Moving Frame Based Strategies for Reduction of Ordinary Differential/Recurrence Systems using their Expanded Lie Point Symmetries.

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ABSTRACT

When an ordinary differential/recurrence system presents a m-parameters solvable group of symmetries, Lie group theory states that its number of variables could be *reduced* by m. This *reduction* process is classically done by rewriting original problem in an invariant coordinates set for these symmetries. We show how to use computational strategies using non explicit (*infinitesimal*) data representation in the reduction process and thus, how to avoid—for differential systems—the explicit expansive computation of these invariants. Thus, these strategies lead to efficient algorithms that were used in the MAPLE implementation [13].

Categories and Subject Descriptors

J.2 [**Computer Applications**]: Physical Sciences and Engineering; G.4 [**Mathematics of Computing**]: Mathematical Software—*Algorithm design and analysis*

General Terms

Reduction of ordinary differential/recurrence systems

Keywords

Moving frame method, Computer algebra.

1. INTRODUCTION

This note presents strategies used in the implementation [13] for the *reduction* process of some parametric ordinary systems that are invariant under a Lie group of transformation.

EXAMPLES 1. *In order to give an example of such a reduction, let us consider the Verhulst's logistic growth model with linear predation (see § 1.1 in [10]):*

$$\Sigma \; : \quad \frac{\mathrm{d}t}{\mathrm{d}t} = 1, \quad \frac{\mathrm{d}x}{\mathrm{d}t} = (a - b\,x)\,x, \quad \frac{\mathrm{d}a}{\mathrm{d}t} = \frac{\mathrm{d}b}{\mathrm{d}t} = 0. \qquad (1)$$

One can represent the flow (t, x) of (1) using the flow $(\widetilde{t}, \widetilde{x})$ of the reduced *differential equation $\mathrm{d}\widetilde{x}/\mathrm{d}\widetilde{t} = (1 - \widetilde{x})\widetilde{x}$ and the fibre $t = \widetilde{t}/a$, $x = a\widetilde{x}/b$. In this formulation of (1), the variables x and t were* nondimensionalised.

This *reduction* process reduces the number of relevant parameters that determine the considered dynamics. This process is used in various applications (for example, as the dimension of the reduced system is smaller, the qualitative analysis—determination of fixed points, etc.—is simplified).

The process applied above to an ordinary differential system could be used on an ordinary recurrence system; as this type of model is widely encountered in biological modelisation for example, we gather the treatment of these two cases in our presentation. The literature on *reduction* and similar simplification procedures is far too vast to be reviewed properly here; thus, the next section just evokes works that are closely related to the standpoint adopted in this note.

1.1 Related Works

In [2], the authors introduce a new version of the so-called Cartan's method of *moving frames* that leads—among other applications—to the construction of invariants for a given finite-dimensional Lie group action on a manifold.

This work produces a growing interest in the field of computer algebra; in particular, it inspired some works concerning the effective computation of invariants in the polynomial (resp. differential) case [4] (resp. [9]) implemented in MAPLE [3] (resp. [8]). In—apparently—a completely different application field, there are many implementations dealing with exact parametric system simplification (like dimensional analysis for example; see [7] and references therein).

This note is devoted to present the relationship between these two domains and to explicit some algorithmic improvements occurring when the general moving frame method is used in this—ordinary system simplification—particular application field.

1.2 Main Contribution

Several methods of parametric system simplification (dimensional analysis, lumping, etc.) amount to rewrite the considered system in an invariant coordinates set (see [12] that follows this strategies). This rewriting (a.k.a. *reduction*) process could be efficiently done using the moving frame method. In fact, we show below how to avoid—as much as possible—the explicit computation of invariants. Instead we explicit computational strategies using non explicit (*infinitesimal*) data representation in the reduction process (for differential systems). As the explicit computation of invariants is computationally expansive, the strategies presented in this notes lead to efficient algorithms that were implemented in the MAPLE package [13].

1.3 Outline

In the next section, we recall some basic definitions concerning considered dynamical systems, their encoding as pseudo-derivations and the notion of continuous system's rectification. Then, we give the definition of dynamical systems' symmetries and their determining infinitesimal criteria. Finally, this section presents the classical reduction of dynamical systems based on the explicit computation of their symmetries' invariants. In Section 3, we present how symmetries determination is handled by the package [13]. Then, we recall briefly the *moving frame* based invariant computation and we show that the same strategy could be applied in order to reduce a continuous dynamical system without explicitly computing the invariants associated to its symmetries.

2. CLASSICAL FRAMEWORK

The next section introduces inputs of our reduction process.

2.1 Infinitesimal Generator, Recurrence Operator and Associated Dynamical System

In the sequel, we consider:

- Σ an explicit ordinary differential system (ODS) having t as its continuous independent variable, bearing on k state variables $X := (x_1, \ldots, x_k)$ and depending on ℓ parameters $\Theta := (\theta_1, \ldots, \theta_\ell)$:
$$\Sigma \ : \quad \frac{\mathrm{d}t}{\mathrm{d}t} = 1, \quad \frac{\mathrm{d}\Theta}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}X}{\mathrm{d}t} = F(t, X, \Theta). \tag{2}$$
The finite set $F := (f_1, \ldots, f_k)$ denotes rational functions in $\mathbb{K}(t, X, \Theta)$ where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$;
- Ω an explicit first order ordinary recurrence system (ORS) with the same notations except that the independent variable n is discrete in that case:
$$\Omega \ : \quad \Theta_{n+1} = \Theta_n, \quad X_{n+1} = F(n, X_n, \Theta_n). \tag{3}$$

In above formulations, a parameter is considered as a constant state space variable; thus, it is worthwhile to work in an *expanded* state space i.e. to denote the coordinates set (t, X, Θ) or (n, X, Θ) by $Z := (z_1, \ldots, z_m)$—and its cardinal by $m := 1 + k + \ell$—in order to avoid any distinction between parameters and variables.

EXAMPLES 2. *Hence, the Verhulst's logistic growth model presented in Example 1 is an example of ODS and the following recurrence system is a simple example of ORS with two state variables x, y and one parameter c:*

$$\Omega \ : \ x_{n+1} = c_n \, x_n, \quad y_{n+1} = y_n + x_n, \quad c_{n+1} = c_n. \tag{4}$$

2.1.1 Algebraic Standpoint

In order to simplify forthcoming presentation, let us introduce the following algebraic definitions:

DEFINITION 3. *A pseudo-derivation δ of $\mathbb{K}[Z]$ is a linear operator $\delta : \mathbb{K}[Z] \to \mathbb{K}[Z]$ that has $\mathbb{K}$ in its kernel and that satisfies the twisted Leibniz rule:*

$$\forall \, (f_1, f_2) \in \mathbb{K}[Z]^2, \quad \delta(f_1 f_2) = f_1 \, \delta f_2 + \sigma(f_2) \, \delta f_1, \tag{5}$$

where σ is an endomorphism of $\mathbb{K}[Z]$. In the sequel, we denote by $\mathrm{Ore}\mathbb{K}[Z]$ the set of all such pseudo-derivations (see [1] for a complete description and more references).

