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# EFFICIENT INTERFACE CONDITIONS FOR THE COUPLING OF OCEAN MODELS

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## 1 INTRODUCTION

The ocean circulation is notably heterogeneous, both in time and space: models have to deal with phenomena as varied as mesoscale eddies, fronts, and the general circulation. A single ocean general circulation model (OGCM) cannot describe completely such a various physics. A way to improve modelling is to couple OGCMs with high resolution regional ocean models (ROMs). The use of ROMs has increased in recent years, in particular due to the development of operational oceanography and coastal oceanography. Although the two-way nesting is the most usual coupling method in the ocean community, it does not address the correct problem but an approximation, and does not ensure enough regularity through the interface between the two models[1]. The correct approach consisting in a full coupling is more difficult and expensive than the two-way nesting, since it requires to find and implement an algorithm ensuring that the solutions in each domain satisfy the regularity conditions through the interface. The *global-in-time non-overlapping Schwarz algorithm* is particularly well suited for such a coupling, and can lead to improved physical results[2].

Let $\Omega$ be the ocean domain, decomposed into two non-overlapping subdomains $\Omega_1$ and $\Omega_2$, separated by a single interface $\Gamma$ (Figures 1a and 2a). Given a problem $Lu = f$ in $\Omega \times [0,T]$, with an initial condition $u|_{t=0} = u_0$ in $\Omega$, the Schwarz iterative algorithm writes as follows:

$$\begin{cases} Lu_1^{n+1} = f & \text{in } \Omega_1 \times [0,T], \\ u_1^{n+1} = u_0 & \text{in } \Omega_1 \text{ at } t=0, \\ B_1 u_1^{n+1} = B_1 u_2^n & \text{on } \Gamma \times [0,T], \end{cases} \quad \begin{cases} Lu_2^{n+1} = f & \text{in } \Omega_2 \times [0,T], \\ u_2^{n+1} = u_0 & \text{in } \Omega_2 \text{ at } t=0, \\ B_2 u_2^{n+1} = B_2 u_1^n & \text{on } \Gamma \times [0,T], \end{cases} \tag{1}$$

where the superscripts denote the number of iterations. In order to lessen the cost of this algorithm, it is necessary to find efficient interface operators $B_1$ and $B_2$. For example note that if they are chosen in such a way that $B_1 u_2^1 = B_2 u_1^1 = 0$, the errors $\mathsf{e}_i^n = u|_{\Omega_i} - u_i$ defined in each subdomain converge towards zero in only two iterations: these are the so-called *absorbing boundary conditions*.

Our work thus aims at improving the ocean coupling by determining efficient interface conditions for the usual ocean equations (the so-called 3-D *primitive equations*). These ones are composed of *advection-diffusion equations for tracers* such as temperature and salinity, and dynamics equations, which can be approximated in the 2-D case by the *shallow-water equations*.

## 2 ADVECTION-DIFFUSION EQUATIONS FOR TRACERS

The evolution of tracers is described by scalar advection-diffusion equations. Two different parameterizations of diffusion, laplacian and bi-laplacian, are usually considered in order to represent mesoscale turbulence as a function of large scale features.

