Grazing Analysis for Synchronization of chaotic hybrid systems

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Abstract

In this paper, a Grazing bifurcation analysis is proposed and a way to chaos is presented. Moreover, based on this analysis an observer design for the synchronization of chaotic hybrid system is given.

1 Introduction

The aim of this paper is to generate chaos for piecewise smooth systems which undergo a specific relied bifurcation named grazing bifurcation. The grazing phenomena and non-smooth bifurcations in dynamic systems are well studied at least since thirty years (Feigin, 1978). Mario di Bernardo and collaborators have contributed in the theory development regarding to the sliding bifurcations (di Bernardo & *al.*, 2000) and they have introduced successfully the relied Poincaré map (di Bernardo & *al.*, 2001). Independently from this, some authors have studied the existence of dynamic models's solutions near the approximate linear one (Benmerzouk & *al.*, 2004). In this paper, we mixed the two approaches in order to analyze the same problem but around a periodic solution with grazing behavior. From this, a way to chaos is highlighted by period-doubling solution. This analysis is based on the topological degree theory and will be apllied to synchronization of chaotic systems, note that from the well known paper of Nijmeijer and Mareels (Nijmeijer & *al.*, 1997), a synchronization of chaotic systems may be studied as an observer design problem. Thus our proposed analysis is finally used in order to design an observer for chaotic hybrid system. These illustrate the efficiency of the proposed approach and it's practical interest. The paper is organized as follow: In section II, the problem statement is established. The problem analysis is presented in section III and a way to chaos is proposed in section IV. Thank to results of previous sections, an observer analysis and design, with simulation results, are given in section V.

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2 Some recalls and problem statement

Let us consider the following piecewise smooth system:

$$\dot{x} = \begin{cases} F_1(x) \text{ if } H(x) \geq 0 \\ F_2(x) \text{ if } H(x) < 0 \end{cases} \tag{2.1}$$

where $x : I \longrightarrow D$, D is an open bounded connex domain of R^n with $n \geq 2$,(generally, as I is the time interval $I \subset R^+$).
And $F_1, F_2 : C_{abs}(I, D) \longrightarrow C(I, D)$ where $C(I, D)$ is the set of continuous functions defined on I and having values in R^n, the norm for $C(I, D)$ is defined as follows:
$x \in C(I, D) \qquad : \|x\| = \sup\limits_{t \in I} \|x(t)\|_n$
$\|.\|_n$ being a norm defined on R^n.

Remark 1 *Throughout the paper, for all definition, proposition and theorem, D is considered as an open bounded and connex domain even if it is not a necessary assumption for the considered purpose but this assumption is necessary for the global result.*

$C_{abs}(I, D)$ is the set of absolutely continuous functions defined on I and having values in D provided with the same norm as $C(I, R^n)$.
According to (Bresis, 1999), $(C(I, D), \|\|)$ and $(C_{abs}(I, D), \|\|)$ are Banach spaces.
H is a phase space boundary between regions of smooth dynamics, H defines the set:
$S = \{x(t) \in D : H(x(t)) = 0\}$ witch is termed the switching manifold.
S divide the phase space into two regions:
$S^+ = \{x(t) \in D : H(x(t)) \geq 0\}$
$S^- = \{x(t) \in D : H(x(t)) < 0\}$
Both vector fields F_1 and F_2 are defined on both sides of S, the flows $\Phi_i, i = 1, 2$ generated by each vector field are defined as the operators that satisfy:
$\frac{\partial \Phi_i(x,t)}{\partial t} = F_i(\Phi_i(x, t))$ and $\Phi_i(x, 0) = x, i = 1, 2$. In (di Bernardo & *al.*, 2001), di Bernardo and coauthors chow that a grazing (denoted grazing point) occurs at $x = 0$ if the following conditions are satisfied:
A-1) $H(0) = 0$.
A-2) $\nabla H(0) \neq 0$.
A-3) for $i = 1, 2$.
$< \nabla H(0), \frac{\partial \Phi_i}{\partial t}(0, 0) >=< \nabla H(0), F_i^0 > = 0,$
A-4) for $i = 1, 2$.
$\frac{\partial^2 H(\Phi(0,0))}{\partial t^2} = (< \nabla H(0), \frac{\partial F_i^0}{\partial x} F_i^0 >$
$\qquad + < \frac{\partial^2 H(\Phi(0,0))}{\partial x^2} F_i^0, F_i^0 >) > 0,$
A-5) $(< L, F_1^0 >< L, F_2^0 >) > 0$.
where $F_i^0 = F_i(\Phi_i(0, 0)), i = 1, 2$, L is the unit vector perpendicular to $\nabla H(0)$ and $< ., . >$ is a usual scalar product on R^n.
The first two conditions state that H well defines the switching manifold.
Condition A-3) states that grazing takes place at the origin of $x = 0$ i.e. the vector field is tangent to S.
Condition A-4) ensures that the curvature of the trajectories in $S^+$and S^-

