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# HOLONOMY REPRESENTATIONS WHICH ARE A DIAGONAL DIRECT SUM OF TWO FAITHFUL REPRESENTATIONS

TOM KRANTZ

ABSTRACT. We study holonomy representations admitting a pair of supplementary faithful sub-representations. In particular the cases where the sub-representations are isomorphic respectively dual to each other are treated. In each case we have a closer look at the classification in small dimension.

## 1. INTRODUCTION

The notion of torsion-free connection on the tangent bundle  $TM$  of a smooth manifold  $M$  gives rise to the notion of parallel transport along paths contained in the manifold, and it is well known that if one considers all closed contractible paths based in the a point  $o$  of the manifold  $M$ , the set of the corresponding parallel transports forms a Lie subgroup of  $Gl(T_oM)$  called the restricted holonomy group of the connection. To this linear Lie group corresponds its linear Lie algebra which will be noted  $\mathfrak{g}$  throughout this text. Seen as a representation we will refer to it as the holonomy representation. For a general study of connections and holonomy and precise definitions we refer for example to [KN].

Holonomy representations  $\mathfrak{g} \subset \mathfrak{gl}(T_oM)$  have been completely studied in the Riemannian case through the classical result of the De Rham-theorem reducing the classification problem to the one of the classification irreducible metric holonomy representations. In the pseudo-Riemannian case Wu's generalization of the de Rham-theorem allows to reduce the classification problem to the one of weakly-irreducible representations. All irreducible holonomy representations are by now known, but in the weakly irreducible case mainly signature  $(1, n)$  (see [BBI], [I2], [Boub], [Le1], [Le2], [Le3], [Le4], [Gal1]),  $(2, n)$  (see [BBI], [I3], [Gal2]) are explored. For signature  $(n, n)$  see ([BBI]). Lionel Bérard Bergery has classified indecomposable semi-simple non simple pseudo-Riemannian holonomy representations. These are examples of  $V \oplus V^*$  representations which we examine in this paper generally.

If the connection is not supposed to preserve a metric very little is known, we have no longer a general result like the De Rham-Wu theorem. In my

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thesis([K])(under the direction of Lionel Bérard Bergery), I studied representations admitting two pairs of supplementary invariant spaces(see also [BBK]). In the pseudo-Riemannian or symplectic case appear naturally factors of type  $V \oplus V^*$ . When the representation is weakly irreducible and admits a decomposition into two supplementary non trivial subspaces, it is of this type as shown in the paper. In the general setting appear factors of type  $V \otimes \mathbb{R}^2$  (with  $\mathbb{R}^2$  being the trivial representation). Here we examine more closely both types of holonomy representations.

The paper is structured as follows: We start by formulating Berger-criteria for representations which are a diagonal direct sum of two faithful representations. We explore then the local geometric structure of torsion-free connections with this holonomy and calculate curvature in an adapted basis. Finally we examine closely holonomy representations of type  $V \oplus V^*$  and  $V \otimes \mathbb{R}^2$  and give classifications in dimension  $2 \times 2$  (which was known for the case  $V \oplus V^*$  and is new for the case  $V \otimes \mathbb{R}^2$ ).

**1.1. Notation.** We will say a representation  $\mathfrak{g} \subset \mathfrak{gl}(V)$  is *decomposable along the direct sum*  $V = V_1 \oplus V_2$  if  $\mathfrak{g} = \mathfrak{g} \cap V_1^* \otimes V_1 \oplus \mathfrak{g} \cap V_2^* \otimes V_2$ .

## 2. BERGER-TYPE CRITERIA

**2.1. The first Berger criterion.** Let  $\mathfrak{g} \subset \mathfrak{gl}(V)$  be a finite-dimensional representation.

Note

$$\mathcal{K}(\mathfrak{g}) := \{R \in (V^* \wedge V^*) \otimes \mathfrak{g} \mid R(x, y)z + R(y, z)x + R(z, x)y = 0 \text{ for } x, y, z \in V\},$$

$$\underline{\mathfrak{g}} := \langle R(x, y) \mid x, y \in V, R \in \mathcal{K}(\mathfrak{g}) \rangle.$$

Since the work of Marcel Berger([Ber]) it is known that:

**Proposition 1.** *If  $\mathfrak{g} \subset \mathfrak{gl}(V)$  is a holonomy algebra of a torsion-free connection then  $\underline{\mathfrak{g}} = \mathfrak{g}$ .*

Recall also the second criterion of Berger:

Note

$$\mathcal{K}^1(\mathfrak{g}) := \{D \in V^* \otimes \mathcal{K}(\mathfrak{g}) \mid Dx(y, z) + Dy(z, x) + Dz(x, y) = 0 \text{ for } x, y, z \in V\}.$$

**Proposition 2.** *If  $\mathfrak{g} \subset \mathfrak{gl}(V)$  is a holonomy algebra of a torsion-free connection which is non locally symmetric then  $\mathcal{K}^1(\mathfrak{g}) \neq \{0\}$ .*

In this text we are interested representations  $V$  of a Lie algebra  $\mathfrak{g}$  such that  $V = V_1 \oplus V_2$  where  $V_1$  and  $V_2$  are two faithful representations of  $\mathfrak{g}$ .

Let's write  $\mathfrak{g}_i \subset \mathfrak{gl}(V_i)$  for the restriction of the action of  $\mathfrak{g}$  to  $V_i$ .

In the context of these particular representations the first Berger criterion can be specialized to the following.

