A distribution framework for the fast identification of linear systems with delays

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A DISTRIBUTION FRAMEWORK FOR THE FAST IDENTIFICATION OF LINEAR SYSTEMS WITH DELAYS

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Abstract: This paper deals with on line identification of delay systems. The work is based on the approach initiated in (Fliess M., 2003) and extended to delays identification. Copyright (C) 2006 IFAC

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1. INTRODUCTION

The delay phenomenon constitutes one of the major complexity components of networked control, since actuators, sensors, computers, field networks and wireless communications that are involved in feedback loops unavoidably introduce variable deadtimes. Despite numerous advances in this field, delay remains a theoretical and practical challenge (see, e.g. the survey (Richard, 2003)). Among these open problems, the online delay identification has a particular part: On the one hand, the delay knowledge can benefit many of the control techniques suited for such systems (predictors, finite spectrum assignments, observers...). On the other hand, the existing identification techniques for time delay systems (TDS) (see (Ren *et al.*, 2005) for a modified least squares technique and a survey in (Belkoura L., 2004)) generally suffer from poor speed performance. The authors regard this paper as a first step to the TDS-adaptation of the fast identification techniques that were recently proposed in (Fliess M., 2003) in the framework of nonlinear, finite-dimensional models. Let us recall that those techniques are not asymptotic, and do not need any statistical knowledge of the noises corrupting the data. (See, e.g., (Fliess and Sira-Ramirez, 2004) for applications to nonlinear state estimation, (Fliess *et al.*, 2004), (Fliess *et al.*, 2005) for linear and nonlinear diagnosis, and (Fliess *et al.*, 2003) for signal processing.

We adopt in this paper a distributional formulation from which the parameters as well as the input delays can be easily identified. The plant is assumed to be linear with respect to its parameters, but not necessarily with respect to its inputs and outputs. The identification procedure mainly consists in the following three steps: (1) Differentiations, (2) Multiplication with an appropriate $\mathcal{C}^\infty$ function and (3) Integrations.

We first recall here some standard definitions and results from distribution theory and fix the notations we shall use in the sequel. Let Ω be an open subset of $\mathbb{R}$. The space of C^∞ functions having compact support in Ω is denoted by $\mathcal{D}(\Omega)$, and $\mathcal{D}'(\Omega)$ is the space of distributions on Ω, *i.e.* the space of continuous linear forms on $\mathcal{D}(\Omega)$.

The complement of the largest open subset of Ω in which a distribution T vanishes is called the support of T and is denoted by $\operatorname{supp} T$. $\mathcal{D}'_+$ (resp. $\mathcal{E}'$) is the space of distributions with support contained in $[0, \infty)$ (resp. compact support). It is an algebra with respect to convolution with identity δ, the Dirac distribution. When concentrated at a point $\{\tau\}$, the latter distribution $\delta(t - \tau)$ is denoted δ_τ. A distribution is said to be of order r if it acts continuously on C^r-functions but not on C^{r-1}-functions. Measures and functions are of order 0. Functions are considered through the distributions they define and are therefore indefinitely differentiable. If y is a continuous function except at a point a with a finite jump σ_a, its derivative dy/dt writes

$$dy/dt = \dot{y} + \sigma_a \, \delta_a \qquad (1)$$

where $\dot{y}$ is the distribution defined from the usual derivative of y. Derivation, integration and translation can be formed from the convolution products

$$\dot{y} = \delta^{(1)} * y, \quad \int y = H * y, \quad y(t-\tau) = \delta_\tau * y \quad (2)$$

where $\delta^{(1)}$ is the derivative of the Dirac distribution, and H denotes the Heaviside function. With a slight abuse of notations, we shall write $H^k y$ the iterated integration of y and more generally T^k the iterated convolution product of order k. For S, T in $\mathcal{D}'_+$, one has:

$$\operatorname{supp} S * T \subset \operatorname{supp} S + \operatorname{supp} T \qquad (3)$$

where the sum in the right hand side is defined by $\{x + y \, ; \, x \in \operatorname{supp} S, \, y \in \operatorname{supp} T\}$. Finally, when there is no danger of confusion, we shall sometimes denote $T(s)$, $s \in \mathbb{C}$, the Laplace transform of T.

The paper is organized as follows. Section 2 gives the main theoretical tools we shall need for our identification problem. Section 3 is devoted to specific cases involving piecewise constants inputs, and Section 4 considers the general case with arbitrary inputs.

