Robust sampled-data control: An input delay approach

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1 Introduction

Modelling of continuous-time systems with digital control in the form of continuous-time systems with delayed control input was introduced by Mikheev, Sobolev & Fridman [19], Astrom & Wittenmark [1] and further developed by Fridman (1992). The digital control law may be represented as delayed control as follows:

$$u(t) = u_d(t_k) = u_d(t-(t-t_k)) = u_d(t-\tau(t)), \quad t_k \le t < t_{k+1}, \ \tau(t) = t-t_k, \quad (1)$$

where u_d is a discrete-time control signal and the time-varying delay $\tau(t) = t-t_k$ is piecewise-linear with derivative $\dot{\tau}(t) = 1$ for $t \ne t_k$. Moreover, $\tau \le t_{k+1} - t_k$. Based on such a model, for small enough sampling intervals $t_{k+1} - t_k$ asymptotic approximations of the trajectory [19] and of the optimal solution to the sampled-data LQ finite horizon problem [7] were constructed.

Since the middle of 90's years of the last century different LMI conditions for robust stability of linear systems with uncertain, but bounded constant delay with a given upper bound have been derived (see e.g. [18],[17] and [20]). For systems with time-varying delays such conditions were obtained via Lyapunov-Krasovskii functionals in the case where the derivative of the delay is less than one (see e.g. [15]). The stability issue in the cases of time-varying delay without any restrictions on the derivative of the delay has been treated mainly via Lyapunov-Razumikhin functions, which usually lead to conservative results (see e.g. [14],[16],[20] and [13]). Only recently for the first time this case was treated by Lyapunov-Krasovskii technique [11]. This became possible due to a new descriptor model representation of the delay system introduced by Fridman [8].

The main approach to the sampled-data robust stabilization problem (see e.g. [6], [21]) is based on the lifting technique ([2],[26]) in which the problem is transformed to equivalent finite-dimensional discrete problem. However, this approach does not work in the cases with uncertain sampling times or uncertain system matrices. In the present paper we suggest a new approach to the robust sampled-data stabilization. We find a solution by solving the problem for

a continuous-time system with uncertain but bounded (by the maximum sampling interval) time-varying delay in the control input. We verify that the LMI sufficient conditions for stability of [11] are valid also in the case of piecewise-continuous delay and derive LMIs for the feedback gain. The conditions which we obtain are robust with respect to different samplings with the only requirement that the maximum sampling interval is not greater than h. As a by-product we show that for $h \rightarrow 0$ the conditions coincide with the necessary and sufficient conditions for the continuous-time stabilization. Such convergence in H_2 framework and related results were proved in [19], [4], [7], [22], [25] and [21].

For the first time the new approach allows to develop different robust control methods for the case of sampled-data control. The LMIs are affine in the system matrices and thus for the systems with polytopic type uncertainty the quadratic stabilization conditions, where the common Lyapunov functional for different vertices of the polytope is used, readily follow. We derive a parameter dependent solution, where different Lypunov functionals are used for different vertices by modifying results of (Fridman & Shaked, 2003). We also consider the regional stabilization by sampled-data saturated state-feedback, where we give an estimate on the domain of attraction. For continuous-time stabilization of state-delayed systems by saturated-feedback see e.g. [5], [24], [3] and [10].

Notation: Throughout the paper the superscript 'T' stands for matrix transposition, $\mathcal{R}^n$ denotes the n dimensional Euclidean space with vector norm $|\cdot|$, $\mathcal{R}^{n \times m}$ is the set of all $n \times m$ real matrices, and the notation $P > 0$, for $P \in \mathcal{R}^{n \times n}$ means that P is symmetric and positive definite. Given $\bar{u} = [\bar{u}_1, ..., \bar{u}_m]^T$, $0 < \bar{u}_i$, $i = 1, ..., m$, for any $u = [u_1, ..., u_m]^T$ we denote by $sat(u, \bar{u})$ the vector with coordinates $sign(u_i)min(|u_i|, \bar{u}_i)$. By stability of the system we understand the asymptotic stability of it.

2 Sampled-Data Stabilization of Systems with Polytopic Type Uncertainty

2.1 Problem Formulation

Consider the system
$$\dot{x}(t) = Ax(t) + Bu(t), \tag{2}$$

where $x(t) \in \mathcal{R}^n$ is the state vector, $u(t) \in \mathcal{R}^m$ is the control input.

