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Abstract: Non-linear state estimation and some related topics, like parametric estimation, fault diagnosis, and perturbation attenuation, are tackled here via a new methodology in numerical differentiation. The corresponding basic system theoretic definitions and properties are presented within the framework of differential algebra, which permits to handle system variables and their derivatives of any order. Several academic examples and their computer simulations, with on-line estimations, are illustrating our viewpoint.

Keywords: Non-linear systems, observability, parametric identifiability, closed-loop state estimation, closed-loop parametric identification, closed-loop fault diagnosis, closed-loop fault tolerant control, closed-loop perturbation attenuation, numerical differentiation, differential algebra.

Biographical notes: M. Fliess is a Research Director at the *Centre National de la Recherche Scientifique* and works at the *École Polytechnique* (Palaiseau, France). He is the head of the INRIA project called ALIEN, which is devoted to the study and the development of new techniques in identification and estimation. In 1991 he invented with J. Lévine, P. Martin, and P. Rouchon, the notion of *differentially flat* systems which is playing a major rôle in control applications.
C. Join received his Ph.D. degree from the University of Nancy, France, in 2002. He is now an Associate Professor at the University of Nancy and is a member of the INRIA project ALIEN. He is interested in the development of estimation technics for linear and non-linear systems with a peculiar emphasis in fault diagnosis and accommodation. His research involves also signal and image processing.
H. Sira-Ramírez obtained the Electrical Engineer's degree from the *Universidad de Los Andes* in Mérida (Venezuela) in 1970. He later obtained the MSc in EE and the Electrical Engineer degree, in 1974, and the PhD degree, also in EE, in 1977, all from the Massachusetts Institute of Technology (Cambridge, USA). Dr. Sira-Ramírez worked for 28 years at the Universidad de Los Andes where he held the positions of: Head of the Control Systems Department, Head of the Graduate Studies in Control Engineering and Vicepresident of the University. Currently, he is a Titular Researcher in the *Centro de Investigación y Estudios Avanzados del Instituto Politécnico Nacional* (CINVESTAV-IPN) in México City (México). Dr Sira-Ramírez is a Senior Member of the Institute of Electrical and Electronics Engineers (IEEE), a Distinguished Lecturer from the same Institute and a Member of the IEEE International Committee. He is also a member of the Society for Industrial and Applied Mathematics (SIAM), of the International Federation of Automatic Control (IFAC) and of the American Mathematical Society (AMS). He is a coauthor of the books, *Passivity Based Control of Euler-Lagrange Systems* published by Springer-Verlag, in 1998, *Algebraic Methods in Flatness, Signal Processing and State Estimation*, Lagares 2003, *Differentially Flat Systems*, Marcel Dekker, 2004, *Control de Sistemas No Lineales* Pearson-Prentice Hall 2006, and of *Control Design Techniques in Power Electronics Devices*, Springer, 2006. Dr. Sira-Ramírez is interested in the theoretical and practical aspects of feedback regulation of nonlinear dynamic systems with special emphasis in Variable Structure feedback control techniques and its applications in Power Electronics.

1 Introduction

1.1 General overview

Since fifteen years non-linear flatness-based control (Fliess, Lévine, Martin & Rouchon (1995, 1999)) has been quite effective in many concrete and industrial applications (see also Lamnabhi-Lagarrigue & Rouchon (2002b); Rudolph (2003); Sira-Ramírez & Agrawal (2004)). On the other hand, most of the problems pertaining to non-linear state estimation, and to related topics, like

- parametric estimation,
- fault diagnosis and fault tolerant control,
- perturbation attenuation,

remain largely open in spite of a huge literature[1]. This paper aims at providing simple and effective design methods for such questions. This is made possible by the following facts:

According to the definition given by Diop & Fliess (1991a,b), a non-linear input-output system is *observable* if, and only if, any system variable, a state variable for instance, is a *differential function* of the control and output variables, i.e., a function of those variables and their derivatives up to some finite order. This definition is easily generalized to parametric identifiability and fault isolability. We will say more generally that an unknown quantity may be determined if, and only if, it is expressible as a differential function of the control and output variables.

It follows from this conceptually simple and natural viewpoint that non-linear estimation boils down to numerical differentiation, i.e., to the derivatives estimations of noisy time signals[2]. This classic ill-posed mathematical problem has been already attacked by numerous means[3]. We follow here another thread, which started in Fliess & Sira-Ramírez (2004b) and Fliess, Join, Mboup & Sira-Ramírez (2004, 2005): derivatives estimates are obtained via integrations. This is the explanation of the quite provocative title of this paper[4] where non-linear asymptotic estimators are replaced by differentiators, which are easy to implement[5].

Remark 1.1. *This approach to non-linear estimation should be regarded as an extension of techniques for linear closed-loop parametric estimation (Fliess & Sira-Ramírez (2003, 2007)). Those techniques gave as a byproduct linear closed-loop fault diagnosis (Fliess, Join & Sira-Ramírez (2004)), and linear state reconstructors (Fliess & Sira-Ramírez (2004a)), which offer a promising alternative to linear asymptotic observers and to Kalman's filtering.*

1.2 Numerical differentiation: a short summary of our approach

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Let us start with the first degree polynomial time function $p_1(t) = a_0 + a_1 t$, $t \geq 0$, $a_0, a_1 \in \mathbb{R}$. Rewrite thanks to classic operational calculus (see, e.g., Yosida (1984)) p_1 as $P_1 = \frac{a_0}{s} + \frac{a_1}{s^2}$. Multiply both sides by s^2:

$$s^2 P_1 = a_0 s + a_1 \tag{1}$$

Take the derivative of both sides with respect to s, which corresponds in the time domain to the multiplication by $-t$:

$$s^2 \frac{dP_1}{ds} + 2sP_1 = a_0 \tag{2}$$

The coefficients a_0, a_1 are obtained via the triangular system of equations (1)-(2). We get rid of the time derivatives, i.e., of sP_1, $s^2 P_1$, and $s^2 \frac{dP_1}{ds}$, by multiplying both sides of Equations (1)-(2) by s^{-n}, $n \geq 2$. The corresponding iterated time integrals are low pass filters which attenuate the corrupting noises, which are viewed as highly fluctuating phenomena (cf. Fliess (2006)). A quite short time window is sufficient for obtaining accurate values of a_0, a_1.

The extension to polynomial functions of higher degree is straightforward. For derivatives estimates up to some finite order of a given smooth function $f : [0, +\infty) \to \mathbb{R}$, take a suitable truncated Taylor expansion around a given time instant t_0, and apply the previous computations. Resetting and utilizing sliding time windows permit to estimate derivatives of various orders at any sampled time instant.

Remark 1.2. *Note that our differentiators are not of asymptotic nature, and do not require any statistical knowledge of the corrupting noises. Those two fundamental features remain therefore valid for our non-linear estimation*[6]*. This is a change of paradigms when compared to most of today's approaches*[7]*.*

[1] See, e.g., the surveys and encyclopedia edited by Aström, Blanke, Isidori, Schaufelberger & Sanz (2001); Lamnabhi-Lagarrigue & Rouchon (2002a,b); Levine (1996); Menini, Zaccarian & Abdallah (2006); Nijmeijer & Fossen (1999); Zinober & Owens (2002), and the references therein.

[2] The origin of flatness-based control may also be traced back to a fresh look at controllability (Fliess (2000)).

[3] For some recent references in the control literature, see, e.g., Braci & Diop (2001); Busvelle & Gauthier (2003); Chitour (2002); Dabroom & Khalil (1999); Diop, Fromion & Grizzle (2001); Diop, Grizzle & Chaplais (2000); Diop, Grizzle, Moraal & Stefanopoulou (1994); Duncan, Madl & Pasik-Duncan (1996); Ibrir (2003, 2004); Ibrir & Diop (2004); Kelly, Ortega, Ailon & Loria (1994); Levant (1998, 2003); Su, Zheng, Mueller & Duan (2006). The literature on numerical differentiation might be even larger in signal processing and in other fields of engineering and applied mathematics.

[4] There are of course situations, for instance with a very strong corrupting noise, where the present state of our techniques may be insufficient. See also Remark 2.5.

[5] Other authors like Slotine (1991) had already noticed that "good" numerical differentiators would greatly simplify control synthesis.

[6] They are also valid for the linear estimation questions listed in Remark 1.1.

[7] See, e.g., Schweppe (1973); Jaulin, Kiefer, Didrit & Walter

1.3 Analysis and organization of our paper

Our paper is organized as follows. Section 2 deals with the differential algebraic setting for nonlinear systems, which was introduced in Fliess (1989, 1990). When compared to those expositions and to other ones like Fliess, Lévine, Martin & Rouchon (1995); Delaleau (2002); Rudolph (2003); Sira-Ramírez & Agrawal (2004), the novelty lies in the two following points:

1. The definitions of observability and parametric identifiability are borrowed from Diop & Fliess (1991a,b).

2. We provide simple and natural definitions related to non-linear diagnosis such as *detectability*, *isolability*, *parity equations*, and *residuals*, which are straightforward extensions of the module-theoretic approach in Fliess, Join & Sira-Ramírez (2004) for linear systems.

The main reason if not the only one for utilizing differential algebra is the absolute necessity of considering derivatives of arbitrary order of the system variables. Note that this could have been also achieved with the differential geometric language of infinite order prolongations (see, e.g., Fliess, Lévine, Martin & Rouchon (1997, 1999))[8].

Section 3 details Subsection 1.2 on numerical differentiation.

Illustrations are provided by several academic examples[9] and their numerical simulations[10] which we wrote in a such a style that they are easy to grasp without understanding the algebraic subtleties of Section 2:

1. Section 4 is adapting a paper by Fan & Arcak (2003) on a non-linear observer. We only need for closing the loop derivatives of the output signal. We nevertheless present also a state reconstructor of an important physical variable.

2. Closed-loop parametric identification is achieved in Section 5.

3. Section 6 deals with closed-loop fault diagnosis and fault tolerant control.

4. Perturbation attenuation is presented in Section 7, via linear and non-linear case-studies.

We end with a brief conclusion. First drafts of various parts of this paper were presented in Fliess & Sira-Ramírez (2004b); Fliess, Join & Sira-Ramírez (2005).

2 Differential algebra

Commutative algebra, which is mainly concerned with the study of commutative rings and fields, provides the right tools for understanding algebraic equations (see, e.g., Hartshorne (1977); Eisenbud (1995)). *Differential algebra*, which was mainly founded by Ritt (1950) and Kolchin (1973), extends to differential equations concepts and results from commutative algebra[11].

2.1 Basic definitions

A *differential ring* R, or, more precisely, an *ordinary differential ring*, (see, e.g., Kolchin (1973) and Chambert-Loir (2005)) will be here a commutative ring[12] which is equipped with a single *derivation* $\frac{d}{dt} : R \to R$ such that, for any $a, b \in R$,

- $\frac{d}{dt}(a+b) = \dot{a} + \dot{b}$,

- $\frac{d}{dt}(ab) = \dot{a}b + a\dot{b}$.

where $\frac{da}{dt} = \dot{a}$, $\frac{d^\nu a}{dt^\nu} = a^{(\nu)}$, $\nu \geq 0$. A *differential field*, or, more precisely, an *ordinary differential field*, is a differential ring which is a field. A *constant* of R is an element $c \in R$ such that $\dot{c} = 0$. A *(differential) ring* (resp. *field*) *of constants* is a differential ring (resp. field) which only contains constants. The set of all constant elements of R is a subring (resp. subfield), which is called the *subring* (resp.*subfield*) *of constants*.

A *differential ring* (resp. *field*) *extension* is given by two differential rings (resp. fields) R_1, R_2, such that $R_1 \subseteq R_2$, and qthe derivation of R_1 is the restriction to R_1 of the derivation of R_2.

Notation Let S be a subset of R_2. Write $R_1\{S\}$ (resp. $R_1\langle S\rangle$) the differential subring (resp. subfield) of R_2 generated by R_1 and S.

Notation Let k be a differential field and $X = \{x_\iota | \iota \in I\}$ a set of *differential indeterminates*, i.e., of indeterminates and their derivatives of any order. Write $k\{X\}$ the differential ring of *differential polynomials*, i.e., of polynomials belonging to $k[x_\iota^{(\nu_\iota)} | \iota \in I;\ \nu_\iota \geq 0]$. Any differential polynomial is of the form $\sum_{\text{finite}} c \prod_{\text{finite}} (x_\iota^{(\mu_\iota)})^{\alpha_{\mu_\iota}}$, $c \in k$.