Note

$$k(\mathfrak{g}) := \left\{ R \in V_1^* \otimes V_2^* \otimes \mathfrak{g} \mid \begin{array}{l} R(x, y')z' = R(x, z')y' \\ R(x, y')t = R(t, y')x \end{array} \text{ for } x, t \in V_1, y', z' \in V_2 \right\},$$

We have then:

**Proposition 3.** *If  $\mathfrak{g} \subset \mathfrak{gl}(V)$  is a finite dimensional representation and  $V_1$  and  $V_2$  are two faithful representations of  $\mathfrak{g}$  such that  $V = V_1 \oplus V_2$  then:*

$$\underline{\mathfrak{g}} = \langle R(x, y') \mid x \in V_1, y' \in V_2, R \in k(\mathfrak{g}) \rangle .$$

*Proof.* This follows from the fact that if  $R$  is in  $K(\mathfrak{g})$  then  $R(x, y) \in \mathfrak{gl}(V_i)$  for  $x, y \in V_i$ , from which follows  $R(x, y) = 0$  by the condition that  $V_1$  and  $V_2$  are two faithful representations of  $\mathfrak{g}$ .  $\square$

Note

$$k^1(\mathfrak{g}) := \left\{ D \in V^* \otimes k(\mathfrak{g}) \mid \begin{array}{l} Dx(y, z') = Dy(x, z') \\ Dt'(y, z') = Dz'(x, t') \end{array} \text{ for } x, y \in V_1, z', t' \in V_2 \right\},$$

Berger's second criterion reads then:

**Proposition 4.** *Let  $\mathfrak{g} \subset \mathfrak{gl}(V)$  be a representation and  $V_1$  and  $V_2$  two faithful representations of  $\mathfrak{g}$  such that  $V = V_1 \oplus V_2$ .*

*If  $\mathfrak{g}$  is a holonomy algebra of a torsion-free connection which is non locally symmetric then  $k^1(\mathfrak{g}) \neq \{0\}$ .*

**2.2. A general weak criterion.** We write

$$\mathfrak{g}_i^{(1)} := V_i^* \otimes \mathfrak{g}_i \cap S^2(V_i^*) \otimes V_i,$$

the first prolongation of  $\mathfrak{g}_i$ .

Let

$$\underline{\underline{\mathfrak{g}}}_i := \langle r(x) \mid x \in V_i, r \in \mathfrak{g}_i^{(1)} \rangle .$$

**Proposition 5.**  *$\mathfrak{g}$  verifies the first Berger criterion implies  $\underline{\underline{\mathfrak{g}}}_i = \mathfrak{g}_i$  for  $i = 1, 2$ .*

Weaker than the first Berger criterion this criterion considers only the  $\mathfrak{g}_i$  and ignores their coupling.

If  $W_i$  is a sub-representation of  $V_i$ , note  $\mathfrak{g}_{i/W_i} \subset \mathfrak{gl}(V_i/W_i)$  the representation obtained from  $\mathfrak{g}_i$  by passing to the quotient.

The following result shows some properties of holonomy representations we are interested in this article in.

**Proposition 6.** *For  $i = 1, 2$ ,  $\underline{\underline{\mathfrak{g}}}_i = \mathfrak{g}_i$  implies that for a sub-representation  $W_i$  of  $V_i$ ,  $\underline{\underline{\mathfrak{g}}}_{i/W_i} = \mathfrak{g}_{i/W_i}$*

### 3. LOCAL GEOMETRIC STRUCTURE

**3.1. Basic results.** Assume we have a connection  $\nabla$  on a manifold  $\mathcal{M}$  admitting a holonomy representation  $\mathfrak{g} \subset \mathfrak{gl}(T_o\mathcal{M})$  (at the point  $o \in \mathcal{M}$ ) for which  $T_o\mathcal{M} = V_1 \oplus V_2$ , where  $V_1$  and  $V_2$  are two faithful representations of  $\mathfrak{g}$ .

One can transport parallelly the direct sum  $V_1 \oplus V_2$  to any point of the manifold. Each  $V_i$  gives a distribution on  $\mathcal{M}$  which is integrable and gives rise to a foliation  $\mathcal{F}_i$  of the manifold with flat leaves. On the leaves  $\mathcal{F}_1(o)$  (resp.  $\mathcal{F}_2(o)$ ) we can choose coordinates  $x_o^i$  ( $i = 1 \dots n_1$ ) (resp.  $y_o^j$  ( $j = 1 \dots n_2$ )) for which  $\nabla_{\partial_{x_o^i}} \partial_{x_o^j} = 0, \forall i, j$ , and  $\nabla_{\partial_{y_o^i}} \partial_{y_o^j} = 0, \forall i, j$ . The coordinates are in bijection with the basis  $(\partial_{x_o^1}, \dots, \partial_{x_o^{n_1}}, \partial_{y_o^1}, \dots, \partial_{y_o^{n_2}})$  in  $o$ .

The manifold  $\mathcal{M}$  - as a differential manifold - locally a product and to a point  $p$  of the manifold one can (locally) associate coordinates  $(x_o^i(p), i = 1 \dots n_1, y_o^j(p), j = 1 \dots n_2)$  by considering the coordinates of the intersection point of  $\mathcal{F}_2(p)$  with  $\mathcal{F}_1(o)$  (respectively the ones of the intersection of  $\mathcal{F}_1(p)$  with  $\mathcal{F}_2(o)$ ).

On the other hand one can equip each leaf  $\mathcal{F}_1(r)$  containing the point  $r \in \mathcal{F}_2(o)$ , because it is flat, with coordinates  $x_r^i, i = 1 \dots n_1$  such that  $\nabla_{\partial_{x_r^i}} \partial_{x_r^j} = 0, \forall i, j$  on the leaf  $\mathcal{F}_1(r)$  and such that  $\partial_{x_r^i}(r)$  is obtained by parallelly transporting  $\partial_{x_o^i}(o)$  along the leaf  $\mathcal{F}_2(o)$ . Analogously one defines  $y_q^j, j = 1 \dots n_2$  on the leaf  $\mathcal{F}_2(q)$ .

