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F-STABLE SUBMODULES OF TOP LOCAL COHOMOLOGY MODULES OF GORENSTEIN RINGS

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ABSTRACT. This paper applies G. Lyubeznik's notion of F-finite modules to describe in a very down-to-earth manner certain annihilator submodules of some top local cohomology modules over Gorenstein rings. As a consequence we obtain an explicit description of the test ideal of Gorenstein rings in terms of ideals in a regular ring.

1. Introduction

Throughout this paper $(R, \mathfrak{m})$ *will denote a regular local ring of characteristic* p*, and* A *will be a surjective image of* R*. We also denote the injective hull of* R/m *with* E *and for any* R*-module* N *we write* $\operatorname{Hom}_R(N, E)$ *as* $N^\vee$*.* We shall always denote with $f : R \to R$ the Frobenius map, for which $f(r) = r^p$ for all $r \in R$ and we shall denote the eth iterated Frobenius functor over R with $F^e_R(-)$. As R is regular, $F^e_R(-)$ is exact (cf. Theorem 2.1 in [K].)

For any commutative ring S of characteristic p, the skew polynomial ring $S[T; f]$ associated to S and the Frobenius map f is a non-commutative ring which as a left R-module is freely generated by $(T^i)_{i\geq 0}$, and so consists of all polynomials $\sum_{i=0}^n s_i T^i$, where $n \geq 0$ and $s_0, \ldots, s_n \in S$; however, its multiplication is subject to the rule

$$Ts = f(s)T = s^p T \quad \text{ for all } s \in S.$$

Any $A[T; f]$-module M is a $R[T; f]$-module in a natural way and, as R-modules, $F^e_R(M) \cong RT^e \otimes_R M$.

It has been known for a long time that the local cohomology module $\mathrm{H}^{\dim A}_{\mathfrak{m}A}(A)$ has the structure of an $A[T; f]$-module and this fact has been employed by many authors to study problems related to tight closure and to Frobenius closure. Recently R. Y. Sharp has described in [S] the parameter test ideal of F-injective rings in terms of certain $A[T; f]$-submodules of $\mathrm{H}^{\dim A}_{\mathfrak{m}A}(A)$ and it is mainly this work which inspired us to look further into the structure of these $A[T; f]$-modules.

The main aim of this paper is to produce a description of the $A[T; f]$-submodules of $\mathrm{H}^{\dim A}_{\mathfrak{m}A}(A)$ in terms of ideals of R with certain properties. We first do this when A is a complete intersection. The F-injective case is described by Theorem 3.5 and as a corollary we obtain a description of the parameter test ideal of A. Notice that for Gorenstein rings the test ideal the parameter test ideal coincide (cf. Proposition 8.23(d) in [HH1] and Proposition 4.4(ii) in [Sm1].) We then proceed to describe the parameter test ideal in the non-F-injective case (Theorem 5.3.) We generalise these results to Gorenstein rings in section 6.

2. Preliminaries: F-finite modules

The main tool used in this paper is the notion of F-modules, and in particular F-finite modules. These were introduced in G. Lyubeznik's seminal work [L] and provide a very fruitful point of view of local cohomology modules in prime characteristic p.

One of the tools introduced in [L] is a functor $\mathcal{H}_{R,A}$ from the category of $A[T;f]$-modules which are Artinian as A-modules to the category of F-finite modules. For any $A[T;f]$-module M which is Artinian as an A-module the F-finite structure of $\mathcal{H}_{R,A}(M)$ is obtained as follows. Let $\gamma : RT \otimes_R M \to M$ be the R-linear map defined by $\gamma(rT \otimes m) = rTm$; apply the functor $^\vee$ to obtain $\gamma^\vee : M^\vee \to F_R(M)^\vee$. Using the isomorphism between $F_R(M)^\vee$ and $F_R(M^\vee)$ (Lemma 4.1 in [L]) we obtain a map $\beta : M^\vee \to F_R(M^\vee)$ which we adopt as a generating morphism of $\mathcal{H}_{R,A}(M)$.

We shall henceforth assume that the kernel of the surjection $R \to A$ is minimally generated by $\mathbf{u} = (u_1, \ldots, u_n)$. We shall also assume until section 6 that A is a complete intersection. We shall write $u = u_1 \cdot \ldots \cdot u_n$ and for all $t \geq 1$ we let $\mathbf{u}^t R$ be the ideal $u_1^t R + \cdots + u_n^t R$.

To obtain the results in this paper we shall need to understand the F-finite module structure of

$$\mathcal{H}_{R,A}\left(\mathrm{H}^{\dim A}_{\mathfrak{m}A}(A)\right) \cong \mathrm{H}^{\dim R - \dim A}_{\mathbf{u}R}(R);$$

this has generating root

$$\frac{R}{\mathbf{u}R} \xrightarrow{u^{p-1}} \frac{R}{\mathbf{u}^p R}$$

(cf. Remark 2.4 in [L].)

