Traffic Grooming in Unidirectional WDM Rings with Bounded Degree Request Graph

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Résumé : Le groupage de trafic est un des problèmes les plus importants dans les réseaux optiques. Il constiste à grouper des signaux de bas débit dans des flux de plus grande capacité, avec l'objectif de réduire le coût du réseau. Dans les réseaux SONET WDM, ce coût est donné principalement par le nombre d'ADMs. Nous considérons l'anneau unidirectionnel comme graphe physique. En termes de théorie des graphes, le groupage de trafic revient à trouver une partition des arêtes du graphe de requêtes en sous-graphes avec un nombre maximum d'arêtes, en minimisant le nombre total de sommets dans la partition.

Nous considérons un graphe de requêtes de degré maximum Δ, et le but est de concevoir un réseau qui soit capable de satisfaire *tous* les graphes de requêtes tels que chaque sommet peut établire au plus Δ communications. Un modèle permettant cette flexibilité n'existait pas dans la littérature. Nous formalisons le problème et trouvons la solution exacte pour $\Delta = 2$ et $\Delta = 3$ (sauf le cas $C = 4$). Nous donnons des bornes inférieures et supérieures pour le cas géneral.

Mots-clés : Réseaux optiques, SONET, groupage de trafic, ADM, décomposition des graphes, graphe cubique, graphes sans bridges.

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Traffic Grooming in Unidirectional WDM Rings with Bounded Degree Request Graph ¶

Abstract: Traffic grooming is a major issue in optical networks. It refers to grouping low rate signals into higher speed streams, in order to reduce the equipment cost. In SONET WDM networks, this cost is mostly given by the number of electronic terminations, namely ADMs. We consider the case when the topology is a unidirectional ring. In graph-theoretical terms, the traffic grooming problem in this case consists in partitioning the edges of a request graph into subgraphs with a maximum number of edges, while minimizing the total number of vertices of the decomposition.

We consider the case when the request graph has bounded maximum degree Δ, and our aim is to design a network being able to support any request graph satisfying the degree constraints. The existing theoretical models in the literature are much more rigid, and do not allow such adaptability. We formalize the problem, and solve the cases $\Delta = 2$ (for all values of C) and $\Delta = 3$ (except the case $C = 4$). We also provide lower and upper bounds for the general case.

Key-words: Optical networks, SONET over WDM, traffic grooming, ADM, graph decomposition, cubic graph, bridgeless graph.

1 Introduction

Traffic grooming is the generic term for packing low rate signals into higher speed streams (see the surveys [3, 9, 15, 16, 20]). By using traffic grooming, it is possible to bypass the electronics at the nodes which are not sources or destinations of traffic, and therefore reducing the cost of the network. Typically, in a WDM (Wavelength Division Multiplexing) network, instead of having one SONET Add Drop Multiplexer (ADM) on every wavelength at every node, it may be possible to have ADMs only for the wavelengths used at that node (the other wavelengths being optically routed without electronic switching).

The so called traffic grooming problem consists in minimizing the total number of ADMs to be used, in order to reduce the overall cost of the network. The problem is easily seen to be NP-complete for an arbitrary set of requests. See [11, 10, 1] for hardness and approximation results of traffic grooming in rings, trees and star networks.

Here we consider unidirectional SONET/WDM ring networks. In that case the routing is unique and we have to assign to each request between two nodes a wavelength and some bandwidth on this wavelength. If the traffic is uniform and if a given wavelength can carry at most C requests, we can assign to each request at most $\frac{1}{C}$ of the bandwidth. C is known as the *grooming ratio* or *grooming factor*. Furthermore if the traffic requirement is symmetric, it can be easily shown (by exchanging wavelengths) that there always exists an optimal solution in which the same wavelength is given to a pair of symmetric requests. Then without loss of generality we will assign to each pair of symmetric requests, called a *circle*, the same wavelength. Then each circle uses $\frac{1}{C}$ of the bandwidth in the whole ring. If the two end-nodes are i and j, we need one ADM at node i and one at node j. The main point is that if two requests have a common end-node, they can share an ADM if they are assigned the same wavelength.

The traffic grooming problem for a unidirectional SONET ring with n nodes and a grooming ratio C has been modeled as a graph partition problem in both [2] and [14] when the request graph is given by a symmetric graph R. To a wavelength λ is associated a subgraph $B_\lambda \subset R$ in which each edge corresponds to a pair of symmetric requests (that is, a circle) and each node to an ADM. The grooming constraint, i.e. the fact that a wavelength can carry at most C requests, corresponds to the fact that the number of edges $|E(B_\lambda)|$ of each subgraph B_λ is at most C. The cost corresponds to the total number of vertices used in the subgraphs, and the objective is therefore to minimize this number.