DEFINITION 4. *An* infinitesimal generator *δ is a derivation i.e. a pseudo-derivation where σ is the identity map* Id*; in the canonical basis $\{\partial/\partial z_1, \ldots, \partial/\partial z_m\}$ of derivations, it is written as follows $\delta = \sum_{\imath=1}^{m} \delta z_\imath \partial/\partial z_\imath$ where $\delta z_\imath$ is in $\mathbb{K}[Z]$. In particular, the infinitesimal generator $\delta_{\mathcal{D}}$ associated to the ODS (2) is defined by $\delta_{\mathcal{D}} := \partial/\partial t + \sum_{\imath=1}^{k} f_\imath \partial/\partial x_\imath$.*

DEFINITION 5. *Given an algebra $\mathbb{K}[Z]$, a* recurrence operator *σ associated to the ORS (3) is a $\mathbb{K}[Z]$-endomorphism defined by the relations $\sigma\kappa = \kappa$ for all κ in $\mathbb{K}$ and by the images $\sigma n = n + 1$, $\sigma\Theta = \Theta$ and $\sigma X = F(n, X, \Theta)$.*

One can associate a pseudo-derivation δ to every ORS, defined in terms of its recurrence operator σ as follows:

$$\forall f \in \mathbb{K}[Z] \, , \quad \delta f := \sigma f - f. \tag{6}$$

EXAMPLES 6. *Using the notations introduced in the Examples 2, we could illustrate the above definitions as follow:*

- *the infinitesimal generator $\delta_{\mathcal{D}}$ associated to the system Σ is $\partial/\partial t + (a - b\,x)\,x\,\partial/\partial x$;*
- *the recurrence operator σ associated to the system Ω is defined by relations $\sigma n = n + 1$, $\sigma x = c\,x$, $\sigma y = y + x$ and $\sigma c = c$.*

2.1.2 Dynamical System

In the sequel, we consider algebraic groups of transformation represented by *dynamical systems* associated to an infinitesimal generator δ (resp. a recurrence operator σ):

DEFINITION 7. *A* dynamical system *$\mathcal{D}$ is composed by a group G with internal operator $\star$ and neutral element e, a state space $\mathbb{K}^m$ and an evolution function $\mathcal{D}$:*

$$\begin{aligned} \mathcal{D} : G \times \mathbb{K}^m &\rightarrow \mathbb{K}^m, \\ (\nu, Z) &\rightarrow \mathcal{D}(\nu, Z) = (\mathcal{D}_{z_1}(\nu, Z), \ldots, \mathcal{D}_{z_m}(\nu, Z)). \end{aligned} \tag{7}$$

This evolution function verifies for all ν_1 and ν_2 in G and all Z in $\mathbb{K}^m$ the following relations:

$$\mathcal{D}\,(e, Z) = Z \ \text{and} \ \mathcal{D}\,(\nu_1 \star \nu_2, Z) = \mathcal{D}\big(\nu_1, \mathcal{D}\,(\nu_2, Z)\big). \tag{8}$$

An orbit *of a dynamical system $\mathcal{D}$ associated to an initial point Z in $\mathbb{K}^m$ is the following set of points:*

$$\mathrm{Orb}_Z := \{\mathcal{D}\,(\nu, Z) \mid \nu \in G\}. \tag{9}$$

HYPOTHESIS 8. *We suppose that $\mathcal{D}$ is at least of differentiability class C^1 w.r.t. initial conditions Z in $\mathbb{K}^m$ and thus, the following relation holds for ϵ in the neighbourhood of* 0 *in $\mathbb{R}$ and for all H in $\mathbb{K}^m$:*

$$\mathcal{D}\,(\nu, Z + \epsilon H) = \mathcal{D}\,(\nu, Z) + \epsilon \frac{\partial \mathcal{D}}{\partial Z}(\nu, Z)\, H + O\big(\epsilon^2\big). \tag{10}$$

In the next paragraphs, we explicit the relationship between dynamical systems and their encoding (associated pseudo-derivation).

Continuous case. For a continuous dynamical system $\mathcal{D}$, G is a Lie group i.e. an—additive—continuous group identified to a subgroup of $\mathbb{K}$. In that case, a continuous dynamical system defined by an ODS and associated to the derivation $\delta_{\mathcal{D}}$ is continuous w.r.t. group parameter ν; thus, it verifies the following property for all ν in G and ϵ in a neighbourhood of e in G:

$$\mathcal{D}\left(\nu+\epsilon, Z\right) = \mathcal{D}\left(\nu, Z\right) + \epsilon\, \delta_{\mathcal{D}} \mathcal{D}\left(\nu, Z\right) + O\left(\epsilon^2\right). \tag{11}$$

This analytic remark leads to the following algebraic relations in the framework of formal power series:

$$\delta_{\mathcal{D}} \cdot = \left.\frac{\partial \mathcal{D}(\tau, \cdot)}{\partial \tau}\right|_{\tau=0}, \mathcal{D}\left(\tau, Z\right) = e^{\tau \delta_{\mathcal{D}}} Z := \sum_{\imath \in \mathbb{N}} \frac{\tau^{\imath} \delta_{\mathcal{D}}{}^{\imath} Z}{\imath!}. \tag{12}$$

Discrete case. For a discrete dynamical system $\mathcal{D}$, G is a discrete group identified to $\mathbb{Z}$ and the flow is given by the discrete exponential map defined by iteration of the endomorphism σ:

$$\mathcal{D}\left(\nu, Z\right) = \sigma^{[\nu]} Z \quad \text{with} \quad \sigma^{[\nu]} = \sigma^{[\nu-1]} \circ \sigma,\ \sigma^{[0]} = \mathrm{Id}. \tag{13}$$

EXAMPLES 9. *For the examples 2:*

- *the continuous dynamical system associated to the system Σ is given by:*

$$\begin{array}{rcl} \mathcal{D}: \ (\mathbb{R},+) \times \mathbb{R}^4 & \rightarrow & \mathbb{R}^4, \\ \left(\tau, (t,x,a,b)\right) & \rightarrow & \left(t+\tau, \frac{a}{b+e^{-a\,\tau} x\, a}, a, b\right). \end{array} \tag{14}$$

- *the discrete dynamical system associated to the system Ω is given by:*

$$\begin{array}{rcl} \mathcal{D}: \ (\mathbb{Z},+) \times \mathbb{R}^4 & \rightarrow & \mathbb{R}^4, \\ \left(\eta, (n,x,y,c)\right) & \rightarrow & \left(n+\eta, x c^{\eta}, \frac{c^{\eta}-1}{c-1} x + y, c\right). \end{array} \tag{15}$$

REMARK 10. *In the sequel, we are going to avoid—as much as possible—to work with the evolution function $\mathcal{D}$ because we do not know in most cases how to compute one of its explicit—closed form—representation even if in theory, it is always possible by exponentiating the associated (pseudo-)derivation (see Theorem* 1.57 *in [11] for example). Instead, we work with the (pseudo-)derivation and consider it as an implicit representation of the evolution function.*

2.1.3 Rectification of an Infinitesimal Generator

DEFINITION 11. *Given an infinitesimal generator δ:*

- *a* principal element *p of δ is a function of the coordinates Z that satisfies the relation $\delta p = 1$;*
- *an* invariant *I of δ is a function of the coordinates Z satisfying the relation $\delta I = 0$.*

To obtain a *rectified form* of a derivation δ consists to find a new coordinates set composed by a *principal element* p and $m-1$ independent invariants $(I_1, \ldots, I_{m-1})$ satisfying:

$$\delta p = 1 \quad \text{and} \quad \delta I_{\imath} = 0,\ \forall \imath \in \{1, \ldots, m-1\}. \tag{16}$$

Such local coordinates always exist (see [11]) and in these *rectifying* coordinates, δ acts as the *translation* $\delta = \partial/\partial p$.

