le premier résultat d'inapproximabilité pour le problème du groupage, en affirmant la conjecture de [CL04], selon laquelle le groupage est APX-complet. On étudie aussi une version amortie du problème de sous-graphe le plus dense dans un graphe donné: trouver le sous graphe de taille minimum et de degrée minimum au moins d, $d \geq 3$. On démontre que ce dernier n'a pas d'approximation à un facteur constant.

Keywords: traffic grooming, optical networks, SONET ADM, APX-hardness, PTAS, inapproximability.

1 Introduction

Traffic grooming in networks refers to group low rate traffic into higher speed streams, with the objective of minimizing the equipment cost. In WDM optical networks the most accepted criterium is to minimize the number of electronic terminations (namely the number of SONET ADMs), rather than the number of wavelengths (namely W). The problem, in the particular case where the communication network is a ring (which is also the most practical case), can be formally stated as follows.

TRAFFIC GROOMING ON THE RING
Input: A cycle C_n on n vertices (network), a digraph R (set of requests), and a grooming factor g.
Output: Find for each arc $r \in R$ a path $P(r) \in C_n$, and a partition of the arcs of R into subgraphs R_ω, $1 \leq \omega \leq W$, such that for all $e \in E(C_n)$ and for all ω, the number of paths using e in R_ω is at most g.
Objective: Minimize $\sum_{\omega=1}^{W} |V(R_\omega)|$, and this minimum is denoted $A(n,R,g)$.

Traffic Grooming has been widely studied in the literature, see [ML01, DR02] for some recent surveys. The problem has been proved to be NP-complete [CL04]. Many heuristics have been done, but exact solutions have been found only for particular cases in unidirectional ring [BC06], bidirectional ring [BCMS06], and path topologies [BC06]. On the other hand, the best approximation algorithm [FMSZ05] achieves an approximation factor of $\log(g)$, but the problem is that the running time is exponential on g, and thus it is only useful when g is a small constant. There was no result on the inapproximability of the problem. In [CL04] the authors conjecture that the traffic grooming should be MAX SNP-hard (or equivalently, APX-hard, modulo PTAS-reductions). Here we answer affirmatively to this question in Theorem 3.1, providing the first hardness result for the traffic grooming problem. To prove our results, we also prove the hardness of two related problems. The first one is the problem of finding the maximum number of edge-disjoint triangles in a graph with bounded degree B: MAXIMUM BOUNDED EDGE COVERING BY TRIANGLES. We will use MECT-B for short. The second one is the problem of finding a subset of vertices of minimum size in a given graph with minimum induced degree at least d, $d \geq 3$: MINIMUM SUBGRAPH OF MINIMUM DEGREE $\geq d$. We will use MSMD_d for short.

2 APX-hardness of MECT-B

MECT-B has been proved to be NP-complete [Hol81], and the APX-hardness when requiring node-disjoint triangles was proved in [Kan91]. For convenience, we prove the MAX SNP-hardness, which is known to be the same as the APX-hardness modulo PTAS-reductions.

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Theorem 2.1 MECT-B, $B \geq 12$ *is* MAX SNP-*complete.*

Proof*:* Sketch. L-reduction from MAX 3SC-B[‡] and L-reduction to INDEP. SET-B[§]:
To prove that MECT-B is in MAX SNP, define $h:$ MECT-B $\rightarrow$ INDEP. SET - (3/2(B-2)) such that there is a node in the independent set graph for every triangle in the original graph, and there is an edge in the independent set graph if the two corresponding triangles have at least one edge on common. Then $OPT(h(I)) = OPT(I)$ and solutions can be translated from one problem to another.

In order to see that MECT-B is MAX SNP-complete, define f : MAX 3SC-B $\rightarrow$ MECT-4B in the following way: we are given as instance I, a collection C of 3-element subsets of a set X with bounded occurrence of elements. The problem on I is to find the maximal number $OPT(I)$ of disjoint subsets. We shall construct an instance $f(I)$ of MECT-B, that is a graph $G = (V,E)$, and we want to find the maximum number $OPT(f(I))$ of edge-disjoint triangles in G. The local replacement f substitutes for each subset $c_i = \{x_i, y_i, z_i\} \in C$, the graph $G_i = (V_i, E_i)$ depicted in Figure 1.

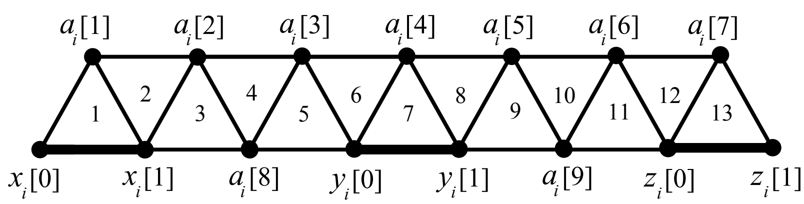


Fig. 1: Gadget used in the reduction of Theorem 2.1

To avoid confusion, note by t_i any element in c_i, i.e. $t \in \{x,y,z\}$. Remark that, for each element t_i, the nodes $t_i[0]$ and $t_i[1]$, and the edge $t_i[0]t_i[1]$ (corresponding to the thick edges in Figure 1) appear only once, regardless of the number of occurrences of t_i. On the other hand, we add 9 new vertices $a_i[j]$, $1 \leq j \leq 9$ for each subset c_i, $1 \leq i \leq |C|$. More precisely, $G = (V,E) = \cup_{i=1}^{|C|} G_i$, where $V = \bigcup_{i=1}^{|X|}\{t_i[0], t_i[1]\} \cup \bigcup_{i=1}^{|C|}\{a_i[j] : 1 \leq j \leq 9\}$ and $E = \bigcup_{i=1}^{|C|} E_i$.