This problem has been well studied when the network is a unidirectional ring [3, 4, 7, 8, 9, 14, 12, 13, 15, 18, 19]. With the all-to-all set of requests, optimal constructions for a given grooming ratio C were obtained using tools of graph and design theory, in particular for grooming ratio $C = 3, 4, 5, 6$ and $C \geq N(N-1)/6$ [3].

Most of the research efforts in this grooming problem have been devoted to find the minimum number of ADMs required either for a given traffic pattern or set of connection requests (typically uniform all-to-all communication pattern), or either for a general traffic pattern. However in most cases the traffic pattern has been considered as an input for the problem for placing the ADMs. In this paper we consider the traffic grooming problem from a different point of view : Assuming a given network topology it would be desirable to place the minimum number of ADMs as possible at each node in such a way that they could be configured to handle different traffic patterns or graphs of requests. One cannot expect to change the equipment of the network each time the traffic requirements change.

Without any restriction in the graph of requests, the number of required ADMs is given by the worst case, i.e. when the Graph of Requests is the complete graph. However, in many cases some restrictions on the graph of requests might be assumed. From a practical point of view, it is interesting to design a network being able to support any request graph with maximum degree not exceeding a given constant. This situation is usual in real optical networks, since due to technology constraints the number of allowed communications for each node is usually bounded. This flexibility can also be thought from another point of view : if we have a limited number of available ADMs to place at the nodes of the network, then it is interesting to know which is the maximum degree of a request graph that our network is able to support, depending on the grooming factor. Equivalently, given a maximum degree and a number of available ADMs, it is useful to know which values of the grooming factor the network will support.

The aim of this article is to provide a theoretical framework to design such networks with dynamically changing traffic. We study the case when the physical network is given by an unidirectional ring, which is a widely used topology (for instance, SONET rings). In [6] the authors consider this problem from a more practical point of view : they call *t-allowable* a traffic matrix where the number of circuits terminated at each node is at most t, and the objective is also to minimize the number of electronic terminations. They give lower bounds on the number of ADMs and provide some heuristics.

In addition, we also suppose that each pair of communicating nodes establishes a two-way communication. That is, each pair (i, j) of communicating nodes in the ring represents two requests : from i to j, and from j to i. Thus, such a pair uses all the edges of the ring, therefore inducing one unity of load. Hence, we can use the notation introduced in [4] and consider each request as an edge, and then again the grooming constraint, i.e. the fact that a wavelength can carry at most C requests, corresponds to the fact that the number of edges $|E(B_\lambda)|$ of each subgraph B_λ is at most C. The cost corresponds to the total number of vertices used in the subgraphs.

Namely, we consider the problem of placing the minimum number of ADMs in the nodes of the ring in such a way that the network could support *any* request graph with maximum degree bounded by a constant Δ. Note that using this approach, as far as the degree of each node does not exceed Δ, the network can support a wide range of traffic demands without reconfiguring the electronics placed at the nodes. The problem can be formally stated as follows :

TRAFFIC GROOMING IN UNIDIRECTIONAL RINGS WITH BOUNDED-DEGREE REQUEST GRAPH
INPUT : Three integers n, C, and Δ.

Output : An assignment of $A(v)$ ADMs to each node $v \in V(C_n)$, in such a way that *for any request graph* R with maximum degree at most Δ, it exists a partition of $E(R)$ into subgraphs B_λ, $1 \leq \lambda \leq \Lambda$, such that :
(i) $|E(B_\lambda)| \leq C$ for all λ ; and
(ii) each vertex $v \in V(C_n)$ appears in at most $A(v)$ subgraphs.

Objective : Minimize $\sum_{v \in V(C_n)} A(v)$, and the optimum is denoted $A(n, C, \Delta)$.

When the request graph is restricted to belong to a subclass of graphs $\mathcal{C}$ of the class of graphs with maximum degree at most Δ, then the optimum is denoted $A(n, C, \Delta, \mathcal{C})$. Obviously, for any subclass of graph $\mathcal{C}$, $A(n, C, \Delta, \mathcal{C}) \leq A(n, C, \Delta)$.

In this article we solve the cases corresponding to $\Delta = 2$ and $\Delta = 3$ (giving a conjecture for the case $C = 4$), and give lower bounds for the general case. The remainder of the article is structured as follows : in Section 2 we give some properties of the function $A(n, C, \Delta)$, to be used in the following sections. In Section 3 we focus on the case $\Delta = 2$, giving a closed formula for all values of C. In Section 4 we study the case $\Delta = 3$, solving all cases except the case $C = 4$, for which we conjecture the solution. Finally, Section 5 is devoted to conclusions and open problems.

2 Behavior of $A(n, C, \Delta)$

In this section we describe some properties of the function $A(n, C, \Delta)$.