has the same sign with respect to H (and without loss of generality, this sign is assumed to be positive).
Condition A-5) ensures that in a sufficiently small neighborhood of the grazing point, there is no sliding behavior.
Now, the system (2.1) is assumed to depend smoothly explicitly or implicitly on a parameter μ. Moreover, at $\mu = 0$ and $x(0) = 0$, there is a periodic orbit $x(t)$ that grazes at the point $x(0)$. The solution is hyperbolic and hence isolated such that there is no points of grazing along the orbit other than $x(.) = 0$. As the previous conditions are defined on an open set then there exist two sufficiently small neighborhoods of $\mu = 0$ and $x(t)$ such that A-i)$_{(1,2,3,4,5)}$ are verified.
If the vector field is continuous at grazing i.e. $F_1^0 = F_2^0 = F$ but has discontinuous first derivatives, then the Poincaré map is given by (di Bernardo & *al.*, 2001):

$$P(x,\mu) = \begin{cases} P_1(x,\mu) & \text{if} < \nabla H, x > \quad > 0 \\ P_2(x,\mu) & \text{if} < \nabla H, x > \quad < 0 \end{cases}$$

Where:

$$\begin{aligned} P_1(x,\mu) &= Nx + M\mu + o(\|x\|_{n-1}, \mu) \\ P_2(x,\mu) &= N(x + v_1| < \nabla H, x > |)^{\frac{3}{2}} \\ &+ v_3(< \nabla H, \frac{\partial F_2}{\partial x} x >)(| < \nabla H, x > |)^{\frac{1}{2}} \\ &+ v_2 x(| < \nabla H, x > |)^{\frac{1}{2}} + M\mu + o(\|x\|_{n-1}^2, \mu)) \end{aligned}$$

N is a nonsingular matrix $(n-1) \times (n-1)$, M is a nonzero $n-1$ dimensional vector.

$$\begin{aligned} v_1 &= \frac{2\sqrt{2}}{(< \nabla H, \frac{\partial F_1}{\partial x} F >)^{\frac{3}{2}}} (\frac{1}{3}(\frac{\partial^2 F_2}{\partial x^2} - \frac{\partial^2 F_1}{\partial x^2})F^2 \\ &+ \frac{\partial F_2}{\partial x}\frac{\partial F_1}{\partial x}F - \frac{1}{3}((\frac{\partial F_1}{\partial x})^2 + 2(\frac{\partial F_2}{\partial x})^2)F \\ &- \frac{1}{\sqrt{< \nabla H, \frac{\partial F_2}{\partial x} F >}}(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}) \\ &F(\frac{1}{3} < \nabla H, (\frac{\partial^2 F_2}{\partial x^2})F^2 > + < \nabla H, (\frac{\partial F_2}{\partial x}\frac{\partial F_1}{\partial x})F > \\ &- \frac{2}{3} < \nabla H, (\frac{\partial^2 F_2}{\partial x^2})^2 F >))) \\ v_2 &= \frac{2\sqrt{2}}{\sqrt{< \nabla H, \frac{\partial F_1}{\partial x} F >}}(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}) \\ v_3 &= \frac{2\sqrt{2}}{\sqrt{< \nabla H, \frac{\partial F_1}{\partial x} F >< \nabla H, \frac{\partial F_2}{\partial x} F >}}(\frac{\partial F_2}{\partial x}F - \frac{\partial F_1}{\partial x}F) \end{aligned}$$

and $o(\alpha,\mu) \longrightarrow 0$ when $(\alpha,\mu) \longrightarrow (0,0)$.
For the sake of compactness, $\frac{\partial F_1}{\partial x}$, $\frac{\partial F_2}{\partial x}$ and ∇H stand respectively for $\frac{\partial F_1}{\partial x}(0)$,

$\frac{\partial F_2}{\partial x}(0)$ and $\nabla H(0)$.
In order to avoid the $\frac{3}{2}$ type singularity in the Poincaré map, the topological degree theory is considered (Katok & *al.*, 1997). Therefore, the main definitions and results are recalled in this context:
Considering $C^1(I,D)$ the set of C^1 functions defined on a domain I having values on a domain D of R^n, the norm for $C^1(I,D)$ is defined as follows:

$$x \in C^1(I,D) \quad : \|x\|_1 = \sup_{t\in I} \|x(t)\|_n + \sup_{t\in I} \|\dot{x}(t)\|_n$$

$(C^1(I,D), \|.\|_{C^1})$ is a Banach space (Bresis, 1999).