Let $\Omega = \mathbb{R}^3$. For the sake of simplicity, $\Omega_1$ and $\Omega_2$ are defined as the subdomains separated by the interface $\Gamma = \{x = 0\}$ (negative and positive part resp.). When the diffusion operator is a laplacian, the system operator $L$ which appears in (1) is defined as $L = \partial_t + a\,\partial_x + b\,\partial_y + c\,\partial_z - \nu_h(\partial_{xx} + \partial_{yy}) - \nu_z \partial_{zz}$, where $\nu_h$ and $\nu_z$ are positive constants. After a Fourier transform in space and time, one solves the characteristic polynomial and obtains two roots $\sigma_i$ of opposite signs. The error Fourier transforms thus satisfy in each subdomain $\partial_x \widehat{\mathsf{e}}_i - \sigma_i \widehat{\mathsf{e}}_i = 0$, $i = 1, 2$, which leads to the following ideal absorbing conditions: $B_1 = \partial_x - S_2$ and $B_2 = \partial_x - S_1$,where $S_i$ are the inverse Fourier transforms of $\sigma_i$. It is clear that these boundary conditions are non-local both in space and time. Therefore an approximation is required in order to obtain local interface conditions and allow their numerical implementation. We propose the following approximated conditions: $B_{1(0)} = \partial_x - \frac{1}{2\nu_h}(a-p)$, $B_{2(0)} = \partial_x - \frac{1}{2\nu_h}(a+p)$ at order 0 and $B_{2(1)} = \partial_x - \frac{1}{2\nu_h}(a-p) + q\,\partial_t + b\,q\,\partial_y + c\,q\,\partial_z$, $B_{2(1)} = \partial_x - \frac{1}{2\nu_h}(a+p) - q\,\partial_t - b\,q\,\partial_y - c\,q\,\partial_z$ at order 1. It can be demonstrated that the algorithms (1) with these interface conditions converge toward the general solution on the condition that $p > 0$ at order 0, and $q > 0$ and $p - a^2 q/2 > 0$ at order 1. If frequencies are low, the $p$ and $q$ parameters can be determined by a Taylor approximation, but this hypothesis is not always true in ocean modelling. A more general approach consists in minimizing the convergence rate of the Schwarz algorithm for any frequency (see [3] for further details). The consequent algorithms were implemented in the 2-D case[3,4] (see Figure 1).

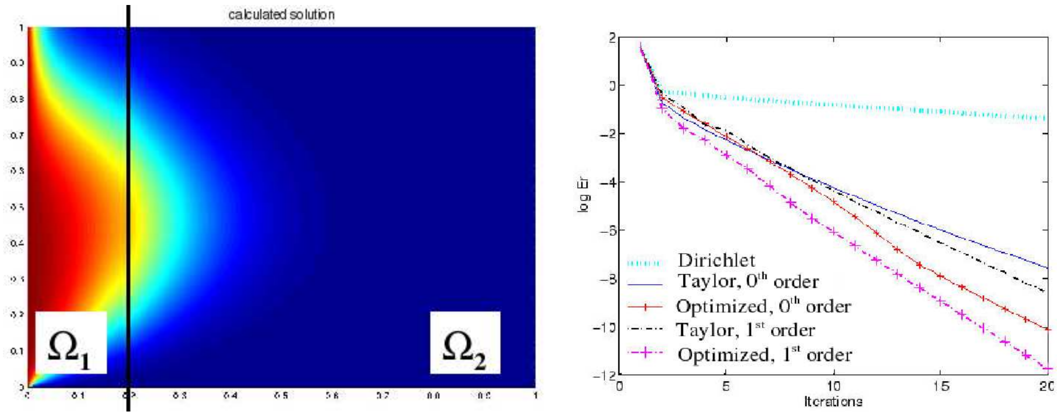


Figure 1: Coupling of two advection-diffusion models with a Schwarz algorithm: (a) snapshot of the solution, (b) $L^2$-norm of the error as a function of the iteration number for different interface conditions: from top to bottom, Dirichlet, Taylor and optimized conditions at zero-th and first order.

When the diffusion is parameterized with a bi-laplacian operator, specific difficulties arise. In two dimensions, the fourth order advection-diffusion operator writtes $L = \partial_t + a\,\partial_x + b\,\partial_y + \nu\,(\partial_{4x} + \partial_{4y} + 2\,\partial_{2x2y})$. The variational formulation induces that at least two boundary conditions have to be specified on the second and third normal derivatives. Thus the absorbing boundary conditions can no longer be expressed as $B = \partial_x - S$. Moreover the four roots of the characteristic polynomial associated to the Fourier transform can be determined, but the sign of their real part is not constant. This problem is presently under investigation.