We are looking for a piecewise-constant control law of the form $u(t) = u_d(t_k), t_k \leq t < t_{k+1}$, where u_d is a discrete-time control signal and $0 = t_0 < t_1 < \cdots < t_k < \cdots$ are the sampling instants. Our objective is to find a state-feedback controller given by
$$u(t) = Kx(t_k), \ t_k \leq t < t_{k+1}, \tag{3}$$

which stabilizes the system.

We represent a piecewise-constant control law as a continuous-time control with a time-varying piecewise-continuous (continuous from the right) delay

$\tau(t) = t - t_k$ as given in (1). We will thus look for a state-feedback controller of the form:

$$u(t) = Kx(t - \tau(t)). \tag{4}$$

Substituting (4) into (2), we obtain the following closed-loop system:

$$\dot{x}(t) = Ax(t) + BKx(t - \tau(t)), \quad \tau(t) = t - t_k, \ t_k \leq t < t_{k+1}. \tag{5}$$

We assume that
A1 $t_{k+1} - t_k \leq h \ \forall k \geq 0$.

From A1 it follows that $\tau(t) \leq h$ since $\tau(t) \leq t_{k+1} - t_k$. We will further consider (5) as the system with uncertain and bounded delay.

2.2   Stability of the Closed-Loop System

Similarly to [11], where the continuous delay was considered, we obtain for the case of piecewise-continuous delay the following result:

Lemma 1. *Given a gain matrix K, the system (5) is stable for all the samplings satisfying A1, if there exist $n \times n$ matrices $0 < P_1, \ P_2, \ P_3, \ Z_1, \ Z_2, Z_3$ and $R > 0$ that satisfy the following LMIs:*

$$\Psi_1 < 0, \quad and \quad \begin{bmatrix} R & [0 \ K^T B^T]P \\ * & Z \end{bmatrix} \geq 0, \tag{6}$$

where

$$P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \quad Z = \begin{bmatrix} Z_1 & Z_2 \\ * & Z_3 \end{bmatrix}, \quad \Psi_1 = \Psi_0 + hZ + \begin{bmatrix} 0 & 0 \\ 0 & hR \end{bmatrix},$$

$$\Psi_0 = P^T \begin{bmatrix} 0 & I \\ A + BK & -I \end{bmatrix} + \begin{bmatrix} 0 & I \\ A + BK & -I \end{bmatrix}^T P.$$

Proof is based on the following descriptor representation of (5) [8]:

$$\dot{x}(t) = y(t), \quad 0 = -y(t) + (A + BK)x(t) - BK \int_{t-\tau(t)}^{t} y(s)ds, \tag{7}$$

which is valid in the case of piecewise-continuous delay $\tau(t)$ for $t \geq 0$. Given a matrix K and initial condition $x(t) = \phi(t)(t \in [-h, 0])$, where ϕ is a piecewise continuous function, $x(t)$ satisfies (5) for $t \geq 0$ iff it satisfies (7). Note that the descriptor system (7) has no impulsive solutions since in (7) $y(t)$ is multiplied by the nonsingular matrix I [9].

We apply the Lyapunov-Krasovskii functional of the form:

$$V(t) = V_1 + V_2, \tag{8}$$

where

$$\bar{x}(t) = col\{x(t), \ y(t)\}, \quad E = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \quad P_1 = P_1^T > 0, \tag{9a-d}$$

and

$$V_1 = \bar{x}^T(t)EP\bar{x}(t), \ V_2 = \int_{-h}^{0}\int_{t+\theta}^{t} y^T(s)Ry(s)dsd\theta, \qquad (9\text{e-f})$$

which satisfies the following inequalities

$$a|x(t)|^2 \le V(t) \le b \sup_{s\in[-h,0]} |\bar{x}(t+s)|^2, \quad a > 0, \ b > 0. \qquad (10)$$

Differentiating $V(t)$ along the trajectories of (7) for $t \ge h$ we find (see [11]) that

$$\dot{V}(t) < \bar{x(t)}^T \Psi_1 \bar{x(t)} < -c|x(t)|^2, \quad c > 0, \qquad (11)$$