Lemma 2.1 *The following statements hold :*

(i) $A(n, C, 1) = n$.
(ii) $A(n, 1, \Delta) = \Delta n$.
(iii) *If* $C' \geq C$, *then* $A(n, C', \Delta) \leq A(n, C, \Delta)$.
(iv) *If* $\Delta' \geq \Delta$, *then* $A(n, C, \Delta') \geq A(n, C, \Delta)$.
(v) $A(n, C, \Delta) \geq n$ *for all* $\Delta \geq 1$.
(vi) *If* $C \geq \frac{n\Delta}{2}$, $A(n, C, \Delta) = n$.

Proof:

(i) The request graph can consist in a perfect matching, so any solution uses 1 ADM per node.
(ii) A Δ-regular graph can be partitioned into $\frac{n\Delta}{2}$ disjoint edges, and we cannot do better.
(iii) Any solution for C is also a solution for C'.
(iv) If $\Delta' \geq \Delta$, the subgraphs with maximum degree at most Δ are a subclass of the class of graphs with maximum degree at most Δ'.
(v) Combine *(i)* and *(iv)*.
(vi) In this case all the edges of the request graph fit into one subgraph. $\square$

Since we are interested in the number of ADMs required at each node, let us consider the following definition :

Definition 2.1 *Let $M(C, \Delta)$ be the least positive number M such that, for any $n \geq 1$, the inequality $A(n, C, \Delta) \leq Mn$ holds.*

Lemma 2.2 *$M(C, \Delta)$ is a natural number.*

Proof: First of all, we know by Lemma 2.1 that, for any $C \geq 1$, $A(n, C, \Delta) \leq A(n, 1, \Delta) = \Delta n$. Thus $A(n, C, \Delta)$ is upper-bounded by Δn. On the other hand, since any vertex may appear in the request graph, $A(n, C, \Delta)$ is lower-bounded by n.

Suppose now that M is not a natural number. That is, suppose that $r < M < r + 1$ for some positive natural number r. This means that, for each n, there exists at least a fraction $\frac{r}{M}$ of the vertices with at most r ADMs. For each n, let $V_{n,r}$ be the subset of vertices of the request graph with at most r ADMs. Then, since $\frac{r}{M} > 0$, we have that $\lim_{n \to \infty} |V_{n,r}| = \infty$. In other words, there is an arbitrarily big subset of vertices with at most r ADMs per vertex. But we can consider a request graph with maximum degree at most Δ on the set of vertices $V_{n,r}$, and this means that with r ADMs per node is enough, a contradiction with the optimality of M. □

If the request graph is restricted to belong to a subclass of graphs $\mathcal{C}$ of the class of graphs with maximum degree at most Δ, then the corresponding positive integer is denoted $M(C, \Delta, \mathcal{C})$. Again, for any subclass $\mathcal{C}$, $M(C, \Delta, \mathcal{C}) \leq M(C, \Delta)$.

Combining Lemmas 2.1 and 2.2, we know that $M(C, \Delta)$ decreases by integer hops when C increases. One would like to have a better knowledge of those hops. The following lemma gives a sufficient condition to assure than $M(C, \Delta)$ decreases by at most 1 when C increases by 1.

Lemma 2.3 *If $C > \Delta$, then $M(C + 1, \Delta) \geq M(C, \Delta) - 1$.*

Proof: Suppose that $M(C + 1, \Delta) \leq M(C, \Delta) - 2$, and let us arrive at contradiction. Beginning with a solution for $C+1$, we will see that adding n ADMs (i.e. increasing M by 1) we obtain a solution for C, a contradiction with the assumption $M(C, \Delta) \geq M(C+1, \Delta)+2$.

The request graph has at most $\frac{\Delta n}{2}$ edges, and then in a solution for $C + 1$ the number of subgraphs with exactly $C+1$ edges is at most $\frac{\Delta n}{2(C+1)}$. All the subgraphs with C or less edges can also be used in a solution for C. We remove an edge from each one of the subgraphs with $C + 1$ edges, obtaining at most $\frac{\Delta n}{2(C+1)}$ edges, or equivalently at most $\frac{\Delta n}{C+1}$ additional ADMs. We want this number to be at most n, i.e.

$$\frac{\Delta n}{C + 1} \leq n \ ,$$

which is equivalent to $\Delta \leq C + 1$, and this is true by hypothesis. □

We provide now a lower bound on $M(C, \Delta)$.

Proposition 2.1 (General Lower Bound) $M(C, \Delta) \geq \left\lceil \frac{C+1}{C} \frac{\Delta}{2} \right\rceil$.

