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Abstract: This paper deals with on-line identification of delay systems. Based on non-asymptotic techniques, the estimation approach reduces to solving polynomials or eigenvalue problems. Numerical simulations with noisy data but also with slowly time varying parameters and delay are provided.

1. INTRODUCTION

The real time delay identification is one of the most crucial open problems in the field of delay systems (see, e.g., Richard [2003]). On the one hand, various powerful control techniques (predictors, flatness-based predictive control, finite spectrum assignments, observers, ...) may be applied if the dead-time is known. On the other hand, most existing identification techniques for time-delay systems (see, e.g., Orlov et al. [2006], Drakunov et al. [2006] for adaptive techniques or Ren et al. [2005] for a modified least squares technique) generally suffer from poor speed performance. Recent developments in Belkoura and Richard [2006], Belkoura et al. [2006] have considered the on line identification of delay systems with particular (structured) inputs. This paper considers the identification problem from general input-output trajectories. Although the parameter estimation technique is still inspired from the fast identification techniques that were proposed Fliess M. [2003] for linear, finite-dimensional models, this paper considers a new approach to deal with the delay estimation. Let us recall that those techniques are not asymptotic, and do not need statistical knowledge of the noises corrupting the data (See, e.g., Fliess et al. [2007] for linear and non-linear diagnosis, Fliess et al. [2003] for signal processing, and Beltran-Carvajal et al. [2005] for successful laboratory experiments).

General Framework

The approach used in this paper is mainly based on well known facts about the convolution product. In a general context, the space of distributions with left bounded support is an algebra with respect to the convolution product, with identity δ, the Dirac distribution. A delayed signal can be formed from the convolution product $u(t-\tau) = \delta_\tau * u$, with δ_τ the Dirac measure concentrated at $\{\tau\}$. The derivative or the integral (from 0 to t) of a convolution product admits the respective equivalents forms $\dot{u}*v = u*\dot{v} = \dot{\delta}*u*v$, and $(\int u)*v = u*\int v = \int(u*v)$. The complement of the largest open subset in which u vanishes is called the support of u and will be denoted $\mathrm{supp}\, u$. The following property allows local considerations, $\mathrm{supp}\, u*v \subset \mathrm{supp}\, u + \mathrm{supp}\, v$, where the sum in the right hand side is defined by $\{x+y\,;\, x \in \mathrm{supp}\ u,\ y \in \mathrm{supp}\ v\}$. Finally, with no danger of confusion, we shall sometimes denote $u(s)$, $s \in \mathbb{C}$, the Laplace transform of u, and let $*$ denote for the convolution product.

The paper is organized as follows. Section 2 focuses on parameters and delay identification starting from the equilibrium or rest position, hence assuming zero initial conditions. Section 3 considers the general case where the measurements are not assumed to start with the experiment, but may run from an arbitrary starting point. Most of our developments are illustrated on the single delayed integrator, although extension to higher order systems with state delay is generally straightforward (see section 3.1).

2. IDENTIFICATION FROM A REST POSITION

2.1 Single delay identification

We first focus on a single delay identification regardless of the process dynamics. When considered on the whole real line, a delay between an input u and an output y reads $y(t) = u(t-\tau)$, rewritten as in (1) in a convolution framework, and leading to (2) once multiplied by $(t-\tau)$.

$$y = \delta_\tau * u, \tag{1}$$

$$(t-\tau)y = \delta_\tau * tu. \tag{2}$$

A convolution product derived from these two relations results in equation (3) with no deviated argument, and from which a non asymptotic and explicit delay formulation (4) is obtained:

$$(t-\tau)y * u = tu * y, \quad (3)$$

$$\Rightarrow \tau = \frac{ty * u - y * tu}{u * y}. \quad (4)$$

Provided the involved convolution products are well defined, this delay formula holds for all nonzero values of $(u * y)(t)$. More precisely, if the input u consists in measurements on $(0,\infty)$, then by virtue of the convolution support property, $\operatorname{supp} y \subset (\tau,\infty)$ and hence both numerator and denominator of (4) have their support within (τ,∞). Therefore, the delay is not identifiable for $t < \tau$. However, as in the finite dimensional case (see, e.g., Fliess and Sira-Ramirez [2007]), the input signal u being used in this algebraic approach does not necessarily exhibit the classical "persistency of excitation" requirement. Although a local loss of identifiability may occur due to the zero crossing of the denominator, only non trivial trajectories are required. This point is considered in the next paragraph for a delayed integrator. For open loop structures, a constructive method for the design of sufficiently rich inputs for delay systems has been considered in Belkoura [2005].

2.2 Taking a dynamic into account

When facing derivatives, one of the nice features of multiplication by polynomial (as in (2)) or exponential functions lies in the ability to use simple integration by parts formulas to avoid any derivation in the identification algorithm. This paragraph illustrates the time lag identification for the delayed integrator:

$$\dot{y} = \delta_\tau * ku. \quad (5)$$

Taking into account the integration by parts $\int_0^t \theta \dot{y} d\theta = ty - \int_0^t y$, as well as the convolution product properties of the introductory section, the previous approach combined with a integration results in:

$$\tau = \frac{\int_0^t (\theta \dot{y} * u - \dot{y} * \theta u) d\theta}{\int_0^t (u * \dot{y}) d\theta} \quad (6)$$

$$= \frac{ty * u - y * tu - \int_0^t u * y}{u * y}. \quad (7)$$

Note that as in the free dynamic case of the previous paragraph, the static gain value k is not required nor identified. Also, and in order to avoid multiplications by unbounded functions (polynomials), and hence the amplification of noise and neglected dynamics, exponential functions may be considered as well. Setting $\lambda = e^{-\gamma\tau}$ for some tunable positive parameter γ, $\eta(t) = e^{-\gamma t}$, and using $\eta(t)u(t-\tau) = \lambda\, \delta_\tau * \eta u$, one obtains from (5),

$$\eta \times (5) \quad \Rightarrow \quad \eta\dot{y} = \lambda\, k\, \delta_\tau * \eta u, \quad (8)$$

$$(5) * (8) \quad \Rightarrow \lambda\, \eta u * \dot{y} = \eta\dot{y} * u. \quad (9)$$

With a slight abuse of notations, the integration by parts of (9) reads in this case, $\int \eta\dot{y} = \eta y + \gamma \int \eta y$, yielding

$$\lambda = \frac{\eta y * u + \gamma \int (\eta y * u)}{\eta u * y}, \quad (10)$$

while the delay is obtained from $\tau = \log(\lambda)/\gamma$. For this simple example, and since only a constant delay has to be identified, an additional step considering the integral of the square of equation (10) (*i.e.* $\int (10)^2$) avoids the possible singularities resulting from the zero crossing of the denominator $\eta u * y$. This finally results in the delay estimation:

$$\lambda = \left[\frac{\int_0^t \left[u * \eta y + \gamma \int_0^\theta (u * \eta y) \right]^2 d\theta}{\int_0^t (\eta u * y)^2} \right]^{\frac{1}{2}}. \quad (11)$$

A simulation result with noisy data is depicted in Figure 1, for an input $u(t) = \cos(2t)(0.2 + \sin(7t))$, $\gamma = 0.2$, and a delay $\tau = 0.3\, s$. The simulation step size has been fixed to $0.05\, s$, and the integrals involved in the convolutions have been approximated by simple sums.