One has finally as well the following coordinates:  $(x^i(p), i = 1 \dots n_1, y^j(p), j = 1 \dots n_2)$ , with  $x^i := x_o^i$  and  $y^j(p) := y_q^j(p)$ .

**Proposition 7.** *Let  $\mathcal{M}$  be a manifold equipped with a torsion-free connection  $\nabla$  such that the holonomy algebra (in  $o$ )  $\mathfrak{g}$  verifies  $T_oM = V_1 \oplus V_2$  with  $V_1$  and  $V_2$  faithful representations of  $\mathfrak{g}$ . For  $o$  in  $\mathcal{M}$  let  $x_o^i$  (respectively  $y_o^j$ ) designate local flat coordinates on  $\mathcal{F}_1(o)$  (resp.  $\mathcal{F}_2(o)$ ) corresponding to a choice of basis  $(b_1)$  of  $V_1$  and  $(b_2)$  of  $V_2$ . To  $(b_1)$  (resp.  $(b_2)$ ) is associated for any point  $r$  of  $\mathcal{F}_2(o)$  (resp.  $q$  of  $\mathcal{F}_1(o)$ ) flat coordinates  $x_r^i$  (respectively  $y_q^j$ ) of  $\mathcal{F}_1(r)$  (resp.  $\mathcal{F}_2(q)$ ).*

*Let  $p$  be a point of  $\mathcal{M}$  close to  $o$  and  $q$  (resp.  $r$ ) the unique intersection point of  $\mathcal{F}_1(o)$  with  $\mathcal{F}_2(p)$  respectively  $\mathcal{F}_2(o)$  with  $\mathcal{F}_1(p)$ ,  $\gamma_1$  a path contained in  $\mathcal{F}_1(o)$  from  $o$  to  $q$ ,  $\gamma_2$  a path contained in  $\mathcal{F}_2(p)$  from  $q$  to  $p$ ,  $\gamma_3$  a path contained in  $\mathcal{F}_1(p)$  from  $p$  to  $r$ ,  $\gamma_4$  a path contained in  $\mathcal{F}_2(o)$  from  $r$  to  $o$ ,  $\gamma$  the composition  $\gamma_1\gamma_2\gamma_3\gamma_4$ . Then  $\tau_\gamma$  depends only on  $p$ , and one has:  $\tau_{\gamma_1\gamma_2}(\partial_{x_o^i,o}) = \partial_{x_o^i,p}$  and  $\tau_{\gamma_3\gamma_4}^{-1}(\partial_{x_o^i,o}) = \partial_{x_r^i,p}$ . Similarly  $\tau_{\gamma_1\gamma_2}(\partial_{y_o^j,o}) = \partial_{y_q^j,p}$  et  $\tau_{\gamma_3\gamma_4}^{-1}(\partial_{y_o^j,o}) = \partial_{y_o^j,p}$ .*

*More precisely one has:*

$$\tau_\gamma(\partial_{x_o^i,o}) = \sum_j \frac{\partial x_r^j}{\partial x_o^i}(p) \partial_{x_o^j,o},$$

*et*

$$\tau_\gamma(\partial_{y_o^j,o}) = \sum_i \frac{\partial y_o^j}{\partial y_q^i}(p) \partial_{y_o^i,o}.$$

*Moreover the holonomy group is generated by the  $\tau_\gamma$  of this type.*

*Proof.* The first part of the proposition can be shown by evaluating parallel transport of the given basis vectors along the given paths.

For the latter statement we will show that one can approximate any path  $\gamma$  contained in a neighbourhood of  $o$  by a sequence of paths  $\gamma_n$  such that  $\tau_{\gamma_n}$  is generated by the parallel transports of curves of the type described in the proposition (which we will call "rectangles" for simplicity) and such that  $\tau_\gamma$  is the limit of the  $\tau_{\gamma_n}$ .

Let  $p_{1,p}$  be the local projection of  $\mathcal{M}$  onto the leaf  $\mathcal{F}_1(p)$  along the leaves of  $\mathcal{F}_2$  and  $p_{2,p}$  be the local projection of  $\mathcal{M}$  onto the leaf  $\mathcal{F}_2(p)$  along the leaves of  $\mathcal{F}_1$ .

For a given curve  $\mu : [t_0, t_1] \rightarrow M$  define the curve  $\mu_{(1)} : [t_0, t_1] \rightarrow M$  by  $\mu_{(1)} = (p_{1,\mu(t_0)} \circ \mu)$  and the curve  $\mu_{(2)} : [t_0, t_1] \rightarrow M$  by  $\mu_{(2)} = (p_{2,\mu(t_1)} \circ \mu)$ .

Finally let  $\mu' : [t_0, t_1] \rightarrow M$  be the curve defined by  $\mu'(\xi t_0 + (1 - \xi)\frac{t_0+t_1}{2}) = \mu_{(1)}(\xi t_0 + (1 - \xi)t_1)$  for  $0 \leq \xi \leq 1$  and by  $\mu'(\xi\frac{t_0+t_1}{2} + (1 - \xi)t_1) = \mu_{(2)}(\xi t_0 + (1 - \xi)t_1)$  for  $0 \leq \xi \leq 1$ .

For a given sequence of reals  $0 = t_0 < t_1 < t_2 < \dots < t_n < t_{n+1} = 1$  one considers the curve  $\nu : [0, 1] \rightarrow M$  such that  $\nu|_{[t_i, t_{i+1}]} = (\gamma|_{[t_i, t_{i+1}]})'$  is obtained by the preceding method.