Definition 1 *(Katok* & al.*, 1997) Let $\beta \in C^1(D, R^n)$, D the domain of definition of β, $\bar{D}$ the corresponding closed domain, $Z_\beta = \left\{x \in D : \frac{\partial \beta(x)}{\partial x} = 0\right\}$ and consider a point $p \in R^n - \beta(\partial D \cup Z_\beta)$, where ∂D is the borderline of D. The topological degree of β at point p defined on D is given by:*

$$deg(\beta, p, D) := \sum_{x\in D \& \beta(x)=p} (sign\frac{\partial \beta(x)}{\partial x})$$

The following result permits to construct a topological degree for a continuous function:

Theorem 1 *(Katok* & al.*, 1997) Let $h \in C(D)$, $p \in R^n - h(\partial D)$ and the distance*

$$r = \frac{1}{2}\inf_{x\in D}(\|p - h(x)\|_n) > 0,$$

then there exists a function $g \in C^1(D)$: $\|g - h\| < r$ such that

$$deg(\beta, p, D) = deg(g, p, D).$$

The following theorem allows to compute the topological degree of some function instead of another one (generally more complicated).

Theorem 2 *(Poincaré-Bohl) (Katok* & al.*, 1997) Let β and g two continuous functions defined on $\bar{D}$ such that the following assumption is satisfied:*
A-6) For every $x \in \partial D$: the line joining $\beta(x)$ to $g(x)$ (i.e. $y = (1-\lambda)\beta(x) + \lambda g(x)$, $\lambda \in (0,1)$) does not contain the origin, then:

$$deg(\beta, p, D) = deg(g, p, D)$$

Remark 2 *Assumption A-6) is verified if $\beta^T(x)g(x) > 0$ for every $x \in \partial D$.*

Now, the importance of this notion is highlighted in the following proposition:

Proposition 1 *(Katok* & al.*, 1997) Let D be open, bounded and connex domain, β is $C^1(D, R^n)$, $Z_\beta = \varnothing$ and $p \in R^n - \beta(\partial D)$, then the number of equation's solutions $\beta(x) = p$ in D is equal to $|deg(\beta, p, D)|$.*

3 Problem analysis

Recall that a fixed point of $P(., \mu)$ represents an mT periodic orbit, where m is a natural number and T is the period. Therefore searching a solution of (2.1) is equivalent to resolve the following equation:

$$P(x, \mu) = x \tag{3.2}$$

Two possibilities appear:
First case: $<\nabla H, x> > 0$
As, the flow evolves entirely in S^+, the solution's analyze only considers the first equation of the system (2.1) and the problem becomes to analyze the equation given by:

$$\gamma(x, \mu) = Nx + M\mu + o(||x||_{n-1}, \mu) - x = 0 \tag{3.3}$$

If the following assumption is satisfied:
A-7) $(N - I)$ is nonsingular, where I is the identity matrix.
The Implicit functions theorem gives:

PROPOSITION 2 *Under assumptions A-i)$_{(i=1,2,3,4,5,7)}$, there exist a neighborhood $V_{\mu=0}$ in R, a neighborhood $V_{x=0}$ in R^{n-1} and an unique application x^* :$V_{\mu=0} \longrightarrow V_{x=0}$ solution of $\gamma(x^*(\mu), \mu) = 0$ such that $x^*(0) = 0$. Furthermore, the periodic solution of (2.1) x^* depends continuously on μ.*

Second case: $< \nabla H, x > < 0$.
The problem of finding a periodic solution of problem (2.1) is equivalent to analyze the following equation:

$$\beta(x, \mu) = P_2(x, \mu) - x = 0 \tag{3.4}$$

where

$$\begin{aligned} P_2(x, \mu) = N(x + v_1(\delta(x))^{\frac{3}{2}} + v_2 x(\delta(x))^{\frac{1}{2}} \\ +v_3(< \nabla H, \tfrac{\partial F_2}{\partial x} x >)(\delta(x))^{\frac{1}{2}}) \\ +M\mu + o(\|x\|^2_{n-1}, \mu) \end{aligned} \tag{3.5}$$

And $\delta(x) = - < \nabla H, x >$.
For continuity reason $\beta(x, \mu)$ is considered equal to 0 for all x such that $< \nabla H, x > \geq 0$.
In order to approach β by a quadratic function having the same solutions number as β, the following "alternative" function $\tilde{\beta}$ is considered:

$$\begin{aligned} \tilde{\beta}(x, \mu) \quad = \quad & Nx + M\mu - x + (\delta(x))^{\frac{1}{2}}(N(v_1(\delta(x))^{\frac{3}{2}} \\ & +v_2 x(\delta(x))^{\frac{1}{2}} + v_3(< \nabla H, \tfrac{\partial F_2}{\partial x} x >)(\delta(x))^{\frac{1}{2}})) \\ & +o(\|x\|^2_{n-1}, \mu) \end{aligned} \tag{3.6}$$

The linear part of $\tilde{\beta}$ is:
$v(x, \mu) = Nx + M\mu - x$
and the quadratic part is:
$w(x, \mu) = N(v_1(\delta(x))^2 + v_2 x(\delta(x)) + v_3(< \nabla H, \frac{\partial F_2}{\partial x} x >)(\delta(x))) + o(\|x\|^2, \mu)$