Definition 2.1. Define $\mathcal{I}(R, \mathbf{u})$ to be the set of all ideals $I \subseteq R$ containing $(u_1, \ldots, u_n)R$ with the property that

$$u^{p-1}\left(I + \mathbf{u}R\right) \subseteq I^{[p]} + \mathbf{u}^p R.$$

Lemma 2.2. *Consider the F_R-finite F-module $M = \mathrm{H}^n_{\mathbf{u}R}(R)$ with generating root*

$$\frac{R}{\mathbf{u}R} \xrightarrow{u^{p-1}} \frac{R}{\mathbf{u}^p R}.$$

(a) *For any* $I \in \mathcal{I}(R, \mathbf{u})$ *the* F_R*-finite module with generating root*

$$\frac{I + \mathbf{u}R}{\mathbf{u}R} \xrightarrow{u^{p-1}} \frac{I^{[p]} + \mathbf{u}^p R}{\mathbf{u}^p R} \cong F_R\left(\frac{I + \mathbf{u}R}{\mathbf{u}R}\right)$$

is an F*-submodule of* M *and every* F_R*-finite* F*-submodule of* M *arises in this way.*

(b) *For any* $I \in \mathcal{I}(R, \mathbf{u})$ *the* F_R*-finite module with generating morphism*

$$\frac{R}{I + \mathbf{u}R} \xrightarrow{u^{p-1}} \frac{R}{I^{[p]} + \mathbf{u}^p R} \cong F_R\left(\frac{R}{I + \mathbf{u}R}\right)$$

is an F*-module quotient of* M *and every* F_R*-finite* F*-module quotient of* M *arises in this way.*

Proof. (a) For any $I \in \mathcal{I}(R, \mathbf{u})$, the map

$$\frac{I + \mathbf{u}R}{\mathbf{u}R} \xrightarrow{u^{p-1}} \frac{I^{[p]} + \mathbf{u}^p R}{\mathbf{u}^p R}$$

is well defined and is injective; now the first statement follows from Proposition 2.5(a) in [L]. If N is any F_R-finite F-submodule of M, the root of N is a submodule of the root of M, i.e., the root of N has the form $(I + \mathbf{u}R)/\mathbf{u}R$ for some ideal $I \subseteq R$ (cf. [L], Proposition 2.5(b)) and the structure morphism of N is induced by that of M, i.e., by multiplication by u^{p-1}, so we must have $u^{p-1} I \subseteq I^{[p]} + \mathbf{u}^p R$, i.e., $I \in \mathcal{I}(R, \mathbf{u})$.

(b) For any $I \in \mathcal{I}(R, \mathbf{u})$, the map

$$\frac{R}{I + \mathbf{u}R} \xrightarrow{u^{p-1}} \frac{R}{I^{[p]} + \mathbf{u}^p R} \cong F_R\left(\frac{R}{I + \mathbf{u}R}\right)$$

is well defined and we have the following commutative diagram with exact rows

(1)
$$\begin{array}{ccccccccc}
0 & \longrightarrow & \dfrac{I + \mathbf{u}R}{\mathbf{u}R} & \longrightarrow & \dfrac{R}{\mathbf{u}R} & \longrightarrow & \dfrac{R}{I + \mathbf{u}R} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u^{p-1}} & & \downarrow{\scriptstyle u^{p-1}} & & \downarrow{\scriptstyle u^{p-1}} & & \\
0 & \longrightarrow & F_R\left(\dfrac{I + \mathbf{u}R}{\mathbf{u}R}\right) & \longrightarrow & F_R\left(\dfrac{R}{\mathbf{u}R}\right) & \longrightarrow & F_R\left(\dfrac{R}{I + \mathbf{u}R}\right) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u^{p(p-1)}} & & \downarrow{\scriptstyle u^{p(p-1)}} & & \downarrow{\scriptstyle u^{p(p-1)}} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Taking direct limits of the vertical maps we obtain an exact sequence $0 \to M' \to M \to M'' \to 0$ which establishes the first statement of (b).

Conversely, if M'' is a F-module quotient of M, say, $M'' \cong M/M'$ for some F-submodule M' of M use (a) to find a generating root of M' of the form

$$\frac{I + \mathbf{u}R}{\mathbf{u}R} \xrightarrow{u^{p-1}} \frac{I^{[p]} + \mathbf{u}^p R}{\mathbf{u}^p R}$$

for some $I \in \mathcal{I}(R, \mathbf{u})$. Looking again at the direct limits of the vertical maps in (1) we establish the second statement of (b).

□

Definition 2.3. For all $I \in \mathcal{I}(R, \mathbf{u})$ we define $\mathcal{N}(I)$ to be the F-module quotient of $\mathrm{H}^n_{\mathbf{u}R}(R)$ with generating morphism

$$\frac{R}{I + \mathbf{u}R} \xrightarrow{u^{p-1}} \frac{R}{I^{[p]} + \mathbf{u}^p R} \cong F_R\left(\frac{R}{I + \mathbf{u}R}\right).$$

Lemma 2.4. *Assume that R is complete. Let H be an Artinian $A[T; f]$-module and write $M = \mathcal{H}_{R,A}(H)$. Let N be a homomorphic image of M with generating morphism N_0. Then $N_0^\vee$ is an $A[T; f]$-submodule of H and $N \cong \mathcal{H}_{R,A}(N_0^\vee)$.*