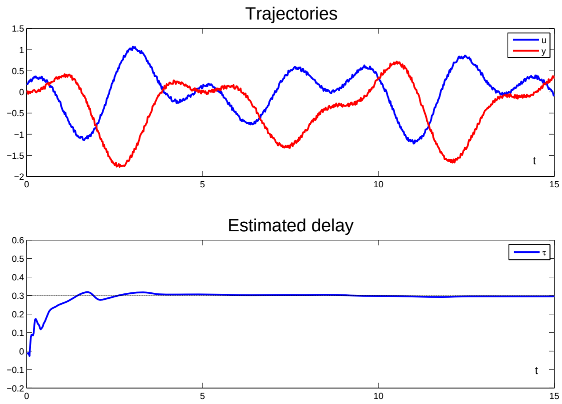

Fig. 1. Trajectories and estimated delay of Eq.(5).

2.3 Simultaneous parameters and delay identification

The aim of this paragraph is to provide a structure of the estimation problem when simultaneous parameters and delay identification is required. We focus on a first order system,

$$\dot{y} + a\, y = \delta_\tau * ku, \quad (12)$$

although a generalization to higher order processes is straightforward. According to the previous procedure (multiplication by $\eta(t) = e^{-\gamma t}$ followed by a convolution product), one gets:

$$\eta\, \dot{y} + a\, \eta y = k\, \lambda \delta_\tau * \eta u, \quad (13)$$

$$\lambda(\eta u * \dot{y} + a\, \eta u * y) = u * \eta\dot{y} + a\, \eta y * u \quad (14)$$

This is a non linear relation w.r.t. the unknown coefficients $\lambda = e^{-\gamma\tau}$ and a. A direct method may consists in $k \geq 3$ successive integrations by parts of (14), leading to a least square estimation of the vector $\theta = (\lambda, a, \lambda a)^t$. On the other hand, and in order to avoid redundancy in the estimated parameters, the following spectral formulation may be be considered,

$$[A - \lambda\, B] \begin{pmatrix} a \\ 1 \end{pmatrix} = 0, \quad (15)$$

where, using a Matlab-like notation, the entries of the trajectory-dependent matrices A and B are given by

$$A(1,:) = (\int u * \eta\dot{y} \;\; \int u * \eta y), \;\; A(2,:) = \int A(1,:),$$

$$B(1,:) = (\int \eta u * \dot{y} \;\; \int \eta u * y), \;\; B(2,:) = \int B(1,:).$$

Hence, for each value of t, the unknown delay consists in one eigenvalue of (15), while the parameter a is obtained from the associated normalized eigenvector. Nevertheless, the problem of selecting the appropriate eigenpair with the help of additional integrations leads to an eigenvalue problem for non square pencils. As mentioned in Wright and Trefethen [2002], "(15) has the awkward feature that most matrices have no eigenvalues at all, whilst for those that do, an infinitesimal perturbation will in general remove them". In the next section where a similar structure is proposed, a simple approach is proposed to overcome this difficulty.

2.4 General remarks

Formulas for higher order systems This paragraph provides some useful relations and examples one can use for systems with higher order derivatives. Equation (16) gives the Leibniz formula related to the product of a Dirac derivative with a smooth function α, while equation (17) shows a property related to a relation involving both convolution product and multiplication par polynomials (see e.g. Schwartz [1966]).

$$\alpha\, \delta_a^{(n)} = \sum\nolimits_{k=0}^{n} (-1)^{(n-k)} C_n^k\, \alpha^{(n-k)}(a)\, \delta_a^{(k)}, \quad (16)$$

$$t^n\, (S * T) = \sum_{k=0}^{n} C_n^k\, (t^k\, S) * (t^{n-k}\, T) \quad (17)$$

Setting $z_i = t^i\, y$, the following example illustrates the application of the theses relations to the product $t^3\, y^{(2)}$. Note that integrating twice this expression results in nothing but the integration by parts formula.

$$t^3\, y^{(2)} = t^3\, (\delta^{(2)} * y) = -6\, z_1 + 6\, z_2^{(1)} - z_3^{(2)} \quad (18)$$