Provided that (6a,b) hold. Integrating (11) we have

$$V(t) - V(h) \le -c \int_{-h}^{t} |x(s)|ds \qquad (12)$$

and, hence, (10) yields $|x(t)|^2 \le V(t)/a \le V(h)/a < b/a \sup_{s\in[-h,0]} |\bar{x}(h+s)|^2$. Since $\sup_{s\in[-h,0]} |\bar{x}(h+s)| \le c_1 \sup_{s\in[-h,0]} |\phi(s)|, c_1 > 0$ (cf. Hale & Lunel, 1993, p168) and thus $\dot{x}$, defined by the right-hand side of (5), satisfy $\sup_{s\in[-h,0]} |\dot{x}(h+s)| \le c_2 \sup_{s\in[-h,0]} |\phi(s)|$, $c_2 > 0$, we obtain that

$$|x(t)|^2 \le c_3 \sup_{s\in[-h,0]} |\phi(s)|^2, c_3 > 0. \qquad (13)$$

Hence (5) is stable (i.e. $x(t)$ is bounded and small for small ϕ). To prove asymptotic stability we note that $x(t)$ is uniformly continuous on $[0,\infty)$ (since $\dot{x}(t)$ defined by the right-hand side of (5 is uniformly bounded). Moreover, (12) yields that $|x(t)|^2$ is integrable on $[0,\infty)$. Then, by Barbalat's lemma, $x(t) \to 0$ for $t \to \infty$ □

Consider now the continuous state-feedback

$$u(t) = Kx(t) \qquad (14)$$

and the closed-loop system (2), (14)

$$\dot{x}(t) = (A+BK)x(t). \qquad (15)$$

It is clear that the stability of the latter system is equivalent to the stability of its equivalent descriptor form

$$\dot{x}(t) = y(t), \quad 0 = -y(t)+(A+BK)x(t), \qquad (16)$$

which coincides with (7) for $h = 0$. It is well-known ([23]) that the stability of the latter system is equivalent to the condition $\Psi_0 < 0$.

If there exists P of the form (9c,d) which satisfies $\Psi_0 < 0$, then for small enough $h > 0$ LMIs of Lemma 2.1 are feasible (take e.g. $Z = I_{2n}$ and $R = [0 \ K^T B^T]P^T P[0 \ K^T B^T]^T$). We, therefore, obtain the following result:

Corollary 1. *If the continuous-time state-feedback (14) stabilizes the linear system (2), then the sampled-data state-feedback (3) with the same gain K stabilizes (2) for all small enough h.*

In the case where the matrices of the system are not exactly known, we denote $\Omega = \begin{bmatrix} A & B \end{bmatrix}$ and assume that $\Omega \in Co\{\Omega_j, \; j = 1, ...N\}$, namely,

$$\Omega = \sum_{j=1}^{N} f_j \Omega_j \quad \text{for some} \quad 0 \le f_j \le 1, \; \sum_{j=1}^{N} f_j = 1, \tag{17}$$

where the N vertices of the polytope are described by

$$\Omega_j = \begin{bmatrix} A^{(j)} & B^{(j)} \end{bmatrix}.$$

In order to guarantee the stability of (2) over the entire polytope one can use the result of Lemma 2.1 by applying the same matrices P_2 and P_3 for all the points in the polytope and solving (6a,b) for the N vertices only. A quadratic stability type criterion is then obtained:

Corollary 2. *Given a gain matrix K, the system (5) is stable, over the entire polytope Ω, if there exist $n \times n$ matrices $0 < P_1^{(j)}$, P_2, P_3, $Z_1^{(j)}$, $Z_2^{(j)}, Z_3^{(j)}$ and $R^{(j)} > 0$ that satisfy the following LMIs:*

$$\Psi_1^{(j)} < 0, \quad and \quad \begin{bmatrix} R^{(j)} & [0 \; K^T B^{(j)T}]P^{(j)} \\ * & Z^{(j)} \end{bmatrix} \ge 0, \; j = 1, ..., N, \tag{18}$$

where

$$P^{(j)} = \begin{bmatrix} P_1^{(j)} & 0 \\ P_2 & P_3 \end{bmatrix}, \quad Z^{(j)} = \begin{bmatrix} Z_1^{(j)} & Z_2^{(j)} \\ * & Z_3^{(j)} \end{bmatrix}, \quad \Psi_1^{(j)} = \Psi_0^{(j)} + hZ^{(j)} + \begin{bmatrix} 0 & 0 \\ 0 & hR^{(j)} \end{bmatrix},$$

$$\Psi_0^{(j)} = P^{(j)T} \begin{bmatrix} 0 & I \\ A^{(j)} + B^{(j)}K & -I \end{bmatrix} + \begin{bmatrix} 0 & I \\ A^{(j)} + B^{(j)}K & -I \end{bmatrix}^T P^{(j)}.$$