2.2 Lie Point Symmetries of Dynamical Systems and their Determining System

First, let us precise the definition of a symmetry:

DEFINITION 12. *Let us consider a dynamical system $\mathcal{D}$. A continuous dynamical system $\mathcal{S}$ is a* Lie point symmetry *of $\mathcal{D}$, if and only if, $\mathcal{S}$ sends an orbit of $\mathcal{D}$ on another orbit of $\mathcal{D}$. For the sake of simplicity, we do not consider in this note the symmetries acting on independent variables (see Remark 15). Hence, the Definition 12 reduces to the commutation of the diagram presented in Figure 1 i.e. to the relation:*

$$\mathcal{D}\left(\nu, \mathcal{S}\left(t, Z\right)\right) = \mathcal{S}\left(t, \mathcal{D}\left(\nu, Z\right)\right), \tag{17}$$

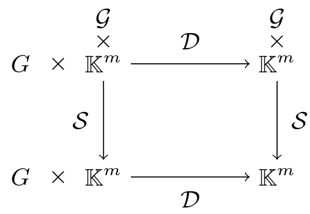

Figure 1: Diagram involving a dynamical system $\mathcal{D}$ and one of its Lie symmetries $\mathcal{S}$.

The two next sections show how the Definition 12 leads to partial differential equation (PDE) systems that determine existing symmetries in the discrete and continuous cases.

2.2.1 Discrete Case

In this section, we consider a discrete dynamical system $\mathcal{D}$ and its associated recurrence operator σ. In this situation, the group G is equal to $(\mathbb{Z},+)$. In any case $\mathcal{S}$ is a continuous dynamical system and thus, we always consider $\delta_{\mathcal{S}}$ its associated infinitesimal generator.

We saw above that $\mathcal{S}$ is an expanded Lie point symmetry of $\mathcal{D}$, if and only if, the defining system (17) holds. Following Lie's standpoint, one can deduce from the defining system (17), a *determining* system of PDEs by applying a linearisation process. As we suppose that $\mathcal{D}$ is of class C^1 w.r.t. initial conditions Z, the following relations hold:

$$\begin{array}{rcl} \mathcal{D}\left(n, \mathcal{S}\left(\epsilon, Z\right)\right) & = & \mathcal{D}\left(n, Z + \epsilon\, \delta_{\mathcal{S}} Z + O\left(\epsilon^2\right)\right), \\ & = & \mathcal{D}\left(n, Z\right) + \epsilon\, \frac{\partial \mathcal{D}(n,Z)}{\partial Z} \delta_{\mathcal{S}} Z + O\left(\epsilon^2\right). \end{array} \tag{18}$$

Moreover, as $\mathcal{S}$ is a continuous dynamical symmetry, the relation (11) leads to the following relations:

$$\begin{array}{rcl} \mathcal{D}\left(n, \mathcal{S}\left(\epsilon, Z\right)\right) & = & \mathcal{S}\left(\epsilon, \mathcal{D}(n, Z)\right), \\ & = & \mathcal{D}(n, Z) + \epsilon\, \delta_{\mathcal{S}} \mathcal{D}(n, Z) + O\left(\epsilon^2\right). \end{array} \tag{19}$$

Hence, for all ν in G the following equation holds:

$$\frac{\partial \mathcal{D}(\nu, Z)}{\partial Z} \delta_{\mathcal{S}} Z = \delta_{\mathcal{S}} \mathcal{D}(\nu, Z). \tag{20}$$

In the case of a discrete dynamical system, these relations are equivalent to $(\partial \sigma^{[n]} Z / \partial Z) \delta_{\mathcal{S}} Z = \sigma^{[n]} \delta_{\mathcal{S}} Z$. Thus, the *determining system* of a Lie point symmetry of an ORS is given for $n = 1$ by the equation:

$$\frac{\partial \sigma Z}{\partial Z} \delta_{\mathcal{S}} Z = \sigma\, \delta_{\mathcal{S}} Z. \tag{21}$$

A further linearisation does not bring new information in discrete case but we show in the next section that this is not true in the continuous case.

2.2.2 Continuous Case

When the considered dynamical system $\mathcal{D}$ is continuous, the group G is supposed to be isomorphic to a subgroup of $(\mathbb{K},+)$ and we work with the infinitesimal generator $\delta_{\mathcal{D}}$ associated to $\mathcal{D}$.

As we do not use the discrete behaviour of the dynamical system $\mathcal{D}$ in the relations of commutation (20), they are also valid for a continuous dynamical system. But, as $\mathcal{D}$ is also a continuous dynamical system, we could preform another linearisation and thus, show that the following relations hold:

$$\begin{aligned} \tfrac{\partial \mathcal{D}(\epsilon,Z)}{\partial Z}\delta_{\mathcal{S}} Z &= \left(\tfrac{\partial}{\partial Z}\left(Z+\epsilon\,\delta_{\mathcal{D}} Z + O(\epsilon^2)\right)\right)\delta_{\mathcal{S}} Z, \\ &= \delta_{\mathcal{S}} Z + \epsilon\,\tfrac{\partial \delta_{\mathcal{D}} Z}{\partial Z}\delta_{\mathcal{S}} Z + O(\epsilon^2), \end{aligned} \tag{22}$$

$$\begin{aligned} \delta_{\mathcal{S}}\mathcal{D}(\epsilon, Z) &= \delta_{\mathcal{S}}\left(Z+\epsilon\,\delta_{\mathcal{D}} Z + O(\epsilon^2)\right), \\ &= \delta_{\mathcal{S}} Z + \epsilon\,\tfrac{\partial \delta_{\mathcal{S}} Z}{\partial Z}\delta_{\mathcal{D}} Z + O(\epsilon^2). \end{aligned} \tag{23}$$

Finally, we deduce from the relations (22) and (23) the system that determines the Lie point symmetries of an ODS:

$$\frac{\partial \delta_{\mathcal{D}} Z}{\partial Z}\delta_{\mathcal{S}} Z = \frac{\partial \delta_{\mathcal{S}} Z}{\partial Z}\delta_{\mathcal{D}} Z. \tag{24}$$

For a slightly different presentation of the discrete case and more references, see [5, 6]; we refer to [11] for the general treatment of the continuous case. Usually in the literature—and contrary to above presentation, the definitions of determining systems (21) and (24) are disconnected. The next section is devoted to show how these determining systems could be summarised in a single algebraic relation.

2.2.3 Algebraic Definition of Determining System

A bracket could be defined on the set of pseudo-derivations as follows:

DEFINITION 13. *The* bracket *of two pseudo-derivations is defined by the* $\mathbb{K}$*-bilinear skew-symmetric map:*

$$\begin{aligned} [\ ,\ ]:\ \operatorname{Ore}\mathbb{K}[Z]\times\operatorname{Ore}\mathbb{K}[Z] &\rightarrow \operatorname{Ore}\mathbb{K}[Z], \\ (\delta_1,\delta_2) &\rightarrow \delta_1\delta_2-\delta_2\delta_1. \end{aligned} \tag{25}$$

In continuous and discrete cases, systems (21) and (24) that determine a continuous symmetry could be summarised by a simple algebraic relation involving the associated pseudo-derivations and their bracket. In fact, the computations made in above sections show the following result:

LEMMA 14. *Given a dynamical system $\mathcal{D}$ and its associated pseudo-derivation $\delta_{\mathcal{D}}$, a derivation $\delta_{\mathcal{S}}$ is associated to a continuous dynamical system that is a symmetry of $\mathcal{D}$ if, and only if, the relation $[\delta_{\mathcal{D}},\delta_{\mathcal{S}}]=0$ holds.*

REMARK 15. *For the sake of simplicity, we restrict ourselves in this note to the study of symmetries that do not affect independent variable of the considered dynamical system (i.e. the bracket $[\delta_{\mathcal{D}},\delta_{\mathcal{S}}]$ is equal to 0); the same study could be made and the results presented in the sequel remain valid when the independent variable is affected by the symmetry (i.e. when $[\delta_{\mathcal{D}},\delta_{\mathcal{S}}]=\lambda\delta_{\mathcal{D}}$ with $\delta_{\mathcal{D}}\lambda=0$; see [12] for an example in the continuous case).*

2.3 Classical Reduction Process

Hereafter, we make the following assumptions (see [11]).