The next relations give the analog formulation in case of multiplication by exponential functions as well as an application to $e^{-\gamma t}\, y^{(2)}$, where we have denoted $z = e^{-\gamma t}\, y$ and $\lambda = e^{\gamma\tau}$.

$$e^{-\gamma t}\, (S * T) = e^{-\gamma t}\, S * e^{-\gamma t}\, T \quad (19)$$

$$e^{-\gamma t}\, y^{(2)} = e^{-\gamma t}\, (\delta^{(2)} * y) = \gamma^2\, z + 2\gamma\, z^{(1)} + z^{(2)} \quad (20)$$

Operational formulation The convolutional approach used in this section may be equivalently formulated in the operational domain (Laplace transform), recalling that multiplying a function $w(t)$ respectively by t and $e^{-\gamma t}$ reads in the operational form $-\frac{dw}{ds}$ and $w(s+\gamma)$. Hence, one can easily get the equivalent form for the integration by parts

$$\int t\dot{y} \leftrightarrow -\frac{dy}{ds} + \frac{y}{s} \quad , \int e^{-\gamma t}\dot{y} \leftrightarrow (1 + \frac{\gamma}{s})y(s+\gamma),$$

and, from (7) and (10), some simple manipulations lead to the delay formulation counterparts,

$$\tau = \left[\frac{dy}{ds}\, u(s) - y\, \frac{du}{ds} - u(s)y(s)/s\right] / u(s)y(s),$$

$$\lambda = \left[(1 + \frac{\gamma}{s})y(s+\gamma)u(s)\right] / u(s+\gamma)y(s).$$

Limits of this approach Although non asymptotic, this method assumes zero initial conditions, which means that the process is initially at rest and that the measurements start with the experiment. In the general case, and due to the convolution products, one can not disregard the effects of non zero initial conditions. On the other hand, the formed algorithms are based, at each time t, on the knowledge of the entire trajectory. Also, and by construction, the static gain is not directly (or need not be) identified. All these drawbacks are considered in the next section.

3. IDENTIFICATION FROM ARBITRARY INITIAL CONDITIONS

3.1 Theoretical approach

Let us consider again the delayed integrator of the previous section. Taking into account the memory effect, equation (5) is rewritten now as

$$\dot{y} = \delta_\tau * ku + \psi_0, \quad (21)$$

in which the "initial condition term" reads $\psi_0(\theta) = y(0)\delta + k\, u(-\tau+\theta), \theta \in (0,\tau)$, and for which $\operatorname{supp}\psi_0 \subset (0,\tau)$. With this formulation, the measurements are not assumed to start with the experiment, but may run from an arbitrary starting point t_0. It should be stressed that in (21) we may consider as well $r - y$ for some reference signal r instead of a control u, which corresponds to the closed loop case with state delay. For the sake of our demonstration, let us assume that we are given an upper bound $\bar{\tau}$ of the unknown delay, and consider $T > \bar{\tau}$ and a smooth function α such that

$$\operatorname{supp}\alpha \subset (\bar{\tau}, T - \bar{\tau}) \subset (0,T). \quad (22)$$

Typically, α may be viewed as an element of the test functions used in the distribution framework. The identification procedure is based on the convolution product of $\alpha^{(p)}$, $p = 0, \cdots, n_p$, with equation (21). By virtue of (22) and the integration by parts formula, the left hand side of this product reads:

$$\alpha^{(p)} * \dot{y} = \alpha^{(p+1)} * y = \int_0^T \alpha^{(p+1)}(\theta) y(t-\theta)\, d\theta. \quad (23)$$

Unlike the previous section, this relation, still based on on-line computations, only requires the output measurements on $(t, t-T)$. Using again the support of α, one gets for the first term of the right hand side:

$$\begin{aligned} k\, \alpha^{(p)} * \delta_\tau * u &= k \int_0^T \alpha^{(p)}(\theta) u(t-\tau-\theta)\, d\theta \\ &= k \int_{-\tau}^{T-\tau} \alpha^{(p)}(\theta) u(t-\tau-\theta)\, d\theta \\ &= k \int_0^T \alpha^{(p)}(\sigma-\tau) u(t-\sigma)\, d\sigma \quad (24) \end{aligned}$$