2.3 Quadratic Stabilization

LMIs of Lemma 2.1 are bilinear in P and K. In order to obtain LMIs we use P^{-1}. It is obvious from the requirement of $0 < P_1$, and the fact that in (6) $-(P_3 + P_3^T)$ must be negative definite, that P is nonsingular. Define

$$P^{-1} = Q = \begin{bmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{bmatrix} \quad \text{and} \quad \Delta = diag\{Q, \; I\} \tag{19}$$

Applying Schur formula to the term hR in (6a), we multiply (6a,b) by Δ^T and Δ, on the left and on the right, respectively. Denoting $\bar{R} = R^{-1}$ and $\bar{Z} = Q^T Z Q$ we obtain, similarly to [11], the following

Theorem 1. *The control law of (3) stabilizes (2) for all the samplings with the maximum sampling interval not greater than h and for all the system parameters that reside in the uncertainty polytope Ω, if there exist: $Q_1 > 0$, $Q_2^{(j)}$, $Q_3^{(j)}$, $\bar{R}$, $\bar{Z}_1^{(j)}$, $\bar{Z}_2^{(j)}$, $\bar{Z}_3^{(j)} \in \mathcal{R}^{n \times n}$, $\bar{Y} \in \mathcal{R}^{q \times n}$ that satisfy the following LMIs:*

$$\begin{bmatrix} Q_2^{(j)} + Q_2^{(j)T} + h\bar{Z}_1^{(j)} & \hat{\bar{\Xi}}^{(j)} & hQ_2^{(j)T} \\ * & -Q_3^{(j)} - Q_3^{(j)T} + h\bar{Z}_3^{(j)} & hQ_3^{(j)T} \\ * & * & -h\bar{R} \end{bmatrix} < 0,$$

and nonlinear matrix inequalities

$$\begin{bmatrix} Q_1 \bar{R}^{-1} Q_1 & 0 & \bar{Y}^T B^{(j)T} \\ * & \bar{Z}_1^{(j)} & \bar{Z}_2^{(j)} \\ * & * & \bar{Z}_3^{(j)} \end{bmatrix} \geq 0, \tag{20}$$

where

$$\hat{\bar{\Xi}}^{(j)} = Q_3^{(j)} - Q_2^{(j)T} + Q_1 A^{(j)T} + h\bar{Z}_2^{(j)} + \bar{Y}^T B^{(j)T}, \quad j = 1, 2, ..., N. \tag{21}$$

The state-feedback gain is then given by

$$K = \bar{Y} Q_1^{-1}. \tag{22}$$

For solving (20) there exist *two methods*. The *first* uses the assumption

$$\bar{R} = \epsilon Q_1, \quad \epsilon > 0, \tag{23}$$

and thus leads to $2N$ LMIs with tuning parameter ϵ:

$$\begin{bmatrix} Q_2^{(j)} + Q_2^{(j)T} + h\bar{Z}_1^{(j)} & \hat{\bar{\Xi}}^{(j)} & hQ_2^{(j)T} \\ * & -Q_3^{(j)} - Q_3^{(j)T} + h\bar{Z}_3^{(j)} & hQ_3^{(j)T} \\ * & * & -\epsilon h Q_1 \end{bmatrix} < 0, \tag{24}$$

$$\begin{bmatrix} \epsilon Q_1 & 0 & \epsilon \bar{Y}^T B^{(j)T} \\ * & \bar{Z}_1^{(j)} & \bar{Z}_2^{(j)} \\ * & * & \bar{Z}_3^{(j)} \end{bmatrix} \geq 0, \tag{25}$$

where $\hat{\bar{\Xi}}^{(j)}$ and j are given by (21).

Similarly to Corollary 2.2 we can show that if the system (2) is quadratically stabilizable by a continuous-time state-feedback (14), then for all small enough h the latter LMIs are feasible and the sampled-data state-feedback with the same gain stabilizes the system.

The *second* method for solving the matrix inequalities of Lemma 2.4 is based on the iterative algorithm developed recently by Gao and Wang (2003). This method is preferable in the cases of comparatively large h, since it leads to less conservative results. However it may take more computer time due to iterative

process. In the sequel we shall adopt the first method for solving the matrix inequalities of Lemma 2.4.