Notation If R_1 and R_2 are differential fields, the corresponding field extension is often written R_2/R_1.

(2001), and the references therein, for other non-statistical approaches.

[8] The choice between the algebraic and geometric languages is a delicate matter. The formalism of differential algebra is perhaps suppler and more elegant, whereas infinite prolongations permit to take advantage of the integration of partial differential equations. This last point plays a crucial rôle in the theoretical study of flatness (see, e.g., Chetverikov (2004); Martin & Rouchon (1994, 1995); van Nieuwstadt, Rathinam & Murray (1998); Pomet (1997); Sastry (1999), and the references therein) but seems to be unimportant here. Differential algebra on the other hand permitted to introduce *quasi-static* state feedbacks (Delaleau & Pereira da Silva (1998a,b)), which are quite helpful in feedback synthesis (see also Delaleau & Rudolph (1998); Rudolph & Delaleau (1998)). The connection of differential algebra with constructive and computer algebra might be useful in control (see, e.g., Diop (1991, 1992); Glad (2006), and the references therein).

[9] These examples happen to be flat, although our estimation techniques are not at all restricted to such systems. We could have examined as well uncontrolled systems and/or non-flat systems. The control of non-flat systems, which is much more delicate (see, e.g., Fliess, Lévine, Martin & Rouchon (1995); Sira-Ramírez & Agrawal (2004), and the references therein), is beyond the scope of this article.

[10] Any interested reader may ask C. Join for the corresponding computer programs (`Cedric.Join@cran.uhp-nancy.fr`).

[11] Algebraic equations are differential equations of order 0.

[12] See, e.g., Atiyah & Macdonald (1969); Chambert-Loir (2005) for basic notions in commutative algebra.

A *differential ideal* $\mathfrak{I}$ of R is an ideal which is also a differential subring. It is said to be *prime* if, and only if, $\mathfrak{I}$ is prime in the usual sense.

2.2 Field extensions

All fields are assumed to be of characteristic zero. Assume also that the differential field extension K/k is *finitely generated*, i.e., there exists a finite subset $S \subset K$ such that $K = k\langle S \rangle$. An element a of K is said to be *differentially algebraic* over k if, and only if, it satisfies an algebraic differential equation with coefficients in k: there exists a non-zero polynomial P over k, in several indeterminates, such that $P(a, \dot{a}, \ldots, a^{(\nu)}) = 0$. It is said to be *differentially transcendental* over k if, and only if, it is not differentially algebraic. The extension K/k is said to be *differentially algebraic* if, and only if, any element of K is differentially algebraic over k. An extension which is not differentially algebraic is said to be *differentially transcendental*.

The following result is playing an important rôle:

Proposition 2.1. *The extension K/k is differentially algebraic if, and only if, its transcendence degree is finite.*

A set $\{\xi_\iota \mid \iota \in I\}$ of elements in K is said to be *differentially algebraically independent* over k if, and only if, the set $\{\xi_\iota^{(\nu)} \mid \iota \in I, \nu \geq 0\}$ of derivatives of any order is algebraically independent over k. If a set is not differentially algebraically independent over k, it is *differentially algebraically dependent over k*. An independent set which is maximal with respect to inclusion is called a *differential transcendence basis*. The cardinalities, i.e., the numbers of elements, of two such bases are equal. This cardinality is the *differential transcendence degree* of the extension K/k; it is written diff tr deg (K/k). Note that this degree is 0 if, and only if, K/k is differentially algebraic.

2.3 Kähler differentials

Kähler differentials (see, e.g., Hartshorne (1977); Eisenbud (1995)) provide a kind of analogue of infinitesimal calculus in commutative algebra. They have been extended to differential algebra by Johnson (1969). Consider again the extension K/k. Denote by

- $K[\frac{d}{dt}]$ the set of linear differential operators $\sum_{\text{finite}} a_\alpha \frac{d^\alpha}{dt^\alpha}$, $a_\alpha \in K$, which is a left and right principal ideal ring (see, e.g., McConnell & Robson (2000));
- $\Omega_{K/k}$ the left $K[\frac{d}{dt}]$-module of Kähler differentials of the extension K/k;
- $d_{K/k}x \in \Omega_{K/k}$ the (Kähler) differential of $x \in K$.

Proposition 2.2. *The next two properties are equivalent:*

1. *The set $\{x_\iota \mid \iota \in I\} \subset K$ is differentially algebraically dependent (resp. independent) over k.*
2. *The set $\{d_{K/k}x_\iota \mid \iota \in I\}$ is $K[\frac{d}{dt}]$-linearly dependent (resp. independent).*

The next corollary is a direct consequence from Propositions 2.1 and 2.2.

Corollary 2.1. *The module $\Omega_{K/k}$ satisfies the following properties:*

- *The* rank[13] *of $\Omega_{K/k}$ is equal to the differential transcendence degree of K/k.*
- *$\Omega_{K/k}$ is* torsion[14] *if, and only if, K/k is differentially algebraic.*
- $\dim_K(\Omega_{K/k}) = \text{tr deg}(L/K)$. *It is therefore finite if, and only if, L/K is differentially algebraic.*
- $\Omega_{K/k} = \{0\}$ *if, and only if, L/K is algebraic.*

2.4 Nonlinear systems

2.4.1 Generalities

Let k be a given differential ground field. A *(nonlinear) (input-output) system* is a finitely generated differential extension K/k. Set $K = k\langle S, \mathbf{W}, \boldsymbol{\pi} \rangle$ where

1. S is a finite set of system variables, which contains the sets $\boldsymbol{u} = (u_1, \ldots, u_m)$ and $\boldsymbol{y} = (y_1, \ldots, y_p)$ of control and output variables,
2. $\mathbf{W} = \{\mathbf{w}_1, \ldots, \mathbf{w}_q\}$ denotes the *fault* variables,
3. $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_r)$ denotes the *perturbation*, or *disturbance*, variables.

They satisfy the following properties:

- The control, fault and perturbation variables do not "interact", i.e., the differential extensions $k\langle \boldsymbol{u} \rangle/k$, $k\langle \mathbf{W} \rangle/k$ and $k\langle \boldsymbol{\pi} \rangle/k$ are *linearly disjoint*[15].
- The control (resp. fault) variables are assumed to be *independent*, i.e., $\boldsymbol{u}$ (resp. $\mathbf{W}$) is a differential transcendence basis of $k\langle \boldsymbol{u} \rangle/k$ (resp. $k\langle \mathbf{W} \rangle/k$).
- The extension $K/k\langle \boldsymbol{u}, \mathbf{W}, \boldsymbol{\pi} \rangle$ is differentially algebraic.
- Assume that the differential ideal $(\boldsymbol{\pi}) \subset k\{S, \boldsymbol{\pi}, \mathbf{W}\}$ generated by $\boldsymbol{\pi}$ is prime[16]. Write

$$k\{S^{\text{nom}}, \mathbf{W}^{\text{nom}}\} = k\{S, \boldsymbol{\pi}, \mathbf{W}\}/(\boldsymbol{\pi})$$

the quotient differential ring, where the *nominal* system and fault variables S^{nom}, $\mathbf{W}^{\text{nom}}$ are the canonical images of S, $\mathbf{W}$. To those nominal variables corresponds the *nominal system*[17] K^{nom}/k,

[13] See, e.g., McConnell & Robson (2000).

[14] See, e.g., McConnell & Robson (2000).

[15] See, e.g., Eisenbud (1995).

[16] Any reader with a good algebraic background will notice a connection with the notion of *differential specialization* (see, e.g., Kolchin (1973)).

[17] Let us explain those algebraic manipulations in plain words. Ignoring the perturbation variables in the original system yields the nominal system.

where $K^{\text{nom}} = k\langle S^{\text{nom}}, \mathbf{W}^{\text{nom}}\rangle$ is the quotient field of $k\{S^{\text{nom}}, \mathbf{W}^{\text{nom}}\}$, which is an *integral domain*, i.e., without zero divisors. The extension $K^{\text{nom}}/k\langle \boldsymbol{u}^{\text{nom}}, \mathbf{W}^{\text{nom}}\rangle$ is differentially algebraic.

- Assume as above that the differential ideal $(\mathbf{W}^{\text{nom}}) \subset k\{S^{\text{nom}}, \mathbf{W}^{\text{nom}}\}$ generated by $\mathbf{W}^{\text{nom}}$ is prime. Write

$$k\{S^{\text{pure}}\} = k\{S^{\text{nom}}, \mathbf{W}^{\text{nom}}\}/(\mathbf{W}^{\text{nom}})$$

where the *pure* system variables S^{pure} are the canonical images of S^{nom}. To those pure variables corresponds the *pure system*[18] K^{pure}/k, where $K^{\text{pure}} = k\langle S^{\text{pure}}\rangle$ is the quotient field of $k\{S^{\text{pure}}\}$. The extension $K^{\text{pure}}/k\langle \boldsymbol{u}^{\text{pure}}\rangle$ is differentially algebraic.

Remark 2.1. *We make moreover the following natural assumptions:* $\text{diff tr deg } (k\langle \boldsymbol{u}^{\text{pure}}\rangle/k) = \text{diff tr deg } (k\langle \boldsymbol{u}^{\text{nom}}\rangle/k) = \text{diff tr deg } (k\langle \boldsymbol{u}\rangle/k) = m$, $\text{diff tr deg } (k\langle \boldsymbol{W}^{\text{nom}}\rangle/k) = \text{diff tr deg } (k\langle \boldsymbol{W}\rangle/k) = q$

Remark 2.2. *Remember that differential algebra considers algebraic differential equations, i.e., differential equations which only contain polynomial functions of the variables and their derivatives up to some finite order. This is of course not always the case in practice. In the example of Section 4, for instance, appears the transcendental function* $\sin\theta_l$. *As already noted in Fliess, Lévine, Martin & Rouchon (1995), we recover algebraic differential equations by introducing* $\tan\frac{\theta_l}{2}$.

2.4.2 State-variable representation

We know, from proposition 2.1, that the transcendence degree of the extension $K/k\langle \mathbf{u}, \mathbf{W}, \boldsymbol{\pi}\rangle$ is finite, say n. Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a transcendence basis. Any derivative $\dot{x}_i$, $i = 1, \ldots, n$, and any output variable y_j, $j = 1, \ldots, p$, are algebraically dependent over $k\langle \mathbf{u}, \mathbf{W}, \boldsymbol{\pi}\rangle$ on $\boldsymbol{x}$:

$$\begin{array}{ll} A_i(\dot{x}_i, \boldsymbol{x}) = 0 & i = 1, \ldots, n \\ B_j(y_j, \boldsymbol{x}) = 0 & j = 1, \ldots, p \end{array} \tag{3}$$

where $A_i \in k\langle \mathbf{u}, \mathbf{W}, \boldsymbol{\pi}\rangle[\dot{x}_i, \boldsymbol{x}]$, $B_j \in k\langle \mathbf{u}, \mathbf{W}, \boldsymbol{\pi}\rangle[y_j, \boldsymbol{x}]$, i.e., the coefficients of the polynomials A_i, B_j depend on the control, fault and perturbation variables and on their derivatives up to some finite order.

Eq. (3) becomes for the nominal system

$$\begin{array}{ll} A_i^{\text{nom}}(\dot{x}_i^{\text{nom}}, \boldsymbol{x}^{\text{nom}}) = 0 & i = 1, \ldots, n_{\text{nom}} \leq n \\ B_j^{\text{nom}}(y_j^{\text{nom}}, \boldsymbol{x}^{\text{nom}}) = 0 & j = 1, \ldots, p \end{array} \tag{4}$$

where $A_i^{\text{nom}} \in k\langle \mathbf{u}^{\text{nom}}, \mathbf{W}^{\text{nom}}\rangle[\dot{x}_i^{\text{nom}}, \boldsymbol{x}^{\text{nom}}]$, $B_j^{\text{nom}} \in k\langle \mathbf{u}^{\text{nom}}, \mathbf{W}^{\text{nom}}\rangle[y_j^{\text{nom}}, \boldsymbol{x}^{\text{nom}}]$, i.e., the coefficients of A_i^{nom} and B_j^{nom} depend on the nominal control and fault variables and their derivatives and no more on the perturbation variables and their derivatives.