Let us prove that f is an L-reduction. In each G_i there are 13 different triangles, numbered from 1 to 13 in Figure 1. The only way to choose 7 edge-disjoint triangles in G_i is by taking all the "odd" triangles, and thus by picking the three edges $x_i[0]x_i[1]$, $y_i[0]y_i[1]$, and $z_i[0]z_i[1]$. All other choices of triangles yield at most 6 edge-disjoint triangles. The key observation is that we are able to choose 7 triangles exactly $OPT(I)$ times (because each time we choose 7 triangles, we cover the edges corresponding to 3 new elements of the set X, and this can be done exactly $OPT(I)$ times). Hence:

$$OPT(f(I)) = 7 \cdot OPT(I) + 6(|C| - OPT(I)) \leq OPT(I) + 6B \cdot OPT(I) = (6B+1)OPT(I)$$

Both f and h are L-reductions and MAX 3SC-B, $B \geq 3$ and INDEP. SET-B, $B \geq 5$ are MAX SNP-complete [Kan91]. Thus, MECT-B, $B \geq 12$ is MAX SNP-complete. □

3 APX-hardness of TRAFFIC GROOMING

Theorem 3.1 *For all g and bounded number B,* $B \geq 12$*, of requests per node, the* RING TRAFFIC GROOMING *is* MAX SNP-*complete. Thus, it does not accept a* PTAS *unless* P=NP.

Proof*:* Sketch. Consider a set of requests R made of a tripartite graph. Wlog, we prove the result for $g = 1$[¶]. In any solution, the only possible involved subgraphs are P_2, P_3, P_4, and K_3. It is clear that the best

[‡] MAXIMUM BOUNDED COVERING BY 3-SETS: Given a collection of 3-subsets of a given set, each element appearing in at most B subsets, find the maximum number of disjoint subsets.

[§] MAXIMUM BOUNDED INDEPENDENT SET: Given a graph of maximum degree $\leq B$, find a maximum independent set.

[¶] for $g > 1$, take a $(2g+1)$-partite graph, in such a way that each cycle makes at least g tours.

we can do is to groom the requests into triangles, because triangles have the best ratio number of edges over number of nodes. From this we derive that $|R|$ is a lower bound for the number of ADMs, and that each path used in the solution adds an additional unity of cost. This additional cost is at least $4/3|R|$, if we only use P_4's. Thus the number A of ADMs used by any solution satisfies $A \geq (1-\varepsilon)|R| + \varepsilon\frac{4}{3}|R| > |R|$, for all $\varepsilon > 0$, where ε is the percentage of the triangles in R that we have not found in the solution. Minimizing A corresponds to be able to find the required edge-disjoint triangles for all $\varepsilon > 0$. By Theorem 2.1 we know that MECT-B in tripartite graphs ($B \geq 12$) is MAX SNP-complete, hence there exists ε_0 such that we can not find in polynomial time a fraction $(1-\varepsilon_0)$ of the triangles of R. Thus, RING TRAFFIC GROOMING is MAX SNP-complete for bounded number of requests per node $B \geq 12$. □

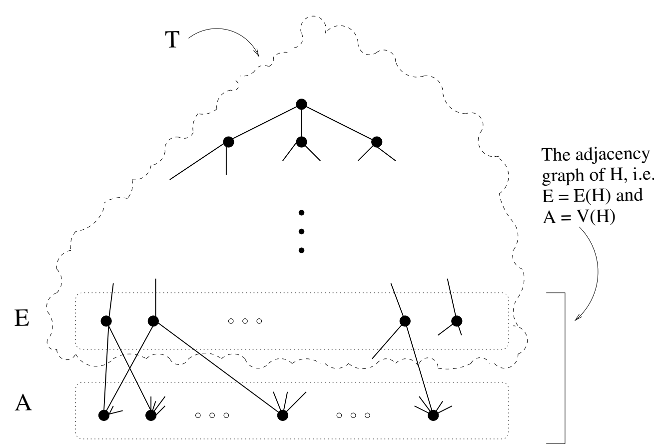
Remark 3.1 *The* PATH TRAFFIC GROOMING *is known to be in* P *for $g = 1$ [BC06], and extending the techniques used in Theorem 3.1 it can be proved that it is* APX*-hard for $g \geq 2$.*