So, from the theorem 2, if the line joining $\beta(x,\mu)$ to $\tilde{\beta}(x,\mu)$ does not contain 0 i.e. if the following assumption is satisfied:
A-8) $v^T(x,\mu)w(x,\mu) > 0$ for all $x \in \partial D$ and any real μ in the neighborhood of zero, then:

$$deg(\beta, 0, D) = deg(\tilde{\beta}, 0, D)$$

where $deg(\beta, 0, D)$ and $deg(\tilde{\beta}, 0, D)$ are respectively the degree of the application β and $\tilde{\beta}$ defined on D at point 0.
Thus the problem to analyze becomes:

$$\tilde{\beta}(x,\mu) = 0 \tag{3.7}$$

The Implicit functions theorem gives:

Proposition 3 *Under assumptions A-i) for $i = 1, 2, 3, 4, 5, 7, 8$), there exist a neighborhood $v_{\mu=0}$ in R, a neighborhood $v_{x=0}$ in R^{n-1} and an unique application x^{**}: $V_{\mu=0} \longrightarrow V_{x=0}$ solution of $\tilde{\beta}(x^{**}(\mu),\mu) = 0$ such that $x^{**}(0) = 0$.*

4 Way to Chaos

Roughly speaking, a chaotic system is characterize by two properties, the first one is a great sensitivity with respect to the initial conditions. This implies that a long term predictions are almost impossible despite the deterministic nature of the system. The second property is "the strange" structure of its attractor. Moreover, according to (Glendinning, 1994) or (Wiggins, 1990), one way to characterized the chaotic behavior of the system (2.1) (and so for the corresponding equation (3.7)) is to determine three distinct points x, y and z such that: $\tilde{\beta}(x,\mu) = y$, $\tilde{\beta}(y,\mu) = z$ and $\tilde{\beta}(z,\mu) = x$. This well be done in three steeps.
First step: analyze of the equation:

$$\tilde{\beta}(x,\mu) = y = x + \eta_1 \tag{4.8}$$

where for a sake of simplicity, η_1 stands for a vector defined in R^{n-1}, having only one component equal to some fixed value(noted also η_1) and the others components are nulls.
Thus, the equation (4.8) is equivalent to:

$$\Psi_1(x,\mu,\eta_1) = \tilde{\beta}(x,\mu) - x - \eta_1 = 0 \tag{4.9}$$

And the next proposition is obtained with the same arguments that the previous one:

Proposition 4 *Under assumptions A-i)for $i = 1, 2, 3, 4, 5, 7, 8$), there exist a neighborhood $v_{\mu=0}$ in R, a neighborhood $v_{\eta_1=0}$ in R , a neighborhood $v_{x=0}$ in R^{n-1} and an unique application x^{***}: $v_{\mu=0} \times v_{\eta_1=0} \longrightarrow v_{x=0}$ solution of $\Psi_1(x^{***}(\mu,\eta_1),\mu,\eta_1) = 0$ such that $x^{***}(0,0) = 0$.*

Second step: analyze of the equation:

$$\tilde{\beta}(\tilde{\beta}(x,\mu),\mu) = z = y + \eta_2 = x^{***}(\mu,\eta_1) + \eta_1 + \eta_2 \tag{4.10}$$

where η_2 stands for a vector defined on R^{n-1}, having only one component equal to some fixed value(noted also η_2) and the others are nulls. Thus, equation (4.10) is equivalent to:

$$\begin{aligned} \Psi_2(\mu,\eta_1,\eta_2) &= \tilde{\beta}(x^{***}(\mu,\eta_1) + \eta_1,\mu) \\ &- x^{***}(\mu,\eta_1) - \eta_1 - \eta_2 \\ \Psi_2(\mu,\eta_1,\eta_2) &= 0 \end{aligned} \tag{4.11}$$

The following assumption:
A-9) $\frac{\partial \Psi_2}{\partial \eta_1}(0,0,0) \neq 0$.
is necessary to obtain:

PROPOSITION 5 *Under assumptions A-i) for i = (1, 2, 3, 4, 5, 7, 8, 9), there exist a neighborhood* $\nu_{\mu=0}$ *in* $\upsilon_{\mu=0}$, *a neighborhood* $\nu_{\eta_1=0}$ *in* $\upsilon_{\eta_1=0}$, *a neighborhood* $\nu_{\eta_2=0}$ *in* R *and an unique application* η_1^*:$\nu_{\mu=0} \times \upsilon_{\eta_2=0} \longrightarrow \nu_{\eta_1=0}$ *solution of* $\Psi_2(\eta_1^*(\mu,\eta_2),\mu,\eta_2) = 0$ *such that* $\eta_1^*(0,0) = 0$.