Proof: Since we have to consider all the graphs with maximum degree at most Δ, we can restrict ourselves to Δ-regular graphs with girth greater than C. Then, the best one could do is to partition the edges of the request graph into trees with C edges. In this case, the sum of the degrees of all the vertices in each subgraph is $2C$. Thus, the average degree of the vertices in all the subgraphs is at most $\frac{2C}{C+1}$, hence it exists at least one vertex v with average degree not greater than $\frac{2C}{C+1}$. Therefore, v must appear in at least M_v subgraphs, with $\frac{2C}{C+1}M_v \geq \Delta$. Thus, $M(C, \Delta) \geq \lceil \frac{C+1}{C}\frac{\Delta}{2} \rceil$. □

Corollary 2.1 *If the set of requests is given by a Δ-regular graph of girth greater than C, then*

$$M(C, \Delta) \geq \left\lceil \frac{\Delta}{2} \right\rceil$$

Proof: Trivial from Proposition 2.1. □

If the value of C is large in comparison to n the number of ADMs required per node may be less than $M(C, \Delta)$ as stated in the following lemma :

Lemma 2.4 $A(n, C, \Delta) \leq \lceil \frac{n\Delta}{2C} \rceil n$.

Proof: The number of edges of a request graph with degree Δ is at most $\frac{n\Delta}{2}$. We can partition this edges greedily into subsets of at most C edges, obtaining at most $\lceil \frac{n\Delta}{2C} \rceil$ subgraphs. Thus, in this partition each vertex appears in at most $\lceil \frac{n\Delta}{2C} \rceil$ subgraphs, as we wanted to prove. □

Notice that this is not in contradiction with Corollary 2.1, since the inequality of the definition of $M(C, \Delta)$ must hold for *all* values of n.

3 Case $\Delta = 2$

Proposition 3.1 $A(n, C, 2) = 2n - (C - 1)$.

Proof: Consider the case when the request graph is 2-regular and has girth greater than C. Then, a feasible solution is obtained by placing 2 ADMs at each vertex. What we do is to count in how many ADMs we can assure that we can place only one ADM.
Let us see first that we cannot use 1 ADM in more than $C-1$ vertices. Suppose this, i.e. that we have 1 ADM in C vertices and 2 in all the others. Then, consider a set of requests given by a cycle H of length $C + 1$ containing all the C vertices with 1 ADM inside it, and other cycles containing the remaining vertices. In this situation, we are forced to use 2 subgraphs for the vertices of H, and at least 2 vertices of H must appear in both subgraphs. Hence we will need more than 1 ADM in some vertex that had initially only 1 ADM.

Now, let us see that we can always save $C - 1$ ADMs. Let $\{a_0, a_1, \ldots, a_{C-2}\}$ be the set of vertices with only 1 ADM, that we can choose arbitrarily. We will see that we can decompose the set of requests in such a way that the vertices a_i always lie in the middle of a path or a cycle, covering in this way both requests of each vertex with only 1 ADM. Indeed, suppose first that two of these vertices (namely, a_i and a_j) do not appear consecutively in one of the disjoint cycles of the set of requests. Let b_i be the nearest vertex to a_i in the cycle in the direction of a_j, and conversely for b_j (b_i may be equal to b_j if a_i and a_j differ only on one vertex). Then, consider two paths (eventually, cycles) of the form $\{b_i, a_i, \ldots\}$ and $\{b_j, a_j, \ldots\}$, to assure that both a_i and a_j lie in the middle of the subgraph. We do the same construction for each pair of non-consecutive vertices.

Now, consider all the vertices $\{a_0, \ldots, a_i, \ldots, a_{t-1}\}$ which are adjacent in the same cycle of the request graph, with $t \leq C-1$. Let b_0 be the nearest vertex to a_0 different from a_1, and let b_{t-1} be the nearest vertex to a_{t-1} different from a_{t-2}. Then, consider a subgraph with the path (or cycle, if $b_0 = b_{t-1}$) $\{b_0 a_0 a_1 \ldots a_{t-1} b_{t-1}\}$. □

4 Case $\Delta = 3$

We study the cases $C = 3$ and $C \geq 5$ in Sections 4.1 and 4.2, respectively. We discuss the open case $C = 4$ in Section 5.

4.1 Case $C = 3$

We study first the case when the request graph is a bridgeless cubic graph in Section 4.1.1, and then the case of a general request graph in Section 4.1.2.

4.1.1 Bridgeless Cubic Request Graph

We will need some preliminary graph theoretical concepts. Let $G = (V, E)$ be a graph. For $A, B \subseteq V$, an *A-B path* in G is a path from x to y, with $x \in A$ and $y \in B$. If $A, B \subseteq V$ and $X \subseteq V \cup E$ are such that every A-B path in G contains a vertex or an edge from X, we say that X *separates* the sets A and B in G. More generally we say that X *separates* G if $G - X$ is disconnected, that is, if X separates in G some two vertices that are not in X. A separating set of vertices is a *separator*. A vertex which separates two other vertices of the same component is a *cut-vertex*, and an edge separating its ends is a *bridge*. Thus, the bridges in a graph are precisely those edges that do not lie on any cycle. A set M of independent edges in a graph $G = (V, E)$ is called a *matching*. A k-regular spanning subgraph is called a k-factor. Thus, a subgraph $H \subseteq G$ is a 1-factor of G if and only if $E(H)$ is a matching of V. We recall a well known result from matching theory :

Theorem 4.1 (Petersen, 1981) *Every bridgeless cubic graph has a 1-factor.*

Then, if we remove a 1-factor from a cubic graph, what it remains is a disjoint set of cycles.