Proof. Notice that M (and hence N) are F-finite modules (cf. [L], Theorems 2.8 and 4.2). Let N_0 be root of N and M_0 a root of M so that we have a commutative diagram with exact rows

$$\begin{array}{ccccc} M_0 & \twoheadrightarrow & N_0 & \longrightarrow & 0 \\ \downarrow \mu & & \downarrow \nu & & \\ F_R(M_0) & \twoheadrightarrow & F_R(N_0) & \longrightarrow & 0 \end{array}$$

where the vertical arrows are generating morphisms. Apply the functor $\mathrm{Hom}(-, E)$ to the commutative diagram above to obtain the following commutative diagram with exact rows

$$\begin{array}{ccccc} 0 & \longrightarrow & F_R(N_0)^\vee & \longrightarrow & F_R(M_0)^\vee \\ & & \downarrow \nu^\vee & & \downarrow \mu^\vee \\ 0 & \longrightarrow & N_0^\vee & \longrightarrow & M_0^\vee \end{array}$$

and recall that M_0 is isomorphic to $H^\vee$ (cf. [L], Theorem 4.2). Since R is complete, $(H^\vee)^\vee \cong H$ and we immediately see that $N_0^\vee$ is a R-submodule of H. We now show that $N_0^\vee$ is an $A[T; f]$ submodule of H by showing that $TN_0^\vee \subseteq N_0^\vee$.

The construction of the functor $\mathcal{H}_{R,A}(-)$ is such that for any $h \in H \cong M_0^\vee$, Th is the image of $T \otimes_R h$ under the map

$$F_R(M_0)^\vee \xrightarrow{\mu^\vee} M_0^\vee$$

and so for $h \in N_0^\vee$, Th is the image of $T \otimes_R h$ under the map

$$F_R(N_0)^\vee \xrightarrow{\nu^\vee} N_0^\vee$$

and hence $Th \in N_0^\vee$.

Now the fact that $N \cong \mathcal{H}_{R,A}(N_0^\vee)$ follows the construction of the functor $\mathcal{H}_{R,A}(-)$. □

Notation 2.5. Let M be a left $A[T,f]$-module. We shall write $AT^\alpha M$ for the A-module generated by $T^\alpha M$. Note that $AT^\alpha M$ is a left $A[T,f]$-module. We shall also write $M^\star = \bigcap_{\alpha \geq 0} AT^\alpha M$.

Lemma 2.6. *Assume that R is complete. Let H be an $A[T;f]$-module and assume that H is T-torsion-free. Let $I, J \subseteq A$ be ideals. If, for some $\alpha \geq 0$,*

$$AT^\alpha \operatorname{ann}_H IA[T;f] = AT^\alpha \operatorname{ann}_H JA[T;f]$$

then $\operatorname{ann}_H IA[T;f] = \operatorname{ann}_H JA[T;f]$.

Proof. Both $AT^\alpha \operatorname{ann}_H IA[T;f]$ and $AT^\alpha \operatorname{ann}_H JA[T;f]$ are left $A[T;f]$-submodules. Now for every T-torsion-free $A[T;f]$-module M, and every ideal $K \subseteq A$, if

$$\left(\bigoplus_{i \geq 0} KT^i \right) AT^\alpha M = \left(\bigoplus_{i \geq 0} KT^{i+\alpha} \right) M$$

vanishes then so does

$$\left(\bigoplus_{i \geq 0} K^{[p^\alpha]} T^{i+\alpha} \right) M = \left(\bigoplus_{i \geq 0} T^\alpha KT^i \right) M = T^\alpha \left(\bigoplus_{i \geq 0} KT^i \right) M$$

and since M is T-torsion-free,

$$\left(\bigoplus_{i \geq 0} KT^i \right) M = 0.$$

We deduce that $\operatorname{gr-ann} AT^\alpha M = \operatorname{gr-ann} M$. Now

$$\operatorname{gr-ann} AT^\alpha \big(\operatorname{ann}_H IA[T;f] \big) = \operatorname{gr-ann} \operatorname{ann}_H IA[T;f],$$

$$\operatorname{gr-ann} AT^\alpha \big(\operatorname{ann}_H JA[T;f] \big) = \operatorname{gr-ann} \operatorname{ann}_H JA[T;f]$$

and Lemma 1.7 in [S] shows that $\operatorname{ann}_H IA[T;f] = \operatorname{ann}_H JA[T;f]$. □

3. The $A[T;f]$ module structure of top local cohomology modules of F-injective Gorenstein rings

Definition 3.1. As in [Sm1] we say that an ideal $I \subseteq A$ is an *F-ideal* if $\operatorname{ann}_{\mathrm{H}^{\dim(A)}_{\mathfrak{m}A}(A)} I$ is a left $A[T;f]$-module, i.e., if $\operatorname{ann}_{\mathrm{H}^{\dim(A)}_{\mathfrak{m}A}(A)} I = \operatorname{ann}_{\mathrm{H}^{\dim(A)}_{\mathfrak{m}A}(A)} IA[T;f]$.

Theorem 3.2. *Assume that R is complete. Consider the F_R-finite F-module $M = \mathrm{H}^n_{\mathbf{u}R}(R)$ with generating root*

$$\frac{R}{\mathbf{u}R} \xrightarrow{u^{p-1}} \frac{R}{\mathbf{u}^p R}$$

and consider the Artinian $A[T;f]$ module $H = \mathrm{H}^{\dim(A)}_{\mathfrak{m}A}(A)$. Let N be a homomorphic image of M.

(a) *$M=\mathcal{H}_{R,A}(-)(H)$ and has generating root* $H^\vee \cong R/\mathbf{u}R \xrightarrow{u^{p-1}} R/\mathbf{u}^p R \cong F_R(H^\vee)$.