One can show that the parallel transport  $\tau_\nu$  is equal to a product of parallel transports along "rectangles": In fact if  $\rho = \rho_1 \rho_2$  is a closed curve then  $\rho'_1 \rho'_2$  is a "rectangle". One decomposes parallel transport  $\tau_\nu$  then into a product of parallel transports along the "rectangles":  $R_1^{-1}, R_2, R_3^{-1}, R_4, \dots, R_{2(n-1)}^{-1}, R_{2n}$  defined as follows:

By setting  $\omega_{2i} = \gamma|_{[t_0, t_i]}$  and  $\omega_{2i+1} = \gamma|_{[t_0, t_i]}(\gamma|_{[t_i, t_{i+1}]})_{(1)}$  for  $i < n$ , let  $R_r$  be the "rectangle"  $\omega'_r(\omega_r^{-1})'$  for  $r < 2n$  and let  $R_{2n}$  be the "rectangle"  $(\gamma|_{[t_0, t_n]})'(\gamma|_{[t_n, t_{n+1}]})'$ .

When  $|t_{i+1} - t_i|$  tends towards 0,  $\nu$  tends towards  $\gamma$  in the sense of the compact-open topology which implies that  $\tau_\nu$  tends towards  $\tau_\gamma$ .

So the parallel transports along "rectangles" generate the whole holonomy group.  $\square$

### 3.2. Generic calculus.

**Proposition 8.** *If the holonomy representation of the manifold  $\mathcal{M}$  equipped with the connection  $\nabla$  is of type  $T_oM = V_1 \oplus V_2$  with  $V_1$  and  $V_2$  two faithful representations of the holonomy algebra  $\mathfrak{g}$ , using the notations of pre preceding subsection and writing  $a_i^j$  and  $\tilde{a}_l^m$  the coefficients defined by the relations  $\partial_{y_q^k, p} = \sum_l a_k^l(p) \partial_{y_o^l, p}$ , and  $\partial_{y_o^l, p} = \sum_m \tilde{a}_l^m(p) \partial_{y_q^m, p}$  where  $q$  is the intersection point of  $\mathcal{F}_2(p)$  with  $\mathcal{F}_1(o)$ , defining  $\Gamma$  by the relations*

$$\nabla_{\partial_{x_o^i}} \partial_{x_o^k} = \sum_m \Gamma_{i,k}^m \partial_{x_o^m},$$

one has:

$$\begin{aligned} R_p(\partial_{x_o^i}, \partial_{y_o^j}) \partial_{x_o^k} &= - \sum_m \partial_{y_o^j} (\Gamma_{i,k}^m) \partial_{x_o^m}, \\ R_p(\partial_{x_o^i}, \partial_{y_o^j}) \partial_{y_q^k} &= - \sum_m \partial_{y_o^j} \left( \sum_l \partial_{x_o^i} a_k^l \tilde{a}_l^m \right) \partial_{y_q^m}. \end{aligned}$$

*Proof.* One can write (using in the point  $p$  the equality:  $dy_o^i = \sum_j (\partial_{y_o^j} y_o^i) dy_o^j$ ):  $a_k^l(p) = \partial_{y_q^k, p} y_o^l$ .

Clearly one has:  $\sum_j a_k^j \tilde{a}_j^l = \delta_k^l$  and  $\sum_j \tilde{a}_k^j a_j^l = \delta_k^l$ . By applying a derivation  $\partial$  to these relations one obtains:

$$\sum_j (\partial a_k^j) \tilde{a}_j^l = - \sum_j a_k^j (\partial \tilde{a}_j^l) \text{ et } \sum_j (\partial \tilde{a}_k^j) a_j^l = - \sum_j \tilde{a}_k^j (\partial a_j^l).$$

The equality  $\nabla_{\partial_{y_o^j}} \partial_{y_q^k} = 0$  (which is verified because the leaf  $\mathcal{F}_2(q)$  is flat), can be written when defining  $\Gamma'$  by the relations

$$\nabla_{\partial_{y_o^i}} \partial_{y_o^k} = \sum_m \Gamma'_{i,k}^m \partial_{y_o^m} :$$

$$\begin{aligned}
0 &= \nabla_{\partial_{y_o^j}} \partial_{y_o^k} \\
&= \nabla_{\partial_{y_o^j}} \sum_l a_k^l \partial_{y_o^l} \\
&= \sum_l \left( (\partial_{y_o^j} a_k^l) \partial_{y_o^l} + a_k^l \nabla_{\partial_{y_o^j}} \partial_{y_o^l} \right) \\
&= \sum_l \left( (\partial_{y_o^j} a_k^l) \partial_{y_o^l} + a_k^l \sum_p \Gamma_{j,l}^p \partial_{y_o^p} \right)
\end{aligned}$$

By separating the basis vectors one has for any  $j, k, l$ :

$$\partial_{y_o^j} a_k^l + \sum_p a_k^p \Gamma_{j,p}^l = 0. \quad (A)$$

One can isolate  $\Gamma_{j,p}^l$  by writing:

$$\Gamma_{j,p}^l = - \sum_k \tilde{a}_p^k \partial_{y_o^j} a_k^l. \quad (B)$$

Deriving relation (A) by  $\partial_{x_o^i}$  one obtains:

$$\partial_{x_o^i} \partial_{y_o^j} a_k^l + \sum_p (\partial_{x_o^i} a_k^p) \Gamma_{j,p}^l + \sum_p a_k^p (\partial_{x_o^i} \Gamma_{j,p}^l) = 0. \quad (*)$$

By choosing a path  $\gamma_1$  from  $o$  to  $q$  contained in  $\mathcal{F}_2(o)$  and a path  $\gamma_2$  from  $q$  to  $p$  contained in  $\mathcal{F}_1(p)$  and noting  $\gamma = \gamma_1 \gamma_2$  and  $\tau_\gamma$  the parallel transport along the path  $\gamma$ ,  $\tau_\gamma(\partial_{x_o^k}) = \partial_{x_o^k}$  and  $\tau_\gamma(\partial_{y_o^l}) = \partial_{y_o^l}$ .