Third step: analyze of the equation:

$$\tilde{\beta}(\tilde{\beta}(\tilde{\beta}(x,\mu),\mu),\mu) = x \tag{4.12}$$

The equation (4.12) is equivalent to:

$$\begin{aligned} \Psi_3(\mu,\eta_2) = \; & \tilde{\beta}(x^{***}(\mu,\eta_1^*(\mu,\eta_2)) + \eta_1^*(\mu,\eta_2) + \eta_2),\mu) \\ & -x^{***}(\mu,\eta_1^*(\mu,\eta_2)) = 0 \end{aligned} \tag{4.13}$$

and the following assumption :
A-10) $\frac{\partial \Psi_3}{\partial \eta_2}(0,0) \neq 0$.
is necessary to have:

PROPOSITION 6 *Under assumptions A-i) for (i = 1, 2, 3, 4, 5, 7, 8, 9, 10), there exist a neighborhood* $\theta_{\mu=0}$ *in* $\nu_{\mu=0}$, *a neighborhood* $\theta_{\eta_2=0}$ *in* $\nu_{\eta_2=0}$ *and an unique application* η_2^*: $\theta_{\mu=0} \longrightarrow \theta_{\eta_2=0}$ *solution of* $\Psi_3(\mu,\eta_2^*(\mu)) = 0$ *such that* $\eta_2^*(0) = 0$.

The next corollary sums up the previous results:

COROLLARY 1 *Under assumptions A-i) for (i= 1, 2, 3, 4, 5, 7, 8, 9, 10), the system (2.1) admits a chaotic solution.*

The aim of the proposed approach is to generate a chaotic system in order to propose a type of emitter which is chaotic and hybrid. The number of assumptions may appear too important but the first five ones are the standard assumptions for the grazing. Now, in order to highlight the interest of the proposed method, an analyze of the way to chaos and approximated observer for synchronization goal are realized on an example.

5 Observer analysis and design:

Let us consider the following system defined in $D \subset R^3$:

$$\dot{x} = \begin{cases} F_1(x,\alpha,\varepsilon) \text{ if } x_2 \leq 0 \\ F_2(x,\alpha,\varepsilon) \text{ if } \ x_2 > 0 \end{cases} \tag{5.14}$$

where $x = (x_1, x_2, x_3)^T$ and α, ε are real parameters. Moreover, $F_1(x,\alpha,\varepsilon)$ is the vector field such that:

$$\begin{cases} F_{11}(x,\alpha,\varepsilon) &= & -(x_2+1) + \varepsilon((x_1 + 3\sin x_3)^2 \\ & & +(x_2+1)^2 - (1+\alpha))(-x_1 - 3\sin x_3) \\ F_{12}(x,\alpha,\varepsilon) &= & (x_1 + 3\sin x_3) + \varepsilon((x_1 + 3\sin x_3)^2 \\ & & +(x_2+1)^2 - (1+\alpha))(-x_2 - 1)) \\ F_{13}(x,\alpha,\varepsilon) &= & 0 \end{cases} \tag{5.15}$$

And $F_2(x,\alpha,\varepsilon)$ is the vector field such that:

$$\begin{cases} F_{21}(x,\alpha,\varepsilon) &= & -(x_2+1) + \varepsilon((x_1 + 3\sin x_3)^2 \\ & & +(x_2+1)^2 - (1+\alpha))(-x_1 - 3\sin x_3) \\ F_{22}(x,\alpha,\varepsilon) &= & (x_1 + 3\sin x_3) + \varepsilon((x_1 + 3\sin x_3)^2 \\ & & +(x_2+1)^2 - (1+\alpha))(-x_2 - 1)) \\ F_{23}(x,\alpha,\varepsilon) &= & -9x_2 + \varepsilon((x_1 + 3\sin x_3)^2 \\ & & +(x_2+1)^2 - (1+\alpha))(-x_1 - 3\sin x_3) \\ & & -\varepsilon(x_3 - 9\cos\omega t - x_1) \end{cases} \tag{5.16}$$

where $S = \{x \in R^3 : x_2 = 0\}$, $S^+ = \{x \in R^3 : x_2 \geq 0\}$ and $S^- = \{x \in R^3 : x_2 < 0\}$. In the next $\alpha = 0.1$ and $\omega 1.0001$, moreover *sinwt* came from a bounded oscillatory and thus all states are bounded.

Remark 3 *In order to guaranty that the behavior of (5.14) stays on bounded domain the case $\varepsilon < 0$ is not considered.*

The associated Poincaré map, defined in $D \bigcap \mathbb{R}^2$ (i.e. corresponding to $x_1 = 0$), is:

$$P(x_2, x_3, \mu, \varepsilon) = \begin{cases} (P_2, P_3)^T = & ((e^{-4\varepsilon\pi} x_2, x_3)^T \\ & -(\frac{1}{2}(1 - e^{-4\varepsilon\pi})\mu, 0)^T \\ & +o(\|x\|, \mu)) \textbf{ if} (-x_2 \geq 0) \\ (P_2, P_3)^T = & ((e^{-4\varepsilon\pi} x_2, x_3)^T \\ & -(\frac{1}{2}(1 - e^{-4\varepsilon\pi})\mu, 0)^T \\ & +(4\varepsilon x_2^{\frac{3}{2}}, (-\frac{8}{9}\varepsilon + 8\varepsilon^2) x_2^{\frac{3}{2}} \\ & -2x_3 \varepsilon x_2^{\frac{1}{2}} - 2\varepsilon x_2^{\frac{1}{2}})^T \\ & +o(\|x\|^2, \mu)) \textbf{ if} (-x_2 < 0) \end{cases}$$

