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ON THE ARITHMETIC OF TIGHT CLOSURE

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0. Introduction

This paper deals with a question regarding tight closure in characteristic zero which we now review. Let R be a commutative ring of prime characteristic p and let $I \subseteq R$ be an ideal. Recall that for $e \geq 0$, the e-th Frobenius power of I, denoted $I^{[p^e]}$, is the ideal of R generated by all p^e-th powers of elements in I. We say that $f \in I^*$, the tight closure of I, if there exists a c not in any minimal prime of R with the property that $cf^{p^e} \in I^{[p^e]}$ for all large $e \geq 0$. This notion, due to M. Hochster and C. Huneke, is now an important tool in commutative algebra and algebraic geometry, particularly since it gives a systematic framework for reduction to positive characteristic. We refer the reader to [17] for the basic properties of tight closure in characteristic p.

How does the containment $f \in I^*$ depend on the prime characteristic? To make sense of this question suppose that $R_{\mathbb{Z}}$ is a finitely generated ring extension of $\mathbb{Z}$ and that $I \subseteq R_{\mathbb{Z}}$ is an ideal, $f \in R_{\mathbb{Z}}$. Then we may consider for every prime number p the specialization $R_{\mathbb{Z}/(p)} = R_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}/(p)$ of characteristic p together with the extended ideal $I_p \subseteq R_{\mathbb{Z}/(p)}$, and one may ask whether $f_p \in I_p^*$ holds or not. We refer to this question about the dependence on the prime numbers as the "arithmetic of tight closure".

Many properties in commutative algebra exhibit an arithmetically nice behaviour: for example, $R_{\mathbb{Q}}$ is smooth (normal, Cohen-Macaulay, Gorenstein) if and only if $R_{\mathbb{Z}/(p)}$ is smooth (normal, Cohen-Macaulay, Gorenstein) for almost all prime numbers (i.e., for all except for at most finitely many). In a similar way we have for an ideal $I \subseteq R_{\mathbb{Z}}$ that $I_{\mathbb{Q}} = IR_{\mathbb{Q}}$ is a parameter ideal or a primary ideal if and only if this is true for almost all specializations I_p. Furthermore, $f \in I$ if and only if $f_p \in I_p$ holds for almost all prime characteristics: see [16, Chapter 2.1] and appendix 1 in [18] for these kinds of results.

When R is a finitely generated $\mathbb{Q}$-algebra, Hochster and Huneke define the tight closure of an ideal $I \subseteq R$, in the same spirit as the examples above, with the help of a $\mathbb{Z}$-algebra $R_{\mathbb{Z}}$ where $R = R_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$, as the set of all $f \in R$ for which $f_p \in (I_p)^*$ holds for almost all p. This definition is independent of the chosen model $R_{\mathbb{Z}}$. The reader should consult [16] for properties of tight closure in characteristic zero. This definition works well, because the most important features from tight closure theory in positive characteristic, like F-regularity of regular rings, colon capturing,

Briançon-Skoda theorems, and persistence, behave well arithmetically, so that these properties pass over to the characteristic zero situation with full force.

M. Hochster and C. Huneke (see appendix 1 in [18] or Question 11 in the appendix of [16] or Question 13 in [15]) and the second author (see §4 in [20]) raise the following natural question: if R is a finitely generated $\mathbb{Z}$-algebra of characteristic zero and $I \subseteq R$ is an ideal which is tightly closed, i.e. $I^* = I$ in $R_{\mathbb{Q}}$, must one have $(I_p)^* = I_p$ for almost all primes p? Or, using the terminology of [20], must tightly closed ideals be fiberwise tightly closed?

As often in tight closure theory, the situation for parameter ideals is better understood than the general case, but even for parameter ideals a complete answer is not known. There are however results due to N. Hara and K. Smith (see [10], [11], [19, Theorem 6.1], [27], [28, Theorem 2.10, Open Problem 2.24]) which imply that for a normal standard-graded Cohen-Macaulay domain with an isolated singularity and for a normal Gorenstein algebra of finite type over a field the answer is affirmative.

The main theorem in this paper (Theorem 4.1) provides, however, a negative answer to this question by showing that for the homogeneous primary ideal $I = (x^4, y^4, z^4)$ in $\mathbb{Z}[x, y, z]/(x^7 + y^7 - z^7)$ one has $x^3 y^3 \in (I_p)^*$ for $p \equiv 3 \bmod 7$ but $x^3 y^3 \notin (I_p)^*$ for $p \equiv 2 \bmod 7$.

Our example has also interesting implications for the dependence of the cohomological dimension on the characteristics of ground fields. The ideal $\mathfrak{a} = (x, y, z)$ inside the forcing algebra $A = \mathbb{K}[x, y, z, u, v, w]/(x^7 + y^7 - z^7, ux^4 + vy^4 + wz^4 + x^3 y^3)$ is such that the open subset $D(\mathfrak{a}) \subset \operatorname{Spec} A$ is affine for infinitely many but not for almost all prime reductions. This means that its cohomological dimension fluctuates arithmetically between 0 and 1; see 4.7 for this relation via solid closure and 4.8 for an interpretation in terms of projective varieties.

Moreover, our example has also consequences for the study of Hilbert-Kunz multiplicities which we discuss in 4.9 and for the non-standard tight closure of H. Schoutens (see 4.10).

1. REDUCTION TO FROBENIUS POWERS

In this section we show where to look for candidates (R, I, f) with the property that $f_p \in I_p^*$ holds for infinitely many but not for almost all prime numbers p. This approach rests on the geometric interpretation of tight closure in terms of bundles, which we now recall briefly. Let R denote a geometrically normal two-dimensional standard-graded domain over a field $\mathbb{K}$. A set of homogeneous generators $f_1, \ldots, f_n \in R$ of degrees $d_1, \ldots, d_n$ of an R_+-primary ideal gives rise to the short exact sequence of locally free sheaves on the smooth projective curve $C = \operatorname{Proj} R$,

$$0 \longrightarrow \operatorname{Syz}(f_1, \ldots, f_n)(m) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_C(m - d_i) \stackrel{f_1, \ldots, f_n}{\longrightarrow} \mathcal{O}_C(m) \longrightarrow 0 \, .$$

A homogeneous element $f \in R$ of degree m defines via the connecting homomorphism a cohomology class $\delta(f) \in H^1(C, \operatorname{Syz}(f_1, \ldots, f_n)(m))$ in this syzygy sheaf. It was shown in [1], [4] how this cohomology class is related to the question as to whether f belongs to the tight closure (in positive characteristic) of the ideal $(f_1, \ldots, f_n)$ or not. The cohomology class $c \in H^1(C, \mathcal{S}) = \operatorname{Ext}^1(\mathcal{O}_C, \mathcal{S})$ corresponds to an extension $0 \to \mathcal{S} \to \mathcal{S}' \to \mathcal{O}_C \to 0$ and to a geometric torsor $\mathbb{P}(\mathcal{S}'^{\vee}) - \mathbb{P}(\mathcal{S}^{\vee})$.