We get for the pure system

$$\begin{array}{ll} A_i^{\text{pure}}(\dot{x}_i^{\text{pure}}, \boldsymbol{x}^{\text{pure}}) = 0 & i = 1, \ldots, n_{\text{pure}} \leq n_{\text{nom}} \\ B_j^{\text{pure}}(y_j^{\text{pure}}, \boldsymbol{x}^{\text{pure}}) = 0 & j = 1, \ldots, p \end{array} \tag{5}$$

[18] Ignoring as above the fault variables in the nominal system yields the pure system.

where $A_i^{\text{pure}} \in k\langle \mathbf{u}^{\text{pure}}\rangle[\dot{x}_i^{\text{pure}}, \boldsymbol{x}^{\text{pure}}]$, $B_j^{\text{pure}} \in k\langle \mathbf{u}^{\text{pure}}\rangle[y_j^{\text{pure}}, \boldsymbol{x}^{\text{pure}}]$, i.e., the coefficients of A_i^{pure} and B_j^{pure} depend only on the pure control variables and their derivatives.

Remark 2.3. *Two main differences, which are confirmed by concrete examples (see, e.g., Fliess & Hasler (1990); Fliess, Lévine & Rouchon (1993)), can be made with the usual state-variable representation*

$$\begin{array}{l} \dot{\boldsymbol{x}} = F(\boldsymbol{x}, \boldsymbol{u}) \\ \boldsymbol{y} = H(\boldsymbol{x}) \end{array}$$

1. *The representations (3), (4), (5) are implicit.*
2. *The derivatives of the control variables in the equations of the dynamics cannot be in general removed (see Delaleau & Respondek (1995)).*

2.5 Variational system[19]

Call $\Omega_{K/k}$ (resp. $\Omega_{K^{\text{nom}}/k}$, $\Omega_{K^{\text{pure}}/k}$) the *variational*, or *linearized, system* (resp. *nominal system*, *pure system*) of system K/k. Proposition 2.2 yields for pure systems

$$A \begin{pmatrix} d_{K^{\text{pure}}/k} y_1^{\text{pure}} \\ \vdots \\ d_{K^{\text{pure}}/k} y_p^{\text{pure}} \end{pmatrix} = B \begin{pmatrix} d_{K^{\text{pure}}/k} u_1^{\text{pure}} \\ \vdots \\ d_{K^{\text{pure}}/k} u_m^{\text{pure}} \end{pmatrix} \tag{6}$$

where

- $A \in K[\frac{d}{dt}]^{p\times p}$ is of full rank,
- $B \in K[\frac{d}{dt}]^{p\times m}$.

The *pure transfer matrix*[20] is the matrix $A^{-1}B \in K(s)^{p\times m}$, where $K(s)$, $s = \frac{d}{dt}$, is the skew quotient field[21] of $K[\frac{d}{dt}]$.

2.6 Differential flatness[22]

The system K/k is said to be *(differentially)* flat if, and only if, the pure system K^{pure}/k is (differentially) flat (Fliess, Lévine, Martin & Rouchon (1995)): the algebraic closure $\bar{K}^{\text{pure}}$ of K^{pure} is equal to the algebraic closure of a purely differentially transcendental extension of k. It means in other words that there exists a finite subset $\boldsymbol{z}^{\text{pure}} = \{z_1^{\text{pure}}, \ldots, z_m^{\text{pure}}\}$ of $\bar{K}^{\text{pure}}$ such that

- $z_1^{\text{pure}}, \ldots, z_m^{\text{pure}}$ are differentially algebraically independent over k,
- $z_1^{\text{pure}}, \ldots, z_m^{\text{pure}}$ are algebraic over K^{pure},

[19] See Fliess, Lévine, Martin & Rouchon (1995) for more details.

[20] See Fliess (1994) for more details on transfer matrices of time-varying linear systems, and, more generally, Fliess, Join & Sira-Ramírez (2004), Bourlès (2006) for the module-theoretic approach to linear systems.

[21] See, e.g., McConnell & Robson (2000).

[22] For more details see Fliess, Lévine, Martin & Rouchon (1995); Rudolph (2003); Sira-Ramírez & Agrawal (2004).

- any pure system variable is algebraic over $k\langle z_1^{\text{pure}}, \ldots, z_m^{\text{pure}}\rangle$.

$\boldsymbol{z}^{\text{pure}}$ is a *(pure) flat*, or *linearizing*, *output*. For a flat dynamics, it is known that the number m of its elements is equal to the number of independent control variables.

2.7 Observability and identifiability

Take a system K/k with control $\boldsymbol{u}$ and output $\boldsymbol{y}$.

2.7.1 Observability

According to Diop & Fliess (1991a,b) (see also Diop (2002)), system K/k is said to be *observable* if, and only if, the extension $K^{\text{pure}}/k\langle \boldsymbol{u}^{\text{pure}}, \boldsymbol{y}^{\text{pure}}\rangle$ is algebraic.

Remark 2.4. *This new definition*[23] *of observability is "roughly" equivalent (see Diop & Fliess (1991a,b) for details*[24]*) to its usual differential geometric counterpart due to Hermann & Krener (1977) (see also Conte, Moog & Perdon (1999); Gauthier & Kupka (2001); Isidori (1995); Nijmeijer & van der Schaft (1990); Sontag (1998)).*

2.7.2 Identifiable parameters[25]

Set $k = k_0\langle \boldsymbol{\Theta}\rangle$, where k_0 is a differential field and $\boldsymbol{\Theta} = \{\theta_1, \ldots, \theta_r\}$ a finite set of *unknown parameters*, which might not be constant. According to Diop & Fliess (1991a,b), a parameter θ_ι, $\iota = 1, \ldots, r$, is said to be *algebraically* (resp. *rationally*) *identifiable* if, and only if, it is algebraic over (resp. belongs to) $k_0\langle \boldsymbol{u}, \boldsymbol{y}\rangle$:

- θ_ι is rationally identifiable if, and only if, it is equal to a differential rational function over k_0 of the variables $\boldsymbol{u}$, $\boldsymbol{y}$, i.e., to a rational function of $\boldsymbol{u}$, $\boldsymbol{y}$ and their derivatives up to some finite order, with coefficients in k_0;
- θ_ι is algebraically identifiable if, and only if, it satisfies an algebraic equation with coefficients in $k_0\langle \boldsymbol{u}, \boldsymbol{y}\rangle$.

2.7.3 *Determinable* variables

More generally, a variable $\Upsilon \in K$ is said to be *rationally* (resp. *algebraically*) *determinable* if, and only if, Υ^{pure} belongs to (resp. is algebraic over) $k\langle \boldsymbol{u}^{\text{pure}}, \boldsymbol{y}^{\text{pure}}\rangle$. A system variable χ is then said to be *rationally* (resp. *algebraically*) *observable* if, and only if, χ^{pure} belongs to (resp. is algebraic over) $k\langle \boldsymbol{u}^{\text{pure}}, \boldsymbol{y}^{\text{pure}}\rangle$.

Remark 2.5. *In the case of algebraic determinability, the corresponding algebraic equation might possess several roots which are not easily discriminated (see, e.g., Li, Chiasson, Bodson & Tolbert (2006) for a concrete example).*

Remark 2.6. *See Sedoglavic (2002) and Ollivier & Sedoglavic (2002) for efficient algorithms in order to test observability and identifiability. Those algorithms may certainly be extended to determinable variables and to various questions related to fault diagnosis in Section 2.8.*

2.8 Fundamental properties of fault variables[26]

2.8.1 Detectability

The fault variable $\mathbf{w}_\iota$, $\iota = 1, \ldots, q$, is said to be *detectable* if, and only if, the field extension $K^{\text{nom}}/k\langle \boldsymbol{u}^{\text{nom}}, \mathbf{W}_\iota^{\text{nom}}\rangle$, where $\mathbf{W}_\iota^{\text{nom}} = \mathbf{W}^{\text{nom}} \backslash \{\mathbf{w}_\iota^{\text{nom}}\}$, is differentially transcendental. It means that $\mathbf{w}_\iota$ is indeed "influencing" the output. When considering the variational nominal system, formula (6) yields

$$\begin{pmatrix} d_{K^{\text{nom}}/k} y_1^{\text{nom}} \\ \vdots \\ d_{K^{\text{nom}}/k} y_p^{\text{nom}} \end{pmatrix} = T_{\boldsymbol{u}} \begin{pmatrix} d_{K^{\text{nom}}/k} u_1^{\text{nom}} \\ \vdots \\ d_{K^{\text{nom}}/k} u_m^{\text{nom}} \end{pmatrix} + T_{\mathbf{W}} \begin{pmatrix} d_{K^{\text{nom}}/k} \mathbf{w}_1^{\text{nom}} \\ \vdots \\ d_{K^{\text{nom}}/k} \mathbf{w}_q^{\text{nom}} \end{pmatrix}$$

where $T_{\boldsymbol{u}} \in K(s)^{p\times m}$, $T_{\mathbf{W}} \in K(s)^{p\times q}$. Call $T_{\mathbf{W}}$ the *fault transfer matrix*. The next result is clear:

Proposition 2.3. *The fault variable $\boldsymbol{w}_\iota$ is detectable if, and only if, the ι^{th} column of the fault transfer matrix $T_{\boldsymbol{W}}$ is non-zero.*

2.8.2 Isolability, parity equations and residuals

A subset $\mathbf{W}' = (\mathbf{w}_{\iota_1}, \ldots, \mathbf{w}_{\iota_{q'}})$ of the set $\mathbf{W}$ of fault variables is said to be

- *Differentially algebraically isolable* if, and only if, the extension $k\langle \boldsymbol{u}^{\text{nom}}, \boldsymbol{y}^{\text{nom}}, \mathbf{W}'^{\text{nom}}\rangle / k\langle \boldsymbol{u}^{\text{nom}}, \boldsymbol{y}^{\text{nom}}\rangle$ is differentially algebraic. It means that any component of $\mathbf{W}'^{\text{nom}}$ satisfies a *parity differential equation*, i.e., an algebraic differential equations where the coefficients belong to $k\langle \boldsymbol{u}^{\text{nom}}, \boldsymbol{y}^{\text{nom}}\rangle$.
- *Algebraically isolable* if, and only if, the extension $k\langle \boldsymbol{u}^{\text{nom}}, \boldsymbol{y}^{\text{nom}}, \mathbf{W}'^{\text{nom}}\rangle / k\langle \boldsymbol{u}^{\text{nom}}, \boldsymbol{y}^{\text{nom}}\rangle$ is algebraic. It

[23] See Fliess & Rudolph (1997) for a definition via infinite prolongations.

[24] The differential algebraic and the differential geometric languages are not equivalent. We cannot therefore hope for a "one-to-one bijection" between definitions and results which are expressed in those two settings.

[25] Differential algebra has already been employed for parametric identifiability and identification but in a different context by several authors (see, e.g., Ljung & Glad (1994); Ollivier (1990); Saccomani, Audoly & D'Angio (2003)).

[26] See, e.g., Chen & Patton (1999); Blanke, Kinnaert, Lunze & Staroswiecki (2003); Gertler (1998); Vachtsevanos, Lewis, Roemer, Hess & Wu (2006) for introductions to this perhaps less well known subject. The definitions and properties below are clear-cut extensions of their linear counterparts in Fliess, Join & Sira-Ramírez (2004). Some of them might also be seen as a direct consequence of Section 2.7.3. Differential algebra has already been employed but in a different context by several authors (see, e.g., Martinez-Guerra & Diop (2004); Martìnez-Guerra, González-Galan, Luviano-Juárez & Cruz-Victoria (2007); Staroswiecki & Comtet-Varga (2001); Zhang, Basseville & Benveniste (1998)).

means that the parity differential equation is of order 0, i.e., it is an algebraic equation with coefficients $k\langle \boldsymbol{u}^{\mathrm{nom}}, \boldsymbol{y}^{\mathrm{nom}} \rangle$.

- *Rationally isolable* if, and only if, $\mathbf{W}'^{\mathrm{nom}}$ belongs to $k\langle \boldsymbol{u}^{\mathrm{nom}}, \boldsymbol{y}^{\mathrm{nom}} \rangle$. It means that the parity equation is a linear algebraic equation, i.e., any component of $\mathbf{W}'^{\mathrm{nom}}$ may be expressed as a rational function over k in the variables $\boldsymbol{u}^{\mathrm{nom}}$, $\boldsymbol{y}^{\mathrm{nom}}$ and their derivatives up to some finite order.