## 3 2-D DYNAMICS: SHALLOW WATER EQUATIONS

The ocean dynamics can be described in 2-D by the linearized shallow-water equations: $\partial_t W + A_1 \partial_x W + A_2 \partial_y W - \nu P \Delta W + BW$, and $W|_{t=0} = W_0$, where $W = (u, v, \phi)^t$, with $(u, v)$ the horizontal velocity, $\phi$ the free surface height and $\nu$ the viscosity. $A_1$ and $A_2$ are matrices, $P$ is a diagonal projection matrix on the horizontal plane, and $B$ is an antisymmetric matrix which expresses the rotation in the horizontal plane due to the Coriolis force. After a Laplace-Fourier transform, the solution of the system is assumed to be $\widehat{W} = \Phi e^{-\xi x}$, with $\Phi$ a vector and $\xi$ a root to be determined. This leads to solve an equation of the fifth order in $\xi$ in the general case. In the simplified case without advection, this reduces to a fourth order bi-squared equation, the roots of which can be easily determined. The variational formulation provides transmission conditions. By combining these two approaches, one obtains the non-local ideal absorbing conditions. In the case without advection, it is sufficient to impose conditions on the horizontal variables[4]: $B = (\tilde{B}, 0)^t$, $\tilde{B}_i = -\nu\partial_x + c(\phi, 0)^t - \Lambda_i$, with $\Lambda_i$ a pseudo-differential operator. An approximation $\Lambda_{i(j)}$ of the matrices $\Lambda_i$, $i = 1, 2$, at order 0 or 1 can provide local interface conditions. Martin[4] demonstrated that these problems are well-posed, and that the algorithms converge towards the global solution, for diagonal $\Lambda_{i(j)}$ matrices. Numerical simulations have also been realized (not shown). In the case with advection, the fifth order equation cannot be solved explicitely and approximations have to be made[5]. This is an ongoing work.

## 4 TOWARDS 3-D

Our final aim is to derive and implement efficient interface conditions in actual 3-D oceanic applications. Preliminary experiments were performed in a regional model of the bay of Biscay, either alone using 0th order absorbing open boundary conditions (OBCs), or coupled with a large scale model of the North Atlantic with a non-optimized Schwarz algorithm[2]. Both testcases lead to encouraging results. The derivation of optimized interface conditions for these full 3-D equations is presently under investigation.

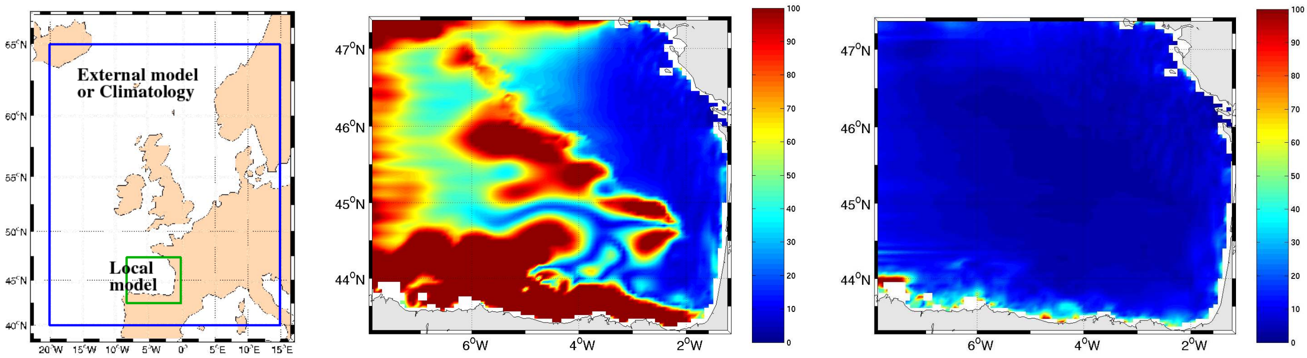


Figure 2: A 3-D high resolution ocean model of the Bay of Biscay is considered alone, forced by the results of a large model of the Northern Atlantic (a). The snapshots of the RMS errors are different depending on the tested OBCs : (b) clamped (Dirichlet) and (c) characteristic conditions (which are particular 0th order absorbing conditions).

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