At point  $p$  one can calculate:

$$R_p(\partial_{x_o^i}, \partial_{y_o^j}) \partial_{x_o^k} = \sum_m (-\partial_{y_o^j} \Gamma_{i,k}^m) \partial_{x_o^m}.$$

The more one has:

$$\begin{aligned}
R_p(\partial_{x_o^i}, \partial_{y_o^j}) \partial_{y_o^k} &= \underbrace{\nabla_{\partial_{x_o^i}} \nabla_{\partial_{y_o^j}} \partial_{y_o^k}}_0 - \nabla_{\partial_{y_o^j}} \nabla_{\partial_{x_o^i}} \partial_{y_o^k} \\
&= -\nabla_{\partial_{y_o^j}} \nabla_{\partial_{x_o^i}} \sum_l a_k^l \partial_{y_o^l} \\
&= -\nabla_{\partial_{y_o^j}} \sum_l (\partial_{x_o^i} a_k^l) \partial_{y_o^l} - \underbrace{\sum_l a_k^l \nabla_{\partial_{x_o^i}} \partial_{y_o^l}}_0 \\
&= -\sum_l \left( (\partial_{y_o^j} \partial_{x_o^i} a_k^l) \partial_{y_o^l} + \partial_{x_o^i} a_k^l \nabla_{\partial_{y_o^j}} \partial_{y_o^l} \right) \\
&= -\sum_l \left( (\partial_{y_o^j} \partial_{x_o^i} a_k^l) \partial_{y_o^l} + \partial_{x_o^i} a_k^l \sum_p \Gamma_{j,l}^p \partial_{y_o^p} \right)
\end{aligned}$$

By the relation (\*) and by the fact that  $\partial_{x_o^i}$  and  $\partial_{y_o^j}$  commute one obtains:

$$\begin{aligned}
 R_p(\partial_{x_o^i}, \partial_{y_o^j})\partial_{y_q^k} &= \sum_l \left( \sum_p a_k^p(\partial_{x_o^i} \Gamma_{j,p}^l) \right) \partial_{y_l^k} \\
 &= \sum_{m,l,p} \left( a_k^p(\partial_{x_o^i} \Gamma_{j,p}^l) \tilde{a}_l^m \right) \partial_{y_q^m},
 \end{aligned}$$

The coefficient of  $\partial_{y_q^m}$  can be written:

$$\begin{aligned}
 \sum_{p,l} \left( a_k^p(\partial_{x_o^i} \Gamma_{j,p}^l) \tilde{a}_l^m \right) &= \sum_{p,l} \left( a_k^p \partial_{x_o^i} \left( - \sum_n \tilde{a}_p^n \partial_{y_o^j} a_n^l \right) \tilde{a}_l^m \right) \\
 &= - \sum_{p,n,l} \left( a_k^p \partial_{x_o^i} \tilde{a}_p^n \partial_{y_o^j} a_n^l \tilde{a}_l^m + a_k^p \tilde{a}_p^n \partial_{x_o^i} \partial_{y_o^j} a_n^l \tilde{a}_l^m \right) \\
 &= - \sum_{p,n,l} \left( \partial_{x_o^i} a_k^p \tilde{a}_p^n a_n^l \partial_{y_o^j} \tilde{a}_l^m + a_k^p \tilde{a}_p^n \partial_{x_o^i} \partial_{y_o^j} a_n^l \tilde{a}_l^m \right) \\
 &= - \sum_l \left( \partial_{x_o^i} a_k^l \partial_{y_o^j} \tilde{a}_l^m + \partial_{x_o^i} \partial_{y_o^j} a_k^l \tilde{a}_l^m \right) \\
 &= - \partial_{y_o^j} \left( \sum_l \partial_{x_o^i} a_k^l \tilde{a}_l^m \right)
 \end{aligned}$$

□

#### 4. $V \oplus V^*$ HOLONOMY

In the para-Kähler case there is a bilinear symmetric non degenerate form  $b$  on  $T_oM$  which is invariant by the action on the holonomy algebra and  $T_oM$  is equal to the direct sum of two totally isotropic invariant subspaces  $V_1$  and  $V_2$ . In particular the signature of  $b$  is neutral.

Remark that the mapping  $\Psi : V_2 \rightarrow V_1^*$ ,  $v \mapsto b(v, \cdot)|_{V_1}$  is injective, and surjective for dimension reasons. In addition  $\Psi$  is a morphism of representations. As a consequence we have:  $T_oM = V_1 \oplus V_1^*$ .

Inversely if one has the equality of representations  $E = V_1 \oplus V_1^*$ , one can associate to  $E$  the symmetric bilinear non degenerate form  $b$  defined for  $x, y \in V_1, u, v \in V_1^*$ , by  $b(x+u, y+v) = u(y) + v(x)$ .  $V_1$  and  $V_1^*$  are totally isotropic for this form. As a conclusion we are in the para-Kähler case.

So it is equivalent to be in the para-Kähler case or to have a holonomy of type  $V \oplus V^*$ .

Note that in this case the form  $\omega(x+u, y+v) = u(y) - v(x)$  for  $x, y \in V_1, u, v \in V_1^*$  which is non degenerate bilinear antisymmetric is also invariant.

##### 4.1. Berger criteria.

**Proposition 9.** *When  $E = V \oplus V^*$ , note  $\tilde{k}(\mathfrak{g}) := (S^2(V^*) \otimes S^2(V)) \cap (V^* \otimes \mathfrak{g} \otimes V)$ , and  $\underline{\mathfrak{g}} := \langle \{r(x, \cdot, \cdot, z') \mid r \in \tilde{k}(\mathfrak{g}), x \in V, z' \in V^*\} \rangle$ . If  $\mathfrak{g} \subset \mathfrak{gl}(V)$  acting on  $V \oplus V^*$  is a holonomy representation of a torsion-free connection then  $\mathfrak{g} = \underline{\mathfrak{g}}$ .*



*Proof.* Recall that for  $x, y \in V, z', t' \in V^*$  and  $R$  a formal curvature tensor one has:  $R(x, y) = 0$  and  $R(z', t') = 0$ .  $R$  is entirely known by the data of  $R(x, z')y$  for  $x, y \in V$  and  $z' \in V^*$ . By the first Bianchi identity necessarily  $x, y \in V$  and  $z' \in V^*$ ,  $R(x, z')y = R(y, z')x$ . We can restate this by saying:  $R(\cdot, z') \cdot|_{V \times V} \in \mathfrak{g}^{(1)}$ .