(b) *If N has generating morphism*

$$\frac{R}{I+\mathbf{u}R} \xrightarrow{u^{p-1}} \frac{R}{I^{[p]}+\mathbf{u}^p R}$$

then IA is an F-ideal, $N \cong \mathcal{H}_{R,A}\big(\operatorname{ann}_H IA[T;f]\big)$. If, in addition, H is T-torsion free then $\operatorname{gr-ann}\operatorname{ann}_H IA[T;f] = IA[T;f]$ *and I is radical.*

(c) *Assume that H is T-torsion free (i.e., $H_r = H$ in the terminology of* [L]*). For any ideal $J \subset R$, the F-finite module $\mathcal{H}_{R,A}\big(\operatorname{ann}_H JA[T;f]\big)$ has generating morphism*

$$\frac{R}{I+\mathbf{u}R} \xrightarrow{u^{p-1}} \frac{R}{I^{[p]}+\mathbf{u}^p R}$$

for some ideal $I \in \mathcal{I}(R,\mathbf{u})$ with $\operatorname{ann}_H IA[T;f] = \operatorname{ann}_H JA[T;f]$.

Proof. The first statement is a restatement of the discussion at the beginning of section 2.

Notice that Lemma 2.2 implies that N must have a generating morphism of the form given in (b) for some $I \in \mathcal{I}(R,\mathbf{u})$.

Since A is Gorenstein, H is an injective hull of $A/\mathfrak{m}A$ which we denote $\overline{E}$. Lemma 2.4 implies that $N \cong \mathcal{H}_{R,A}(L)$ where $L = \left(\frac{R}{I+\mathbf{u}R}\right)^\vee$ is a $A[T;f]$-submodule of $H=\overline{E}$. But

$$\begin{aligned}\left(\frac{R}{I+\mathbf{u}R}\right)^\vee &= \operatorname{ann}_E (I+\mathbf{u}R)\\ &= \operatorname{ann}_{(\operatorname{ann}_{\mathbf{u}R} E)} I\\ &= \operatorname{ann}_{\overline{E}} I.\end{aligned}$$

But L is a $A[T;f]$-submodule of $\overline{E}$ and so IA is an F-ideal and $L=\operatorname{ann}_{\overline{E}} IA[T;f]$. Also,

$$\begin{aligned}(0:_R \operatorname{ann}_{\overline{E}} IA[T;f]) &= (0:_R \operatorname{ann}_E I)\\ &= (0:_R (R/I)^\vee)\\ &= (0:_R (R/I))\\ &= I\end{aligned}$$

(where the third equality follows from 10.2.2 in [BS]) If H is T-torsion free, Proposition 1.11 in [S] implies that $I=\operatorname{gr-ann}\operatorname{ann}_{\overline{E}} IA[T;f]$ and Lemma 1.9 in [S] implies that I is radical.

To prove part (c) we recall Lemma 2.2 which states that $\mathcal{H}_{R,A}\big(\operatorname{ann}_H JA[T;f]\big)$ has generating morphism

$$\frac{R}{I+\mathbf{u}R} \xrightarrow{u^{p-1}} \frac{R}{I^{[p]}+\mathbf{u}^p R}$$

for some $I \in \mathcal{I}(R,\mathbf{u})$ and we need only show that $\operatorname{ann}_H IA[T;f] = \operatorname{ann}_H JA[T;f]$.

Part (b) implies that $\mathcal{H}_{R,A}\big(\operatorname{ann}_H JA[T;f]\big) = \mathcal{H}_{R,A}\big(\operatorname{ann}_H IA[T;f]\big)$ for some $I \in \mathcal{I}(R, \mathbf{u})$ and Theorem 4.2 (iv) in [L] implies

$$\bigcap_{i=0}^{\infty} AT^i\big(\operatorname{ann}_H JA[T;f]\big) = \bigcap_{i=0}^{\infty} AT^i\big(\operatorname{ann}_H IA[T;f]\big)$$

and since H is Artinian there exists an $\alpha \geq 0$ for which $AT^\alpha\big(\operatorname{ann}_H JA[T;f]\big) = AT^\alpha\big(\operatorname{ann}_H IA[T;f]\big)$ and the result follows from Lemma 2.6. □

Remark 3.3. Theorem 3.2 can provide an easy way to show that $H = \mathrm{H}^{\dim(A)}_{\mathfrak{m}A}(A)$ is *not* T-torsion free. As an example consider $R = \mathbb{K}[\![x, y, a, b]\!]$, $u = x^2a - y^2b$ and $A = R/uR$. Its easy to verify that $(x, y, a^2)R \in \mathcal{I}(R, x^2a - y^2b)$ when $\mathbb{K}$ has characteristic 2, and we deduce that $\mathrm{H}^3_{(x,y,a,b)A}(A)$ is not T-torsion free.