Example 1. We consider (2) with the following matrices:

$$A = \begin{bmatrix} 1 & 0.5 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

It is verified by using Theorem 2.4 that the system is stabilizable by a sampled-data state-feedback with the maximum sampling interval $h \leq 0.69$. Thus, for $h = 0.69$ the resulting $K = [-1.048 \ \ 0.2511]$ (with $\epsilon = 0.34$). Simulation results (for uniform samplings with the sampling interval less than 0.7) show that the closed-loop system is stable.

2.4 Parameter Dependent Stabilization

The requirement for the quadratic stabilization imposes a serious constraint on the solution, where the same matrices Q_1 should satisfy the matrix inequalities in all the vertices of the polytope. To alleviate this difficulty a parameter dependent solution with different matrices $Q_1^{(j)}$ was derived in [12] for the case of state delay. We modify the results of Corollary 4 of [12] for the case of input delay by assuming that $\bar{R} = \epsilon G_1$ and obtain

Theorem 2. *Consider the system (2) and assume that its parameters lie in the polytope $\bar{\Omega}$. The system is stabilized, over the entire polytope Ω, by the controller of (3), for all the samplings with the maximum sampling interval not greater than h, if for some tuning positive scalar parameters ϵ and α there exist $2n \times 2n$ matrices: Q_j, G_j and H_j, of the form*

$$Q_j = \begin{bmatrix} Q_1^{(j)} & 0 \\ Q_2^{(j)} & Q_3^{(j)} \end{bmatrix}, \ G_j = \begin{bmatrix} G_1 & 0 \\ G_2^{(j)} & G_3^{(j)} \end{bmatrix}, \ H_j = \begin{bmatrix} \alpha G_1 & 0 \\ H_2^{(j)} & H_3^{(j)} \end{bmatrix}.$$

and $\bar{Z}_j$, $j = 1, ..., \bar{N}$, a $m \times n$ matrix $\bar{Y}$ and $n \times n$ matrix $\bar{R}$ that satisfy the following LMIs.

$$\begin{bmatrix} M_j & \begin{bmatrix} Q_2^{(j)T} \\ Q_3^{(j)T} \end{bmatrix} & Q_j^T - G_j^T + \bar{A}^{(j)} H_j + \begin{bmatrix} 0 \\ \alpha B^{(j)} \end{bmatrix} [\bar{Y} \ 0] \\ * & -\epsilon h^{-1} G_1 & 0 \\ * & * & -H_j^T - H_j \end{bmatrix} < 0 \quad (26a)$$

$$\begin{bmatrix} \epsilon G_1 & [0 \ \epsilon \bar{Y}^T B^{(j)T}] \\ * & \bar{Z}_j \end{bmatrix} > 0, \quad j = 1, ..., \bar{N}. \quad (26b)$$

where

$$M_j = G_j^T \bar{A}^{(j)T} + \bar{A}^{(j)} G_j + \begin{bmatrix} \bar{Y}^T \\ 0 \end{bmatrix} [0 B^{(j)T}] + \begin{bmatrix} 0 \\ B^{(j)} \end{bmatrix} [\bar{Y} \ 0] + h\bar{Z}_j, \ \bar{A}^{(j)} = \begin{bmatrix} 0 & I \\ A^{(j)} & -I \end{bmatrix}.$$

The state-feedback gain that stabilizes the system over Ω is then given by $K = \bar{Y} G_1^{-1}$.

Example 2. We consider (2) with the following matrices taken from [10], where $h = 0$:

$$A = \begin{bmatrix} 0 & 1 \\ -1 + g_1 & -.5 \end{bmatrix}, \ B = \begin{bmatrix} -1 + g_2 \\ 1 \end{bmatrix}$$

and where $|g_1| \leq 0.53$ and $|g_2| \leq 1.7$. It was shown in [12] that the system is not quadratically stabilizable by continuous state-feedback (i.e. for $h = 0$). It is verified by using Theorem 2.5 that the system is stabilizable by a sampled-data state-feedback with the maximum sampling interval $h \leq 0.299$. Thus, for $h = 0.299$ the resulting $K = [0.0821 \ -0.1487]$ (with $\epsilon = 3.56$ and $\alpha = 1.1$). Simulation results (see e.g. Fig.1 for the case of $g_1 = g_1(t) = 0.53 \sin t$, $g_2 = g_2(t) = 1.7 \cos t, t_{k+1} - t_k = 0.299 \ \forall k \geq 0$ and the initial condition $x(0) = [5 \ -5]^T$) show that the resulting closed-loop solutions converge to origin.