The next property is obvious:

Proposition 2.4. *Rational isolability $\Rightarrow$ algebraic isolability $\Rightarrow$ differentially algebraic isolability.*

When we will say for short that fault variables are *isolable*, it will mean that they are differentially algebraically isolable.

Proposition 2.5. *Assume that the fault variables belonging to $\boldsymbol{W}'$ are isolable. Then* $\mathrm{card}(\boldsymbol{W}') \leq \mathrm{card}(\boldsymbol{y})$.

Proof. The differential transcendence degree of the extension $k\langle \boldsymbol{u}^{\mathrm{nom}}, \boldsymbol{y}^{\mathrm{nom}}, \mathbf{W}'^{\mathrm{nom}} \rangle / k$ (resp. $k\langle \boldsymbol{u}^{\mathrm{nom}}, \boldsymbol{y}^{\mathrm{nom}} \rangle / k$) is equal to $\mathrm{card}(\boldsymbol{u}) + \mathrm{card}(\boldsymbol{W}')$ (resp. is less than or equal to $\mathrm{card}(\boldsymbol{u}) + \mathrm{card}(\boldsymbol{y})$). The equality of those two degrees implies our result thanks to the Remark 2.1. $\square$

3 Derivatives of a noisy signal

3.1 Polynomial time signals

Consider the real-valued polynomial function $x_N(t) = \sum_{\nu=0}^{N} x^{(\nu)}(0) \frac{t^\nu}{\nu!} \in \mathbb{R}[t]$, $t \geq 0$, of degree N. Rewrite it in the well known notations of operational calculus:

$$X_N(s) = \sum_{\nu=0}^{N} \frac{x^{(\nu)}(0)}{s^{\nu+1}}$$

We know utilize $\frac{d}{ds}$, which is sometimes called the *algebraic derivative* (cf. Mikusinski (1983); Mikusinski & Boehme (1987)). Multiply both sides by $\frac{d^\alpha}{ds^\alpha} s^{N+1}$, $\alpha = 0, 1, \ldots, N$. The quantities $x^{(\nu)}(0)$, $\nu = 0, 1, \ldots, N$ are given by the triangular system of linear equations[27]:

$$\frac{d^\alpha s^{N+1} X_N}{ds^\alpha} = \frac{d^\alpha}{ds^\alpha} \left(\sum_{\nu=0}^{N} x^{(\nu)}(0) s^{N-\nu} \right) \tag{7}$$

The time derivatives, i.e., $s^\mu \frac{d^\iota X_N}{ds^\iota}$, $\mu = 1, \ldots, N$, $0 \leq \iota \leq N$, are removed by multiplying both sides of Eq. (7) by $s^{-\bar{N}}$, $\bar{N} > N$.

Remark 3.1. *Remember (cf. Mikusinski (1983); Mikusinski & Boehme (1987); Yosida (1984)) that $\frac{d}{ds}$ corresponds in the time domain to the multiplication by $-t$.*

[27] Following Fliess & Sira-Ramírez (2003, 2007), those quantities are said to be *linearly identifiable*.

3.2 Analytic time signals

Consider a real-valued analytic time function defined by the convergent power series $x(t) = \sum_{\nu=0}^{\infty} x^{(\nu)}(0) \frac{t^\nu}{\nu!}$, where $0 \leq t < \rho$. Introduce its truncated Taylor expansion

$$x(t) = \sum_{\nu=0}^{N} x^{(\nu)}(0) \frac{t^\nu}{\nu!} + O(t^{N+1}) \tag{8}$$

Approximate $x(t)$ in the interval $(0, \varepsilon)$, $0 < \varepsilon \leq \rho$, by its truncated Taylor expansion $x_N(t) = \sum_{\nu=0}^{N} x^{(\nu)}(0) \frac{t^\nu}{\nu!}$ of order N. Introduce the operational analogue of $x(t)$, i.e., $X(s) = \sum_{\nu \geq 0} \frac{x^{(\nu)}(0)}{s^{\nu+1}}$, which is an *operationally convergent* series in the sense of Mikusinski (1983); Mikusinski & Boehme (1987). Denote by $[x^{(\nu)}(0)]_{e_N}(t)$, $0 \leq \nu \leq N$, the numerical estimate of $x^{(\nu)}(0)$, which is obtained by replacing $X_N(s)$ by $X(s)$ in Eq. (7). The next result, which is elementary from an analytic standpoint, provides a mathematical justification for the computer implementations:

Proposition 3.1. *For $0 < t < \varepsilon$,*

$$\lim_{t \downarrow 0} [x^{(\nu)}(0)]_{e_N}(t) = \lim_{N \to +\infty} [x^{(\nu)}(0)]_{e_N}(t) = x^{(\nu)}(0) \tag{9}$$

Proof. Following (8) replace $x_N(t)$ by $x(t) = x_N(t) + O(t^{N+1})$. The quantity $O(t^{N+1})$ becomes negligible if $t \downarrow 0$ or $N \to +\infty$. $\square$

Remark 3.2. *See Mboup, Join & Fliess (2007)) for fundamental theoretical developments. See also Nöthen (2007) for most fruitful comparisons and discussions.*

3.3 Noisy signals

Assume that our signals are corrupted by additive noises. Those noises are viewed here as highly fluctuating, or oscillatory, phenomena. They may be therefore attenuated by low-pass filters, like iterated time integrals. Remember that those iterated time integrals do occur in Eq. (7) after multiplying both sides by $s^{-\bar{N}}$, for $\bar{N} > 0$ large enough.

Remark 3.3. *The estimated value of $x(0)$, which is obtained along those lines, should be viewed as a denoising of the corresponding signal.*

Remark 3.4. *See Fliess (2006) for a precise mathematical foundation, which is based on* nonstandard analysis. *A highly fluctuating function of zero mean is then defined by the following property: its integral over a finite time interval is* infinitesimal, *i.e., "very small". Let us emphasize that this approach[28], which has been confirmed by numerous computer simulations and several laboratory experiments in control and in signal processing[29], is independent of any probabilistic setting. No knowledge of the statistical properties of the noises is required.*

[28] This approach applies as well to multiplicative noises (see Fliess (2006)). The assumption on the noises being only additive is therefore unnecessary.

[29] For numerical simulations in signal processing, see Fliess, Join, Mboup & Sira-Ramírez (2004, 2005); Fliess, Join, Mboup & Sedoglavic (2005). Some of them are dealing with multiplicative noises.

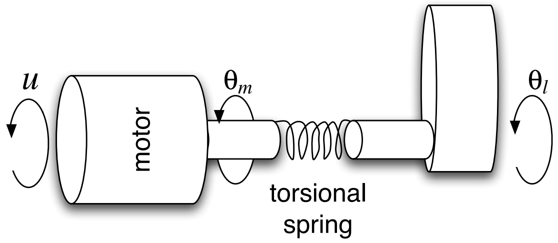

Figure 1: A single link flexible joint manipulator

4 Feedback and state reconstructor

4.1 System description

Consider with Fan & Arcak (2003) the mechanical system, depicted in Figure 1. It consists of a DC-motor joined to an inverted pendulum through a torsional spring:

$$\begin{aligned} J_m\ddot{\theta}_m(t) &= \kappa\big(\theta_l(t)-\theta_m(t)\big) - B\dot{\theta}_m(t) + K_\tau u(t) \\ J_l\ddot{\theta}_l(t) &= -\kappa\big(\theta_l(t)-\theta_m(t)\big) - mgh\sin(\theta_l(t)) \\ y(t) &= \theta_l(t) \end{aligned} \tag{10}$$

where

- θ_m and θ_l represent respectively the angular deviation of the motor shaft and the angular position of the inverted pendulum,
- J_m, J_l, h, m, κ, B, K_τ and g are physical parameters which are assumed to be constant and known.

System (10), which is linearizable by static state feedback, is flat; $y = \theta_l$ is a flat output.

4.2 Control design

Tracking of a given smooth reference trajectory $y^*(t) = \theta_l^*(t)$ is achieved via the linearizing feedback controller

$$\begin{aligned} u(t) = &\tfrac{1}{K_\tau}\Big(\tfrac{J_m}{\kappa}\big[J_l v(t) + \kappa\ddot{y}_e(t) \\ &+ mgh(\ddot{y}_e(t)\cos(y_e(t)) - (\dot{y}_e(t))^2\sin(y_e(t)))\big] \\ &+ J_l\ddot{y}_e(t) + mgh\sin(y_e(t)) \\ &\tfrac{B}{\kappa}\big[J_l y_e^{(3)}(t) + \kappa\dot{y}_e(t) + mgh\dot{y}_e(t)\cos(y_e(t))\big]\Big) \end{aligned} \tag{11}$$

where

$$\begin{aligned} v(t) &= y^{*(4)}(t) - \gamma_4(y_e^{(3)}(t) - y^{*(3)}(t)) \\ &-\gamma_3(\ddot{y}_e(t) - \ddot{y}^*(t)) - \gamma_2(\dot{y}_e(t) - \dot{y}^*(t)) \\ &-\gamma_1(y_e(t) - y^*(t)) \end{aligned} \tag{12}$$

The subscript "e"denotes the estimated value. The design parameters γ_1, ..., γ_4 are chosen so that the resulting characteristic polynomial is Hurwitz.

Remark 4.1. *Feedback laws like (11)-(12) depend, as usual in flatness-based control (see, e.g., Fliess, Lévine, Martin & Rouchon (1995, 1999); Sira-Ramírez & Agrawal (2004)), on the derivatives of the flat output and not on the state variables.*

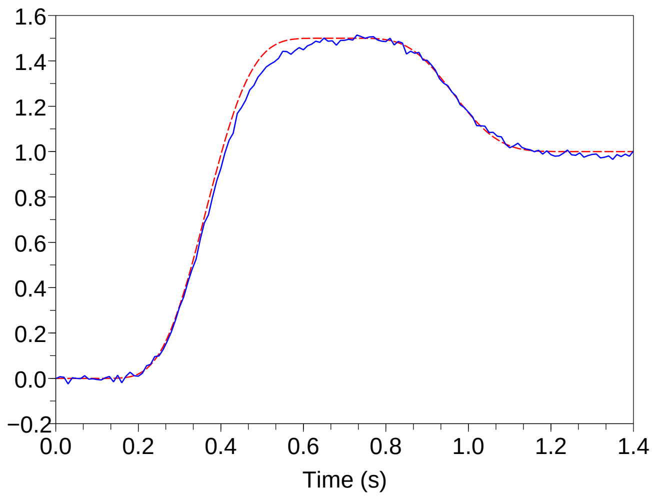

Figure 2: Output (–) and reference trajectory (- -)

4.3 A state reconstructor[30]

We might nevertheless be interested in obtaining an estimate $[\theta_m]_e(t)$ of the unmeasured state $\theta_m(t)$:

$$[\theta_m]_e(t) = \frac{1}{\kappa}\Big(J_l\ddot{y}_e(t) + mgh\sin(y_e(t))\Big) + y_e(t) \tag{13}$$

4.4 Numerical simulations

The physical parameters have the same numerical values as in Fan & Arcak (2003): $J_m = 3.7 \times 10^{-3}\ kgm^2$, $J_l = 9.3 \times 10^{-3}\ kgm^2$, $h = 1.5 \times 10^{-1}\ m$, $m = 0.21\ kg$, $B = 4.6 \times 10^{-2}\ m$, $K_\tau = 8 \times 10^{-2}\ NmV^{-1}$. The numerical simulations are presented in Figures 2 - 9. Robustness has been tested with an additive white Gaussian noise N(0; 0.01) on the output y. Note that the off-line estimations of $\ddot{y}$ and θ_m, where a "small" delay is allowed, are better than the on-line estimation of $\ddot{y}$.

5 Parametric identification

5.1 A rigid body

Consider the fully actuated rigid body, depicted in Figure 10, which is given by the Euler equations

$$\begin{aligned} I_1\dot{w}_1(t) &= (I_2 - I_3)w_2(t)w_3(t) + u_1(t) \\ I_2\dot{w}_2(t) &= (I_3 - I_1)w_3(t)w_1(t) + u_2(t) \\ I_3\dot{w}_3(t) &= (I_1 - I_2)w_1(t)w_2(t) + u_3(t) \end{aligned} \tag{14}$$

where w_1, w_2, w_3 are the measured angular velocities, u_1, u_2, u_3 the applied control input torques, I_1, I_2, I_3 the constant moments of inertia, which are poorly known. System (14) is stabilized around the origin, for suitably chosen design parameters $\lambda_{1\iota}$, $\lambda_{0\iota}$, $\iota = 1, 2, 3$, by the feedback

[30] See Sira-Ramírez & Fliess (2006) and Reger, Mai & Sira-Ramírez (2006) for other interesting examples of state reconstructors which are applied to chaotically encrypted messages.