Now $f \in (f_1, \ldots, f_n)^*$ if and only if the torsor defined by $\delta(f)$ is not an affine scheme.

If the syzygy bundle is strongly semistable in positive characteristic p, then this approach gives a numerical criterion for $(f_1, \ldots, f_n)^*$, where the degree bound which separates inclusion from exclusion is given by $(d_1 + \ldots + d_n)/(n-1)$. So, if we want to find an example where $f \in (f_1, \ldots, f_n)^*$ holds for infinitely many prime numbers but not for almost all, we have to look first for an example where for infinitely many prime numbers the syzygy bundle is not strongly semistable (this is also the reason why such an example cannot exist in the cone over an elliptic curve). That $\mathcal{S}$ is not strongly semistable means that some Frobenius pull-back of it, say $\mathcal{T} = F^{e*}(\mathcal{S})$, is not semistable, and that means that there exists a subbundle $\mathcal{F} \subset \mathcal{T}$ such that $\deg(\mathcal{F})/\operatorname{rk}(\mathcal{F}) > \deg(\mathcal{T})/\operatorname{rk}(\mathcal{T})$.

Examples of such syzygy bundles with the property that they are semistable in characteristic zero but not strongly semistable for infinitely many prime numbers were first given in [5], where it was shown that a question of Miyaoka and Shepherd-Barron ([21], [25]) has a negative answer. The following lemma gives another example of that kind.

Lemma 1.1. *Let $d \in \mathbb{N}$ and let p denote a prime number; write $p = d\ell + r$, $0 < r < d$. Suppose that $d/4 \le r < d/3$. Let $\mathbb{K}$ denote a field of characteristic p and let $C = \operatorname{Proj} \mathbb{K}[x, y, z]/(x^d + y^d - z^d)$ be the Fermat curve of degree d. Then the first Frobenius pull-back of $\operatorname{Syz}(x^4, y^4, z^4)$ on C is not semistable.*

Proof. We have $4p = 4d\ell + 4r = d(4\ell + 1) + (4r - d)$; set $t = 4r - d$. We consider first in $\mathbb{K}[x, y]$ the syzygies for

$$x^{4p} = x^{d(4\ell+1)+t}, \quad y^{4p} = y^{d(4\ell+1)+t}, \quad (x^d + y^d)^{4\ell+1}.$$

We multiply the last term by the $2\ell + 1$ monomials

$$x^t y^t (x^{d2\ell} y^0), \ x^t y^t (x^{d(2\ell-1)} y^d), \ldots, x^t y^t (x^0 y^{d2\ell}).$$

The resulting polynomials are expressible modulo the first two terms as a $\mathbb{K}$-linear combination of the monomials $x^t y^t x^{di} y^{dj}$, where $i + j = 6\ell + 1$ and $i, j \le 4\ell$. Therefore $i = 2\ell+1, \ldots, 4\ell$ and there are only 2ℓ of these. Hence there exists a global non-trivial syzygy (h_1, h_2, h_3) of these polynomials of total degree $d(6\ell + 1) + 2t$. Therefore $(z^t h_1, z^t h_2, h_3)$ is a global non-trivial syzygy for x^{4p}, y^{4p}, z^{4p}, since

$$0 = z^t h_1 x^{4p} + z^t h_2 y^{4p} + z^t h_3 (x^d + y^d)^{4\ell+1} = z^t h_1 x^{4p} + z^t h_2 y^{4p} + h_3 z^{4p}.$$

The total degree of this syzygy is $d(6\ell + 1) + 3t$. The degree of the bundle

$$\operatorname{Syz}(x^{4p}, y^{4p}, z^{4p})(d(6\ell + 1) + 3t)$$

is however (up to the factor $\deg(\mathcal{O}(1))$)

$$2(d(6\ell + 1) + 3t) - 3(d(4\ell + 1) + t) = -d + 3t = 12r - 4d,$$

which is negative due to the assumption that $r < d/3$. But a bundle of negative degree and with a non-trivial section is not semistable. $\square$

The following proposition reduces under suitable conditions the computation of tight closure to the computation of a certain small Frobenius power.

Proposition 1.2. *Let $\mathbb{K}$ denote a field of positive characteristic p and let R denote a two-dimensional geometrically normal standard-graded domain over $\mathbb{K}$. Suppose that $p \ge 2g + 1$, where g denotes the genus of the smooth projective curve $C =$*

$\operatorname{Proj} R$. Let f_1, f_2, f_3 denote homogeneous elements in R which generate an R_+-primary ideal. Let $m \in \mathbb{Z}$ be such that the e-th pull-back of the syzygy bundle $\operatorname{Syz}(f_1, f_2, f_3)(m)$ can be incorporated in a short exact sequence on C,

$$0 \longrightarrow \mathcal{L} \longrightarrow F^{e*}(\operatorname{Syz}(f_1, f_2, f_3)(m)) = \operatorname{Syz}(f_1^q, f_2^q, f_3^q)(qm) \longrightarrow \mathcal{M} \longrightarrow 0\,,$$

where $q = p^e$ and $\mathcal{L}$ is an invertible sheaf of positive degree and $\mathcal{M}$ is an invertible sheaf of negative degree. Let f denote a homogeneous element of degree m. Then $f \in (f_1, f_2, f_3)^*$ if and only if $f^{pq} \in (f_1^{pq}, f_2^{pq}, f_3^{pq})$.