HYPOTHESIS 16. *Let $\mathcal{S}$ be a continuous dynamical system considered as a local group of transformation acting on $\mathbb{K}^m$. We suppose that this action is regular i.e. for all points Z_1 and Z_2 in $\mathbb{K}^m$, there exists exactly one ν in $\mathcal{G}$ such that the relation $\mathcal{S}(\nu, Z_1)=Z_2$ holds. Remark that in this case, $\mathcal{S}$ acts locally freely on $\mathbb{K}^m$ i.e. the orbits of $\mathcal{S}$ are all of the same dimension as $\mathcal{G}$ itself.*

This assumption is not restrictive for our applications and allows to state the following result:

THEOREM 17. *(see Section 3.4 in [11]). Let $\mathcal{G}$ be of dimension s and so the associated orbits of $\mathcal{S}$. Then, there exists a $(m-s)$-dimensional quotient space of $\mathbb{K}^m$ by the action of $\mathcal{S}$, denoted $\mathbb{K}^m/\mathcal{G}$, together with a smooth projection $\pi:\mathbb{K}^m\rightarrow\mathbb{K}^m/\mathcal{G}$, that satisfy the following properties:*

- *The points Z_1 and Z_2 lie in the same orbit of $\mathcal{S}$ in $\mathbb{K}^m$, if and only if, $\pi(Z_1)$ is equal to $\pi(Z_2)$.*
- *If $\mathcal{L}$ denotes the Lie algebra (see [11]) of infinitesimal generators associated to the action of $\mathcal{S}$, then the linear map $\mathrm{d}\pi: T\mathbb{K}^m|_Z\rightarrow T(\mathbb{K}^m/\mathcal{G})|_{\pi(Z)}$ between tangent spaces is surjective, with kernel $\{v|_Z : v\in\mathcal{L}\}$.*

From a computational standpoint, when a dynamical system $\mathcal{D}$ presents a symmetry, its reduction is performed by the rectification of the symmetry $\mathcal{S}$ as shown below.

EXAMPLE 18. *Let us consider the recurrence system Ω given in the Examples 2. The infinitesimal generator associated to one of its symmetries is:*

$$\delta_{\mathcal{S}} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{26}$$

Classically, the reduction of $\mathcal{D}$ requires two steps:

1. ***Invariants computation.*** *As (26) is a scaling, the computation of its associated invariants $N:=n, C:=c$ and $Y:=y/x$ is straightforward.*
2. ***Elimination step.*** *In order to* eliminate *x, we are going to express σY and σC in terms of these invariants. The recurrence operator σ is an endomorphism, thus we obtain $\sigma C = C, \sigma N = N+1$ and*

$$\sigma Y = \frac{\sigma y}{\sigma x} = \frac{(y/x)+1}{cx}x = \frac{Y+1}{C}. \tag{27}$$

Thus, the reduced ORS associated to Ω is:

$$\widetilde{\Omega}\ :\ \widetilde{\sigma}\,Y = (Y+1)/C, \quad \widetilde{\sigma}\,C = C. \tag{28}$$

The reduced operator $\widetilde{\sigma}$ obtained above, acts on the state-space denoted by $\mathbb{K}^m/\mathcal{G}$ in the Theorem 17 (a.k.a. the state-space $\mathbb{K}^m$ quotiented by the orbits of $\mathcal{S}$). The same type of computations hold for a continuous dynamical system.

EXAMPLE 19. *Let us consider the following ODS:*

$$\frac{\mathrm{d}t}{\mathrm{d}t}=1, \quad \frac{\mathrm{d}x}{\mathrm{d}t}=y, \quad \frac{\mathrm{d}y}{\mathrm{d}t}=\frac{y}{x}. \tag{29}$$

1. ***Invariants Computation.*** *The corresponding infinitesimal generator $\delta_{\mathcal{D}}$ and one of its symmetries $\delta_{\mathcal{S}}$ are:*

$$\delta_{\mathcal{D}} := \frac{\partial}{\partial t} + y\frac{\partial}{\partial x} + \frac{y}{x}\frac{\partial}{\partial y}, \quad \delta_{\mathcal{S}} := t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}. \tag{30}$$

As $\delta_{\mathcal{S}}$ is a scaling, in that special case the computation of its invariants $T:=t/x$ and $Y:=y$ is simple.

2. ***Elimination step.*** *Classical reduction could be performed in order to* eliminate *x exactly with the same type of computation done above (rewrite $\delta_{\mathcal{D}} Y$ and $\delta_{\mathcal{D}} T$ w.r.t. Y and T). But, one could also use the dual form of the infinitesimal generator $\delta_{\mathcal{D}}$:*

$$\mathrm{d}x = y\,\mathrm{d}t, \quad x\,\mathrm{d}y = y\,\mathrm{d}t, \tag{31}$$

as follows. The exterior derivation of T implies that the relation $x\,\mathrm{d}T + T\,\mathrm{d}x = \mathrm{d}t$ holds; using (31), one can eliminate $\mathrm{d}x$ in this relation to obtain:

$$x\,\mathrm{d}T + T\,x\,\mathrm{d}y = \frac{x\,\mathrm{d}y}{y}. \tag{32}$$

Using explicit definition of the invariants T and Y, the variable x can be eliminated from (32) to obtain:

$$Y\,\mathrm{d}T = (1 - T\,Y)\,\mathrm{d}Y. \tag{33}$$

Thus, the associated reduced ODS is associated to the following infinitesimal generator:

$$\widetilde{\delta_{\mathcal{D}}} = (1 - T\,Y)\,\frac{\partial}{\partial T} + Y\,\frac{\partial}{\partial Y}. \tag{34}$$

In the next section, we show how to perform the same type of reduction but without computing explicitly the invariants and avoiding as much as possible nonlinear elimination steps.

3. STRATEGIES TO DEAL WITH HIGH COMPLEXITY PROBLEMS

The previous sections introduced all the definitions necessary to describe each steps of the classical and most general reduction process. Now, let us describe how this process is handled in our work.

REMARK 20. *The input considered in this note is not the explicit action of a transformation group $\mathcal{S}$ as done usually in the literature (see [9, 4] for examples) but only a set of equations representing the considered dynamical system $\mathcal{D}$. In the implementation [13], a continuous dynamical system is encoded by its associated infinitesimal generator and a discrete one by the associated recurrence endomorphism.*

Thus, under these assumptions and using classical strategies,

- given $\mathcal{D}$, we must compute a symmetry of $\mathcal{D}$ first i.e. find a solution to the PDE system (24) or (21); then,
- the rectification of $\mathcal{S}$ requires the resolution of another system of PDE given in (16) in order to determine a new coordinates set composed by a principal element and a set of invariants; finally,
- using elimination techniques, the original problem is rewritten w.r.t. to this new coordinates set.