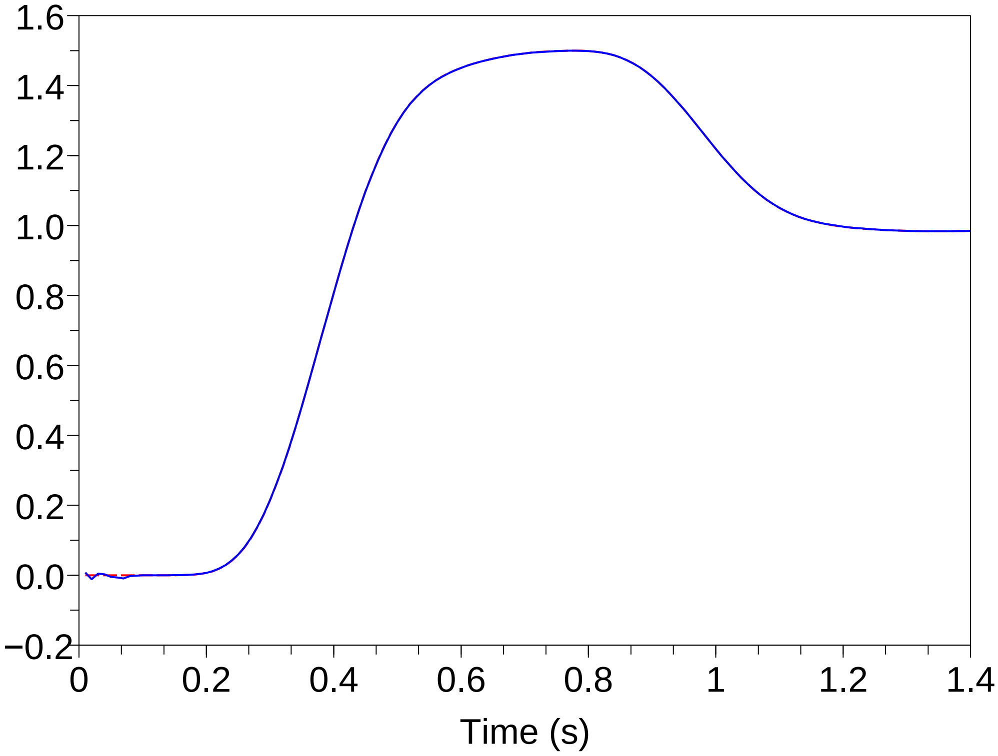

Figure 5: y: (- -); on-line noise attenuation y_e (–)

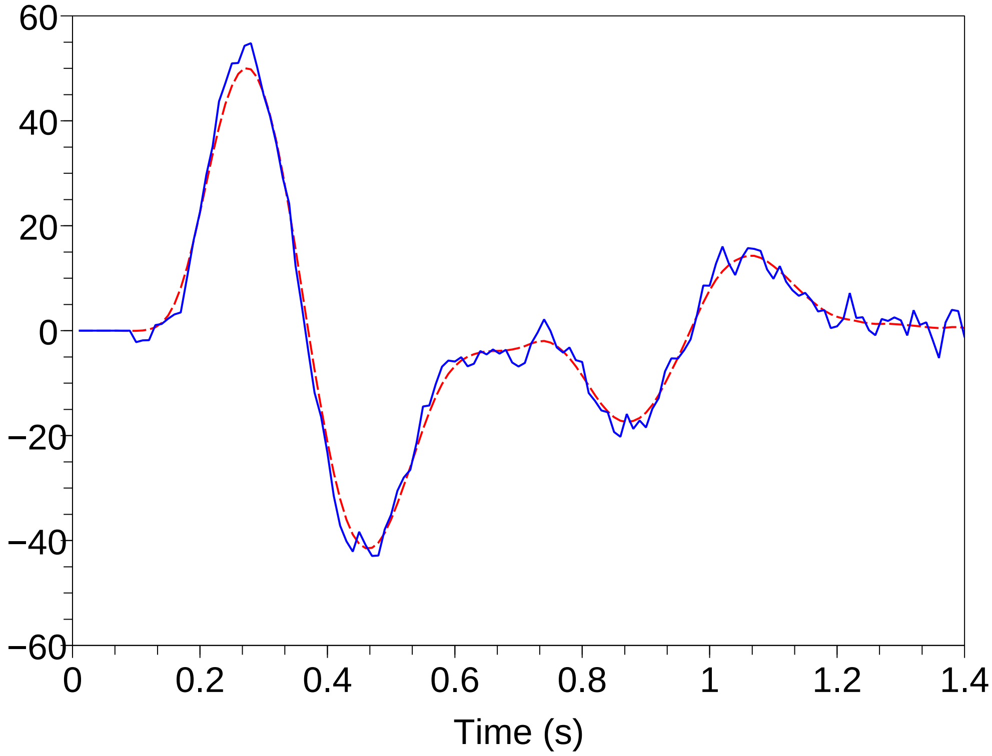

Figure 6: $\ddot{y}$ (- -); on-line estimation $\ddot{y}_e$ (–)

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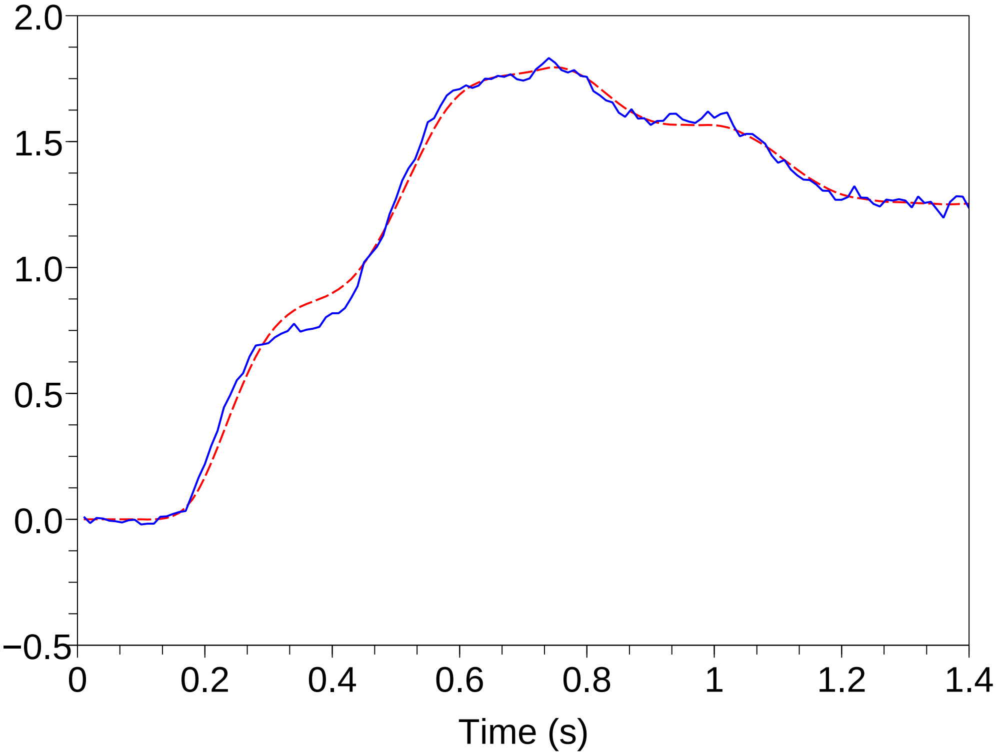

Figure 7: θ_m (- -); on-line estimation $[\theta_m]_e$ (–)

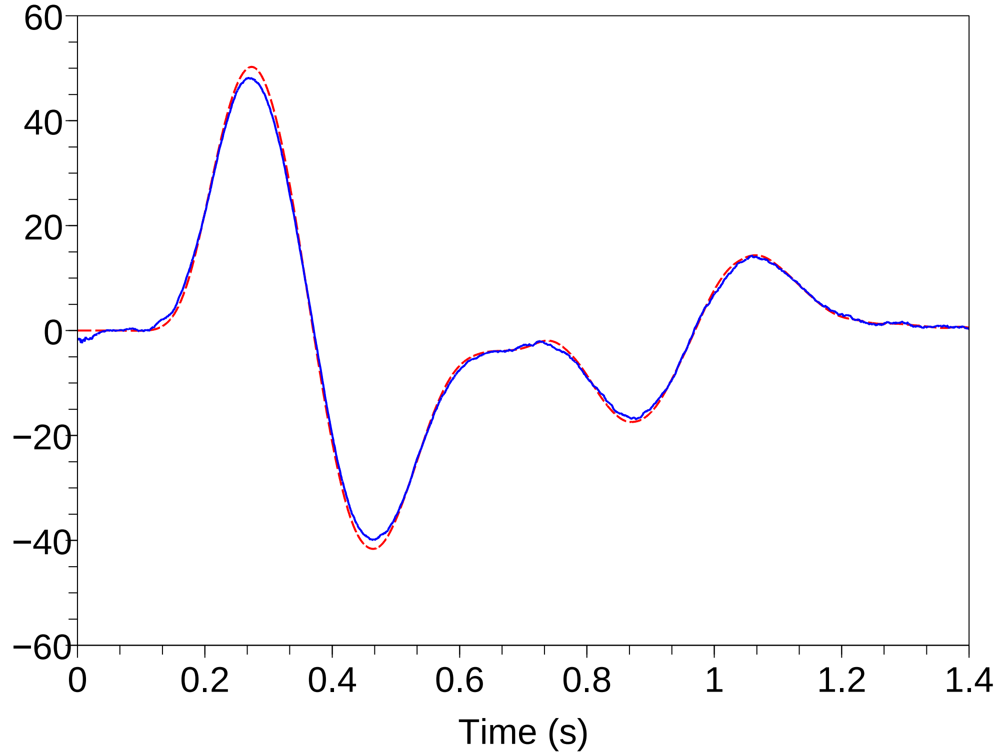

Figure 8: $\ddot{y}$ (- -); off-line estimation $\ddot{y}_e$ (–)

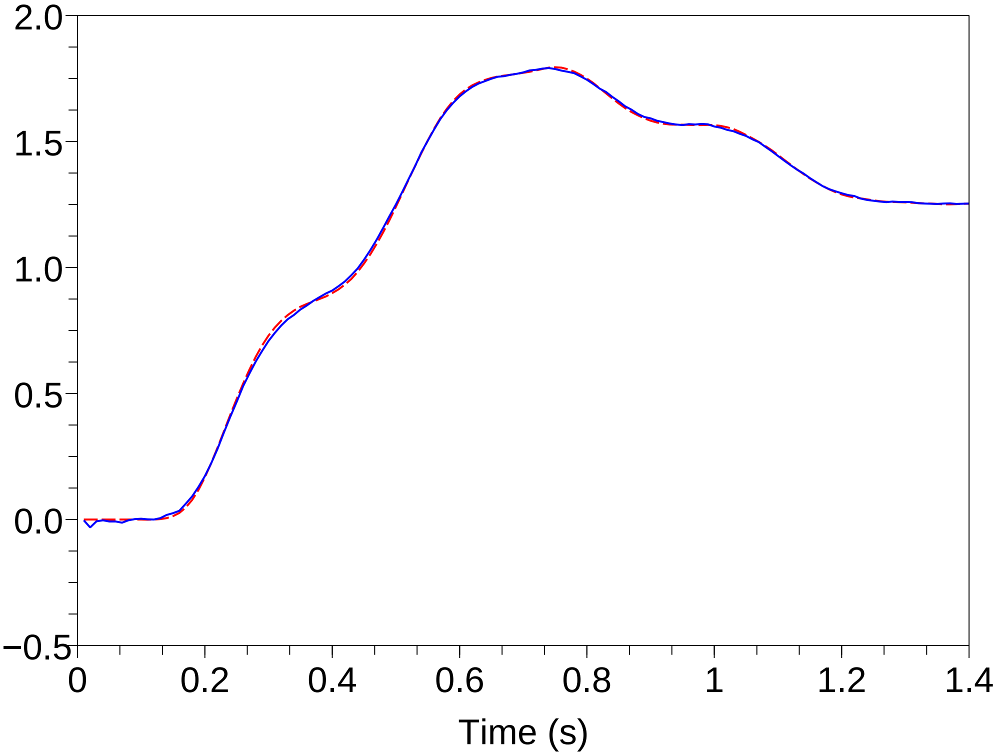

Figure 9: θ_m (- -); off-line estimation $[\theta_m]_e$ (−)