2. MULTIPLICATION AND CONVOLUTION PRODUCTS

In the general case, the multiplication of two distributions (say α and T) is not always defined. However, this operation always make sense when one of the two terms is a smooth function. Several properties can be derived from such product and the most important for our developments are given bellow. The next Theorem is the key result from which most of the parameters (including the delays) can be identified from step input responses.

Theorem 1. (Schwartz, 1966) If T has a compact support K and is of order m (necessarily finite),

$\alpha T = 0$ whenever α and its derivatives of order $\leq m$ vanish on K.

The following examples illustrate this statement in case α is a polynomial and T a singular distribution. Note that, in forming the product αT, the delay τ involved in the argument $T(t-\tau)$ now appears also as a coefficient.

$$t \, \delta = 0, \quad (t - \tau) \, \delta_\tau = 0,$$
$$t^2 (t - \tau) \, (a \, \delta^{(1)} + b \, \delta_\tau) = 0.$$

The usual derivation rule $(\alpha T)' = \alpha' T + \alpha T'$ also applies. Thanks to the previous Theorem and the Leibniz's formula, the statement for the product $t \, \delta$ can be generalized to $t^l \, \delta^{(n)} = 0$ for $l > n$, and

$$t^l \, \delta^{(n)} = (-1)^l \, (n!/(n - l)!) \, \delta^{(n-l)}, \quad l \leq n. \quad (4)$$

We shall make use of another property involving both multiplication with t^n and the convolution product, in case one of the two distributions (S or T) has a compact support.

$$t^n \, (S * T) = \sum_{k=0}^{n} C_n^k \, (t^k \, S) * (t^{n-k} \, T). \qquad (5)$$

The combination of the rules (4) and (5) with $S = \delta^{(p)}$ and $T = y$ allows us to transform terms of the form $t^n \, y^{(p)}$ into linear combinations of derivatives of products $t^k \, y$. Denoting $z_i = t^i y$, one has for example,

$$t^3 \, y^{(2)} = t^3 \, (\delta^{(2)} * y) = -6 \, z_1 + 6 \, z_2^{(1)} - z_3^{(2)}. \quad (6)$$

Note that integrating twice this expression by considering $H^2 t^3 \, y^{(2)}$ results in nothing but the integration by parts formula with available data z_i. When facing delayed terms, relation (5) also applies and leads to expressions involving the delay not only as an argument, but also as a unknown coefficient to be identified. For example, one easily gets

$$t^2 \, \dot{y}(t - \tau) = t^2 \, (\delta_\tau^{(1)} * y)$$
$$= \delta_\tau * [\tau^2 \, z_0^{(1)} - 2\tau \, (z_0 - z_1^{(1)}) - 2z_1 + z_2^{(1)}] \quad (7)$$

Here again, integration with H yields a causal relation with available data z_i.

3. IDENTIFICATION FROM STEP INPUT RESPONSES

Using simple examples, this section shows hows Theorem 1 can be used for both parameters and delays identification in case of piecewise constant inputs.

3.1 First order system with a single input delay

Let us consider a first order system with a delayed input governed by:

$$\dot{y} + a\,y = y(0)\,\delta + \gamma_0\,H + b\,u(t-\tau), \qquad (8)$$

where γ_0 is a constant perturbation, a, b, and τ are constant parameters, and the coefficient a is assumed to be known (for the moment). Consider also a step input $u = u_0 H$. A first order derivation yields

$$\ddot{y} + a\dot{y} = y(0)\,\delta^{(1)} + \gamma_0\,\delta + b\,u_0\,\delta_\tau. \qquad (9)$$

By virtue of Theorem 1, the right hand side of equation (9) can be canceled by means of a multiplication with a function α such that

$$\alpha(0) = \alpha'(0) = 0, \quad \alpha(\tau) = 0, \qquad (10)$$

and the choice of the polynomial $\alpha(t) = t^3 - \tau\,t^2$ results in

$$t^3\,(\ddot{y} + a\dot{y}) = \tau\,t^2\,(\ddot{y} + a\dot{y}). \qquad (11)$$

As an equality of singular distributions, this relation doesn't make sense for any t (otherwise we would have $\tau = t$). However, $k \geq 1$ successive integrations (or a convolution with H^k) result in functions equality from which the delay τ becomes available. More precisely, since $\text{supp}\,H^k\delta_\tau \subset (\tau, \infty)$, we can easily show that all the obtained functions will vanish on $(0, \tau)$ and the delay is consequently not identifiable on this interval. Conversely, being nonzero for all $t > \tau$, the delay is everywhere identifiable on (τ, ∞). We therefore get from (11):