Theorem 3.4. *Assume that R is complete and that $\mathrm{H}^{\dim(A)}_{\mathfrak{m}A}(A)$ is T-torsion free.*

(a) *For all $A[T;f]$-submodules L of $\mathrm{H}^{\dim(A)}_{\mathfrak{m}A}(A)$,*

$$L^\star = \bigcap_{i=0}^{\infty} AT^i L$$

has the form $AT^\alpha M$ where $\alpha \geq 0$ and M is a special annihilator submodule in the terminology of [S]*.*

(b) *The set $\big\{\mathcal{N}(I) \mid I \in \mathcal{I}(R, \mathbf{u})\big\}$ is finite.*

Proof. (a) Let L be a $A[T;f]$-submodule of $\mathrm{H}^{\dim(A)}_{\mathfrak{m}A}(A)$. Pick a $I \in \mathcal{I}(R, \mathbf{u})$ such that $\mathcal{N}(I) = \mathcal{H}_{R,A}(L)$. Now use part (b) of Theorem 3.2 and deduce that $\mathcal{N}(I) \cong \mathcal{H}_{R,A}(\operatorname{ann}_H IA[T;f])$. Now the result follows from Theorem 4.2 (iv) in [L].
(b) Theorem 3.2(b) implies that

$$\big\{\mathcal{N}(I) \mid I \in \mathcal{I}(R, \mathbf{u})\big\} = \Big\{\mathcal{H}_{R,A}\Big(\operatorname{ann}_{\mathrm{H}^{\dim A}_{\mathfrak{m}A}(A)} IA[T;f]\Big) \mid I \in \mathcal{I}(R, \mathbf{u})\Big\};$$

now Corollary 3.11 and Proposition 1.11 in [S] imply that the set on the right is finite.

□

The following Theorem reduces the problem of classifying all F-ideals of A (in the terminology of [Sm1]) or all special $\mathrm{H}^{\dim(A)}_{\mathfrak{m}A}(A)$-ideals (in the terminology of [S]) in the case where A is an F-injective complete intersection, to problem of determining the set $\mathcal{I}(R, \mathbf{u})$.

Theorem 3.5. *Assume $H := \mathrm{H}^{\dim(A)}_{\mathfrak{m}A}(A)$ is T-torsion free and let $\mathcal{B}$ be the set of all H-special A-ideals (cf. §0 in* [S]*)*

(a) *The map $\Psi : \mathcal{I}(R, \mathbf{u}) \to \mathcal{B}$ given by $\Psi(I) = IA$ is a bijection.*

(b) *There exists a unique minimal element* τ *in* $\{I \mid I \in \mathcal{I}(R,\mathbf{u}),\ \mathrm{ht}\, IA > 0\}$ *and that* τ *is a parameter-test-ideal for* A.

(c) A *is* F*-rational if and only if* $\mathcal{I}(R,\mathbf{u}) = \{0, R\}$.

Proof. (a) Assume first that R is complete. Theorem 3.2(b) implies that Ψ is well defined, i.e., $\Psi(I) \in \mathbf{B}$ for all $I \in \mathcal{I}(R,\mathbf{u})$, and, clearly, Ψ is injective. The surjectivity of Ψ is a consequence of Theorem 3.2(c).

Assume now that R is not complete, denote completions with $\widehat{\ }$ and write $\widehat{H} = \mathrm{H}^{\dim(\widehat{A})}_{\mathfrak{m}\widehat{A}}(\widehat{A})$. If I is a $\widehat{H}$-special $\widehat{A}$-ideal, i.e., if there exists an $\widehat{A}[T;f]$-submodule $N \subseteq \widehat{H}$ such that $\operatorname{gr-ann} N = I\widehat{A}[T;f]$ then $I = (0 :_{\widehat{A}} N)$ (cf. Definition 1.10 in [S]). But recall that $\widehat{H} = H$ and N is a $A[T;f]$-submodule of H; now $I = (0 :_{\widehat{A}} N) = (0 :_A N)\widehat{A}$. If we let $\widehat{\mathcal{B}}$ be the set of $\mathrm{H}^{\dim(\widehat{A})}_{\mathfrak{m}\widehat{A}}(\widehat{A})$-special $\widehat{A}$-ideals, we have a bijection $\Upsilon : \mathcal{B} \to \widehat{\mathcal{B}}$ mapping I to $I\widehat{A}$. This also shows that all ideals in $\mathcal{I}(\widehat{R},\mathbf{u})$ are expanded from R, and now since $\widehat{R}$ is faithfully flat over R, we deduce that all ideals in $\mathcal{I}(\widehat{R},\mathbf{u})$ have the form $I\widehat{R}$ for some $I \in \mathcal{I}(R,\mathbf{u})$. We now obtain a chain of bijections

$$\mathcal{I}(R,\mathbf{u}) \longleftrightarrow \mathcal{I}(\widehat{R},\mathbf{u}) \longleftrightarrow \widehat{\mathcal{B}} \longleftrightarrow \mathcal{B}.$$

(b) This is immediate from (a) and Corollary 4.7 in [S].
(c) If A is F-rational, $\mathrm{H}^{\dim(A)}_{\mathfrak{m}A}(A)$ is a simple $A[T;f]$-module (cf. Theorem 2.6 in [Sm2]) and the only H-special A-ideals must be 0 and A. The bijection established in (a) implies now $\mathcal{I}(R,\mathbf{u}) = \{0, R\}$.

Conversely, if $\mathcal{I}(R,\mathbf{u}) = \{0, R\}$, part (b) of the Theorem implies that $1 \in A$ is a parameter-test-ideal, i.e., for all systems of parameters $\mathbf{x} = (x_1, \dots, x_d)$ of A, $(\mathbf{x}A)^* = (\mathbf{x}A)^F = \mathbf{x}A$ where the second equality follows from the fact that $\mathrm{H}^{\dim(A)}_{\mathfrak{m}A}(A)$ is T-torsion free.

□

4. Examples

Throughout this section $\mathbb{K}$ will denote a field of prime characteristic.