Let us introduce the tensor  $r$  defined for  $x, y, z', t' \in V \oplus V^*$  by  $r(x, y, z', t') := \langle R(x, z')y, t' \rangle$  which satisfies the classical relations in pseudo-Riemannian geometry:  $r(x, y, z', t') = -r(z', y, x, t') = r(y, x, t', z')$ . We can restrict to  $x, y \in V$  and  $z', t' \in V^*$ . As a consequence  $r \in S^2(V^*) \otimes S^2(V)$ . Because  $r(x, \cdot, \cdot, z') = R(x, z') \in \mathfrak{g}$  it is clear that  $r \in \tilde{k}(\mathfrak{g})$ .

A condition for  $\mathfrak{g}$  being a holonomy representation is that  $\mathfrak{g}$  is generated by the  $R(x, z') = r(x, \cdot, \cdot, z')$  for  $r$  in  $\tilde{k}(\mathfrak{g})$ ,  $x$  in  $V$  and  $z'$  in  $V^*$ .  $\square$

**Proposition 10.** *When  $E = V \oplus V^*$ , note*

$$\tilde{k}^1(\mathfrak{g}) := (S^3(V^*) \otimes S^2(V)) \cap (S^2(V^*) \otimes \mathfrak{g} \otimes V) \oplus (S^2(V^*) \otimes S^3(V)) \cap (V^* \otimes \mathfrak{g} \otimes S^2(V)).$$

*If  $\mathfrak{g} \subset \mathfrak{gl}(V)$  acting on  $V \oplus V^*$  is a holonomy representation of a torsion-free non locally symmetric connection then  $\tilde{k}^1(\mathfrak{g}) \neq 0$ .*

*Proof.* This is again simply a specialization of the second Berger criterion to this case.  $\square$

**4.2. Classification in dimension  $2 \times 1$  and  $2 \times 2$ .** In dimension  $1 + 1$  the algebra  $\mathfrak{so}(1, 1)$  corresponding to the metric of matrix  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  is precisely  $\left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \mid a \in \mathbb{R} \right\}$ . It is the generic holonomy algebra in the pseudo-Riemannian case of signature  $(1, 1)$ .

The paper [BBI] gives a classification in signature  $(2, 2)$ :

**Proposition 11.** *For  $V$  of dimension 2,  $V \oplus V^*$  is an indecomposable holonomy representation if and only if  $V$  or  $V^*$  is in the following list:*

- (1)  $\left\{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \mid c \in \mathbb{R} \right\}$
- (2) For fixed  $\lambda \in [-1, 1]$ ,  $\left\{ \begin{pmatrix} a & c \\ 0 & \lambda a \end{pmatrix} \mid a, c \in \mathbb{R} \right\}$
- (3)  $\left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$
- (4)  $\mathfrak{sl}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$
- (5)  $\mathfrak{gl}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$
- (6)  $\mathfrak{co}(2) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$

## 5. $V \oplus V$ HOLONOMY

A representation  $V \oplus V$  can be written as well  $V \otimes \mathbb{R} \oplus V \otimes \mathbb{R} = V \otimes \mathbb{R}^2$  where  $\mathbb{R}$  and  $\mathbb{R}^2$  are the trivial representations.

**5.1. Berger criteria.** Let  $V_1 \oplus V_2$  be a holonomy representation of  $\mathfrak{g} \subset \mathfrak{gl}(V_1)$  and let  $\Phi : V_1 \rightarrow V_2, x \mapsto x'$  be an isomorphism of representations. The curvature tensor  $R$  verifies then the following relations due to the Bianchi identity:  $x, y, z \in V_1$ :

$$\begin{aligned} R(x, y) &= 0, \\ R(x', y') &= 0, \\ R(x, y')z &= R(z, y')x, \\ R(x, y')z' &= R(x, z')y'. \end{aligned}$$

In addition because of the isomorphism  $\Phi$  we have:

$$R(x, y')z' = (R(x, y')z)'$$

One can deduce that the tensor  $T$  defined for  $x, y, z \in V_1$  by  $T(x, y, z) := R(x, y')z$  verifies:

$$T \in S^3(V_1^*) \otimes V_1.$$

In addition The Ambrose-Singer theorem gives us for  $x, y \in V_1$ :

$$T(x, y, \cdot) \in \mathfrak{g}.$$

Note

$$\mathfrak{g}^{(2)} := S^3(V_1^*) \otimes V_1 \cap (V_1^*)^{\otimes 2} \otimes \mathfrak{g}$$

and

$$\hat{\mathfrak{g}} := \{T(x, y, \cdot) \mid T \in \mathfrak{g}^{(2)}, x, y \in V_1\}.$$

The first Berger criterion can be formulated in this context simply by

**Proposition 12.** *Let  $\mathfrak{g} \subset \mathfrak{gl}(V_1)$ . A necessary condition for  $\mathfrak{g} \subset \mathfrak{gl}(V_1)$  acting by  $V_1 \otimes \mathbb{R}^2$  is a holonomy representation for a torsion-free connection is that  $\mathfrak{g} = \hat{\mathfrak{g}}$ .*

Note that the second Bianchi identity can be formulated here by: for  $x, y, z \in V_1$ ,

$$(\nabla_x R)(y, z') = (\nabla_y R)(x, z'),$$

so that the tensor  $S$  defined by  $S(x, y, z, t) := (\nabla_x R)(y, z')t$  lives in  $S^4(V_1^*) \otimes V_1$ .