Proof. The implication from right to left is clear. For the other direction we may assume that $\mathbb{K}$ is algebraically closed. We will argue on the smooth projective plane curve $C = \operatorname{Proj} R$ and use the geometric interpretation of tight closure. We apply the Frobenius pull-back to the given short exact sequence and obtain a new exact sequence

$$0 \longrightarrow \mathcal{L}^p \longrightarrow \operatorname{Syz}(f_1^{pq}, f_2^{pq}, f_3^{pq})(pqm) \longrightarrow \mathcal{M}^p \longrightarrow 0\,.$$

The cohomology sequence is

$$\longrightarrow H^1(C, \mathcal{L}^p) \longrightarrow H^1(C, \operatorname{Syz}(f_1^{pq}, f_2^{pq}, f_3^{pq})(pqm)) \longrightarrow H^1(C, \mathcal{M}^p) \longrightarrow 0.$$

The genus of the curve C is g and the canonical sheaf ω_C has degree $2g - 2$. Hence for $p > 2g - 2$ we have that $\deg(\mathcal{L}^{-p} \otimes \omega_C) < 0$ and therefore $H^1(C, \mathcal{L}^p) = 0$ by Serre duality. This gives an isomorphism

$$H^1(C, \operatorname{Syz}(f_1^{pq}, f_2^{pq}, f_3^{pq})(pqm)) \cong H^1(C, \mathcal{M}^p)\,.$$

Suppose now that $f^{pq} \notin (f_1^{pq}, f_2^{pq}, f_3^{pq})$. This means that the corresponding cohomology class $c = \delta(f^{pq}) \in H^1(C, \operatorname{Syz}(f_1^{pq}, f_2^{pq}, f_3^{pq})(pqm))$ is not zero; let $c' \neq 0$ denote the corresponding class in $H^1(C, \mathcal{M}^p)$. To show that f does not belong to the tight closure of (f_1, f_2, f_3) we show that the geometric torsor corresponding to c is an affine scheme [1, Proposition 3.9], and for that it is sufficient to show that the geometric torsor corresponding to c' is an affine scheme. The class $c' \in H^1(C, \mathcal{M}^p) \cong \operatorname{Ext}^1(\mathcal{O}_C, \mathcal{M}^p)$ defines a non-trivial extension

$$0 \longrightarrow \mathcal{M}^p \longrightarrow \mathcal{T} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

with dual sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{T}^\vee \longrightarrow \mathcal{M}^{-p} \longrightarrow 0\,.$$

Here $\mathcal{M}^{-p}$ is ample, since its degree is positive, and therefore by [7, Proposition 2.2] every quotient bundle of $\mathcal{T}^\vee$ has positive degree. Since $\deg \mathcal{T}^\vee = \deg \mathcal{M}^{-p} \geq p > 2 \cdot g$, it follows by [7, Lemma 2.2] that $\mathcal{T}^\vee$ is an ample vector bundle (one can also argue using [12, Corollary 7.7]). But then $C \cong \mathbb{P}(\mathcal{M}^{-p}) \subset \mathbb{P}(\mathcal{T}^\vee)$ is an ample divisor and its complement is affine. $\square$

Remark 1.3. The situation described in Proposition 1.2 occurs in particular for

$$2m = \deg(f_1) + \deg(f_2) + \deg(f_3)$$

under the condition that the syzygy bundle is not strongly semistable. For then some Frobenius pull-back $\mathcal{T} = \operatorname{Syz}(f_1^q, f_2^q, f_3^q)(qm)$ is not semistable, but its degree is

$$q(2m - \deg(f_1) - \deg(f_2) - \deg(f_3)) \deg(\mathcal{O}_C(1)) = 0\,.$$

Then there exists the maximal destabilizing invertible subsheaf $\mathcal{L} \subset \mathcal{T}$ of positive degree, and the quotient sheaf is also invertible of negative degree.

Corollary 1.4. *Let $\mathbb{K}$ denote a field of positive characteristic $p \equiv 2 \bmod 7$, $p \neq 2, 23$, and let $R = \mathbb{K}[x, y, z]/(x^7 + y^7 - z^7)$. Then $x^3 y^3 \in (x^4, y^4, z^4)^*$ if and only if $x^{3p^2} y^{3p^2} \in (x^{4p^2}, y^{4p^2}, z^{4p^2})$.*

Proof. In the notation of Lemma 1.1 we have $p = d\ell + 2$; hence $r = 2$ and clearly $7/4 \leq 2 < 7/3$. Hence the first Frobenius pull-back of $\mathrm{Syz}(x^4, y^4, z^4)$ is not semistable. Therefore via Remark 1.3 we are in the situation of Proposition 1.2 with $e = 1$; hence $pq = p^2$. Since $g = 15$, the condition on the prime number is $p \geq 31$, so only $p = 2$ and 23 are excluded. $\square$

Remark 1.5. The method of Proposition 1.2 works in principle also for small prime numbers p. We only need to find a power $p^u \geq 2g + 1$. If the e-th Frobenius pull-back is not semistable, then we can conclude that $f \in (f_1, f_2, f_3)^*$ if and only if $f^{p^u p^e} \in (f_1^{p^u p^e}, f_2^{p^u p^e}, f_3^{p^u p^e})$. So in Corollary 1.4 we take $u = 2$ for $p = 23$ and $u = 5$ for $p = 2$ to make things work also in these cases.

2. THE CASE $p \equiv 2 \bmod 7$

In this section we want to show that $x^3 y^3 \notin (x^4, y^4, z^4)^*$ in $\mathbb{K}[x, y, z]/(x^7 + y^7 - z^7)$ if $\mathbb{K}$ has characteristic $p \equiv 2 \bmod 7$. We will need the following lemmata on matrices.

Lemma 2.1. *The $r \times s$ matrix A with entries*

$$\left(\binom{a}{b + i - j} \right)_{1 \leq i \leq r, 1 \leq j \leq s}$$

can be brought to the form

$$\left(\binom{a + j - 1}{b + i - 1} \right)_{1 \leq i \leq r, 1 \leq j \leq s}$$

by performing elementary column operations.

Proof. We proceed by induction on s. If $s = 1$ there is nothing to show, so assume $s > 1$. Add the penultimate column of A to the last column; in the result, add the $(s - 2)$-th column to the $(s - 1)$-th, and continue in this way until the first column has been added to the second column. In this way one obtains the matrix

$$\begin{pmatrix} \binom{a}{b} & \binom{a+1}{b} & \binom{a+1}{b-1} & \cdots & \binom{a+1}{b+2-s} \\ \binom{a}{b+1} & \binom{a+1}{b+1} & \binom{a+1}{b} & \cdots & \binom{a+1}{b+3-s} \\ \vdots & \vdots & & & \vdots \\ \binom{a}{b+r-1} & \binom{a+1}{b+r-1} & \binom{a+1}{b+r-2} & \cdots & \binom{a+1}{b+r-s+1} \end{pmatrix}.$$

Now apply the induction hypothesis to the submatrix of this matrix consisting of all its columns except the first. $\square$

Using Lemma 2.1 one can obtain the following result due to V. van Zeipel ([30]; the calculation is described in [22, Chapter XX].)

Lemma 2.2.

$$\det \left(\binom{a}{b + i - j} \right)_{\substack{1 \leq i \leq r \\ 1 \leq j \leq r}} = \prod_{t=0}^{r-1} \frac{\binom{a+r-1-t}{b}}{\binom{b+t}{b}}.$$

We will use these lemmata in the proof of the following result.