Example 4.1. Let R be the localization of $\mathbb{K}[x,y]$ at (x,y), $u = xy$ and $A = R/uR$. Then $\mathrm{H}^1_{xyR}(R) = \mathcal{H}_{R,A}\big(\mathrm{H}^1_{xA+yA}(A)\big)$ ought to have four proper F-finite F-submodules corresponding to the elements 0, xR, yR and $xR + yR$ of $\mathcal{I}(R, xy)$.

We verify this by giving an explicit description the $A[T;f]$-module structure of

$$H := \mathrm{H}^1_{xA+yA}(A) \cong \varinjlim \left(\frac{A}{(x-y)A} \xrightarrow{x-y} \frac{A}{(x-y)^2A} \xrightarrow{x-y} \frac{A}{(x-y)^3A} \xrightarrow{x-y} \dots \right)$$

First notice that in H, for all $n \geq 1$ and $0 < \alpha \leq n$, $x^\alpha + (x-y)^n A = x + (x-y)^{n-\alpha+1}$ and $y^\alpha + (x-y)^n A = y + (x-y)^{n-\alpha+1}$ so H is the $\mathbb{K}$-span of $\{x + (x-y)A\} \cup X \cup Y \cup U$ where

$$\begin{aligned} X &= \left\{x + (x-y)^n A \,|\, n \geq 2\right\}, \\ Y &= \left\{y + (x-y)^n A \,|\, n \geq 2\right\}, \\ U &= \left\{1 + (x-y)^n A \,|\, n \geq 1\right\} \end{aligned}$$

and notice also that the action of the Frobenius map f on H is such that $T\left(x^\alpha + (x-y)^n A\right) = x^{\alpha p} + (x-y)^{np} A$ and $T\left(y^\alpha + (x-y)^n A\right) = y^{\alpha p} + (x-y)^{np} A$ for all $\alpha \geq 0$.

Next notice that any $A[T,f]$-submodule M of H which contains an element $1 + (x-y)^n A \in U$ must coincide with H: for $1 \leq m < n$ we have $(x-y)^{n-m}\left(1 + (x-y)^n A\right) = (x-y)^{n-m} + (x-y)^n A = 1 + (x-y)^m A$, whereas for $m > n$, pick an $e \geq 0$ such that $np^e > m$, write

$$T^e\left(1 + (x-y)^n A\right) = 1 + (x-y)^{np^e} A \in M$$

and use the previous case ($m < n$) to deduce that $1 + (x-y)^m A \in M$. Since now $U \subseteq M$, we see that $M = H$.

We now show that there are only three non-trivial $A[T,f]$-submodules of H, namely $\operatorname{Span}_{\mathbb{K}} X$ and $\operatorname{Span}_{\mathbb{K}} Y$, and $\operatorname{Span}_{\mathbb{K}} \{x + (x-y)A\} \cup X$. By symmetry, it is enough to show that, if M is an $A[T,f]$-submodule of H and $x + (x-y)^n A \in M$ for some $n \geq 2$, then $X \subset M$. If $1 \leq m < n$,

$$x^{n-m}\left(x + (x-y)^n A\right) = x^{n-m+1} + (x-y)^n A = x + (x-y)^{n-(n-m)} A = x + (x-y)^m A$$

whereas, if $m > n \geq 2$, pick an $e \geq 0$ such that $np^e - p^e + 1 > m$ and write

$$T^e\left(x + (x-y)^n A\right) = x^{p^e} + (x-y)^{np^e} A = x + (x-y)^{np^e - p^e + 1} A \in M$$

and using the previous case ($m < n$) we deduce that $x + (x-y)^m A \in M$.

Example 4.2. Let R be the localization of $\mathbb{K}[x,y,z]$ at $\mathfrak{m} = (x,y,z)$, $u = x^2 y + xyz + z^3$ and $A = R/uR$. Fedder's criterion (cf. Propositon 2.1 in [F]) implies that A is F-pure, and Lemma 3.3 in [F] implies that the $A[T;f]$ module $\mathrm{H}^1_{\mathfrak{m}A}(A)$ is T-torsion-free.

Here $\mathcal{I}(R,u)$ contains the ideals 0, $xR + zR$ and $xR + yR + zR$. We deduce that A is not F-rational and that its parameter-test-ideal is $xR + zR$. Also, Theorem 3.5(b) implies that the only proper ideals in $\mathcal{I}(R,u)$ are the ones listed above.

Example 4.3. Let R be the localization of $\mathbb{K}[x,y,z]$ at $\mathfrak{m} = (x,y,z)$ and assume that $\mathbb{K}$ has characteristic 2. Let $u = x^3 + y^3 + z^3 + xyz$ and $A = R/uR$. Notice that we can factor

$u = (x+y+z)(x^2+y^2+z^2+xy+xz+yz)$. Fedder's criterion implies that A is F-pure, and Lemma 3.3 in [F] implies that the $A[T;f]$ module $\mathrm{H}^1_{\mathfrak{m}A}(A)$ is T-torsion-free.