3   Regional Stabilization by Sampled-Data Controller with Saturation

3.1   Problem Formulation

Consider the system (2) with the sampled-data control law (3) which is subject to the following amplitude constraints

$$|u_i(t)| \leq \bar{u}_i, \quad 0 < \bar{u}_i, \ i = 1, ..., m \tag{27}$$

Denote by $x(t, x(0))$ the state trajectory of (2) with the initial condition $x(0) \in R^n$. Then the domain of attraction of the origin of the closed-loop system (2), (3) is the set
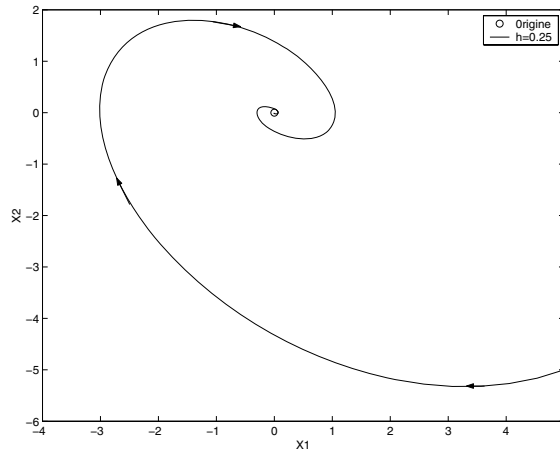
Fig. 1. State trajectories of the closed-loop system for $h = 0.299, g_1 = 0.53 \sin t$, $g_2 = 1.7 \cos t$ and $x(0) = [5 \ -5]^T$

$$\mathcal{A} = \{x(0) \in R^n : \lim_{t \to \infty} x(t, x(0)) = 0\}.$$

We seek conditions for the existence of a gain matrix K which leads to a stable closed-loop. Having met these conditions, a simple procedure for finding the gain K should be presented. Moreover, we obtain an estimate $\mathcal{X}_\beta \subset \mathcal{A}$ on the domain of attraction, where

$$\mathcal{X}_\beta = \{x(0) \in R^n : x^T(0)P_1 x(0) \le \beta^{-1}\}, \tag{28}$$

and where $\beta > 0$ is a scalar and $P_1 > 0$ is an $n \times n$ matrix.

We represent the state-feedback in the delayed form

$$u(t) = sat(Kx(t - \tau(t)), \bar{u}). \tag{29}$$

Reducing the original problem to the problem with input delay, we solve it by modifying derivations of [10], where the case of state delay was considered.

3.2 A Linear System Representation with Polytopic Type Uncertainty

Applying the control law of (29) the closed-loop system obtained is

$$\dot{x}(t) = Ax(t) + Bsat(Kx(t - \tau(t)), \bar{u}), \ \tau(t) = t - t_k, \ t_k \le t < t_{k+1}. \tag{30}$$

Though the closed-loop system has a delay, we keep in mind that in the case of sampled-data control the initial condition is defined in the point $t = 0$ and not on the segment $[-h, 0]$. That is why for the estimation of the domain of attraction we can restrict ourself to the following initial functions $\phi(s)$, $s \in [-h, 0]$:

$$\phi(0) = x(0), \ \phi(s) = 0, s \in [-h, 0). \tag{31}$$

Denoting the i-th row by k_i, we define the polyhedron

$$\mathcal{L}(K, \bar{u}) = \{x \in \mathcal{R}^n : |k_i x| \le \bar{u}_i, \ i = 1, ..., m\}.$$

If the control and the disturbance are such that $x \in \mathcal{L}(K, \bar{u})$ then the system (30) admits the linear representation. Following [3], we denote the set of all diagonal matrices in $\mathcal{R}^{m \times m}$ with diagonal elements that are either 1 or 0 by Υ, then there are 2^m elements D_i in Υ, and for every $i = 1, ..., 2^m$ $D_i^- \triangleq I_m - D_i$ is also an element in Υ.

Lemma 2. *[3] Given K and H in $\mathcal{R}^{m \times n}$. Then*

$$sat(Kx(t), \bar{u}) \in Co\{D_i Kx + D_i^- Hx, \ i = 1, ..., 2^m\}$$

for all $x \in \mathcal{R}^n$ that satisfy $|h_i x| \le \bar{u}_i, \ i = 1, ..., 2^m$.

The following is obtained from Lemma 3.1.