Lemma 2.3. *Let $\mathbb{K}$ denote a field of positive characteristic $p = 7\ell + 2$. Then we have $x^{3p}y^{3p} \notin (x^{4p}, y^{4p}, (x^7 + y^7)^{4\ell+1})$ in the polynomial ring $\mathbb{K}[x, y]$.*

Proof. The case $p = 2$ is checked immediately, so suppose that $\ell > 0$. Since $3p = 21\ell + 6$ and $4p = 28\ell + 8$, we rewrite what we want to show as

$$(1) \qquad x^{21\ell+6}y^{21\ell+6} \notin (x^{28\ell+8}, y^{28\ell+8}, (x^7 + y^7)^{4\ell+1}).$$

We endow $\mathbb{K}[x, y]$ with a $\mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}$ grading by assigning x degree $(\overline{1}, \overline{0}, 1)$ and y degree $(\overline{0}, \overline{1}, 1)$. With this grading the left-hand side of (1) is homogeneous of degree $(\overline{6}, \overline{6}, 42\ell + 12)$ while $(x^7 + y^7)^{4\ell+1}$ is homogeneous of degree $(\overline{0}, \overline{0}, 28\ell + 7)$, so the degree difference is $(\overline{6}, \overline{6}, 14\ell + 5)$. Condition (1) fails to hold if and only if there exist $a_0, \ldots, a_{2\ell-1} \in \mathbb{K}$ such that

$$x^{21\ell+6}y^{21\ell+6} \equiv \left(\sum_{i=0}^{2\ell-1} a_i x^{7i+6}y^{7(2\ell-1-i)+6} \right) \left(\sum_{j=0}^{4\ell+1} \binom{4\ell+1}{j} x^{7j}y^{7(4\ell+1-j)} \right)$$

$$\mod (x^{28\ell+8}, y^{28\ell+8})$$

and we assume that this is the case. Notice that the terms occurring on the right-hand side of this equation have the form $x^{7i+6}y^{7(6\ell-i)+6}$ for $0 \leq i \leq 6\ell$. Since

$$\begin{cases} 7i + 6 & < \quad 28\ell + 8 \\ 7(6\ell - i) + 6 & < \quad 28\ell + 8 \end{cases} \Leftrightarrow \quad 2\ell \leq i \leq 4\ell,$$

we obtain $\mod(x^{28\ell+8}, y^{28\ell+8})$

$$x^{21\ell+6}y^{21\ell+6} \equiv x^6 y^6 \sum_{i=2\ell}^{4\ell} \left(\sum_{j=0}^{2\ell-1} a_j \binom{4\ell+1}{i-j} \right) x^{7i}y^{7(6\ell-i)}$$

$$\equiv x^6 y^6 \sum_{i=1}^{2\ell+1} \left(\sum_{j=1}^{2\ell} a_{j-1} \binom{4\ell+1}{2\ell+i-j} \right) x^{7(2\ell+i-1)}y^{7(4\ell-i+1)}$$

and since no term in the last expression is divisible by $x^{28\ell+8}$ or by $y^{28\ell+8}$, we deduce that

$$x^{21\ell+6}y^{21\ell+6} = x^6 y^6 \sum_{i=1}^{2\ell+1} \left(\sum_{j=1}^{2\ell} a_{j-1} \binom{4\ell+1}{2\ell+i-j} \right) x^{7(2\ell+i-1)}y^{7(4\ell-i+1)}.$$

We may cancel $x^6 y^6$ from both sides of the equation and we write $X = x^7, Y = y^7$ to obtain

$$X^{3\ell}Y^{3\ell} = \sum_{i=1}^{2\ell+1} \left(\sum_{j=1}^{2\ell} a_{j-1} \binom{4\ell+1}{2\ell+i-j} \right) X^{2\ell+i-1}Y^{4\ell-i+1}.$$

If we compare the coefficients of $X^{2\ell+i-1}Y^{4\ell+1-i}$ for $1 \leq i \leq 2\ell + 1$ we obtain the conditions

$$\sum_{j=1}^{2\ell} a_{j-1} \binom{4\ell+1}{2\ell+i-j} = \delta_{i,\ell+1} \quad \text{for all} \quad 1 \leq i \leq 2\ell + 1$$

where $\delta_{i,\ell+1}$ is Kronecker's delta. If we define M_1 to be the $(2\ell+1, 2\ell)$ matrix whose entries are

$$\binom{4\ell+1}{2\ell+i-j}_{1\leq i\leq 2\ell+1, 1\leq j\leq 2\ell}$$

and if $\mathbf{e}_{\ell+1}$ is the $(\ell+1)$-th elementary column vector of size $2\ell+1$, then we are now assuming that $\mathbf{e}_{\ell+1}$ is in the span of the columns of M_1. We have to show that this is not possible. Since M_1 has more rows than columns, its rows are linearly dependent, i.e., there exists a $\rho = (\rho_1, \ldots, \rho_{2\ell+1}) \neq \mathbf{0}$ such that $\rho M_1 = 0$. It is now enough to show that we can choose this ρ with $\rho_{\ell+1} \neq 0$, since for such ρ we have $\rho\mathbf{e}_{\ell+1} = \rho_{\ell+1} \neq 0$ and so $\mathbf{e}_{\ell+1}$ could not be in the span of the columns of M_1. Assume by way of contradiction that we can find a non-zero ρ as above with $\rho_{\ell+1} = 0$. This implies that the rows of M_1 numbered $1, \ldots, \ell, \ell+2, \ldots, 2\ell+1$ are linearly dependent. Use Lemma 2.1 and apply elementary column operations to M_1 to obtain the matrix

$$M_2 = \left(\binom{4\ell+j}{2\ell+i-1}\right)_{\substack{1\leq i\leq 2\ell+1, \\ 1\leq j\leq 2\ell}} = \begin{pmatrix} \binom{4\ell+1}{2\ell} & \binom{4\ell+2}{2\ell} & \cdots & \binom{6\ell}{2\ell} \\ \binom{4\ell+1}{2\ell+1} & \binom{4\ell+2}{2\ell+1} & \cdots & \binom{6\ell}{2\ell+1} \\ \vdots & \vdots & & \vdots \\ \binom{4\ell+1}{4\ell} & \binom{4\ell+2}{4\ell} & \cdots & \binom{6\ell}{4\ell} \end{pmatrix}.$$