Corollary 4.1 *Every bridgeless cubic graph has a decomposition into a 1-factor and disjoint cycles.*

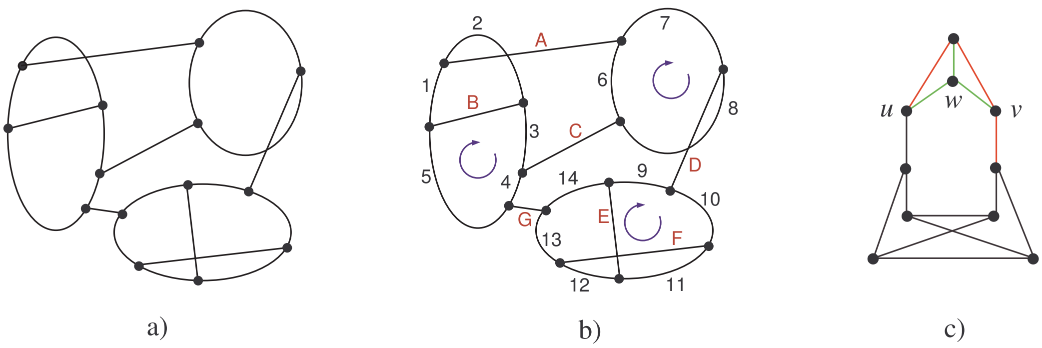

Fig. 1 – **a)** Decomposition of a bridgeless cubic graph into disjoints cycles and a 1-factor. **b)** Decomposition of a bridgeless cubic graph into paths of length 3. **c)** Cubic bridgeless graph used in the proof of Proposition 4.1

An example of a decomposition of a bridgeless cubic graph into disjoints cycles and a 1-factor is depicted in Fig. 1a.

Proposition 4.1 *Let $\mathcal{C}$ be the class of bridgeless cubic graphs. Then,*

$$M(3, 3, \mathcal{C}) = 2.$$

Proof: Let us proof that we can always partition the request graph into paths with 3 edges in such a way that each vertex appears in 2 paths. To do so, we take the decomposition given by Proposition 4.1, together with a clockwise orientation of the edges of each cycle. With this orientation, each edge of the 1-factor has two *incoming* and two *outgoing* edges of the cycles. For each edge of the 1-factor we take its two incoming edges, and form in this way a path of length 3. It is easy to verify that this is indeed a decomposition into paths of length three. For instance, if we do this construction in the graph of Fig. 1a, and we label the edges of the 1-factor as {A,B,. . .,G} and the ones of the cycles as {1,2,. . .,14} (see Fig. 1b), we obtain the following decomposition :

$$\{1, A, 6\}, \{5, B, 2\}, \{3, C, 8\}, \{7, D, 9\}, \{14, E, 11\}, \{10, F, 12\}, \{4, G, 13\}$$

Now let us see that we cannot do better, i.e. with $2n - 1$ ADMs. If such a solution exists, there would be at least one vertex with only 1 ADM, and the average of the number of ADMs of all the other vertices must not exceed 2. In order to see that this is not always possible, consider the cubic bridgeless graph on 10 vertices of Fig. 1c. Let w be the vertex with only 1 ADM. This graph has no triangles except those containing w. Since we can use only 1 ADM in w, we must take all its requests in one subgraph. It is not possible to cover the 4 remaining requests of the nodes u and v in one subgraph, and thus without loss of generality we will need 3 ADMs in u. With these constraints, one can check that the best solution uses 20 ADMs, that is $2n > 2n - 1$. □

Taking a look at the proof we see that the only property that we need from the bridgeless cubic graph is that we can partition it into a 1-factor and disjoint cycles. Hence, we can relax the hypothesis of Proposition 4.1 to obtain the following corollary :

Corollary 4.2 *Let $\mathcal{C}$ be the class of graphs of maximum degree at most 3 that can be partitioned into disjoints cycles and a 1-factor. Then*

(i) $M(3,3,\mathcal{C}) = 2$ *; and*

(ii) $M(C,3,\mathcal{C}) \leq 2$ *for any* $C \geq 4$.

4.1.2 General Request Graph

It turns out that when the request graph is not restricted to be bridgeless we have that $M(3,3) = 3$.

Proposition 4.2 $M(3,3) = 3$.