So, finding a periodic solution of (5.14) is equivalent to analyze:

$$P(x, \mu, \varepsilon) = (x_2, x_3)^T \tag{5.17}$$

The equation (5.17) is equivalent to:

$$\beta(x, \mu, \varepsilon) := P(x, \mu, \varepsilon) - (X_2, x_3)^T = 0 \tag{5.18}$$

The corresponding "alternative" function is:

$$\tilde{\beta}(x,\mu,\varepsilon) = \left(\tilde{\beta}_1(x,\mu,\varepsilon), \tilde{\beta}_2(x,\mu,\varepsilon)\right)^{\mathsf{T}} \tag{5.19}$$

Where, the approximations at order $o(\|x\|^2, \mu)$ are

$$\begin{aligned} \tilde{\beta}_1(x,\mu,\varepsilon) &= e^{-4\varepsilon\pi}x_2 - \frac{1}{2}(1-e^{-4\varepsilon\pi})\mu + 4\varepsilon x_2^2 - x_2 \\ \tilde{\beta}_2(x,\mu,\varepsilon) &= (-\frac{8}{9}\varepsilon + 8\varepsilon^2)x_2^2 - 2x_3\varepsilon x_2 - 2\varepsilon x_2 - x_3 \end{aligned}$$

Following the proposed way to chaos, the assumptions A$-i)_{(i=1,2,3,4,5,7,8,9,10)}$ are satisfied for $x_2 > \frac{2\pi\varepsilon\mu}{1-4\varepsilon\pi}$, any x_3 in the neighborhood of 0 and μ in the neighborhood of 0 (for the simulation the staring state are $x_1 = x_3 = 0$, $x_2 = 0.5$, and $\varepsilon = 0.01$), then a chaotic behavior appears in the 3 dimension representation (see figure 1). Now, in order to design an observer we firstly consider the

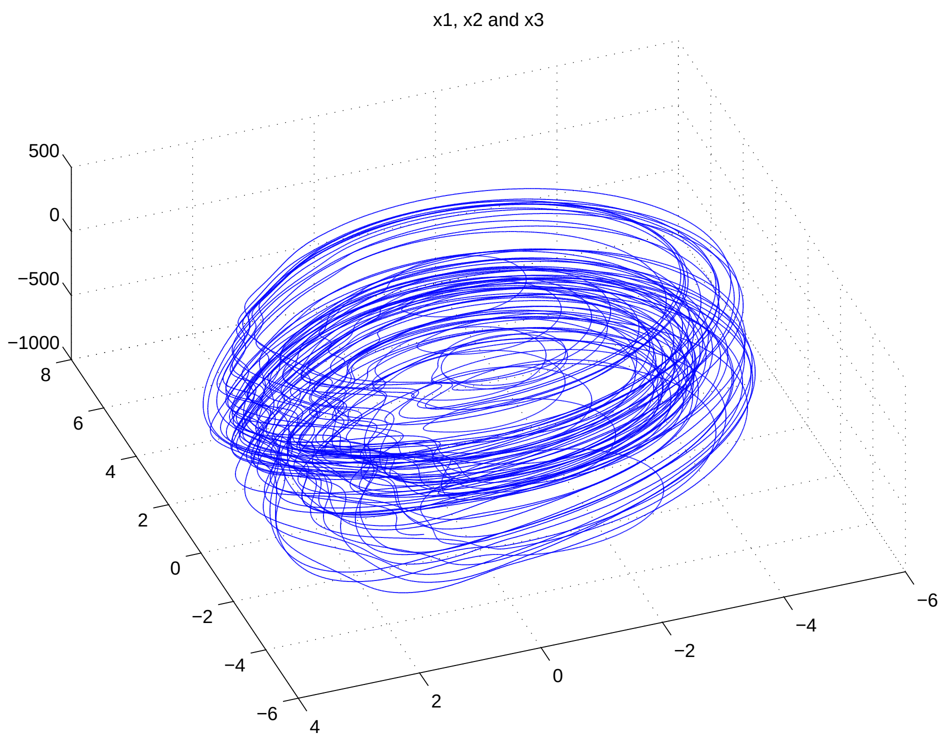

Figure 1: Strange attractor in 3D

unperturbed subsystems associated to (5.14) (i.e. those corresponding to $\varepsilon = 0$ in (5.15) and (5.16)) and given by:

$$\dot{x} = f_1(x) \text{ if } x_2 \leq 0 \tag{5.20}$$

where $x = (x_1, x_2, x_3)^T$ and $f_1(x) = (f_{11}(x), f_{12}(x), f_{13}(x))^T$ is the vector field such that:

$$\left\{ \begin{array}{rcl} f_{11}(x) &=& -(x_2+1) \\ f_{12}(x) &=& x_1 + 3\sin x_3 \\ f_{13}(x) &=& 0 \end{array} \right. \tag{5.21}$$

$$\dot{x} = f_2(x) \text{ if } x_2 > 0 \tag{5.22}$$

And $f_2(x) = (f_{21}(x), f_{22}(x), f_{23}(x))^T$ is the vector field such that:

$$\left\{ \begin{array}{lcl} f_{21}(x) & = & -(x_2+1) \\ f_{22}(x) & = & (x_1 + 3\sin x_3) \\ f_{23}(x) & = & -9\sin\omega t - (x_2+1) \end{array} \right. \tag{5.23}$$

Both the subsystems (5.20) and (5.22) are supposed to have the same output $h(x) = x_1$. Simple computations show that $\dim(\text{span}(dh(0), dL_{f_i}h(0), dL_{f_i}^2 h(0))$ is equal to 3, for $i = 1, 2$, thus from (Isidori, 1999), the observer linearization problem for these subsystems is solvable in the neighborhood v_0 of $x = 0$, furthermore, it is possible to define in v_0, vectors $g_i(x)$, for $i = 1, 2$, witch satisfy:
$L_{g_i}h(x) = L_{g_i}L_{f_i}h(x) = 0$ and $L_{g_i}L_{f_i}^2 h(x) = 1$, for all x in v_0 and $i = 1, 2$.
The choice of the subsystems and also the output permits to obtain the same vector solution $g(x) = \left(0, 0, \frac{-1}{\cos x_3}\right)^{\mathsf{T}}$
Consequently, both subsystems (5.20) and (5.22) have relative degree r at point 0 equal to 3 and thus there exist a diffeomorphism ϕ in the neighborhood v_0 of 0 and new coordinates $z_j = \phi_j(x)$ for $j = 1, 2, 3$ such that they are equivalent to the following system:

$$\left\{ \begin{array}{l} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = b(z) + a(z)u \end{array} \right. \tag{5.24}$$

Where: $a(z) = L_g L_{f_1}^2 h(x(t)), b(z) = L_{f_1}^3 h(x(t))$, u is some real control and $y = z_1$ is the output.

REMARK 4 *In the neighborhood v_0 of 0, the continuous function $a(z)$ is nonzero and thus considering the following feedback control law: $u = \frac{1}{a(z)}(-b(z) + v)$, the system (5.24) is equivalent to:*

$$\left\{ \begin{array}{l} \dot{z}_1 = z_2 \\ \dot{z}_2 = z_3 \\ \dot{z}_3 = v \end{array} \right. \tag{5.25}$$

It is important here to point out that thank to the condition on the relative degree $r = 3$ for all subsystems, the control law which linearizes the system in a canonical form does not depend on the switching sequence. This may be very useful for stabilization purpose and find a Common Control Lyapunov function CCLF (Moulay & al., 2007) which is an extension to Hybrid system of the well-known CLF (Lin & al., 1991).

As the system (5.24) is observable, now, from the work of (Boutat & *al.*, 2004), it is possible to give conditions in order to recover also the discrete states and the unknown input. This analysis and observers design will be presented in more details in forcoming paper and this on the basis of the work presented in (Barbot & *al.*, 2006). Nevertheless, hereafter, in order to highlight the interest of our approach and in the same time the robustness of high gain observer design, such kind of observer is designed on the basis of system (5.24), where the last row is supposed bounded but unknown:

$$\begin{cases} \dot{\hat{z}}_1 = \hat{z}_2 + \frac{1}{\varepsilon}(\hat{z}_1 - z_1) \\ \dot{\hat{z}}_2 = \hat{z}_3 + \frac{100}{\varepsilon}(\hat{z}_1 - z_1) \\ \dot{\hat{z}}_3 = \frac{10000}{\varepsilon}(\hat{z}_1 - z_1) \end{cases} \tag{5.26}$$

The simulation results are correct. In figure 2-a and 3-a the state x_2 is shown and so the switching condition clearly appears. After that the states in the space of the canonical form are shown z_1 and $\hat{z}_1$ in figure 2-b, z_2 and $\hat{z}_2$ in figure 2-c, z_3 and $\hat{z}_3$ in figure 2-d in the same way in figure 3 the observation errors are given $e_1 = z_1 - \hat{z}_1$ in figure 3-b, $e_2 = z_2 - \hat{z}_2$ in figure 3-c and finally $e_3 = z_3 - \hat{z}_3$ in figure 3-d.