Use the fact that $\binom{a+1}{b+1} = \frac{a+1}{b+1}\binom{a}{b}$ and multiply rows $1, 2, \ldots, 2\ell+1$ by $2\ell, 2\ell+1, \ldots, 4\ell$ and divide columns $1, 2, \ldots, 2\ell$ by $4\ell+1, 4\ell+2, \ldots, 6\ell$ to obtain

$$M_2 = \Lambda\left(\frac{1}{2\ell}, \ldots, \frac{1}{4\ell}\right) \begin{pmatrix} \binom{4\ell}{2\ell-1} & \binom{4\ell+1}{2\ell-1} & \cdots & \binom{6\ell-1}{2\ell-1} \\ \binom{4\ell}{2\ell} & \binom{4\ell+1}{2\ell} & \cdots & \binom{6\ell-1}{2\ell} \\ \vdots & \vdots & & \vdots \\ \binom{4\ell}{4\ell-1} & \binom{4\ell+1}{4\ell-1} & \cdots & \binom{6\ell-1}{4\ell-1} \end{pmatrix} \Upsilon(4\ell+1, \ldots, 6\ell)$$

where $\Lambda(a_1, \ldots, a_{2\ell+1})$ is the $(2\ell+1) \times (2\ell+1)$ diagonal matrix with $a_1, \ldots, a_{2\ell+1}$ along its diagonal and $\Upsilon(b_1, \ldots, b_{2\ell})$ is the $2\ell \times 2\ell$ diagonal matrix with $b_1, \ldots, b_{2\ell}$ along its diagonal. We can repeat this process 2ℓ times to obtain

$$(2) \qquad M_2 = \Lambda \begin{pmatrix} \binom{2\ell+1}{0} & \binom{2\ell+2}{0} & \cdots & \binom{4\ell}{0} \\ \binom{2\ell+1}{1} & \binom{2\ell+2}{1} & \cdots & \binom{4\ell}{1} \\ \vdots & \vdots & & \vdots \\ \binom{2\ell+1}{2\ell} & \binom{2\ell+2}{2\ell} & \cdots & \binom{4\ell}{2\ell} \end{pmatrix} \Upsilon$$

where

$$\Lambda = \left(\prod_{t=0}^{2\ell-1} \Lambda\left(\frac{1}{2\ell-t}, \frac{1}{2\ell+1-t}, \ldots, \frac{1}{4\ell-t}\right)\right),$$

$$\Upsilon = \left(\prod_{t=0}^{2\ell-1} \Upsilon(4\ell+1-t, 4\ell+2-t, \ldots, 6\ell-t)\right).$$

We notice that none of the entries in the diagonal matrices above is 0 or $1/0$ modulo p and so, if we denote with M_3 the middle matrix in Equation 2, and, if we write $\rho' = \rho\Lambda$, then $\rho M_2 = 0 \Leftrightarrow \rho' M_3 = 0$, $\rho = 0 \Leftrightarrow \rho' = 0$ and $\rho_{\ell+1} = 0 \Leftrightarrow \rho'_{\ell+1} = 0$. It is now, therefore, sufficient to show that the rows of M_3 numbered

$1, \ldots, \ell, \ell+2, \ldots, 2\ell+1$ are not linearly dependent. Use Lemma 2.1 to perform the inverse elementary column operations on M_3 to bring it to the form

$$M_4 = \left(\binom{2\ell+1}{i-j} \right)_{\substack{1 \le i \le 2\ell+1, \\ 1 \le j \le 2\ell}} = \begin{pmatrix} \binom{2\ell+1}{0} & 0 & 0 & \cdots & 0 \\ \binom{2\ell+1}{1} & \binom{2\ell+1}{0} & 0 & \cdots & 0 \\ & \ddots & & & \\ \binom{2\ell+1}{2\ell-1} & \binom{2\ell+1}{2\ell-2} & \binom{2\ell+1}{2\ell-3} & \cdots & \binom{2\ell+1}{0} \\ \binom{2\ell+1}{2\ell} & \binom{2\ell+1}{2\ell-1} & \binom{2\ell+1}{2\ell-2} & \cdots & \binom{2\ell+1}{1} \end{pmatrix},$$

and we now need to show that the rows of M_4 numbered $1, \ldots, \ell, \ell+2, \ldots, 2\ell+1$ are not linearly dependent. If we delete the $(\ell+1)$-th row from M_4 and perform elementary row operations consisting of adding multiples of rows $1, 2, \ldots, \ell$ to lower rows, we can bring the resulting matrix to the form $I_\ell \oplus M_5$ where I_ℓ is an $\ell \times \ell$ identity matrix and M_5 consists of the lower-rightmost block of size $\ell \times \ell$ in M_4, i.e.,

$$M_5 = \begin{pmatrix} \binom{2\ell+1}{1} & \binom{2\ell+1}{0} & \cdots & 0 \\ & \ddots & & \\ \binom{2\ell+1}{\ell-1} & \binom{2\ell+1}{2\ell-2} & \cdots & \binom{2\ell+1}{0} \\ \binom{2\ell+1}{\ell} & \binom{2\ell+1}{\ell-1} & \cdots & \binom{2\ell+1}{1} \end{pmatrix}.$$

The value of the determinant of M_5 can be computed using Lemma 2.2:

$$\det M_5 = \prod_{t=0}^{\ell-1} \frac{\binom{2\ell+1+\ell-1-t}{1}}{\binom{1+t}{1}} = \prod_{t=0}^{\ell-1} \frac{3\ell-t}{1+t},$$

which is a unit modulo p. Hence the rows of M_5 are linearly independent, and we conclude that $\mathbf{e}_{\ell+1}$ is not in the span of the columns of M_2. $\qquad\square$

Proposition 2.4. *If $p \equiv 2 \bmod 7$, then $x^3 y^3 \notin (x^4, y^4, z^4)^*$ in $R = \mathbb{K}[x, y, z]/(x^7 + y^7 - z^7)$, char $\mathbb{K} = p$.*