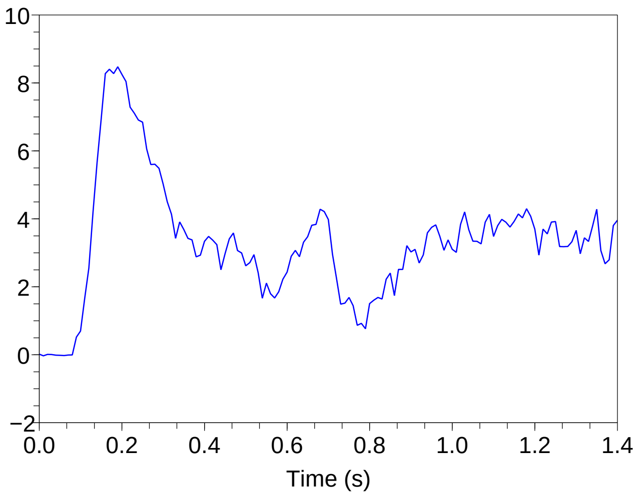

Figure 3: Control

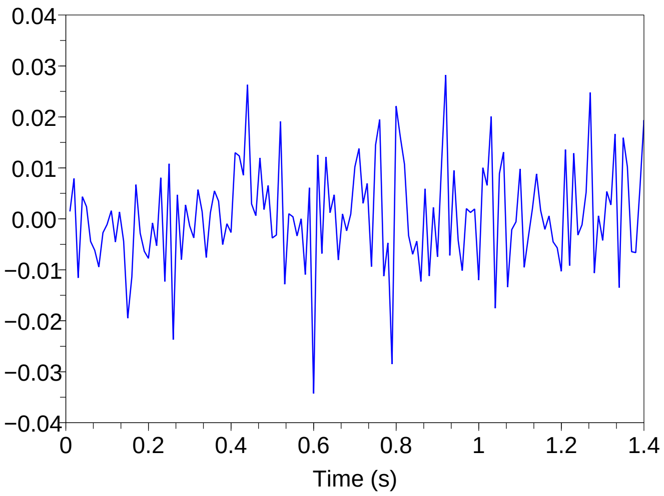

Figure 4: Output noise

controller, which is an obvious extension of the familiar proportional-integral (PI) regulators,

$$\begin{aligned}
u_1(t) = \quad & -(I_2 - I_3)w_2(t)w_3(t) \\
& +I_1\Big(-\lambda_{11}w_1(t) - \lambda_{01}\int_0^t w_1(\sigma)d\sigma\Big) \\
u_2(t) = \quad & -(I_3 - I_1)w_3(t)w_1(t) \\
& +I_2\Big(-\lambda_{12}w_2(t) - \lambda_{02}\int_0^t w_2(\sigma)d\sigma\Big) \\
u_3(t) = \quad & -(I_1 - I_2)w_1(t)w_2(t) \\
& +I_3\Big(-\lambda_{13}w_3(t) - \lambda_{03}\int_0^t w_3(\sigma)d\sigma\Big)
\end{aligned} \tag{15}$$

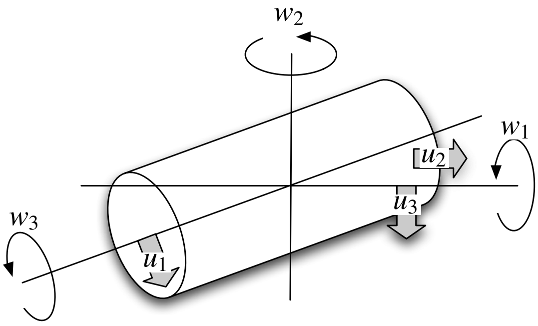

Figure 10: Rigid body

5.2 Identification of the moments of inertia

Write Eq. (14) in the following matrix form:

$$\begin{pmatrix} \dot{w}_1 & -w_2 w_3 & w_2 w_2 \\ w_1 w_3 & \dot{w}_2 & -w_1 w_3 \\ -w_1 w_2 & w_1 w_2 & \dot{w}_3 \end{pmatrix} \times \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

It yields estimates $[I_1]_e$, $[I_2]_e$, $[I_3]_e$ of I_1, I_2, I_3 when we replace w_1, w_2, w_3, $\dot{w}_1$, $\dot{w}_2$, $\dot{w}_3$ by their estimates[31]. The control law (15) becomes

$$\begin{array}{rl} u_1(t) = & -([I_2]_e - [I_3]_e)[w_2]_e(t)[w_3]_e(t) \\ & +[I_1]_e\Big(- \lambda_{11}[w_1]_e(t) - \lambda_{01}\int_0^t [w_1]_e(\sigma)d\sigma\Big) \\ u_2(t) = & -([I_3]_e - [I_1]_e)[w_3]_e(t)[w_1]_e(t) \\ & +[I_2]_e\Big(- \lambda_{12}[w_2]_e(t) - \lambda_{02}\int_0^t [w_2]_e(\sigma)d\sigma\Big) \\ u_3(t) = & -([I_1]_e - [I_2]_e)[w_1]_e(t)[w_2]_e(t) \\ & +[I_3]_e\Big(- \lambda_{13}[w_3]_e(t) - \lambda_{03}\int_0^t [w_3]_e(\sigma)d\sigma\Big) \end{array} \quad (16)$$

5.3 Numerical simulations

The output measurements are corrupted by an additive Gaussian white noise $N(0; 0.005)$. Figure 11 shows an excellent on-line estimation of the three moments of inertia. Set for the design parameters in the controllers (15) and (16) $\lambda_{1\iota} = 2\xi\varpi$, $\lambda_{0\iota} = \varpi^2$, $\iota = 1, 2, 3$, where $\xi = 0.707$, $\varpi = 0.5$. The stabilization with the above estimated values in Figure 12 is quite better than in Figure 13 where the following false values where utilized: $I_1 = 0.2$, $I_2 = 0.1$ and $I_3 = 0.1$.

6 Fault diagnosis and accommodation

6.1 A two tank system[32]

Consider the cascade arrangement of two identical tank systems, shown in Figure 14, which is a popular example

[31] See Remark 3.3.

[32] See Mai, Join & Reger (2007) for another example.

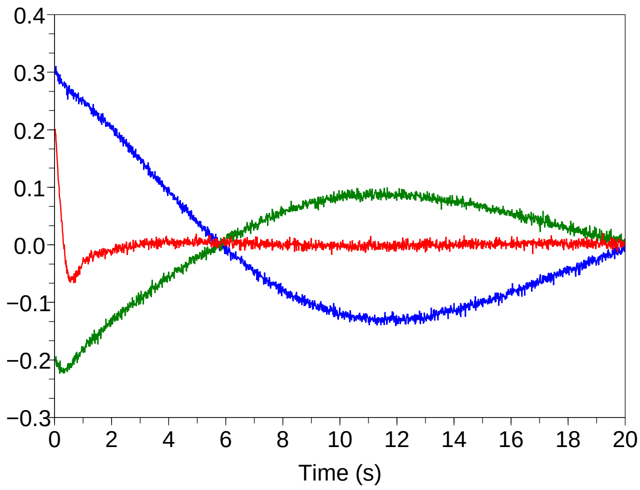

Figure 13: Feedback stabilization without parametric estimation

in fault diagnosis (see, e.g., Blanke, Kinnaert, Lunze & Staroswiecki (2003)).

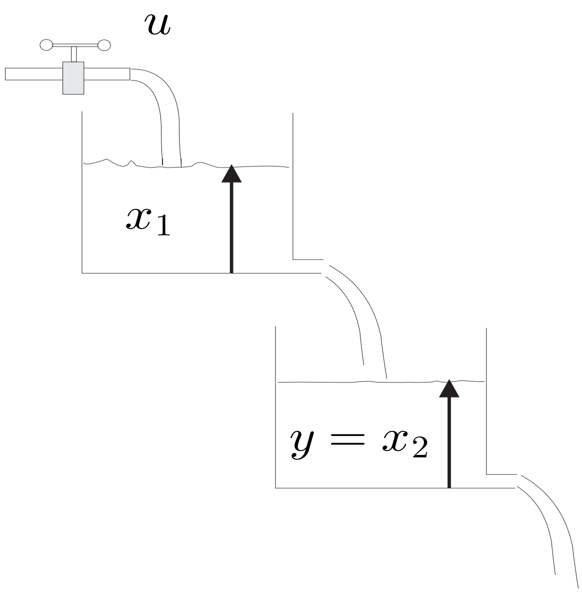

Figure 14: A two tank system

Its mathematical description is given by

$$\begin{array}{rcl} \dot{x}_1(t) & = & -\frac{c}{A}\sqrt{x_1(t)} + \frac{1}{A}u(t)\,(1 - \mathbf{w}(t)) \\ & & +\varpi(t) \\ \dot{x}_2(t) & = & \frac{c}{A}\sqrt{x_1(t)} - \frac{c}{A}\sqrt{x_2(t)} \\ y(t) & = & x_2(t) \end{array} \quad (17)$$

where:

- The constant c and the area A of the tank's bottom are known parameters.

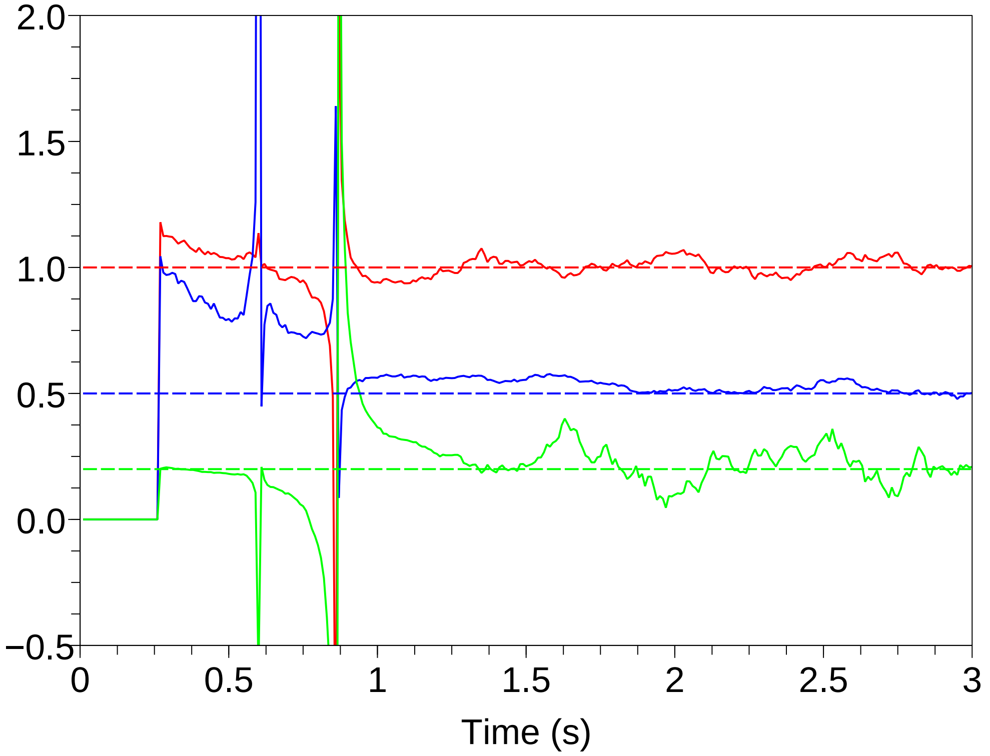

Figure 11: Zoom on the parametric estimation (–) and real values (- -)