Here

$$\begin{aligned}\mathcal{I}(R,u) \quad \supseteq \quad & \big\{0, (x+y+z)R, (x^2+y^2+z^2+xy+xz+yz)R,\\ & (x+z, y+z)R, (x+y+z, y^2+yz+z^2)R,\\ & (x,y,z)R\big\}.\end{aligned}$$

The images in A of the first three ideals have height zero while the images in A of the fourth and fifth ideals have height 1. Using 3.5(b) we conclude that that the parameter test-ideal of A is a sub-ideal of

$$J = (x+z, y+z)A \cap (x+y+z, y^2+yz+z^2)A = (x^2+yx, y^2+xz, z^2+xy)A.$$

But this ideal defines the singular locus of A and Theorem 6.2 in [HH2] implies that the parameter test-element of A contains J, so J is the parameter test-ideal of A.

5. The non-F-injective case

In this section we extend the results of the previous section to the case where A is not F-injective. First we produce a criterion for the F-injectivity of A.

Definition 5.1. Define

$$\mathcal{I}_0(R,\mathbf{u}) = \left\{ L \in \mathcal{I}(R,\mathbf{u}) \,|\, u^{(p-1)(1+p+\cdots+p^{e-1})} \in L^{[p^e]} + \mathbf{u}^{p^e} R \text{ for some } e \geq 1 \right\}.$$

Proposition 5.2. (a) *For any $L \in \mathcal{I}(R,\mathbf{u})$, $\mathcal{N}(L) = 0$ if and only if $L \in \mathcal{I}_0(R,\mathbf{u})$.*
(b) *$\mathrm{H}^{\dim(A)}_{\mathfrak{m}A}(A)$ is T-torsion free if and only if $\mathcal{I}_0(R,\mathbf{u}) = \{R\}$.*

Proof. (a) Recall that the F-finite module $\mathcal{N}(L)$ has generating morphism

$$\frac{R}{L+\mathbf{u}R} \xrightarrow{u^{p-1}} \frac{R}{L^{[p]}+\mathbf{u}^p R} \cong F_R\left(\frac{R}{L+\mathbf{u}R}\right).$$

Proposition 2.3 in [L] implies that $\mathcal{N}(L)=0$ if and only if for some $e \geq 1$ the composition

$$\frac{R}{L+\mathbf{u}R} \xrightarrow{u^{p-1}} \frac{R}{L^{[p]}+\mathbf{u}^p R} \xrightarrow{u^{(p-1)p}} \frac{R}{L^{[p]}+\mathbf{u}^{p^2} R} \cdots \xrightarrow{u^{(p-1)p^{e-1}}} \frac{R}{L^{[p]}+\mathbf{u}^{p^e} R}$$

vanishes, i.e., if and only if $u^{(p-1)(1+p+\cdots+p^{e-1})} \in L^{[p^e]} + \mathbf{u}^{p^e} R$ for some $e \geq 1$.
(b) Write $H = \mathrm{H}^{\dim(A)}_{\mathfrak{m}A}(A)$. If H is T-torsion free, the existence of the bijection described in Theorem 3.5(a) implies that for any non-unit $L \in \mathcal{I}_0(R,\mathbf{u})$, $\operatorname{ann}_H LA[T;f] \neq \operatorname{ann}_H A[T;f] = 0$. Theorem 3.2(b) implies $\mathcal{N}(L) \cong \mathcal{H}_{R,A}\big(\operatorname{ann}_H LA[T;f]\big)$ so $\mathcal{H}_{R,A}\big(\operatorname{ann}_H LA[T;f]\big) = 0$. But Theorem 4.2(ii) in [L] now implies that $\operatorname{ann}_H LA[T;f]$ is nilpotent, a contradiction.

Assume now that H is not T-torsion free, i.e., $H_n \neq 0$. The short exact sequence

$$0 \to H_n \to H \to H/H_n \to 0$$

yields the short exact sequence

$$0 \to \big(H/H_n\big)^\vee \to \frac{R}{\mathbf{u}R} \to H_n^\vee \to 0.$$

Notice that as the functor $\operatorname{Hom}(-, E)$ is faithful, $H_n^\vee \neq 0$, and so $H_n^\vee \cong R/I$ for some ideal $\mathbf{u}R \subseteq I \subsetneq R$. Now $\mathcal{H}_{R,A}\big(H_n\big)$ is the F-finite quotient of H with generating morphism

$$\frac{R}{I} \xrightarrow{u^{p-1}} \frac{R}{I^{[p]}}$$

and this vanishes because of Theorem 4.2(ii) in [L], i.e., $I \in \mathcal{I}_0(R, \mathbf{u})$. □

We now describe the parameter test ideal of A. Henceforth we shall always denote $\mathrm{H}^{\dim(A)}_{\mathfrak{m}A}(A)$ with H.

Theorem 5.3. *Assume that R is complete. The parameter test ideal of A is given by*

$$\bigcap \big\{ I \in \mathcal{I}(R, \mathbf{u}) \,|\, \operatorname{ht} IA > 0 \big\}.$$

Proof. Write $\overline{\tau}$ for the parameter test ideal of A and let τ be its pre-image in R. Recall that $\overline{\tau}$ is an F-ideal (Proposition 4.5 in [Sm1],) i.e., $\operatorname{ann}_H \overline{\tau}$ is an $A[T; f]$-submodule of H, and $\mathcal{H}_{R,A}\big(\operatorname{ann}_H \overline{\tau}\big)$ has generating morphism

$$(\operatorname{ann}_H \overline{\tau})^\vee \xrightarrow{u^{p-1}} F_R\left((\operatorname{ann}_H \overline{\tau})^\vee\right).$$

But

$$(\operatorname{ann}_H \overline{\tau})^\vee \cong \big((A/\overline{\tau})^\vee\big)^\vee \cong R/(\tau + \mathbf{u}R)$$

so the generating morphism of $\mathcal{H}_{R,A}\big(\operatorname{ann}_H \overline{\tau}\big)$ is

$$R/(\tau + \mathbf{u}R) \xrightarrow{u^{p-1}} R/(\tau^{[p]} + \mathbf{u}^p R)$$

and so we must have $\tau \in \mathcal{I}(R, \mathbf{u})$.