Lemma 3. *Given $\beta > 0$, assume that there exists H in $\mathcal{R}^{m \times n}$ such that $|h_i x| \leq \bar{u}_i$ for all $x(t) \in \mathcal{X}_\beta$. Then for $x(t) \in \mathcal{X}_\beta$ the system (30) admits the following representation.*

$$\dot{x}(t) = Ax(t) + \sum_{j=1}^{2^m} \lambda_j(t) A_j x(t - \tau(t)) \tag{32}$$

where

$$A_j = B(D_j K + D_j^- H) \quad j = 1, ..., 2^m, \ \sum_{j=1}^{2^m} \lambda_j(t) = 1, \ 0 \leq \lambda_j(t), \ \forall \, 0 < t, \tag{33}$$

We denote

$$\Omega_\alpha = \sum_{j=1}^{2^m} \lambda_j \Omega_j \quad \text{for all} \quad 0 \leq \lambda_j \leq 1, \ \sum_{j=1}^{2^m} \lambda_j = 1 \tag{34}$$

where the vertices of the polytope are described by $\Omega_j = \begin{bmatrix} A_j \end{bmatrix}$, $j = 1, ..., 2^m$. The problem becomes one of finding $\mathcal{X}_\beta$ and a corresponding H such that $|h_i x| \leq \bar{u}_i$, $i = 1, ... 2^m$ for all $x \in \mathcal{X}_\beta$ and that the state of the system

$$\dot{x}(t) = Ax(t) + A_j x(t - \tau(t)), \ \tau(t) = t - t_k, \ t_k \leq t < t_{k+1}, \tag{35}$$

remains in $\mathcal{X}_\beta$.

3.3   Regional Stabilization

Applying the descriptor model transformation and the Lyapunov-Krasovskii functional of (8) by using the first method for solving the stabilization matrix inequalities (with tuning parameter ϵ) , we obtain the following result:

Theorem 3. *Consider the system (2) with the sampled-data control law (3) which is subject to the constraints (27). The system is stable with $\mathcal{X}_\beta$ inside the domain of attraction for all the samplings with the maximum sampling interval not greater than h, if there exist $0 < Q_1$, $Q_2^{(j)}$, $Q_3^{(j)}$, $Z_1^{(j)}$, $Z_2^{(j)}$, $Z_3^{(j)} \in \mathcal{R}^{n \times n}$, Y, $G \in \mathcal{R}^{m \times n}$ and $\beta > 0$ that satisfy the following set of inequalities:*

$$\begin{bmatrix} Q_2^{(j)} + Q_2^{T(j)} + hZ_1^{(j)} & \Sigma_j & hQ_2^{(j)} \\ * & -Q_3^{(j)} - Q_3^{T(j)} + hZ_3^{(j)} & hQ_3^{(j)} \\ * & * & -\epsilon hQ_1 \end{bmatrix} < 0, \ j = 1, ..., 2^m \tag{36a}$$

$$\begin{bmatrix} \epsilon Q_1 & 0 & \epsilon(Y^T D_j + G^T D_j^-)B^T \\ * & Z_1^{(j)} & Z_2^{(j)} \\ * & * & Z_3^{(j)} \end{bmatrix} \geq 0 \tag{36b}$$

$$\begin{bmatrix} \beta & g_i \\ * & \bar{u}_i^2 Q_1 \end{bmatrix} \geq 0, \ i = 1, ..., m, \tag{37}$$

where

$$\Sigma_j = Q_3^{(j)} - Q_2^{T(j)} + Q_1 A^T + (Y^T D_j + G^T D_j^-)B^T + hZ_2^{(j)}. \tag{38}$$

The feedback gain matrix which stabilizes the system is given by $K = YQ_1^{-1}$.

Proof: For V given by (8) conditions are sought to ensure that $\dot{V} < 0$ for any $x(t) \in \mathcal{X}_\beta$. As in [10], the inequalities (37) guarantee that $|h_i x| \leq \bar{u}_i, \forall x \in \mathcal{X}_\beta$, $i = 1, ..., m$, where $g_i \overset{\Delta}{=} h_i Q_1$, $i = 1, ..., m$ and $Q_1 \overset{\Delta}{=} P_1^{-1}$, and the polytopic system representation of (35) is thus valid. Moreover, (36a,b) guarantee that $\dot{V} < 0$.