$$\tau = \frac{H^k(t^3 y^{(2)} + a\,t^3 y^{(1)})}{H^k(t^2 y^{(2)} + a\,t^2 y^{(1)})}, \qquad t > \tau, \qquad (12)$$

where, by virtue of (5) and recalling the notation $z_i = t^i y$ of the previous section,

$$
\begin{aligned}
t^3\,y^{(2)} &= -6\,z_1 + 6\,z_2^{(1)} - z_3^{(2)}, \\
t^2\,y^{(2)} &= -2\,z_0 + 4\,z_1^{(1)} - z_2^{(2)}, \\
t^2\,y^{(1)} &= 2\,z_1 - z_2^{(1)}, \\
t^3\,y^{(1)} &= 3\,z_2 - z_3^{(1)}.
\end{aligned}
$$

These coefficients show that $k \geq 2$ integrations can avoid any derivation in the delay identification algorithm, and the next figure show a partial realization scheme (limited to $H^k t^3 y^{(2)}$) of the terms involved in (12).

Note that the algorithm given in (12) only requires the values of a and the output y. In case the coefficient a is also unknown, the same approach can be used for a simultaneous identification. The following relation is easily derived from (11)

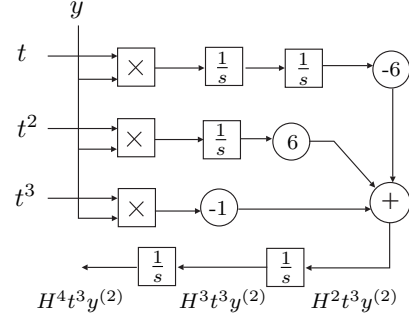$$\tau(t^2 y^{(2)}) + a\,\tau(t^2 y^{(1)}) - a\,(t^3 y^{(1)}) = t^3 y^{(2)} \qquad (13)$$

Fig. 1. Realization scheme of $H^k t^3 y^{(2)}$

and a linear system with unknown parameters $(\tau, a\,\tau, a)$ is obtained by using different integration orders

$$
\begin{pmatrix} H^2 w_1 & H^2 w_2 & H^2 w_3 \\ H^3 w_1 & H^3 w_2 & H^3 w_3 \\ H^4 w_1 & H^4 w_2 & H^4 w_3 \end{pmatrix}
\begin{pmatrix} \tau \\ a\tau \\ -a \end{pmatrix} =
\begin{pmatrix} H^2 w_0 \\ H^3 w_0 \\ H^4 w_0 \end{pmatrix} \qquad (14)
$$

where we have denoted $w_0 = t^3 y^{(2)}$, $w_1 = t^2 y^{(2)}$, $w_2 = t^2 y^{(1)}$, and $w_3 = t^3 y^{(1)}$. A simulation result from equation (14) is given in Figure 2 for $k = 2, 3, 4$ integrations and the parameters

$$y(0) = 0.3,\ a = 2,\ \tau = 0.6,\ \gamma_0 = 2,\ b = 1,\ u_0 = 1.$$

For the previous identifiability reason, the obtained linear system may be not consistent for $t < \tau$. Moreover, and unlike the single delay case, a local loss of identifiability may occur for $t > \tau$ as suggested in Figure 2 for $t \approx 1.5$s. For a complete identifiability study, the reader may refer to (Fliess M., 2003) for nonlinear and free delay cases, and (Belkoura, 2005) for linear convolutional systems.
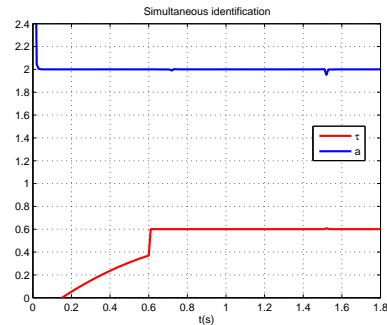

Fig. 2. Simultaneous identification from (14)