By noting

$$\mathfrak{g}^{(3)} := S^4(V_1^*) \otimes V_1 \cap (V_1^*)^{\otimes 3} \otimes \mathfrak{g},$$

the second Berger criterion can be formulated:

**Proposition 13.** *If  $\mathfrak{g} \subset \mathfrak{gl}(V_1)$  acting by  $V_1 \otimes \mathbb{R}^2$  is a holonomy representation for a torsion-free non locally symmetric connection then necessarily  $\mathfrak{g}^{(3)} \neq \{0\}$ .*

**5.2. Geometric structure.** By using the notations and calculus used in the proof of proposition 8 we obtain:

If the holonomy is of type  $V \otimes \mathbb{R}^2$ , at the origin  $o$   $\partial_{x_o^k, o}$  and  $\partial_{y_o^k, o}$  transform by the same endomorphism. Their parallel transport along a path  $\gamma$  from  $o$  to  $p$  defined as the product  $\gamma_1\gamma_2$  of a path  $\gamma_1$  from  $o$  to  $q$  contained in  $\mathcal{F}_1(o)$  and a path  $\gamma_2$  from  $q$  to  $p$  contained in  $\mathcal{F}_2(p)$  gives:  $\tau_\gamma(\partial_{x_o^k}) = \partial_{x_o^k}$  et  $\tau_\gamma(\partial_{y_o^k}) = \partial_{y_q^k}$ .

$R_p(\partial_{x_o^i}, \partial_{y_o^j})$  acts again "identically" on these two vectors, which means that the relation between  $R_p(\partial_{x_o^i}, \partial_{y_o^j})\partial_{x_o^k}$  and  $R_p(\partial_{x_o^i}, \partial_{y_o^j})\partial_{y_q^k}$  obtained in proposition 8 can be translated by:

$$\partial_{y_o^j} \Gamma_{i,k}^m = \partial_{y_o^j} \left( \sum_l \partial_{x_o^i} a_k^l \tilde{a}_l^m \right),$$

or by:

$$\Gamma_{i,k}^m = \sum_l \partial_{x_o^i} a_k^l \tilde{a}_l^m + \text{const.}$$

By evaluating this equality in  $o$  it follows

$$\Gamma_{i,k}^m = \sum_l \partial_{x_o^i} a_k^l \tilde{a}_l^m.$$

So that we can write:

**Proposition 14.** *If the holonomy of the connection  $\nabla$  is of type  $V \otimes \mathbb{R}^2$ ,  $\alpha_i^j$  et  $\tilde{a}_l^m$  being the coefficients defined by  $\partial_{y_q^k} = \sum_l a_k^l(p) \partial_{y_o^l}$ , and  $\partial_{y_o^l} = \sum_m \tilde{a}_l^m(p) \partial_{y_q^m}$  (where  $q$  is the intersection point of  $\mathcal{F}_2(p)$  with  $\mathcal{F}_1(o)$ ), defining  $\Gamma$  by the relations*

$$\nabla_{\partial_{x_o^i}} \partial_{x_o^k} = \sum_m \Gamma_{i,k}^m \partial_{x_o^m},$$

we have:

$$\Gamma_{i,k}^m = \sum_l \partial_{x_o^i} a_k^l \tilde{a}_l^m.$$

The condition that  $\nabla$  is torsion-free gives then the following equalities:

**Proposition 15.** *The coordinates defined before verify the relations (for any  $i, j, k$ ):*

$$\frac{\partial^2 y_o^k}{\partial x^i \partial y^j} = \frac{\partial^2 y_o^k}{\partial x^j \partial y^i}.$$

**5.3. Classification in dimension  $2 \times 2$ .** Remark that the representations  $V \otimes \mathbb{R}^2$  which are decomposable are all holonomy representations (for a torsion-free connection) because all representations of dimension 2 are.

**Proposition 16.** *For  $V$  of dimension 2,  $V \otimes \mathbb{R}^2$  is an indecomposable holonomy representation if and only if  $V$  is in the following list:*

- (1)  $\left\{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \mid c \in \mathbb{R} \right\}$
- (2) For fixed  $\lambda \in \mathbb{R}$ ,  $\left\{ \begin{pmatrix} a & c \\ 0 & \lambda a \end{pmatrix} \mid a, c \in \mathbb{R} \right\}$
- (3)  $\left\{ \begin{pmatrix} 0 & c \\ 0 & a \end{pmatrix} \mid a, c \in \mathbb{R} \right\}$
- (4)  $\left\{ \begin{pmatrix} a & c \\ 0 & b \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$
- (5)  $\mathfrak{sl}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$
- (6)  $\mathfrak{gl}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$
- (7)  $\mathfrak{co}(2) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$

*Proof.* Recall the list of the representations of dimension 2:

	Lie group	Lie algebra
$Gl^+(2, \mathbb{R})$	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc > 0$	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, b, c, d \in \mathbb{R}$
$Sl(2, \mathbb{R})$	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $ad - bc = 1$	$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ with $a, b, c \in \mathbb{R}$
$SO(2)$	$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ with $\theta \in \mathbb{R}$	$\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ with $a \in \mathbb{R}$
$CO(2)$	$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $a^2 + b^2 \neq 0$	$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $a, b \in \mathbb{R}$
$CO(2)_\lambda$	$e^\theta \begin{pmatrix} \cos(\lambda\theta) & -\sin(\lambda\theta) \\ \sin(\lambda\theta) & \cos(\lambda\theta) \end{pmatrix}$ with $\lambda \neq 0$ fixed and $\theta \in \mathbb{R}$	$\begin{pmatrix} \theta & -\lambda\theta \\ \lambda\theta & \theta \end{pmatrix}$ avec $\theta \in \mathbb{R}$
$Id$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	0
Homotheties	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ with $a > 0$	$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ with $a \in \mathbb{R}$
$SO(1, 1)$	$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$ with $a > 0$	$\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}$ with $a \in \mathbb{R}$
$SO_\lambda$	$\begin{pmatrix} a & 0 \\ 0 & a^\lambda \end{pmatrix}$ with $\lambda$ fixed $\neq -1, 0, 1$ and $a > 0$	$\begin{pmatrix} a & 0 \\ 0 & \lambda a \end{pmatrix}$ with $a \in \mathbb{R}$
$SO - 1_1$	$\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}$ with $a > 0$	$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ with $a \in \mathbb{R}$
$CO(1, 1)$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with $a, b > 0$	$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ with $a, b \in \mathbb{R}$
$He$	$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ with $a \in \mathbb{R}$	$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ with $a \in \mathbb{R}$
$Tr - X$	$\begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix}$ with $t \in \mathbb{R}$	$\begin{pmatrix} a & a \\ 0 & a \end{pmatrix}$ with $a \in \mathbb{R}$
$Tr - H$	$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ with $a > 0, b \in \mathbb{R}$	$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ with $a, b \in \mathbb{R}$
$Tr - SO(1, 1)$	$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$ with $a > 0, b \in \mathbb{R}$	$\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$ with $a, b \in \mathbb{R}$
$Tr - SO_\lambda$	$\begin{pmatrix} a & b \\ 0 & a^\lambda \end{pmatrix}$ with $\lambda$ fixed $\neq -1, 0, 1$ and $a > 0, b \in \mathbb{R}$	$\begin{pmatrix} a & b \\ 0 & \lambda a \end{pmatrix}$ with $a, b \in \mathbb{R}$
$Tr - SO - 1_1$	$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a > 0, b \in \mathbb{R}$	$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$ with $a, b \in \mathbb{R}$
$Tr - SO - 1_2$	$\begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix}$ with $a > 0, b \in \mathbb{R}$	$\begin{pmatrix} 0 & b \\ 0 & a \end{pmatrix}$ with $a, b \in \mathbb{R}$
$Tr$	$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $a > 0, c > 0, b \in \mathbb{R}$	$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with $a, b, c \in \mathbb{R}$

By examining the different representations  $V$  in dimension 2, we have:

$CO(1, 1)$ ,  $Id$  et  $SO - 1_1$  are decomposable.

$SO(2)$ ,  $CO(2)_\lambda$  (for  $\lambda \neq 0$ ),  $SO(1, 1)$ ,  $Tr - X$ , the homotheties and  $SO_\lambda$  (for  $\lambda \neq -1, 0, 1$ ) are excluded because  $V \otimes \mathbb{R}^2$  is then a representation of a Lie algebra of dimension 1 whose image is generated by an endomorphism of rank bigger or equal to 3 which can not be a holonomy representation because of the first Bianchi identity.

For the other cases we construct associated connections by making explicit the coordinate relations by proposition 7 and verifying the relations stated in 15. Note that  $y^i$  and  $y_o^i$  have to coincide if  $\forall j, x^j = 0$ .

For  $Gl^+(2, \mathbb{R})$  we take:

$$\begin{aligned} y_o^1 &= x_o^1 y^2 + x_o^2 y^1 + y^1, \\ y_o^2 &= x_o^1 y^2 x_o^2 y^1 + y^2. \end{aligned}$$

$Sl(2, \mathbb{R})$  is an example known from the work of R. Bryant (see [Br1] and [Br2]):  $\mathfrak{su}(1, 1)$  is a holonomy representation.

For  $CO(2)$  we take:  $X = x_o^1 + ix_o^2$ ,  $Y = y^1 + iy^2$  a holomorphic function  $F : U \rightarrow \mathbb{C}$  (with  $U$  an open neighbourhood of  $o$  in  $\mathbb{C}^2$ ) such that  $\frac{\partial^2 F}{\partial X \partial Y}$  is not identically vanishing and such that  $F(0, Y) = Y$  et  $F(X, 0) = X$ , for example  $F(X, Y) = XY + X + Y$ . We note  $F = F_R + iF_I$  with  $F_R$  and  $F_I$  real-valued functions. The functions

$$\begin{aligned} y_o^1 &= F_R, \\ y_o^2 &= F_I \end{aligned}$$

define then a connection of the wanted type.

$He$  is as well of type  $V \oplus V^*$ . One can take:

$$\begin{aligned} y_o^1 &= y^1 + f(x^2, y^2), \\ y_o^2 &= y^2, \end{aligned}$$

$f$  is a function vanishing for  $x^2 = 0$ .

For  $Tr$  one can take

$$\begin{aligned}y_o^1 &= x^1 y^1 + y^1 + y^2, \\y_o^2 &= y^2(1 + x^2),\end{aligned}$$

as an example.

$Tr - H$ ,  $Tr - SO(1, 1)$ ,  $Tr - SO_\lambda$  ( $\lambda \neq -1, 0, 1$ ),  $Tr - SO - 1_1$ , which enter the scheme  $\begin{pmatrix} a & b \\ 0 & a^\lambda \end{pmatrix}$  with  $\lambda$  one can take:

$$\begin{aligned}y_o^1 &= x^1 y^2 + x^2 y^1 + y^1, \\y_o^2 &= y^2(1 + x^2)^\lambda.\end{aligned}$$

For the group  $Tr - SO - 1_2$  one can take

$$\begin{aligned}y_o^1 &= y^1 + y^2, \\y_o^2 &= y^2(1 + x^2).\end{aligned}$$

□

## 6. FINAL REMARK

We evaluated the Berger criteria for both types of representations  $V \oplus V^*$  and  $V \otimes \mathbb{R}^2$  for all representations  $V$  up to dimension 3 using the software Maple. These results might appear in future articles.

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