4 Hardness of approximating MSMD$_d$

In this section we prove a related result on hardness of approximating the minimum size of a subgraph of minimum degree d ($d \geq 3$) in a graph G. Before presenting the result, let us begin by some motivations. Let A and B be two non intersecting intervals on the path. The grooming problem when the requests are only from A to B is equivalent to the following problem: Let $G = (A \cup B, E)$ be a bipartite graph, and g be an integer. We want to find a partition of the edges into $E_1, E_2, \ldots, E_r$ such that the size of each E_i is at most g and $\sum |V(E_i)|$ is minimum possible. This situation is of particular interest in designing approximation algorithms. The reason is that for a general instance of traffic grooming problem, we can decompose the requests to $\log(n)$ classes, such that in each class the length of each request is in the interval $[2^i, 2^{i+1}]$. Then each class contains several subproblems of the form described above. One possible approach to the above problem is to apply a greedy algorithm by choosing at step i, the subset E_i which has the largest possible ratio $\frac{|E_i|}{|V(E_i)|}$ for $|E_i| \leq g$ (this provides a $\log(g) \times \beta$ approximation, where β is the approximation factor of the above problem). This relates our original problem to the problem of finding a densest subgraph of G containing at most g edges. We conjecture that this problem is $\log(g)$ inapproximable. A related problem to our problem is the densest subgraph problem [FPK01]. The density is related to minimum degree by a factor two. We prove below that for a fix $d \geq 3$, the problem of finding the minimum subgraph of G of minimum degree at least d does not contain any constant factor approximation, i.e. is not in APX. Remark that asking the same question but for $d \geq 2$ is essentially finding the shortest cycle in a graph, i.e. the girth problem, which is polynomial. For simplicity we present the result for degree three.

Theorem 4.1 *MSMD$_3$ does not have any constant factor approximation unless $P = NP$.*

Proof: Sketch. The proof is divided in two parts:
MSMD$_3$ is not in PTAS. Gap-preserving reduction from VERTEX COVER: Let H be an instance of VERTEX COVER with n vertices. We will construct an instance G of MSMD$_3$. Wlog, we can suppose that H contains 3×2^m edges, for some m, and also that every vertex of H has degree at least three. Let T be the ternary rooted tree with root r and depth m. It is easy to see that the number of leaves of T is exactly 3×2^m, and that T contains $3 \times 2^{m+1} - 2$ vertices. Let us identify the leaves of T with edges of H. Now add n new vertices A (identified with vertices of H), and join them to the leaves of T according to adjacency relations between the edges and vertices in H, i.e. a leaf ℓ in T is connected to $v \in A$ if the corresponding edge to ℓ in H is adjacent to $v \in V(H)$, see the Figure beside.



Minimum subgraphs of G of minimum degree at least three correspond to minimum vertex covers of H and vice versa[‖]. We have $3 \times 2^m + minvc(H) = min_{d \geq 3}(G)$ ($minvc$ denotes the size of minimum vertex cover in H). This proves the above claim. To finish the proof, remark that VERTEX COVER is APX-hard, even restricted to graphs H of size linear on $minvc(H)$. The existence of a PTAS for $min_{d \geq 3}$ provides a PTAS for $minvc(H)$, which is a contradiction (under assumption APX $\neq$ PTAS).

Amplifying the error: Let G be the transformation graph described above, and let $\alpha > 1$ be the factor of inapproximability of MSMD_3. We construct a sequence of transformation graphs G_k, such that MSMD_3 problem is hard to approximate within a factor $\theta(\alpha^{2^k})$ after these transformations. This proves that MSMD_3 is not in APX. We outline only the construction of G_2, G_k is obtained by repeating the same construction. For every vertex v in G of degree d, construct a graph G_v as follows: first choose d other vertices $x_1, \ldots, x_d$ of degree 3 in $T \subset G$. Replace each of these vertices by a cycle of length 4 and add to three of them the three outgoing edges incident to v. Let G_v be the graph obtained at the end. Remark that G_v has exactly d vertices of degree 2. To construct the transformation graph G_2, first take a copy of G, and then replace each vertex v by G_v, and join the d_v edges incident to v to the d_v vertices of degree 2 in G_v. Remark that in G_2 we have

Claim 1 *1)* $|V(G_2)| = |V(G)|^2 + o(|V(G)|^2)$.
2) It is hard to approximate MSMD_3 *after this transformation within a factor* α^2.

To see the last statement, remark that once a vertex in one G_v is chosen, we should look for MSMD_3 in G, which is hard up to a constant factor α. But approximating the number of v's for which we should touch G_v is also MSMD_3 in G, which is hard up to the same factor α. This proves that approximating MSMD_3 is hard up to a factor α^2. To finish the proof of the theorem, repeat this procedure by applying the same transformation to obtain G_3, and inductively G_k. □

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[‖] To see this, first remark that if U, such a subgraph of G, contains a leaf of T, then it should contain all the vertices of T. As the subgraph on U is minimum then we should be able to cover the leaves of T with a minimum number of vertices in A, and this is exactly VERTEX COVER problem for H.