In whole generality, the resolution of each of these tasks is *difficult* i.e. it could not be done by algorithms with polynomial time complexity in input *size*. The forthcoming sections summarise the strategies used in the MAPLE package [13] to overcome these difficulties i.e. to reduce the computation cost to polynomial time complexity and thus, to be able to treat large problems occurring in practice.

3.1 Restrict the Set of Searched Symmetries

Given a dynamical system, the first task performed by our package is to determine its symmetries. To do so, the class of solutions computed by our implementation is initially restricted to the classical geometric transformations (dilatation, scaling, rotation, etc). Then, larger solutions classes are considered (inversion, Möbius transforms, etc).

This is done by first requesting that infinitesimal generators encoding the searched symmetries have coefficients linear w.r.t. the coordinate set Z (classical geometric transformation); then quadratic coefficients are considered (Möbius transforms), etc.

These incremental restrictions on the searched symmetries are motivated by the type of symmetries occurring in practice and by the fact that the reduction process could be done incrementally using one symmetry after the other (under suitable solvability condition on the resulting Lie algebra of symmetry (see Definition 2.63 in [11]) that are generally fulfilled in applications).

In the next section, we show—on a particular example—that these type of restriction on the solution class of determining system (24)–(21) reduces the computation to linear algebra over a constant effective field using the classical method of undetermined coefficients.

3.1.1 *Example of Affine Infinitesimal Generators*

From now, we focus our attention on symmetries associated with *affine infinitesimal generators* but almost all the results remains valid with more general classes of generators (with quadratic coefficients, etc).

DEFINITION 21. *Let us denote by $\mathrm{AffDer}_{\mathbb{K}}\mathbb{K}[Z]$ the following set of derivations:*

$$\left\{ \sum_{z\in Z} \delta_{\mathcal{S}} z \frac{\partial}{\partial z} \;\middle|\; \delta_{\mathcal{S}} z := b_z + \sum_{z'\in Z} a_{zz'} z',\ (b_z, a_{zz'}) \in \mathbb{K}^{m+1} \right\}. \tag{35}$$

First, let us introduce some notations. Given a derivation $\delta_{\mathcal{S}}$ in $\mathrm{AffDer}_{\mathbb{K}}\mathbb{K}[Z]$, we are going to consider Z in the sequel as a vector and use the following matricial notations:

$$\mathcal{A}_{\delta_{\mathcal{S}}} = (a_{zz'})_{(z,z')\in Z^2} \quad \text{and} \quad \mathcal{B}_{\delta_{\mathcal{S}}} = (b_z)_{z\in Z}. \tag{36}$$

Now, we could explicitly encodes the determining systems considered by our implementation as a linear problem:

LEMMA 22. *Given Σ (resp. Ω) an ODS (resp. ORS) defined by the relations (2) (resp. (3)), the determining system (24) (resp. (21)) of its Lie point symmetries associated to an affine infinitesimal generator $\delta_{\mathcal{S}}$ in $\mathrm{AffDer}_{\mathbb{K}}\mathbb{K}[Z]$ is given by the linear system (37) (resp. (38)):*

$$\begin{pmatrix} \frac{\partial f_1}{\partial z_1} & \dots & \frac{\partial f_1}{\partial z_m} \\ \vdots & & \vdots \\ \frac{\partial f_k}{\partial z_1} & \dots & \frac{\partial f_k}{\partial z_m} \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \begin{pmatrix} \mathcal{A}_{\delta_{\mathcal{S}}} Z \\ + \\ \mathcal{B}_{\delta_{\mathcal{S}}} \end{pmatrix} = \mathcal{A}_{\delta_{\mathcal{S}}} \begin{pmatrix} 1 \\ f_1 \\ \vdots \\ f_k \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{37}$$

$$\begin{pmatrix} \frac{\partial f_1}{\partial n} & \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_k} & \frac{\partial f_1}{\partial \theta_1} & \dots & \frac{\partial f_1}{\partial \theta_\ell} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial f_k}{\partial n} & \frac{\partial f_k}{\partial x_1} & \dots & \frac{\partial f_k}{\partial x_k} & \frac{\partial f_k}{\partial \theta_1} & \dots & \frac{\partial f_k}{\partial \theta_\ell} \\ 0 & 0 & \dots & 0 & 1 & & 0 \\ \vdots & \vdots & & \vdots & & \ddots & \\ 0 & 0 & \dots & 0 & 0 & & 1 \end{pmatrix} \begin{pmatrix} \mathcal{A}_{\delta_{\mathcal{S}}} Z \\ + \\ \mathcal{B}_{\delta_{\mathcal{S}}} \end{pmatrix} = \mathcal{A}_{\delta_{\mathcal{S}}} \begin{pmatrix} n+1 \\ f_1 \\ \vdots \\ f_k \\ \theta_1 \\ \vdots \\ \theta_\ell \end{pmatrix} + \mathcal{B}_{\delta_{\mathcal{S}}}. \tag{38}$$

The next section is devoted to show how the determining systems (37) and (38) are solved in our implementation by classical probabilistic numerical methods.

Algorithm 1 Solving determining systems (37) and (38)

Require: A list L of equations which are linear w.r.t. the coefficients of $\mathcal{A}_{\delta_\mathcal{S}}$ and $\mathcal{B}_{\delta_\mathcal{S}}$. The list of coordinates Z. A list of indeterminates $U := (a_{zz'}, b_z)_{(z,z')\in Z^2}$.

Ensure: A list of affine infinitesimal generators represented by coordinates in basis $\{\partial/\partial z\}$, which are solutions of L.

```
# Rewrite L in the form N tU = 0
N := Matrix (seq (seq (coeff (ℓ, u) , ℓ ∈ L) , u ∈ U));
r := −1;  M := ∅;
repeat
    C := {seq (RandomInteger(), i = 1 . . . m)};
    s := r;
    M := StackMatrix (M, subs (Z = C, N));
    r := Rank (M);
until r ≠ s;
V := Basis (M); # Kernel computation of N
return [seq (subs(u = v, A_{δ_S} Z + B_{δ_S}), v ∈ V)];
```

3.1.2 Numerical Computations

One can rewrite the determining system (37) and (38) in matricial form $N\,{}^tU = 0$ where N is a $(m-1)\times(m+1)\,m$ matrix with coefficients in $\mathbb{K}[Z]$ and U is an unknown list of $(m+1)\,m$ elements composing $\mathcal{A}_{\delta_\mathcal{S}}$ and $\mathcal{B}_{\delta_\mathcal{S}}$. As this system is under-determined, several specialisations of the coordinates Z to random values in $\mathbb{K}$, are necessary to obtain at most $(m+1)\,m$ linear equations. The kernel in $\mathbb{K}$ of the resulting purely numerical system gives the $\mathbb{K}$-vector-space V of affine infinitesimal generators that are solutions of the considered determining system. This kernel is computed using any classical numerical method. This strategy is summarised in Algorithm 1 using a MAPLE like syntax.

The considerations presented in this section allow to compute the infinitesimal generators encoding the symmetries of the dynamical systems considered in a large set of applications. The rest of this note is devoted to expose the strategies used in our implementation to handle the reduction process. But before doing so, the next section presents some necessary definitions.