Note that this simultaneous identification of a and τ can also be formulated as a generalized eigenvalue problem as

$$\left[\begin{pmatrix} H^2 w_3 & H^2 w_0 \\ H^3 w_3 & H^3 w_0 \end{pmatrix} - \tau \begin{pmatrix} H^2 w_2 & H^2 w_1 \\ H^3 w_2 & H^3 w_1 \end{pmatrix} \right] \begin{pmatrix} a \\ 1 \end{pmatrix} = 0 \qquad (15)$$

Finally, others combinations of parameters can be obtained and identified by means of a multiplication of (9) with different $\mathcal{C}^\infty$ functions. For

instance, considering $\alpha(t) = t(t - \tau)$ and using the same previous technique results in

$$H^k t(\ddot{y} + a\dot{y})\,\tau + H^k\,\tau y(0) = H^k t^2(\ddot{y} + a\dot{y}) \quad (16)$$

from which both delay and initial condition can be identified. The only coefficient for which the explicit value of τ is required is its associated parameter b. Due to the fast convergence of the algorithms, we may also consider a separate procedure in which the undelayed terms are firstly identified and reused for the delay identification.

3.2 Second order system with multiple input delays

The previous technique can be extended to multiple input delays by means of one of the three different following approaches: (a) A multiplication with the $\mathcal{C}^\infty$ function $t^2(t - \tau_1)\cdots(t - \tau_K)$ (if K is finite). This may however lead to a large size linear system for which all the delays are not identifiable until $t > \tau_K$, (b) A recursive identification whose main drawback comes from propagation errors, and (c) A local (in time) identification if one assumes an incompressible delay such that

$$\tau_{k+1} - \tau_k > \Delta. \quad (17)$$

We shall focus in this section on delays identification based on the third case and consider the example of a second order system governed by

$$\ddot{y} + a_1\,\dot{y} + a_0\,y = \varphi_0 + \gamma_0\,H + \sum_{k=0}^{4} b_k\,u(t - \tau_k) \quad (18)$$

where $\tau_0 = 0$, γ_0 is a constant perturbation, and φ_0 (of order 1 and support $\{0\}$) contains the initial condition terms. A first order derivation, a multiplication by $t^3(t - \lambda)$, and a "local integration" using $H(s) = (1 - e^{-\Delta s/3})/s$ (instead of $1/s$) result in

$$0 = H^3 t^3(t - \lambda)(y^{(3)} + a_1 y^{(2)} + a_0 y^{(1)})$$
$$= N_1 - \lambda N_0. \quad (19)$$

Terms of the form $H^3 t^p y^{(q)}$ (and hence N_0 and N_1) are formed as in the previous section, while from the right hand side of (18) we also have,

$$N_i = H^3 \sum_{k=1}^{4} b_k\,\tau_k^{3+i}\,\delta_{\tau_k}, \quad i = 0, 1. \quad (20)$$

Since $\operatorname{supp} H^3 \in (0, \Delta)$, and by virtue of (3), we get that $N_0 = N_1 = 0$ on $\mathbb{R}\backslash(\tau_k, \tau_k + \Delta)$ while on the remaining subsets, the delays are obtained from

$$\lambda = \tau_k = N_1/N_0 \quad \text{on} \quad (\tau_k, \tau_k + \Delta). \quad (21)$$

A simulation result is shown in Figure 3 for the parameters $a_0 = 2$, $a_1 = 1$, $y(0) = 1.3$, $\dot{y}(0) = -2.3$, $\gamma_0 = 0.5$, $\Delta = 0.25$, $(b_0, \cdots, b_4) = (0.5, 1, -5, 2, 1)$, $(\tau_1, \cdots, \tau_4) = (2, 4, 6, 9)$.
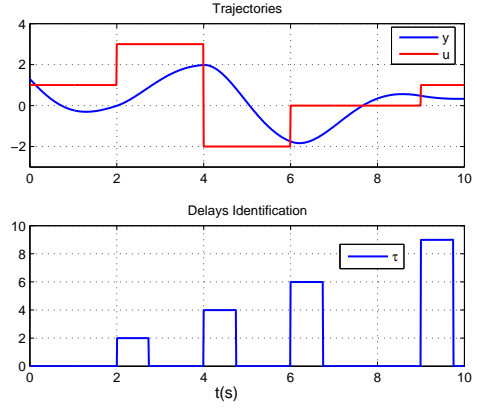

Fig. 3. Multiple delays identification

Due to the non identifiability of the delays on $\mathbb{R}\backslash(\tau_k, \tau_k + \Delta)$, the value of λ in (21) is set to zero until N_0 reaches a significant nonzero value.