Proof: By *(ii)* and *(iii)* of Lemma 2.1 we know that $M(3,3) \leq 3$. We shall exhibit a counterexample showing that $M(3,3) > 2$, proving the result. Consider the cubic graph G depicted in FIG. 2a. We will prove that it is not possible to partition the edges of G into subgraphs with at most 3 edges in such a way that each vertex appears in at most 2 subgraphs.

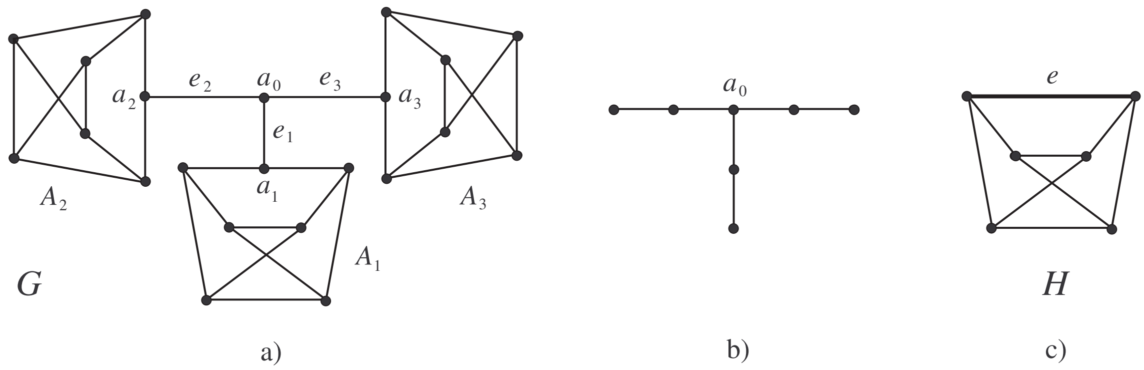

FIG. 2 – **a)** Cubic graph G that can not be edge-partitioned into subgraphs with at most 3 edges in such a way that each vertex appears in at most 2 subgraphs. **b)** Graph that cannot be partitioned into 2 connected subgraphs with at most 3 edges. **c)** Counterexample of Proposition 4.2 showing that $M(3,3) = 3$

Indeed, suppose the opposite, i.e. that we can partition the edges of G into (connected) subgraphs $B_1, \ldots, B_k$ with $|E(B_i)| \leq 3$ in such a way that each vertex appears in at most 2 subgraphs, and let us arrive at a contradiction.

Following the notation illustrated in FIG. 2a, let A_1, A_2, A_3 be the connected components of $G \backslash \{e_1, e_2, e_3\}$. Let also, with abuse of notation, $a_i = A_i \cap e_i$, $i = 1,2,3$, and $a_0 = e_1 \cap e_2 \cap e_3$.

Claim 1 *There exist an index $i^* \in \{1,2,3\}$ and a subgraph B_{k^*} containing a_0, such that $B_{k^*} \cap A_{i^*} = \{a_{i^*}\}$.*

Proof: Among all the subgraphs $B_1, \ldots, B_k$ involved in the decomposition of G, consider the ℓ subgraphs $B_{j_1}, \ldots, B_{j_\ell}$ covering the edges $\{e_1, e_2, e_3\}$. If $\ell = 1$, then the subgraph B_{j_1} is a star with three edges and center a_0, and then $B_{j_1} \cap A_i = \{a_i\}$ for each $i = 1,2,3$. If $\ell \geq 3$, then the vertex a_0 appears in 3 subgraphs, a contradiction. Hence it remains to handle the case $\ell = 2$. If the claim was not true, it would imply that for each $i = 1,2,3$ it would exist $j_{f(i)} \in \{j_1, j_2\}$ such that $B_{f(i)} \cap A_i$ contains at least one edge. In particular, this would imply that the graph depicted in FIG. 2b could be partitioned into 2 connected subgraphs with at most 3 edges, which is clearly not possible. □

Suppose without loss of generality that the index i^* given by Claim 1 is equal to 1. Thus, a_1 appears in a subgraph B_{k^*} that does not contain any edge of A_1. Therefore, the edges of A_1 must be partitioned into connected subgraphs with at most 3 edges, in such a way that a_1 appears in only 1 subgraph, and all its other vertices in at most 2 subgraphs. Let us now see that this is not possible, obtaining the contradiction we are looking for.

Indeed, since a_1 has degree 2 in A_1 and it can appear in only one subgraph, it must have degree two in the subgraph in which it appears, i.e. in the middle of a P_3 or a P_4, because A_1 is triangle-free. It is easy to see that this is equivalent to partitioning the edges of the graph H depicted in FIG. 2c into connected subgraphs with at most 3 edges, in such a way that the thick edge e appears in a subgraph with at most 2 edges, and each vertex appears in at most 2 subgraphs. Observe that H is cubic and triangle-free. Let n_1 be the total number of vertices of degree 1 in all the subgraphs of the decomposition of H. Since each vertex of H can appear in at most 2 subgraphs and H is cubic, each vertex can appear with degree 1 in at most 1 subgraph. Thus, $n_1 \leq |V(H)| = 6$.