3.2 Intermezzo: Moving Frame Based Invariant Computation

3.2.1 Cross-section

DEFINITION 23. *Given an s-dimensional group of transformation $\mathcal{S}$ acting regularly on $\mathbb{K}^m$, a cross-section $\mathcal{H}$[1] of dimension $m-s$ is a variety of $\mathbb{K}^m$ that locally intersects transversely (i.e. $\delta_\mathcal{S}\mathcal{H} \neq 0$) the orbits of $\mathcal{S}$ at least in one point (see [2, 4] and the references therein).*

REMARK 24. *In our implementation, the group of transformation $\mathcal{S}$ is of dimension 1 and is associated to the infinitesimal generator $\delta_\mathcal{S}$. Furthermore, we choose a linear—w.r.t. the variables Z—cross-section $\mathcal{H}$ of dimension $m-1$.*

3.2.2 An Artless Presentation of Moving Frames

Given a group of transformation $\mathcal{S}$ acting regularly on $\mathbb{K}^m$ and one of its cross-section $\mathcal{H}$, we are going to define the notion of *moving frame* in a very restricted framework by considering a compatible projection $\pi : \mathbb{K}^m \to \mathcal{H}$ (see [2] for a general definition). With this projection, a point Z in $\mathbb{K}^m$ is sent to a base point $Z_\mathcal{H} := \pi(Z)$ in the cross-section $\mathcal{H}$ through the orbits of $\mathcal{S}$ (see Figure 3).

[1]Same letters denote cross-sections and their equations.

DEFINITION 25. *Let us consider a continuous dynamical system $\mathcal{S}$ acting regularly on $\mathbb{K}^m$, $\mathcal{H}$ one of its cross-section; let Z be a point in $\mathbb{K}^m$ and Orb_Z the orbit of $\mathcal{S}$ passing by this point. Let $Z_\mathcal{H}$ be a base point associated to Z i.e. the intersection of $\mathcal{H}$ and Orb_Z. The smooth map $\rho : \mathbb{K}^m \to \mathcal{G}$ is a* moving frame *if it satisfies the relation $\mathcal{S}(\rho(Z), Z) = Z_\mathcal{H}$.*

In order to explicit the relationship of this definition and the rectification process, let us remark that a moving frame ρ is a $\mathcal{G}$-equivariant map (see [2]); this means that the following relation holds in $\mathcal{G}$:

$$\forall (Z,\nu) \in \mathbb{K}^m \times \mathcal{G}, \quad \nu \star \rho(Z) = \rho\big(\mathcal{S}(\nu, Z)\big). \tag{39}$$

Thus, the equivariance of ρ implies the commutation property of Figure 2. The following lemma explicit a relationship

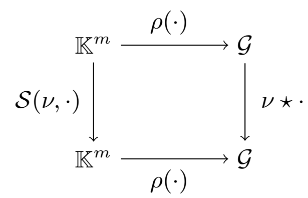

Figure 2: $\mathcal{G}$-equivariance of the moving frame ρ.

between moving frames of a continuous dynamical system $\mathcal{S}$ and its principal elements.

LEMMA 26. *Let $\rho : \mathbb{K}^m \to \mathcal{G}$ be a moving frame associated to continuous dynamical system $\mathcal{S}$ and to its infinitesimal generator $\delta_\mathcal{S}$. The moving frame ρ is a principal element of $\delta_\mathcal{S}$ i.e. the relation $\delta_\mathcal{S}\rho = 1$ holds.*

PROOF. *The definition (39) of equivariance of ρ in $\mathcal{G}$ induces the following one in $\mathbb{K}^m$:*

$$\mathcal{S}\Big(\epsilon, \mathcal{S}\big(\rho(Z), Z\big)\Big) = \mathcal{S}\Big(\rho\big(\mathcal{S}(\epsilon, Z)\big), Z\Big). \tag{40}$$

Again, we are going to consider the linearisation of this relation. The right hand side of relation (40) is equal to:

$$\begin{aligned} &\mathcal{S}\Big(\rho\Big(Z + \epsilon\delta_\mathcal{S}Z + O\big(\epsilon^2\big)\Big), Z\Big), \\ &\mathcal{S}\Big(\rho(Z) + \epsilon\tfrac{\partial\rho}{\partial Z}(Z)\delta_\mathcal{S}Z + O\big(\epsilon^2\big), Z\Big), \\ &\mathcal{S}\big(\rho(Z), Z\big) + \epsilon\tfrac{\partial\rho}{\partial Z}(Z)\delta_\mathcal{S}Z\delta_\mathcal{S}\mathcal{S}\big(\rho(Z), Z\big) + O\big(\epsilon^2\big). \end{aligned} \tag{41}$$

Moreover, the left hand side of the relation (40) is equal to: $\mathcal{S}(\rho(Z), Z) + \epsilon\delta_\mathcal{S}\mathcal{S}(\rho(Z), Z) + O\big(\epsilon^2\big)$. The equality of this expression with the expression (41) implies the following relation $\partial\rho/\partial Z\delta_\mathcal{S}Z = 1$, that is the relation $\delta_\mathcal{S}\rho = 1$. □

In the next section, we recall that the invariants of $\mathcal{S}$ may be computed by a *normalisation method* based on moving frames and presented in [2]. The key point for our purposes is that this method does not require the resolution of a PDE system when a moving frame is known.

3.2.3 Moving based Invariant Computation

We refer to [2, 4] for a complete presentation of the moving based invariant computation process and we just recall it there using the Example 19. To do so, let us consider the symmetry with infinitesimal generator $\delta_{\mathcal{S}}$ given in (30). The explicit form of the associated transformation group $\mathcal{S}$ is

$$(\mathcal{S}_t, \mathcal{S}_x, \mathcal{S}_y) := \mathcal{S}(\nu, (t, x, y)) = (t \exp \nu, x \exp \nu, y), \quad (42)$$

where ν is the group parameter of $\mathcal{G} \simeq \mathbb{R}$. We arbitrarily choose to work with the cross-section $\mathcal{H}$ defined by the hyperplane of equation $x - 1 = 0$. In this situation, the moving frame ρ send the point (t, x, y) to $-\log x$ in $\mathcal{G}$ if x is positive. This moving frame is obtained by solving the *normalisation* equation $\mathcal{S}_x(\rho(\nu), (t, x, y)) - 1 = 0$ with respect to ν. A set of independent invariants is found by substituting this moving frame into the transformation group. Hence, the invariant T of $\mathcal{S}$ is equal to $\mathcal{S}_t(-\log x, (t, x, y))$ that is t/x.

From a naive geometrical standpoint, the moving frame associated to a cross-section $\mathcal{H}$ is a function $\rho(Z)$ such that the generic point $Z := (t, x, y)$ is projected by the application $\mathcal{S}(\rho(\cdot), \cdot)$ on $\mathcal{H}$ and the invariants $\widetilde{Z}$ of $\mathcal{S}$ are the coordinates induced by this projection on $\mathcal{H}$ i.e. $\widetilde{Z} = \mathcal{S}(\rho(Z), Z)$.

Using notations of Theorem 17, $\mathcal{H}$ is a covering variety of the quotient space $\mathbb{K}^m/\mathcal{G}$ and the projection π is given by the map $\mathcal{S}(\rho(\cdot), \cdot)$. Our implementation follows exactly this strategy and the next section is devoted to precise this point. An interesting fact presented below is that, in the case of a continuous dynamical system, the moving frame method leads to a reduction process for which, neither the moving frame, nor the invariants need to be computed.