Proof. For $p = 2, 23$ this was checked with the help of a computer and Remark 1.5, so suppose that $p \ne 2, 23$. Corollary 1.4 then guarantees that $x^3 y^3 \notin (x^4, y^4, z^4)^*$ if and only if $x^{3p^2} y^{3p^2} \notin (x^{4p^2}, y^{4p^2}, z^{4p^2})$. Write $p = 7\ell + 2$ and $p^2 = 7k + 4$ where $k = 7\ell^2 + 4\ell = p\ell + 2\ell$. Now $4p^2 = 7(4k+2) + 2 = 28k + 16$ and so z^{4p^2} equals $(x^7 + y^7)^{4k+2} z^2$, so it is enough to show that

$$x^{3p^2} y^{3p^2} \notin (x^{4p^2}, y^{4p^2}, (x^7 + y^7)^{4k+2}).$$

If this were not the case, then we would have already, since $\mathbb{K}[x, y] \subset R$ is a free extension,

$$x^{3p^2} y^{3p^2} \in (x^{4p^2}, y^{4p^2}, (x^7 + y^7)^{4k+2})\mathbb{K}[x, y],$$

so we have to show that this is not true. By Lemma 2.3 we know that

$$x^{3p} y^{3p} \notin (x^{4p}, y^{4p}, (x^7 + y^7)^{4\ell+1})$$

in $\mathbb{K}[x, y]$. Since $\mathbb{K}[x, y]$ is a regular ring, it is F-pure. Therefore we take a Frobenius power to conclude that

$$x^{3p^2} y^{3p^2} \notin (x^{4p^2}, y^{4p^2}, (x^7 + y^7)^{p(4\ell+1)}).$$

But we have

$$p(4\ell+1) = 4\ell(7\ell+2) + 7\ell + 2 = 4(7\ell^2 + 2\ell) + (7\ell+2) = 4(k-2\ell) + (7\ell+2) = 4k - \ell + 2,$$

which is strictly smaller than $4k + 2$. Therefore replacing the ideal generator $(x^7 + y^7)^{p(4\ell+1)}$ by $(x^7 + y^7)^{4k+2}$ makes the ideal smaller; hence $x^{3p^2}y^{3p^2} \notin (x^{4p^2}, y^{4p^2}, (x^7 + y^7)^{4k+2})$ holds. $\square$

3. The case $p \equiv 3 \bmod 7$

Proposition 3.1. *If* $p \equiv 3 \bmod 7$, *then* $x^3y^3 \in (x^4, y^4, z^4)^F \subseteq (x^4, y^4, z^4)^*$ *in* $\mathbb{K}[x, y, z]/(x^7 + y^7 - z^7)$ *for* char $\mathbb{K} = p$.

Proof. We show indeed that $x^{3p}y^{3p} \in (x^{4p}, y^{4p}, z^{4p})$. Write $p = 7\ell + 3$; notice that $z^2 z^{4p}$ equals $(x^7 + y^7)^{4\ell+2}$, so it is enough to show that

$$x^{7(3\ell+1)+2}y^{7(3\ell+1)+2} = x^{3p}y^{3p} \in (x^{4p}, y^{4p}, (x^7 + y^7)^{4\ell+2})$$
$$= (x^{28\ell+12}, y^{28\ell+12}, (x^7 + y^7)^{4\ell+2}).$$

We will show that $x^{7(3\ell+1)}y^{7(3\ell+1)} \in (x^{28\ell+12}, y^{28\ell+12}, (x^7+y^7)^{4\ell+2})$ holds in $\mathbb{K}[x, y]$. Consider the $(2\ell + 1) \times (2\ell + 1)$ matrix

$$A = \left(\binom{4\ell + 2}{2\ell + 1 + i - j} \right)_{\substack{1 \leq i \leq 2\ell+1 \\ 1 \leq j \leq 2\ell+1}}.$$

Lemma 2.2 shows that

$$\det A = \prod_{t=0}^{2\ell} \frac{\binom{6\ell+2-t}{2\ell+1}}{\binom{2\ell+1+t}{2\ell+1}}$$

and since $2\ell + 1 \leq 6\ell + 2 - t, 2\ell + 1 + t < p$ for $0 \leq t \leq 2\ell$, none of the binomial coefficients in the determinant vanishes modulo p and so $\det A$ is a unit modulo p. Now A, as a matrix with entries in $\mathbb{K}$, is invertible and we can find $a_0, \dots, a_{2\ell} \in \mathbb{K}$ such that

$$A \begin{pmatrix} a_0 \\ \vdots \\ a_{2\ell} \end{pmatrix} = \mathbf{e}_{\ell+1},$$

where $\mathbf{e}_{\ell+1}$ is the $(\ell+1)$th elementary vector of size $2\ell+1$. Consider the polynomial

$$\begin{aligned} f &= \left(\sum_{i=0}^{2\ell} a_i x^{7i} y^{7(2\ell-i)} \right) (x^7 + y^7)^{4\ell+2} \\ &= \left(\sum_{i=0}^{2\ell} a_i x^{7i} y^{7(2\ell-i)} \right) \left(\sum_{j=0}^{4\ell+2} \binom{4\ell + 2}{j} x^{7j} y^{7(4\ell+2-j)} \right) \in \mathbb{K}[x, y] \end{aligned}$$

and notice that the terms occurring in f have the form $x^{7i}y^{7(6\ell+2-i)}$ for $0 \leq i \leq (6\ell + 2)$. Working modulo $x^{28\ell+12}, y^{28\ell+12}$, since

$$\begin{cases} 7i & < & 28\ell + 12 \\ 7(6\ell + 2 - i) & < & 28\ell + 12 \end{cases} \Leftrightarrow 2\ell + 1 \leq i \leq 4\ell + 1,$$

we have

$$f \equiv \sum_{i=2\ell+1}^{4\ell+1} \left(\sum_{j=0}^{2\ell} a_j \binom{4\ell+2}{i-j} \right) x^{7i} y^{7(6\ell+2-i)}$$

$$= \sum_{i=1}^{2\ell+1} \left(\sum_{j=1}^{2\ell+1} a_{j-1} \binom{4\ell+2}{2\ell+1+i-j} \right) x^{7(i+2\ell)} y^{7(4\ell+2-i)} \mod \left(x^{28\ell+12}, y^{28\ell+12} \right)$$

and our choice of $a_0, \ldots, a_{2\ell}$ gives

$$\sum_{i=1}^{2\ell+1} \left(\sum_{j=1}^{2\ell+1} a_{j-1} \binom{4\ell+2}{2\ell+1+i-j} \right) x^{7(i+2\ell)} y^{7(4\ell+2-i)} = x^{7(2\ell+\ell+1)} y^{7(4\ell+2-\ell-1)}$$

$$= x^{7(3\ell+1)} y^{7(3\ell+1)}$$

and so $x^{7(3\ell+1)} y^{7(3\ell+1)} \in (x^{28\ell+12}, y^{28\ell+12}, (x^7+y^7)^{4\ell+2})$.   □

4. CONCLUSIONS AND REMARKS

Putting together the results of the previous sections we obtain the following theorem.