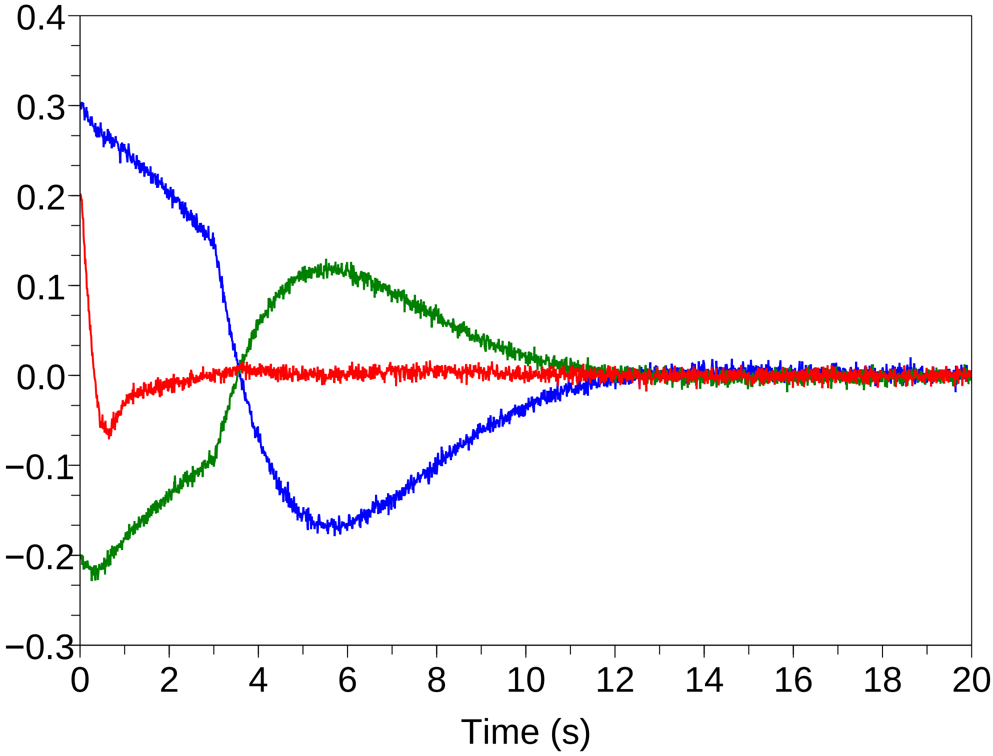

Figure 12: Feedback stabilization with parametric estimation

- The perturbation $\varpi(t)$ is constant but unknown,
- The actuator failure $\mathbf{w}(t)$, $0 \le \mathbf{w}(t) \le 1$, is constant but unknown. It starts at some unknown time $t_I >> 0$ which is not "small".
- Only the output $y = x_2$ is available for measurement.

The corresponding pure system, where we are ignoring the fault and perturbation variables (cf. Section 2.4.1),

$$\begin{array}{l} \dot{x}_1^{\text{pure}} = -\frac{c}{A}\sqrt{x_1^{\text{pure}}} + \frac{1}{A}u^{\text{pure}} \\ \dot{x}_2^{\text{pure}} = \frac{c}{A}\sqrt{x_1^{\text{pure}}} - \frac{c}{A}\sqrt{x_2^{\text{pure}}} \\ y^{\text{pure}} = x_2^{\text{pure}} \end{array}$$

is flat. Its flat output is $y^{\text{pure}} = x_2^{\text{pure}}$. The state variable x_1^{pure} and control variable u^{pure} are given by

$$\begin{aligned} x_1^{\text{pure}} &= \big(\frac{A}{c}\dot{y}^{\text{pure}} + \sqrt{y^{\text{pure}}}\big)^2 && (18) \\ u^{\text{pure}} &= 2A\big(\frac{A}{c}\dot{y}^{\text{pure}} + \sqrt{y^{\text{pure}}}\big)\big(\frac{A}{c}\ddot{y}^{\text{pure}} + \frac{\dot{y}^{\text{pure}}}{2\sqrt{y^{\text{pure}}}}\big) \\ &\quad + c\big(\frac{A}{c}\dot{y}^{\text{pure}} + \sqrt{y^{\text{pure}}}\big) && (19) \end{aligned}$$

6.2 Fault tolerant tracking controller

It is desired that the output y tracks a given smooth reference trajectory $y^*(t)$. Rewrite Formulae (18)-(19) by taking into account the perturbation variable $\varpi(t)$ and the actuator failure $\mathbf{w}(t)$:

$$\begin{aligned} x_1(t) &= \big(\frac{A}{c}\dot{y}(t) + \sqrt{y(t)}\big)^2 && (20) \\ u(t) &= \frac{1}{(1-\mathbf{w}(t))}\Big(- A\varpi \\ &\quad + 2A\big(\frac{A}{c}\dot{y}(t) + \sqrt{y(t)}\big) \times \\ &\quad \big(\frac{A}{c}\ddot{y}(t) + \frac{\dot{y}(t)}{2\sqrt{y(t)}}\big) \\ &\quad + c\big(\frac{A}{c}\dot{y}(t) + \sqrt{y(t)}\big)\Big) \end{aligned}$$

With reliable on-line estimates $\hat{\mathbf{w}}(t)$ and $\hat{\varpi}(t)$ of the failure signal $\mathbf{w}(t)$ and of the perturbation $\varpi(t)$, we design a failure accommodating linearizing feedback controller. It incorporates a classical robustifying integral action:

$$\begin{aligned} u(t) &= \frac{1}{(1-\hat{\mathbf{w}}(t))}\Big(- A\hat{\varpi}(t) \\ &\quad + 2A\big(\frac{A}{c}\dot{y}_e(t) + \sqrt{y_e(t)}\big)\big(\frac{A}{c}v(t) + \frac{\dot{y}_e(t)}{2\sqrt{y_e(t)}}\big) \\ &\quad + c\big(\frac{A}{c}\dot{y}_e(t) + \sqrt{y_e(t)}\big)\Big) \\ v(t) &= \ddot{y}^*(t) - \mathcal{G} \star (y_e(t) - y^*(t)) \end{aligned}$$

This is a *generalized proportional integral (GPI)* controller (cf. Fliess, Marquez, Delaleau & Sira-Ramírez (2002)) where

- $\star$ denotes the convolution product,
- the transfer function of $\mathcal{G}$ is
$$\frac{\lambda_2 s^2 + \lambda_1 s + \lambda_0}{s(s+\lambda_3)}$$
where $\lambda_0, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$,
- $y_e(t)$ is the on-line denoised estimate of $y(t)$ (cf. Remark 3.3),
- $\dot{y}_e(t)$ is the on-line estimated value of $\dot{y}(t)$.

6.3 Perturbation and fault estimation

The estimation of the constant perturbation ϖ is readily accomplished from Eq. (17) before the occurrence of the failure $\mathbf{w}$, which starts at time $t_I >> 0$:

$$\dot{x}_1(t) = -\frac{c}{A}\sqrt{x_1(t)} + \frac{1}{A}u(t) + \varpi \quad \text{if} \quad 0 < t < t_I$$

Multiplying both sides by t and integrating by parts yields[33]

$$\hat{\varpi} = \begin{cases} \text{arbitrary} & 0 < t < \epsilon \\ 2\frac{t\hat{x}_1(t) - \int_0^t \left[\hat{x}_1(\sigma) - \sigma\left(\frac{c}{A}\sqrt{\hat{x}_1(\sigma)} - \frac{1}{A}u(\sigma)\right)\right] d\sigma}{t^2} & \epsilon < t < t_I \end{cases}$$

where $\epsilon > 0$ is "very small". The estimated value $\hat{x}_1(t)$ of $x_1(t)$, which is obtained from Formula (20), needs as in Section 6.2 the on-line estimation $y_e(t)$ and $\dot{y}_e(t)$.

The estimated value $\hat{\mathbf{w}}$ of $\mathbf{w}$, which is detectable and algebraically isolable (cf. Section 2.8.2), follows from

$$\begin{aligned} \hat{\mathbf{w}} &= 1 - \frac{1}{u(t)}\Big(2A\big(\frac{A}{c}\dot{y}_e(t) + \sqrt{y_e(t)}\big) \\ &\quad \times \big(\frac{A}{c}\ddot{y}_e(t) + \frac{\dot{y}_e(t)}{2\sqrt{y_e(t)}}\big) \\ &\quad + c\big(\frac{A}{c}\dot{y}_e(t) + \sqrt{y_e(t)}\big) - A\hat{\varpi}\Big) \end{aligned}$$

6.4 Numerical simulations

Figure 15 shows the closed-loop performance of our trajectory tracking controller. The simulation scenario is the following:

- The actuator fault $\mathbf{w} = 0.7$ occurs at time $t_I = 1.5s$.
- We estimate before the unknown constant perturbation $\varpi = 0.2$ and use it for estimating $\mathbf{w}$.
- The fault tolerant control becomes effective at time $t = 2.5s$.

Robustness is checked via an additive Gaussian white noise $N(0; 0.01)$. Comparison between Figures 16 and 15 confirms the efficiency of our fault accommodation.

[33] We are adapting here linear techniques stemming from Fliess & Sira-Ramírez (2003, 2007).

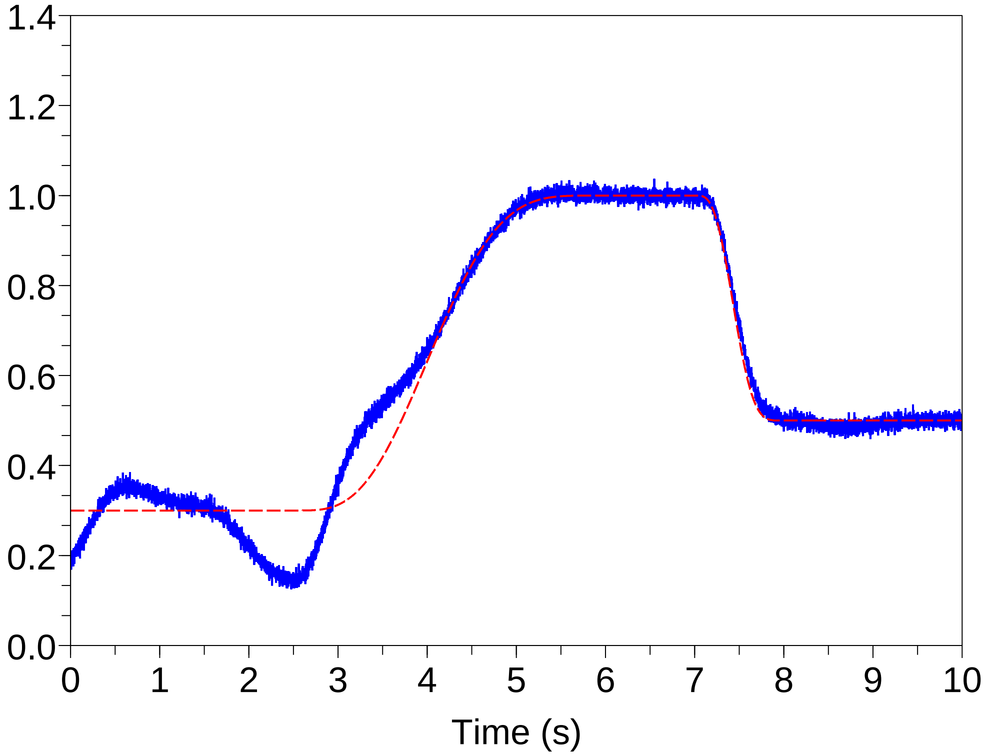

Figure 15: $y^\star(t)$ (- -) and $y(t)$ (–) with fault accommodation

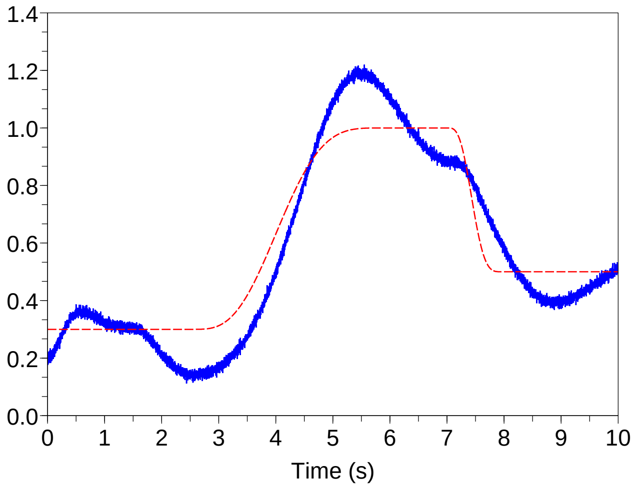

Figure 16: $y^\star(t)$ (- -) and $y(t)$ (–) without fault accommodation

7 Perturbation attenuation

7.0.1 Linear case

Suppose we are given a linear perturbed second order system

$$\ddot{y}(t) + y(t) = u(t) - z(t) + C\mathbf{1}(t - t_I) \qquad (21)$$

where

- $z(t)$ is an unknown perturbation input,
- $\mathbf{1}(t)$ is the Heaviside step function, i.e.,

$$\mathbf{1}(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ 1 & \text{if} \quad t \geq 0 \end{cases}$$

- C is an unknown constant and thus $C\mathbf{1}(t - t_I)$ is a constant bias, of unknown amplitude, starting at time $t_I \geq 0$.