3.3 Moving Frame Based Reduction Process

Let $\mathcal{D}$ be a dynamical system considered as an input of our reduction process (see Remark 20) and suppose that we know $\mathcal{S}$ one of its symmetries (computed for example using the method presented in Section 3.1). We present in this section the reduction process whose output is the reduced system $\widetilde{\mathcal{D}}$ in which a variable z (which is not a constant of $\delta_{\mathcal{S}}$) was *eliminated* (see examples below); this process uses a cross-section $\mathcal{H}$ and the associated moving frame ρ. It is based on the following relation:

$$\forall Z \in \mathcal{H}, \quad \widetilde{\mathcal{D}}(\cdot, Z) = \mathcal{S}\Big(\rho\big(\mathcal{D}(\cdot, Z)\big), \mathcal{D}(\cdot, Z)\Big), \quad (43)$$

which is illustrated by the Figure 3 and is similar to the invariant computation described in Section 3.2.3. Several situations could be considered to illustrate this point.

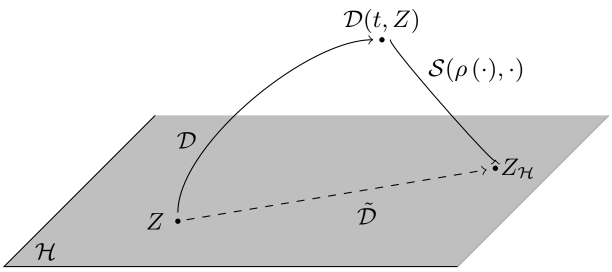

Figure 3: Moving frame based reduction process.

3.3.1 *Elimination of z if $\delta_{\mathcal{D}} z$ is equal to 0*

In this situation[2] z is a constant of $\mathcal{D}$ i.e. $\mathcal{D}_z(\nu, Z)$ is equal to z for all ν in G and z is supposed to be a parameter of $\mathcal{D}$. Hence, given any cross-section $\mathcal{H}$ of $\mathcal{S}$ verifying $\delta_{\mathcal{D}}\mathcal{H} = 0$ and $\partial\mathcal{H}/\partial z \neq 0$, the flow of $\mathcal{D}$ maps any point of $\mathcal{H}$ to another point of this cross-section. Furthermore, the reduction process is a simple specialisation if $\mathcal{H}$ is linear w.r.t. z.

In fact, relation (43) shows that the corresponding reduced ODS/ORS denoted by $\widetilde{\mathcal{D}}$ is the image of $\mathcal{D}$ by the compatible projection $\mathcal{S}(\rho(\cdot), \cdot)$ already considered in Section 3.2.3. The following example illustrate this process:

Example 27. *The Verhulst's model (1) presented in Example 2 presents a symmetry $\mathcal{S}$ with infinitesimal generator:*

$$\delta_{\mathcal{S}} = b\frac{\partial}{\partial b} - x\frac{\partial}{\partial x}. \quad (44)$$

In order to eliminate *the parameter b, we choose arbitrarily the hyperplane of equation $b = 1$ but we do not compute explicitly the projection map $\mathcal{S}(\rho(\cdot), \cdot)$ except for b whose image is* 1 *by definition. In fact, as $\mathcal{S}$ is a symmetry of $\mathcal{D}$, the commutation properties and relation (43) show that the map $\widetilde{\mathcal{D}}(\cdot, Z)$ is equal to $\mathcal{D}(\cdot, \mathcal{S}(\rho(Z), Z))$. The implicit expressions $\mathcal{S}(\rho(Z), Z)$ are renamed as $\widetilde{Z}$. Hence, we just have to specialise b to* 1 *in relation (1) to obtain:*

$$\widetilde{\Sigma} \;:\; \frac{\mathrm{d}\widetilde{t}}{\mathrm{d}t} = 1, \quad \frac{\mathrm{d}\widetilde{x}}{\mathrm{d}t} = (\widetilde{a} - \widetilde{x})\,\widetilde{x}, \quad \frac{\mathrm{d}\widetilde{a}}{\mathrm{d}t} = 0. \quad (45)$$

The data (45) and (44) are the outputs of the reduction process: the first one is the reduced dynamic and the second one encodes the fibre of the map $\mathcal{S}(\cdot, \widetilde{Z})$ that allows to retrieve orbits of the original dynamical system from orbits of the reduced one. In Example 27, the derivation (44) is associated to a differential systems that is an implicit definition of the fibre $\widetilde{x} = bx$, $\widetilde{t} = t$ and $\widetilde{a} = a$ that allows to express the original problem from the reduced one.

Hence, the moving frame method allows to avoid the difficult computations (explicit computation of the fibre a.k.a. invariant computations) and to perform the reduction process using only implicit representations (the infinitesimal generator of the symmetry a.k.a. the map the original problem to the reduced one) that could be numerically exploited.

In next section, we show that the strategies sketched above could be extended to more complicated situations.

3.3.2 *Elimination of z if $\delta_{\mathcal{D}} z$ is different from 0*

Given an input dynamical system $\mathcal{D}$, one of its symmetries $\mathcal{S}$ and any cross-section $\mathcal{H}$ of $\mathcal{S}$ verifying $\delta_{\mathcal{D}}\mathcal{H} = 0$ and $\partial\mathcal{H}/\partial z \neq 0$, we consider now the case when the orbit of $\mathcal{D}$ associated to a point Z in $\mathcal{H}$ does not stay in $\mathcal{H}$ i.e. $\delta_{\mathcal{D}}\mathcal{H} \neq 0$ as shown in the Figure 3.

First, we must precise that we were not able to avoid the computation of invariants in all cases. Hence, the first paragraph of this section states that we failed to obtain a low complexity algorithm for the reduction of ORS. Hence, this part of our code requires explicit integration, invariant computation and elimination methods.

Ordinary Recurrence Systems. The part of our code that allows to eliminate a variable of an ORS follows exactly the strategy explained in the section 3.2.3 and computes local invariants as proposed in [2] and implemented in [3].

We illustrate this point by explaining how our code treats the Example 18. The input of the reduction process is the recurrence endomorphism defined by the ORS (4). Following the strategy presented in Section 3.1, the user must provide $\mathcal{S}$ the group action defined by $\delta_{\mathcal{S}}$ the infinitesimal generator (26) explicitly and not only this infinitesimal generator

[2] Remark that $\mathcal{D}$ (resp. $\delta_{\mathcal{D}}$) could be continuous (resp. a derivation) or discrete (resp. a pseudo-derivation).

as in the other cases. Necessary elimination tools, that handle computations similar to those described in section 2.3, are already implemented in MAPLE and thus, we just use them in our implementation; we gives below a short MAPLE code using AIDA (see [3]) that illustrates theses points:

```
# cs_rat_inv & rewrite are 2 procedures of aida
> z := [n,x,y,c]: # coordinates set
> lambda := [p,q]: # group parameters
> G := [p*q-1]: # Rabinowitz' trick i.e. p<>0
> sigma := [n+1,c*x,y+x,c]: # recurrence operator
> Symmetry := [n,x*p,y*p,c]: # its used symmetry
# invariants computation with cross-section x-1
> inv := cs_rat_inv(G,Symmetry,[x-1],z,lambda,r):
# change data-structure
> map2(subs,zip(`=`,z,sigma),map(rhs,inv[1])):
# elimination producing the reduced recurrence
> lprint(map(rhs,map(rewrite,%,z,inv)));
[1+r1, r2, (1+r3)/r2] # where r1=n, r2=c and r3=Y
```

Some application could not be handled following this strategy because, as mentioned in the Remark 10, in some cases we do not know how to compute the group action induced by the symmetries found by our implementation. Furthermore, the high complexity of the elimination step becomes a real concern when dealing with large applications.