3.3 A particular case with state delay

For some particular cases, a local state delay identification based on step input response is possible as well. The following example illustrates the proposed approach.

$$\dot{y}(t) + a\,y(t - \tau) = b\,u(t), \quad (22)$$
$$y(\theta) = 0 \quad \theta \in (-\tau, 0).$$

Denoting $H_\tau = H(t - \tau)$, the left hand side member writes $(\delta + a\,H_\tau) * \dot{y}$ in which $\delta + a\,H_\tau$ admits the convolution inverse (Hirsh and Lacombes, 1999),

$$(\delta + a\,H_\tau)^{*-1} =$$
$$\delta - a\,H_\tau + \cdots + (-1)^{n-1}a^{n-1}\,H_\tau^{n-1} + \cdots (23)$$

Since, by virtue of (3), $\operatorname{supp} H_\tau^k \subset (k\,\tau, \infty)$, a second order derivation of (22) combined with (23) results in

$$y^{(3)} = b\,\delta^{(1)} - a\,b\,\delta_\tau + \zeta \quad (24)$$
$$\text{where} \quad \operatorname{supp} \zeta \subset (2\,\tau, \infty).$$

Following the line of the previous sections, a cancellation of the singularities using the smooth function $t^2(t - \tau)$ yields

$$t^2(t - \tau)(y^{(3)} - \zeta) = 0. \quad (25)$$

Therefore, when restricted to the interval $(\tau, 2\,\tau)$ the state delay is obtained from

$$\tau = (H^k\,t^3\,y^{(3)})/(H^k\,t^2\,y^{(3)}), \quad t \in (\tau, 2\,\tau). \quad (26)$$

The next simulation shows the realization of (26) with $a = 2$ and $\tau = 0.6$.

As expected, the delay is identified only on the interval $(\tau, 2\,\tau)$ while due to the nonzero term ζ in (25), relation (26) doesn't hold for $t >$
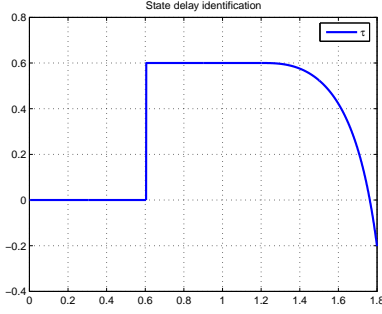
Fig. 4. Local identification of a state delay

2τ. Note also that unlike the input delay case, a simultaneous identification including the state coefficient a can not be considered in this case.

4. IDENTIFICATION WITH ARBITRARY INPUTS

For systems with input/state delays and general inputs, there are mainly two limitations of different nature. The first one concerns the online aspects while the second one is more fundamental. More precisely, the delays can be generally identified by means of convolution products and this may constitute a drawback if online results are expected. However, this constraint can be relaxed if there is no state delay and if the input is not prescribed. On the other hand, for a state delay case and unless particular situations, the initial condition (involving a function) is by definition unknown and depends, via its support within $(-\tau, 0)$, on the delay to be identified.

These observations as well as the identification procedure are illustrated in the following first order example

$$\dot{y} + a_0\, y + a_1\, y(t-\tau) = \varphi_0 + b_0\, u + b_1\, u(t-\tau), \quad (27)$$

in which the initial condition φ_0 writes

$$\varphi_0 = y(0)\, \delta - a_1\, \varphi, \qquad \mathrm{supp}\, \varphi \in (0, \tau),$$

for some function φ. For simplicity reasons and although a simultaneous identification of both the parameters and the delay can be considered, we shall restrict our study to the delay identification. For notational convenience, let us define from (27) the delayed and undelayed terms as

$$\eta \triangleq \dot{y} + a_0\, y - b_0\, u, \qquad \beta \triangleq b_1\, u - a_1\, y. \quad (28)$$

Using the developments given in Section 2-eq.(7), a first order derivation of (27) followed by a multiplication by t and t^2 results in

$$t(\eta - \varphi_0) = t(\delta_\tau * \beta) = \delta_\tau * (\tau\,\beta + t\beta), \qquad (29)$$
$$t^2(\eta - \varphi_0) = t^2(\delta_\tau * \beta) = \delta_\tau * (\tau^2 + 2\tau\, t\beta + t^2\beta).$$

Now, a cross convolution product of these two relation yields the following convolution equation

involving τ, τ^2, and in which the shift operator $(\delta_\tau *)$ has been removed.