Since we have to use at least 1 subgraph with at most 2 edges and $|E(H)| = 9$, there are at least $1 + \lceil \frac{9-2}{3} \rceil = 4$ subgraphs in the decomposition of H. But each subgraph involved in the decomposition of H has at least 2 vertices of degree 1, because H is triangle-free. Therefore, $n_1 \geq 8$, a contradiction. □

4.2 Case $C \geq 5$

For $C \geq 5$ we can easily prove that $M(C,3) = 2$, making use of a conjecture made by Bermond *et al.* in 1984 [5] and proved by Thomassen in 1999 [17] :

Theorem 4.2 ([17]) *The edges of a cubic graph can be 2-colored such that each monochromatic component is a path of length at most 5.*

A *linear k-forest* is a forest consisting of paths of length at most k. The *linear k-arboricity* of a graph G is the minimum number of linear k-forests required to partition $E(G)$, and is denoted by $la_k(G)$ [5]. Theorem 4.2 is equivalent to saying that, if G is cubic, then $la_2(G) = 2$.

Let us now see that Theorem 4.2 implies that $M(C,3) = 2$ for all . Indeed, all the paths of the linear forests have at most 5 edges, and each vertex will appear in exactly 1 of the 2 linear forests, so the decomposition given by Theorem [17] is a partition of the edges of a cubic graph into subgraphs with at most 5 edges, in such a way that each vertex appears in at most 2 subgraphs. In fact the result of [17] is stronger, in the sense that G can be any graph of maximum degree at most 3. Thus, we deduce that

Corollary 4.3 *For any $C \geq 5$, $M(C,3) = 2$.*

Thomassen also proved [17] that 5 cannot be replaced by 4 in Theorem 4.2. This fact do not imply that $M(4,3) = 3$, because of the following reasons : (i) the subgraphs of the decomposition of the request graph are not restricted to be paths, and (ii) it is not necessary to be able to find a 2-coloring of the subgraphs of the decomposition (a *coloring* in this context means that each subgraph receives a color, and 2 subgraphs with the same color must have empty intersection).

5 Conclusions

We have considered the traffic grooming problem in unidirectional WDM rings when the request graph belongs to the class of graph with maximum degree Δ. This formulation allows the network to support dynamic traffic without reconfiguring the electronic equipment at the nodes. We have formally defined the problem, and we have focused mainly on the cases $\Delta = 2$ and $\Delta = 3$, solving completely the former and solving all the cases of the latter, except the case when the grooming value C equals 4. We have proved in Section 4.1.2 that $M(3,3) = 3$, and in Section 4.2 that $M(C,3) = 2$ for all $C \geq 5$. Because of the integrality of $M(C,\Delta)$ and Lemma 2.1, $M(4,3)$ equals either 2 or 3. We conjecture that

Conjecture 5.1 *The edges of a graph with maximum degree at most 3 can be partitioned into subgraphs with at most 4 edges, in such a way that each vertex appears in at most 2 subgraphs.*

If Conjecture 5.1 is true, it clearly implies that $M(4,3) = 2$. Corollary 4.2 states that $M(4,3,\mathcal{C}) = 2$, $\mathcal{C}$ being the class of bridgeless graphs of maximum degree at most 3. Nevertheless, finding the value of $M(4,3)$ remains open. We have also deduced lower and upper bounds in the general case (any value of C and Δ). Tab. 1 summarizes the values of $M(C,\Delta)$ that we have obtained.

$C \setminus \Delta$	1	2	3	4	5	6	$\ldots$	Δ
1	1	2	3	4	5	6	$\ldots$	Δ
2	1	2	3	4	5	6	$\ldots$	Δ
3	1	2	3	≥ 3	≥ 4	≥ 4	$\ldots$	$\geq \lceil \frac{2\Delta}{3} \rceil$
4	1	2	2??	≥ 3	≥ 4	≥ 4	$\ldots$	$\geq \lceil \frac{5\Delta}{8} \rceil$
5	1	2	2	≥ 3	≥ 3	≥ 4	$\ldots$	$\geq \lceil \frac{3\Delta}{5} \rceil$
$\ldots$	1	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
C	1	2	2	$\geq \lceil \frac{C+1}{C} 2 \rceil$	$\geq \lceil \frac{C+1}{C} \frac{5}{2} \rceil$	$\geq \lceil \frac{C+1}{C} 3 \rceil$	$\ldots$	$\geq \lceil \frac{C+1}{C} \frac{\Delta}{2} \rceil$

Tab. 1 – Values of $M(C,\Delta)$. The case $C = 4, \Delta = 3$ is a conjectured value

This problem can find wide applications in the design of optical networks using WDM technology. It would be interesting to continue the study for larger values of Δ, which will certainly rely on graph decomposition results. Another generalization could be to restrict the request graph to belong to other classes of graphs for which there exist powerful decomposition tools, like graphs with bounded tree-width, or families of graphs excluding a fixed graph as a minor.