From $\dot{V} < 0$ it follows that $V(t) < V(0)$ and therefore for the initial conditions of the form (31)

$$x^T(t)P_1 x(t) \leq V(t) < V(0) = x^T(0)P_1 x(0) \leq \beta^{-1}. \tag{39}$$

Then for all initial values $x(0) \in \mathcal{X}_\beta$, the trajectories of $x(t)$ remain within $\mathcal{X}_\beta$, and the polytopic system representation (35) is valid. Hence $x(t)$ is a trajectory of the linear system (35) and $\dot{V} < 0$ along the trajectories of the latter system which implies that $\lim_{t \to \infty} x(t) = 0$.

Example 3. We consider (2) with the following matrices (taken from [3], where $h = 0$):

$$A = \begin{bmatrix} 1.1 & -0.6 \\ .5 & -1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and where $\bar{u} = 5$. Applying Theorem 3.3 a stabilizing gain was obtained for all samplings with the maximum sampling interval $h \leq 0.75$. In order to 'enlarge' the volume of the ellipse we minimized the value of β (to improve the result we also added the inequality $Q_1 > \alpha I$ and chose such $\alpha > 0$ that enlarged the resulting ellipse). The ellipse volume increases when h decreases (see Figure 2). For, say, $h = 0.75$ we obtain $K = [-1.6964 \; 0.5231]$ (with $\epsilon = 0.325, \beta = 0.1261, P_1 = \begin{bmatrix} 0.9132 & -0.2816 \\ -0.2816 & 0.0868 \end{bmatrix}, \alpha = 1$) and we show (see Figure 3) that a trajectory starting on
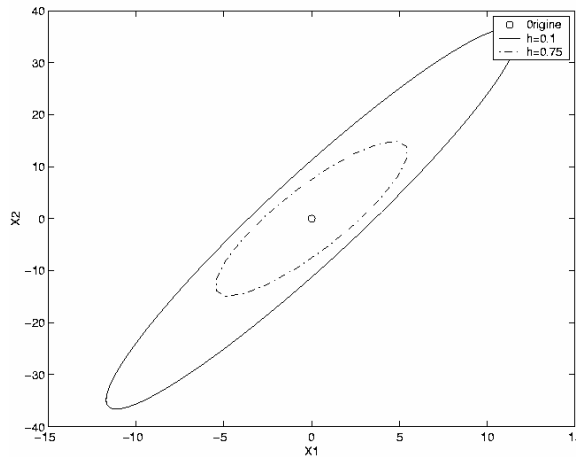
Fig. 2. Ellipsoidal bounds on the domain of attraction: line corresponds to $h = 0.1$; slash line to $h = 0.75$; point line to $h =$
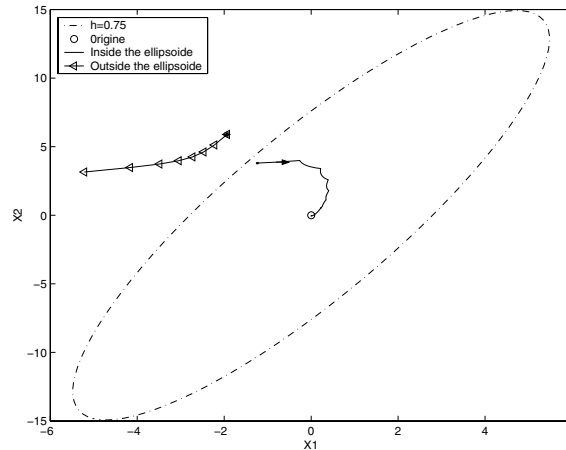
Fig. 3. Stabilization result for $h = 0.75$

the periphery of the ellipse (for the case of the uniform sampling with the sampling period $t_{k+1} - t_k = 0.75$) never leaves this ellipse and converges to the origin, while a trajectory starting not far from the ellipse remains outside the ellipse.

4   Conclusions

A new method for robust sampled-data stabilization of linear continuous-time systems is introduced. This method is based on the continuous-time model with time-varying input delay. Under assumption that the maximum sampling interval is not greater then $h > 0$, the h-dependent sufficient LMIs conditions for stabilization of systems with polytopic type uncertainty and for regional stabilization of systems with sampled-data saturated state-feedback are derived via descriptor system approach to time-delay systems.

The new approach solves the problems for comparatively small h and leads to sufficient conditions only, however these conditions are simple. The method may be applied to a wide spectrum of robust sampled-data control problems (e.g. to guaranteed cost or to $H_\infty control$).

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