$$\tau^2(t\beta * t\eta) + \tau(2t\beta * t\eta - \beta * t^2\eta) =$$
$$(t^2\beta * t\eta - t\beta * t^2\eta) \qquad (30)$$
$$+ (t^2\beta + 2\tau t\beta - \tau^2 t\beta) * t\varphi_0 + (\tau\beta - t\beta) * t^2\varphi_0.$$

It remains to consider the initial condition term $\varphi_0 = y(0)\, \delta - a_1\, \varphi$ that appears in this equation. If $\varphi \equiv 0$ (the plant is initially at rest) or $a_1 = 0$ (there is no state delay), then $t\varphi_0 = t^2\varphi_0 = 0$ and successive integrations lead to the linear system

$$\begin{pmatrix} Hw_1 & Hw_2 \\ H^2w_1 & H^2w_2 \end{pmatrix} \begin{pmatrix} \tau^2 \\ \tau \end{pmatrix} = \begin{pmatrix} Hw_0 \\ H^2w_0 \end{pmatrix}, \qquad (31)$$

where we have denoted

$$w_1 = t\beta * t\eta, \quad w_2 = 2t\beta * t\eta - \beta * t^2\eta, \qquad (32)$$
$$w_0 = t^2\beta * t\eta - t\beta * t^2\eta.$$

Note that by means of successive derivations, the same technique can be used in case of constant or piecewise polynomial function φ. On the other hand, in the free state delay case ($a_1 = 0 \Rightarrow \beta = b_1\, u$), and if the input is free, the above convolution products can be easily realized online by considering simple expressions for $u(s)$. For instance, with $u(t) = He^{-at}$, and recalling that du/ds is the Laplace transform of tu, the coefficient w_1 in (32) also writes

$$w_1(s) = \frac{b_1\, (t\eta)(s)}{(s+a)^2}. \qquad (33)$$

Figure 5 shows a simulation result with $\tau = 2$, $a_1 = 0$, $a_0 = 2$, $y(0) = 0$ $b_0 = b_1 = 1$, and the discontinuous input $u = H(t) - 3H(t-4) + \cos(t)(0.2+\sin(2t))$. This correspond to a single input delay which does not require a multiplication by t^2. Note that a singularity occurred at $t \approx 8.5$s.
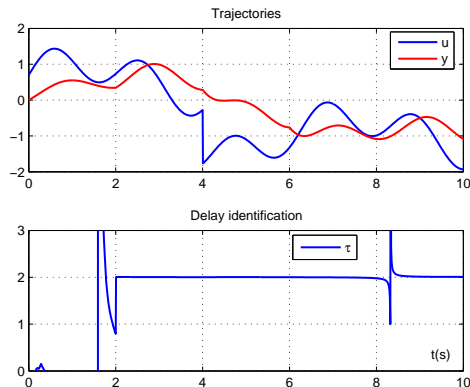

Fig. 5. Input delay identification

In Figure 6, the input/state delay is considered with $a_1 = 0.4$, $a_0 = 2$, $b_0 = b_1 = 1$, $\tau = 2$, and the input $u = \cos(t)(0.2 + \sin(2t))$. The initial condition φ is set to zero although $y(0) = 0.5 \neq 0$. Here again, the small deviations observed for

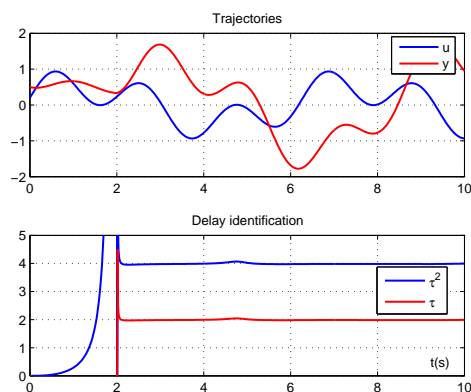$t \approx 4.5$s may result from a local ill-conditioned linear system.

Fig. 6. Input/State delay identification

5. CONCLUSION

This note has presented a simple method for both parameter and delay identification of dynamical systems. As in (Fliess M., 2003), the high speed convergence of the proposed algorithms allows us to consider the problem of on line identification and control of time delay systems. Robustness issues including the noise effects and a deeper study of the possible singularities, multivariable systems with partial state measurements as well as the extension to discrete time processes are under active investigation.

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