Theorem 4.1. *Let $\mathbb{K}$ denote a field of positive characteristic p and let $R = \mathbb{K}[x,y,z]/(x^7+y^7-z^7)$. Then $x^3y^3 \in (x^4,y^4,z^4)^*$ for infinitely many prime numbers p and $x^3y^3 \notin (x^4,y^4,z^4)^*$ for infinitely many prime numbers p.*

Proof. This follows directly from Propositions 2.4 and 3.1, taking into account Dirichlet's theorem on primes in an arithmetic progression; see for example [24, Chapitre VI, §4].   □

We can now settle the question mentioned in the introduction that was posed by M. Hochster, C. Huneke and the second author.

Corollary 4.2. *There exists an ideal $J \subseteq \mathbb{Q}[x,y,z]/(x^7+y^7-z^7)$ which is tightly closed but whose descents $J_p \subseteq \mathbb{Z}/p\mathbb{Z}[x,y,z]/(x^7+y^7-z^7)$ to characteristic p are not tightly closed for infinitely many primes p.*

Proof. Let J be the tight closure in characteristic zero of the ideal (x^4,y^4,z^4) in $\mathbb{Q}[x,y,z]/(x^7+y^7-z^7)$; obviously $J^* = J$. Since there are infinitely many primes p satisfying $p \equiv 2 \bmod 7$, Proposition 2.4 shows that $x^3y^3 \notin J$. For the infinitely many primes p satisfying $p \equiv 3 \bmod 7$ we have however $x^3y^3 \in (J_p)^*$ and so for these primes $(J_p)^* \neq J_p$.   □

Surprisingly, we can also deduce from our considerations in positive characteristic that the syzygy bundle $\operatorname{Syz}(x^4,y^4,z^4)$ is semistable in characteristic zero (we do not know of a single prime number where it is strongly semistable).

Corollary 4.3. *The syzygy bundle $\operatorname{Syz}(x^4,y^4,z^4)$ is semistable on*

$$C = \operatorname{Proj} \mathbb{Q}[x,y,z]/(x^7+y^7-z^7).$$

Proof. Suppose there exists a destabilizing sequence $0 \to \mathcal{L} \to \operatorname{Syz}(x^4,y^4,z^4)(6) \to \mathcal{M} \to 0$, $\mathcal{L}$ of positive and $\mathcal{M}$ of negative degree. Such a sequence may be extended to a sequence on the relative curve over an open subset of $\operatorname{Spec}\mathbb{Z}$. Let $c = \delta(x^3y^3) \in H^1(C, \operatorname{Syz}(x^4,y^4,z^4)(6))$ denote the cohomology class corresponding to x^3y^3 and

let c' denote the image of c in $H^1(C, \mathcal{M})$. If $c' \neq 0$, then its torsor would be affine and x^3y^3 would not belong to the solid closure of (x^4, y^4, z^4) in characteristic zero. But then it would not belong to the tight closure for almost all prime numbers (since affineness is an open property), which contradicts Proposition 3.1. Hence $c' = 0$ and c stems from a class $c'' \in H^1(C, \mathcal{L})$. Modulo p, c''_p is annihilated by a Frobenius power, since $\mathcal{L}$ has positive degree. But that would mean that also c_p would be annihilated by a Frobenius power and hence $x^3y^3 \in (x^4, y^4, z^4)^*$ for almost all prime numbers, which contradicts Proposition 2.4. $\qquad\square$

Remark 4.4. What does $J = (x^4, y^4, z^4)^*$ in characteristic zero look like? Since $\mathrm{Syz}(x^4, y^4, z^4)$ is semistable in characteristic zero by Corollary 4.3, we know that $\mathrm{Syz}(x^4, y^4, z^4)(m)$ is an ample sheaf for $m \geq 7$ and that its dual is ample for $m \leq 5$. Since ampleness is an open property, it follows for almost all prime numbers p that $R_{\geq 7} \subseteq (x^4, y^4, z^4)^*$ (even in the Frobenius closure) and that $R_{\leq 5} \cap (x^4, y^4, z^4)^* \subseteq (x^4, y^4, z^4)$. For degree 6 we know that $x^3y^3, x^3z^3, y^3z^3 \notin J$ by Proposition 2.4. We do not know whether $x^2y^2z^2$ and xy^2z^3 etc. belong to J or not.

Remark 4.5. What can we say in our example about tight closure and Frobenius closure for the other remainders of p modulo 7? There is numerical evidence showing that for $p \equiv 3, 5, 6 \bmod 7$ the element x^3y^3 belongs to the Frobenius closure of (x^4, y^4, z^4), but not for $p \equiv 1, 2, 4 \bmod 7$. Moreover it seems as if $x^3y^3 \in (x^4, y^4, z^4)^*$ for exactly $p \equiv 1, 3, 5, 6 \bmod 7$.

We began this work by looking at the example $xyz \in (x^2, y^2, z^2)^*$ in $\mathbb{K}[x, y, z]/(x^5 + y^5 - z^5)$. Here we have strong computer evidence that $xyz \in (x^2, y^2, z^2)^F$ holds exactly for the remainders $p \equiv 2, 4 \bmod 5$, and we have proved this for $p \equiv 2 \bmod 5$. Moreover, for $p \equiv 3 \bmod 5$ we have proved as in Lemma 1.1 and Corollary 1.4 that the computation of tight closure reduces to the question of whether $(xyz)^{p^2} \notin (x^{2p^2}, y^{2p^2}, z^{2p^2})$, but we were unable to settle this. The difficulty lies in the fact that in reducing the statement to a problem over $\mathbb{K}[x, y]$ (and then to a matrix problem over $\mathbb{K}$), we have to replace z twice, and have to deal with two different kinds of binomial coefficients. For $p \equiv 1 \bmod 5$ it is likely that $xyz \in (x^2, y^2, z^2)^*$ holds without being in the Frobenius closure.

Remark 4.6. It has been known since the early days of tight closure that the Frobenius closure I^F of an ideal I fluctuates arithmetically. The easiest example is that $y^2 \in (x, y)^F$ holds in $\mathbb{K}[x, y, z]/(x^3 + y^3 + z^3)$ for prime characteristic char $\mathbb{K} = p \equiv 2 \bmod 3$, but not for $p \equiv 1 \bmod 3$; see [19, Example 2.2]. It is therefore not surprising that our argument reduces the tight closure question to a question about Frobenius closure.