Remark 7.1. *The difference $C\mathbf{1}(t - t_I) - z(t)$ is a rationally determinable variable according to Section 2.7.3.*

The estimate $z_e(t)$ of $z(t)$ is given up to a piecewise constant error by

$$z_e(t) = -\ddot{y}_e(t) - y_e(t) + u(t)$$

where $y_e(t)$ and $\ddot{y}_e(t)$ are the on-line estimated values of $y(t)$ and $\ddot{y}(t)$. We design a generalized-proportional-integral (GPI) regulator, in order to track asymptotically a given output reference trajectory $y^\star(t)$, i.e.,

$$u(t) = y_e(t) + z_e(t) + \ddot{y}^\star(t) + \mathcal{G} \star (y_e(t) - y^*(t)) \qquad (22)$$

where

- $\mathcal{G}$ is defined via its rational transfer function $\frac{c_2 s^2 + c_1 s + c_0}{s(s+c_3)}$

- $s^4 + c_3 s^3 + c_2 s^2 + c_1 s + c_0$ is the characteristic polynomial of the unperturbed closed-loop system. The coefficients c_0, c_1, c_2, c_3 are chosen so that the imaginary parts of its roots are strictly negative.

Like usual proportional-integral-derivative (PID) regulators, this controller is robust with respect to un-modeled piecewise constant errors

The computer simulations were performed with

$$z(t) = \frac{10t^3 \sin(2t)}{1 + t^2 + t^3}$$

The unknown constant perturbation suddenly appears at time $t_I = 4$ with a permanent value $C = 1.25$. The coefficients of the characteristic polynomial were forced to be those of the desired polynomial $P_d(s) = (s^2 + 2\zeta\omega_n s + \omega_n^2)^2$, with $\zeta = 0.81$, $\omega_n = 4$. We have set $y^\star(t) = \sin \omega t$, $\omega = 2.5$[rad/s].

Figure 17 (resp. 18) shows the reference signal $y^\star(t)$ and the output signal $y(t)$ without estimating $z_e(t)$ (resp. with the estimate $z_e(t)$). We added in the simulations of Figure 18 a Gaussian white noise $N(0; 0.025)$ to the measurement $y(t)$. The results are quite remarkable.

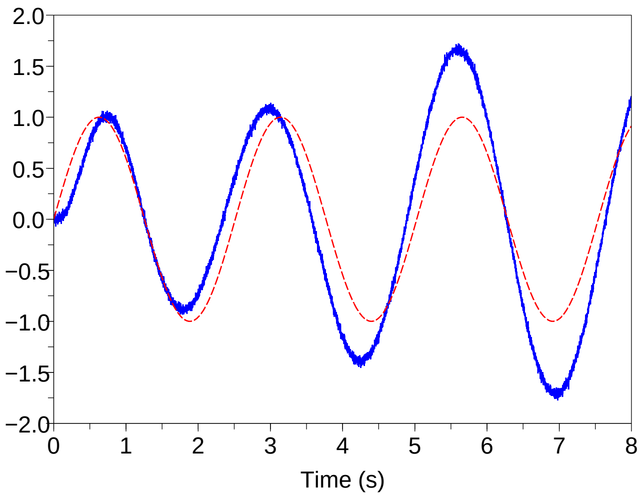

Figure 17: $y^\star(t)$ (- -) and $y(t)$ (–) without perturbation attenuation

Remark 7.2. *The same technique yields an efficient solution to fault tolerant linear control, which completes Fliess, Join & Sira-Ramírez (2004). Just think at $z(t)$ as a fault variable.*

7.0.2 Non-linear extension

Replace the term $y(t)$ in system (21) by the product $y(t)\dot{y}(t)$:

$$\ddot{y}(t) + y(t)\dot{y}(t) = u(t) - z(t) + C\mathbf{1}(t - t_I) \qquad (23)$$

The perturbations $z(t)$ and $C\mathbf{1}(t - t_I)$ are the same as above. The estimate $z_e(t)$ of $z(t)$ up to a piecewise constant is given by

$$z_e(t) = -\ddot{y}_e(t) - y_e\dot{y}_e(t) + u(t)$$

where $y_e(t)$, $\dot{y}_e(t)$ and $\ddot{y}_e(t)$ are the estimates of $y(t)$, $\dot{y}(t)$ and $\ddot{y}(t)$. The feedback law (22) becomes

$$u = y_e(t)\dot{y}_e(t) + z_e(t) + \ddot{y}^\star(t) + \mathcal{G} \star (y_e(t) - y^*(t)) \qquad (24)$$

Remark 7.3. *Rewrite system (23) via the following state variable representation*

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = -x_1(t)x_2(t) + u(t) - z(t) + C\mathbf{1}(t - t_I) \\ y(t) = x_1(t) \end{cases}$$

Applying the feedback law (24) amounts possessing good estimates of the two state variables.

Figures 19 and 20 depict the computer simulations with the same numerical conditions as before. The results are again excellent.

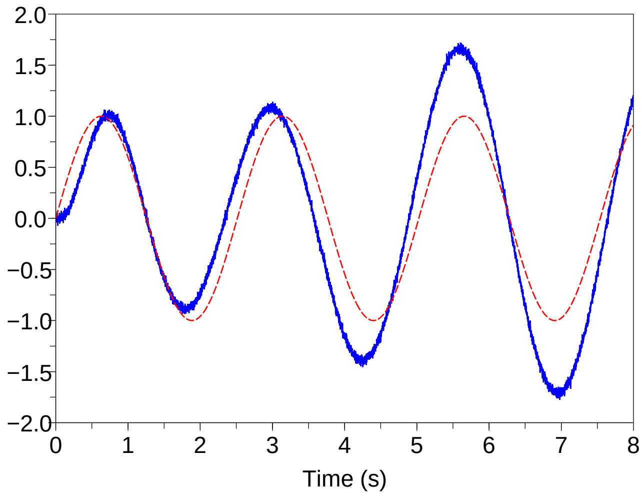

Figure 19: $y^\star(t)$ (- -) and $y(t)$ (–) without perturbation attenuation

8 Conclusion

We have proposed a new approach to non-linear estimation, which is not of asymptotic nature and does not necessitate any statistical knowledge of the corrupting noises[34]. Promising results have already been obtained, which will be supplemented in a near future by other theoretical advances (see, e.g., Barbot, Fliess & Floquet (2007) on observers with unknown inputs) and several concrete case-studies (see already García-Rodríguez & Sira-Ramírez (2005); Nöthen (2007)). Further numerical improvements

[34] Let us refer to a recent book by Smolin (2006), which contains an exciting description of the competition between various theories in today's physics. Similar studies do not seem to exist in control.

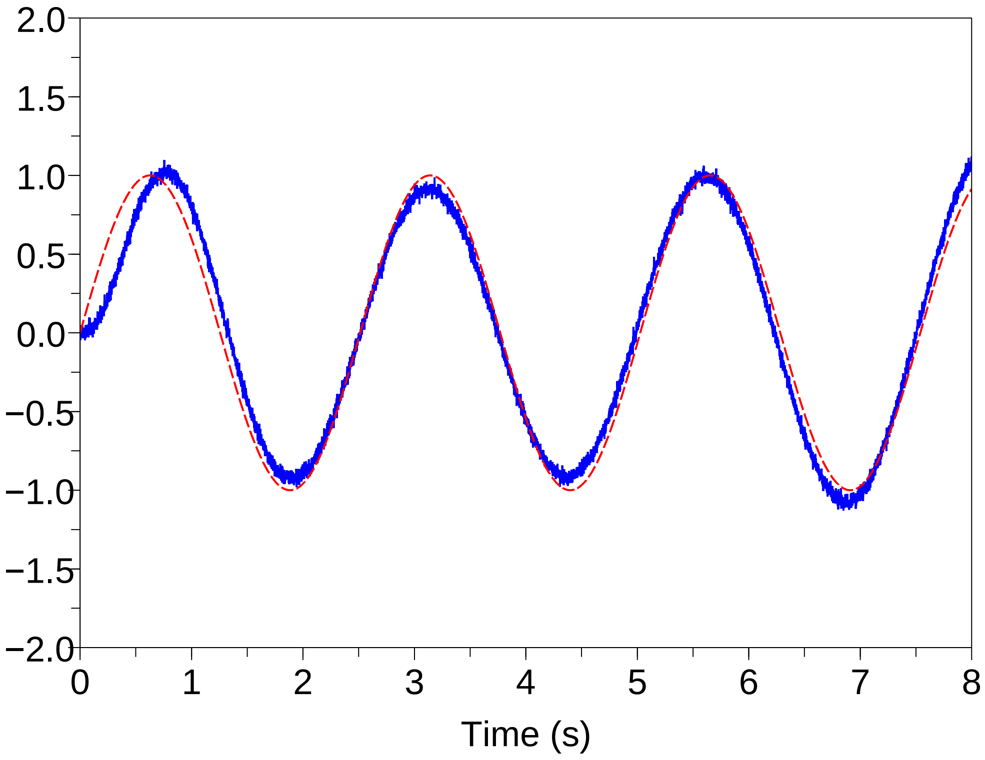

Figure 18: $y^\star(t)$ (- -) and $y(t)$ (–) with perturbation attenuation

will also be investigated (see already Mboup, Join & Fliess (2007)).

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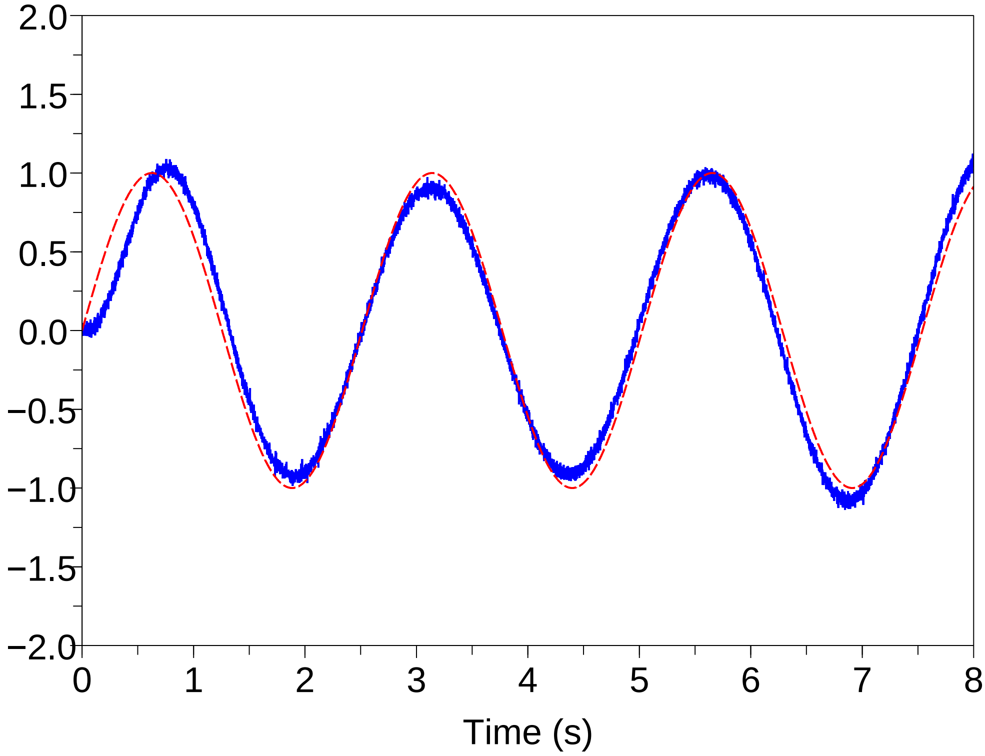

Figure 20: $y^{\star}(t)$ (- -) and $y(t)$ (–) with perturbation attenuation.

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