Références

[1] O. Amini, S. Pérennes, and I. Sau. Hardness and approximation of traffic grooming. In *18th International Symposium on Algorithms and Computation (ISAAC 2007)*, pages 561–573, Sendai, Japan, December 2007.

[2] J.-C. Bermond and D. Coudert. Traffic grooming in unidirectional WDM ring networks using design theory. In *IEEE ICC*, volume 2, pages 1402–1406, Anchorage, Alaska, May 2003.

[3] J.-C. Bermond and D. Coudert. *The CRC Handbook of Combinatorial Designs (2nd edition)*, volume 42 of *Discrete Mathematics and Its Applications*, chapter VI.27, Grooming, pages 493–496. CRC Press, C.J. Colbourn and J.H. Dinitz edition, Nov. 2006.

[4] J.-C. Bermond, D. Coudert, and X. Muñoz. Traffic grooming in unidirectional WDM ring networks : the all-to-all unitary case. In *The 7th IFIP Working Conference on Optical Network Design & Modelling – ONDM'03*, pages 1135–1153, Feb. 2003.

[5] J.-C. Bermond, J.-L. Fouquet, M. Habib, and B. Péroche. On linear k-arboricity. *Discrete Math.*, 52(2-3) :123–132, 1984.

[6] R. Berry and E. Modiano. Reducing electronic multiplexing costs in SONET/WDM rings with dynamically changing traffic . *IEEE J. on Selected Areas in Comm. 18*, pages 1961–1971, 2000.

[7] A. Chiu and E. Modiano. Traffic grooming algorithms for reducing electronic multiplexing costs in WDM ring networks. *IEEE/OSA Journal of Lightwave Technology*, 18(1) :2–12, 2000.

[8] R. Dutta and N. Rouskas. On optimal traffic grooming in WDM rings. *IEEE Journal of Selected Areas in Communications*, 20(1) :1–12, Jan. 2002.

[9] R. Dutta and N. Rouskas. Traffic grooming in WDM networks : Past and future. *IEEE Network*, 16(6) :46–56, November/December 2002.

[10] M. Flammini, G. Monaco, L. Moscardelli, M. Shalom, and S. Zaks. Approximating the traffic grooming problem in tree and star networks. In *32nd International Workshop on*

Graph-Theoretic Concepts in Computer Science (WG), Bergen, Norway, June 2006.

[11] M. Flammini, L. Moscardelli, M. Shalom, and S. Zaks. Approximating the traffic grooming problem. In *the 16th International Symposium on Algorithms and Computation (ISAAC 2005)*, LNCS 3827, Springer-Verlag, pages 915–924, 2005.

[12] O. Gerstel, P. Lin, and G. Sasaki. Wavelength assignment in a WDM ring to minimize cost of embedded SONET rings. In *IEEE Infocom*, pages 94–101, San Francisco, California, 1998.

[13] O. Gerstel, R. Ramaswani, and G. Sasaki. Cost-effective traffic grooming in WDM rings. *IEEE/ACM Transactions on Networking*, 8(5) :618–630, 2000.

[14] O. Goldschmidt, D. Hochbaum, A. Levin, and E. Olinick. The SONET edge-partition problem. *Networks*, 41(1) :13–23, 2003.

[15] E. Modiano and P. Lin. Traffic grooming in WDM networks. *IEEE Communications Magazine*, 39(7) :124–129, July 2001.

[16] A. Somani. Survivable traffic grooming in WDM networks. In D. Gautam, editor, *Broad band optical fiber communications technology – BBOFCT*, pages 17–45, Jalgaon, India, December 2001. Nirtali Prakashan. Invited paper.

[17] C. Thomassen. Two-coloring the edges of a cubic graph such that each monochromatic component is a path of length at most 5. *J. Comb. Theory Ser. B*, 75(1) :100–109, 1999.

[18] P.-J. Wan, G. Calinescu, L. Liu, and O. Frieder. Grooming of arbitrary traffic in SONET/WDM BLSRs. *IEEE Journal of Selected Areas in Communications*, 18(10) :1995–2003, Oct. 2000.

[19] J. Wang, W. Cho, V. Vemuri, and B. Mukherjee. Improved approches for cost-effective traffic grooming in WDM ring networks : Ilp formulations and single-hop and multihop connections. *IEEE/OSA Journal of Lightwave Technology*, 19(11) :1645–1653, Nov. 2001.

[20] K. Zhu and B. Mukherjee. A review of traffic grooming in WDM optical networks : Architectures and challenges. *Optical Networks Magazine*, 4(2) :55–64, March/April 2003.

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