Fortunately, the situation is completely different when dealing with ODS; in that case, the reduction process could be done without computing any explicit group action, moving frame or invariant.

Ordinary Differential Systems. We are going in this section to use the relation (43) expressing the searched reduced continuous dynamical system $\widetilde{\mathcal{D}}$ w.r.t. the original one $\mathcal{D}$. But this time, we are interested in an infinitesimal expression of this relation in order to determine the infinitesimal generator $\delta_{\widetilde{\mathcal{D}}}$ encoding $\widetilde{\mathcal{D}}$.

First, remark that any point Z in $\mathcal{H}$ satisfies the relations $\mathcal{H}(Z) = 0$ and $\mathcal{S}(\rho(Z), Z) = Z$. Furthermore, for any ϵ in the neighbourhood of 0 in $\mathbb{R}$, the following relation holds:

$$\mathcal{H}\Big(\mathcal{S}\Big(\rho\big(\mathcal{D}(\epsilon, Z)\big), \mathcal{D}(\epsilon, Z)\Big)\Big) = 0. \tag{46}$$

Keeping in mind that $\mathcal{S}$ is a symmetry of $\mathcal{D}$ (i.e. these evolution functions commute) and the properties (10) & (11), the right hand side of relation (46) is equal to:

$$\begin{array}{c} \mathcal{H}\big(\mathcal{D}\big(\epsilon, \mathcal{S}\left(\rho\left(\mathcal{D}(\epsilon, Z)\right), Z\right)\big)\big), \\ \mathcal{H}\big(\mathcal{D}\big(\epsilon, \mathcal{S}\left(\rho\left(Z\right) + \epsilon \frac{\partial \rho}{\partial Z}(Z)\, \delta_{\mathcal{D}} Z + O\big(\epsilon^2\big), Z\right)\big)\big), \\ \mathcal{H}\big(\mathcal{D}\big(\epsilon, Z + \epsilon \frac{\partial \rho}{\partial Z}(Z)\, \delta_{\mathcal{D}} Z\, \delta_{\mathcal{S}} Z + O\big(\epsilon^2\big)\big)\big), \\ \mathcal{H}\big(Z + \epsilon \frac{\partial \rho}{\partial Z}(Z)\, \delta_{\mathcal{D}} Z\, \delta_{\mathcal{S}} Z + \epsilon \delta_{\mathcal{D}} Z + O\big(\epsilon^2\big)\big), \\ \mathcal{H}(Z) + \epsilon \frac{\partial \mathcal{H}}{\partial Z}(Z)\left(\frac{\partial \rho}{\partial Z}(Z)\, \delta_{\mathcal{D}} Z\, \delta_{\mathcal{S}} Z + \delta_{\mathcal{D}} Z\right) + O\big(\epsilon^2\big). \end{array} \tag{47}$$

As $\mathcal{H}(Z)$ is equal to 0, the reduced continuous dynamical system $\widetilde{\mathcal{D}}$ has $\delta_{\mathcal{D}} + \mu\delta_{\mathcal{S}}$ as infinitesimal generator, where μ is equal to $\delta_{\mathcal{D}}\rho(Z)$ which is unknown. This shows that z, $\mathcal{H}$ and μ must verify the following conditions:

$$(\delta_{\mathcal{D}} + \mu\delta_{\mathcal{S}})\, z = 0, \quad (\delta_{\mathcal{D}} + \mu\delta_{\mathcal{S}})\, \mathcal{H} = 0. \tag{48}$$

The first condition just states that the variable z was chosen to be eliminated from $\delta_{\mathcal{D}}$. Thus μ is equal to $-\delta_{\mathcal{D}} z/\delta_{\mathcal{S}} z$. The second above condition states that $\widetilde{\mathcal{D}}$ maps a point of $\mathcal{H}$ to another point of this variety. In order to illustrate this reduction process, let us consider again the Example 19:

EXAMPLE 28. *If we choose to eliminate the variable x, then μ is equal to $-y/x$ and $\delta_{\widetilde{\mathcal{D}}}$ is equal to:*

$$\left(1 - \frac{ty}{x}\right)\frac{\partial}{\partial t} + \frac{y}{x}\frac{\partial}{\partial y}. \tag{49}$$

Now if we choose the cross-section $x = 1$ of $\mathcal{S}$, we just have to substitute x by 1 in order to retrieve the reduced infinitesimal generator (34). Capital letters Y and T stand for applications $\mathcal{S}_y(\rho(Z), Z)$ and $\mathcal{S}_t(\rho(Z), Z)$ where $Z = (t, x, y)$. They are invariants of $\mathcal{S}$ but remain implicit in our work and were not computed.

Thus, in actual case we could perform the reduction process without explicitly computing the invariants of $\mathcal{S}$. Nevertheless, these invariants encode the fibre that allows to retrieve orbits of the original dynamical system from orbits of the reduced one; this fibre is implicitly encoded in the differential equation associated to $\delta_{\mathcal{S}}$ and could be numerically computed.

4. REFERENCES

[1] F. Chyzak. *Fonctions holonomes en calcul formel.* PhD thesis, École polytechnique, 1998.

[2] M. Fels and P. J. Olver. Moving coframes II. Regularization and theoretical foundations. *Acta Applicandae Mathematicae*, 55(2):127–208, 1999.

[3] É. Hubert. AIDA (maple package), 2007. Available on author's webpage (valid on january 2008).

[4] É. Hubert and I. Kogan. Rational invariants of an algebraic group action. Construction and rewriting. *J. of Symbolic Computation*, 42(1-2):203–217, 2006.

[5] P. E. Hydon. Symmetries and first integrals of ordinary difference equations. *Proceedings of the Royal Society of London*, A(456):2835–2855, 2000. Available on author's webpage (valid on january 2008).

[6] N. Joshi and P. Vassiliou. The existence of Lie symmetries for first order analytic discrete dynamical systems. *Journal of Mathematical Analysis and Applications*, 195:872–887, November 1995.

[7] R. Khanin. Dimensional Analysis in Computer Algebra. In B. Mourrain, editor, *Proceedings of ISSAC'2001*, pages 201–208, London, Ontario, Canada, July 22–25 2001. ACM, ACM press.

[8] E. Mansfield. Indiff (maple package), 2001. Available on author's webpage (valid on january 2008).

[9] E. L. Mansfield. Algorithms for Symmetric Differential Systems. *Foundations of Computational Mathematics*, 1(4):335–383, 2001.

[10] J. D. Murray. *Mathematical Biology*, volume 17 of *Interdisciplinary Applied Mathematics*. Springer, 2002.

[11] P. J. Olver. *Applications of Lie Groups to Differential Equations*. Springer-Verlag, 2nd edition, 1993.

[12] A. Sedoglavic. Reduction of Algebraic Parametric Systems by Rectification of their Affine Expanded Lie Symmetries. In *Proceedings of Algebraic Biology 2007*. Springer Verlag, 2007. Available at http://hal.inria.fr/inria-00120991.

[13] A. Sedoglavic and A. Ürgüplü. Expanded Lie Point Symmetry (maple package), 2007. Available on authors' webpage (valid on january 2008).