As A is Cohen-Macaulay, $\overline{\tau} = (0 :_A 0^*_H)$ (cf. Proposition 4.4 in [Sm1].)

By Theorem 3.2(b), for each $I \in \mathcal{I}(R, \mathbf{u})$, the ideal IA is an F-ideal and, if $\operatorname{ht} I > 0$, $\operatorname{ann}_H IA = \operatorname{ann}_H IA[T; f] \subseteq 0^*_H$ and so

$$\overline{\tau} = (0 :_A 0^*_H) \subseteq \bigcap \big\{ (0 :_A \operatorname{ann}_H IA) \,|\, IA \in \mathcal{I}(R, \mathbf{u}), \operatorname{ht} IA > 0 \big\}.$$

But H is an injective hull of $A/\mathfrak{m}A$ so

$$(0 :_A \operatorname{ann}_H IA) = (0 :_A \operatorname{Hom}(A/IA, H)) = (0 :_A A/IA) = IA$$

and

$$\overline{\tau} \subseteq \bigcap \big\{ IA \,|\, IA \in \mathcal{I}(R, \mathbf{u}), \operatorname{ht} IA > 0 \big\}.$$

But as $\overline{\tau}$ is one of the ideals in this intersection, we obtain $\overline{\tau} = \bigcap \big\{ IA \in \mathcal{I}(R, \mathbf{u}) \,|\, \operatorname{ht} IA > 0 \big\}$. □

6. The Gorenstein case

In this section we generalise the results so far to the case where A is Gorenstein.

Write $\delta = \dim R - \dim A$ and $\overline{E} = E_A(A/\mathfrak{m}A)$. Local duality implies $\operatorname{Ext}^{\delta}_R(A, R) = \mathrm{H}^{\dim A}_{\mathfrak{m}}(A)^{\vee} \cong \operatorname{Hom}\left(\mathrm{H}^{\dim A}_{\mathfrak{m}A}(A), \overline{E}\right)$ and since A is Gorenstein this is just $A = R/\mathbf{u}R$.

Now $\operatorname{Ext}^{\delta}_R(R/\mathbf{u}R, A) \cong R/\mathbf{u}R$, $\operatorname{Ext}^{\delta}_R(R/\mathbf{u}^p R, A) \cong R/\mathbf{u}^p R$ and $\mathcal{H}_{R,A}\big(\mathrm{H}^{\dim A}_{\mathfrak{m}A}\big) = \mathrm{H}^{\delta}_{\mathfrak{m}}(R)$ has generating morphism $R/\mathbf{u} \to R/\mathbf{u}^p R$ given by multiplication by some element of R which we denote $\varepsilon(\mathbf{u})$ (this is unique up to multiplication by a unit.) Unlike the complete intersection case, the map $R/\mathbf{u} \xrightarrow{\varepsilon(\mathbf{u})} R/\mathbf{u}^p R$ may not be injective, i.e., this generating morphism of $\mathrm{H}^{\delta}_{\mathfrak{m}}(R)$ is not a *root*. However, if define

$$K_{\mathbf{u}} := \bigcup_{e \geq 0} \left(\mathbf{u}^{p^{e+1}} R :_R \varepsilon(\mathbf{u})^{1+p+\cdots+p^e} \right)$$

we obtain a root $R/K_{\mathbf{u}} \xrightarrow{\varepsilon(\mathbf{u})} R/K_{\mathbf{u}}^{[p]}$ (cf. Proposition 2.3 in [L].)

We now extend naturally our definition of $\mathcal{I}(R, \mathbf{u})$ when A is Gorenstein as follows.

Definition 6.1. If $A = R/\mathbf{u}R$ is Gorenstein we define $\mathcal{I}(R, \mathbf{u})$ to be the set of all ideals I of R containing $K_{\mathbf{u}}$ for which $\varepsilon(\mathbf{u}) I \subseteq I^{[p]}$.

Now a routine modification of the proofs of the previous sections gives the following two theorems.

Theorem 6.2. *Assume A is Gorenstein and that $H^{\dim A}_{\mathfrak{m}A}(A)$ is T-torsion-free.*

(a) *The map $I \mapsto IA$ is a bijection between $\mathcal{I}(R, \mathbf{u})$ and the A-special $H^{\dim A}_{\mathfrak{m}A}(A)$-ideals.*
(b) *There exists a unique minimal element τ in $\{I \,|\, I \in \mathcal{I}(R, \mathbf{u}),\ \operatorname{ht} IA > 0\}$ and that τ is a parameter-test-ideal for A.*
(c) *A is F-rational if and only if $\mathcal{I}(R, \mathbf{u}) = \{0, R\}$.*

Theorem 6.3. *Assume that R is complete and that A is Gorenstein. The parameter test ideal of A is given by*

$$\bigcap \big\{ I \in \mathcal{I}(R, \mathbf{u}) \,|\, \operatorname{ht} IA > 0 \big\}.$$

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