Remark 4.7. Our example shows also that tight closure in characteristic zero and in dimension two is not the same as solid closure. Recall that an element f in a local (or graded) excellent domain $(R, \mathfrak{m})$ of dimension d belongs to the solid closure of an $\mathfrak{m}$-primary ideal $(f_1, \ldots, f_n)$ if and only if $H_\mathfrak{m}^d(R[u_1, \ldots, u_n]/(u_1 f_1 + \ldots + u_n f_n + f)) \neq 0$ (see [14] and [1]). In positive characteristic, tight closure and solid closure are the same, and solid closure contains always tight closure. The containment of x^3y^3 inside the solid closure of (x^4, y^4, z^4) in $\mathbb{K}[x, y, z]/(x^7 + y^7 - z^7)$ follows from Proposition 3.1 or from the fact that the syzygy bundle is semistable in characteristic zero.

The example provides also an example of a ring $R_\mathbb{Z} = \mathbb{Z}[x, y, z]/(x^7 + y^7 - z^7)$ and an $R_\mathbb{Z}$-algebra $A = R_\mathbb{Z}[u, v, w]/(ux^4 + vy^4 + wz^4 + x^3y^3)$ such that $H_{\mathfrak{m} R_\mathbb{K}}^2(A_\mathbb{K})$

is zero for infinitely many prime fields $\mathbb{K} = \mathbb{Z}/(p)$ and non-zero for infinitely many prime fields. The ring A together with the ideal $\mathfrak{a} = (x, y, z)A \subset A$ gives an example where the cohomological dimension of the open subset $D(\mathfrak{a})$ varies between 0 and 1 with the characteristic. Classical examples for the dependence on the prime characteristic of the cohomological dimension were given in [13, Example 3] (see also [26, Corollary 2.2]), but as far as we know our example is the first where it varies between 0 and 1, corresponding to $D(\mathfrak{a})$ being affine or not.

Remark 4.8. Let $Y \subset X$ denote a divisor on a smooth projective variety over $\operatorname{Spec} \mathbb{Z}$ and let $Y_p \subset X_p$ denote the specialisations for a prime number p. How do properties of Y_p vary with p? Our example gives a smooth irreducible divisor Y on a smooth projective three-dimensional variety X such that the complement $X_p - Y_p$ is an affine variety for infinitely many but not for almost all p. Indeed, let $C = \operatorname{Proj} \mathbb{Z}[x, y, z]/(x^7 + y^7 - z^7) \to \operatorname{Spec} \mathbb{Z}$ be the relative curve and let

$$0 \longrightarrow \mathcal{S} = \operatorname{Syz}(x^4, y^4, z^4)(6) \longrightarrow \mathcal{S}' = \operatorname{Syz}(x^4, y^4, z^4, x^3 y^3)(6) \longrightarrow \mathcal{O}_C \longrightarrow 0$$

denote the extension on C defined by $x^3 y^3$. Then $Y = \mathbb{P}(\mathcal{S}^\vee) \subset \mathbb{P}((\mathcal{S}')^\vee) = X$ is a projective subbundle of codimension one inside a projective bundle over C of fiber dimension two. Our result says that $X_p - Y_p = \mathbb{P}((\mathcal{S}_p')^\vee) - \mathbb{P}(\mathcal{S}_p^\vee)$ is affine for $p \equiv 2$ mod 7 and not affine for $p \equiv 3 \bmod 7$. We do not know whether such an example exists if X is a surface.

Remark 4.9. Our example is also relevant to the study of Hilbert-Kunz multiplicities. The Hilbert-Kunz multiplicity is an invariant of an ideal I (primary to a maximal ideal) in a ring R of positive characteristic p, defined by $e_{HK}(I) = \lim_{e \in \mathbb{N}} \operatorname{length}(R/I^{[p^e]})/p^{e \dim(R)} \in \mathbb{R}$; see [18, Chapter 6]. It is related to tight closure by the fact that $f \in I^*$ holds if and only if $e_{HK}(I) = e_{HK}((I, f))$. Set $I = (x^4, y^4, z^4)$ and $I' = (x^4, y^4, z^4, x^3 y^3)$ in $\mathbb{Z}[x, y, z]/(x^7 + y^7 - z^7)$. Our results give $e_{HK}(I_p) = e_{HK}(I_p')$ for $p \equiv 3 \bmod 7$ and $e_{HK}(I_p) \neq e_{HK}(I_p')$ for $p \equiv 2 \bmod 7$. In particular, the Hilbert-Kunz multiplicity is not eventually constant as $p \to \infty$. The first examples with this last behaviour were given by C. Han and P. Monsky in [9].

On the other hand, V. Trivedi has shown in [29] that in the two-dimensional graded situation the limit $\lim_{p \mapsto \infty} e_{HK}(I_p)$ exists. Moreover, one can show that this limit is the Hilbert-Kunz multiplicity in characteristic zero as defined in [2]. In our example we have $\lim_{p \mapsto \infty} e_{HK}(I_p) = \lim_{p \mapsto \infty} e_{HK}(I_p')$, because they coincide for infinitely many prime numbers. The equality of these two limits corresponds to the fact that $x^3 y^3$ belongs to the solid closure of I in characteristic zero. This limit is in our example 84 (see [3, Introduction] for the formulas to compute the Hilbert-Kunz multiplicity). Apart from that we only know for $p \equiv 2 \bmod 7$ that $e_{HK}(I_p) \geq 84 + 28/p^2$; we get here only an inequality because the instability of $\operatorname{Syz}(x^{4p}, y^{4p}, z^{4p})$ might be even worse than the instability detected in Lemma 1.1.

Remark 4.10. H. Schoutens defined another variant of tight closure for finitely generated algebras R over $\mathbb{C}$, called non-standard tight closure (see [23]). He uses methods from model theory and an identification $\operatorname{ulim} \overline{\mathbb{Z}/(p)} \cong \mathbb{C}$, where ulim denotes the ultraproduct with respect to a fixed non-principal ultrafilter. Then the ultraproduct of the various Frobenius morphisms of the approximations R_p gives a characteristic zero Frobenius $R \to R_\infty = \operatorname{ulim} R_p$ and yields a new closure operation with several variants. A natural question is whether these closure operations

are independent of the choice of the ultrafilter and whether the several variants coincide or not (Question 1 after Theorem 10.4 in [23]). Our example shows at once that the so-called generic tight closure depends on the choice of the ultra-filter. Moreover, if the parameter theorem of Hara [19, Theorem 6.1] holds for non-standard tight closure for two-dimensional graded $\mathbb{C}$-domains, then it follows that also non-standard tight closure depends on the ultrafilter.

Question 4.11. Suppose that R is a finitely generated extension of $\mathbb{Z}$, let $I \subseteq R$ denote an ideal and let $f \in R$. Set $M = \{p \text{ prime} : f_p \in (I_p)^*\}$. Is it possible to characterize the subsets of the prime numbers which arise in this way? Do there always exist congruence conditions which describe such an M up to finitely many exceptions?

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