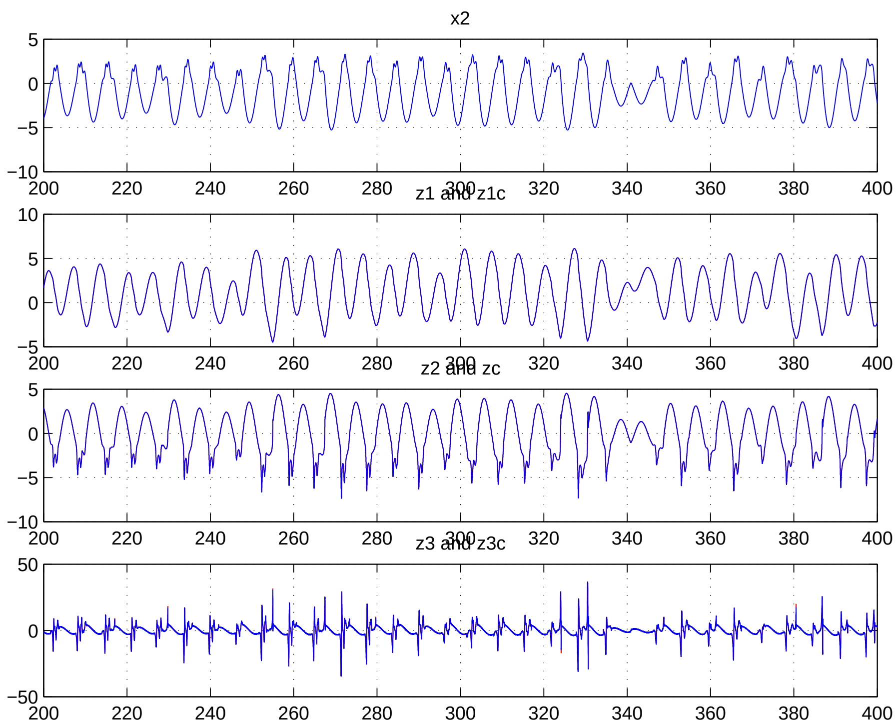

Figure 2: a-x_2, b-z_1 $\hat{z}_1$, c-z_2 $\hat{z}_2$, d-z_3 $\hat{z}_3$

6 CONCLUSION

The proposed method reclaim the resolution of general Poincaré map, this problem may be very difficult without grazing and more tedious with grazing, but as it is shown in the example, this computations may be simplified using some symmetric matrices and usual methods as polar transformations. On another hand, we think that our grazing analysis may be also applied to other type of borderline collisions. Nevertheless, we start our study by grazing phenomena behavior because it is a natural prolongation of the classical smooth ODE

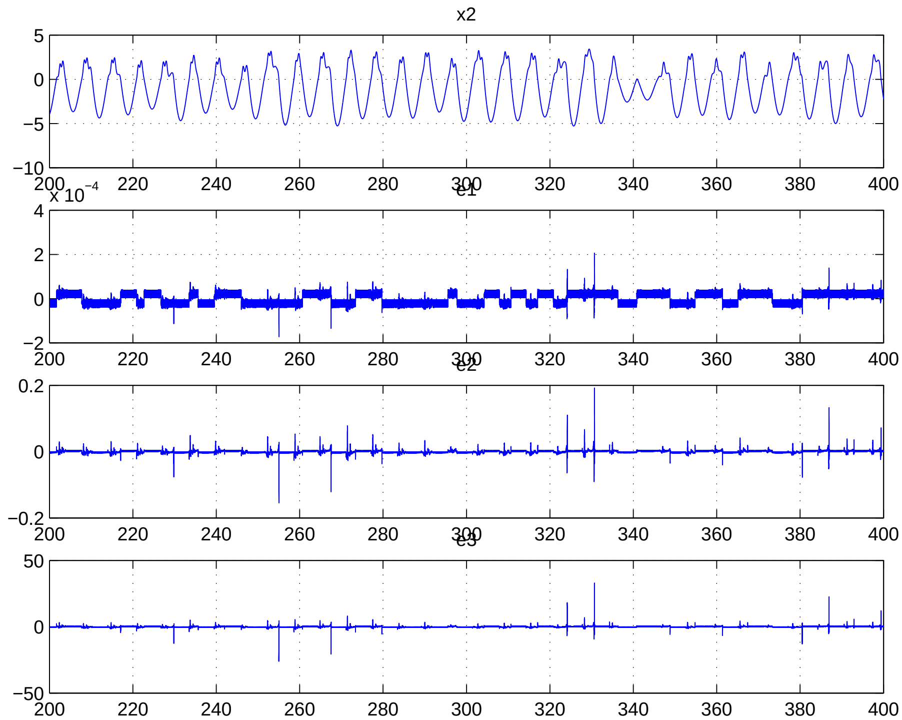

Figure 3: a-x_2, b-$z_1 - \hat{z}_1$, c-$z_2 - \hat{z}_2$, d-$z_3 - \hat{z}_3$

and as it is developed in the numerical example, this approach will be very interesting in chaotic systems synchronization using piecewise smooth systems, it allows particularly to double the crypts security of the concerned circuit, a more detailed work will be proposed in a future work.

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