As for the convolution with the initial condition term ψ_0, let us notice that even if $y(0)$ and the past values of u are known, the support within $(0,\tau)$ of ψ_0, and hence the convolution product $\alpha^{(p)} * \psi_0$ are by nature unknown. However, their support satisfies $\operatorname{supp}\alpha^{(p)} * \psi_0 \subset (0, T+\tau)$, so that for $t > T + \tau$, one has

$$\int_0^T \alpha^{(p+1)}(\theta) y(t-\theta) d\theta = k \int_0^T \alpha^{(p)}(\theta-\tau) u(t-\theta) d\theta. \quad (25)$$

In the next step, the shifted candidate function α and its derivatives are replaced by their Fourier series approximations of order n, yielding

$$\alpha(\theta) \approx \sum_{q=-n}^{n} c_q e^{j\,q\omega\theta}, \tag{26}$$

$$\alpha^{(p)}(\theta-\tau) \approx \sum_{q=-n}^{n} \lambda^q c_{p,q} e^{j\,q\omega\theta}, \tag{27}$$

where we have denoted $c_{p,q} = (jq\omega)^p c_q$, $\omega = 2\pi/T$, and the unknown delay dependent coefficient $\lambda = e^{-j\omega\tau}$. The delay here will be deduced from the phase angle of λ. For each derivation order p, equation (25) is therefore approximated by the following relation with available terms $a_p(y)$ and $b_{q,p}(u)$,

$$a_p(y) \approx k \sum_{q=-n}^{n} \lambda^q\, b_{q,p}(u), \tag{28}$$

$$a_p(y) = \int_0^T \alpha^{(p+1)}(\theta) y(t-\theta) d\theta,$$

$$b_{q,p}(u) = \int_0^T c_{p,q} e^{j\,q\omega\theta} u(t-\theta) d\theta.$$

In case of known static gain k, the delay estimation problem reduces to computing the polynomials roots of (28) for different derivation orders p. For the general case, the simultaneous gain and delay identification problem may be stated as the following generalized eigenvalue formulation,

$$\sum_{q=-n}^{n} A_q \lambda^q \begin{pmatrix} k \\ 1 \end{pmatrix} = 0, \tag{29}$$

$$A_{q,q\neq 0} = \begin{pmatrix} b_{q,0} & 0 \\ \cdots & \cdots \\ b_{q,n_p} & 0 \end{pmatrix}, \quad A_0 = \begin{pmatrix} b_{0,0} & -a_0 \\ \cdots & \cdots \\ b_{0,n_p} & -a_{n_p} \end{pmatrix}.$$

In the next paragraph, the resolution of (29) as well as its practical implementation is discussed.

3.2 Practical implementation

As mentioned earlier, and since the content of the pencil involves noisy measurements, solving a generalized eigenvalue problem with non square matrices is not an easy task, and traditional methods are expected to lead to no solutions in most cases. On the other hand, the recent pseudo-spectra analysis one can find for instance in (Wright and Trefethen [2002]) is not compatible with an online perspective. As an alternative, the following steps are proposed:

(1) solve (29) with square A_q (i.e. with $n_p = 2$). Using for instance the `polyeig` Matlab function provides $4n$ eigenpairs and hence $4n$ estimation pairs (λ, k),
(2) select the pair that minimizes the norm of the left hand side of (29) for $n_k > 2$. This step may be interpreted as the selection of the stationary solution of the problem.

On the other hand, the constraint on the support of α will be relaxed by considering harmonic functions taking small values in the vicinity of T. In return, equation (27) is no more an approximation for some appropriate order n. Figure 2 shows the example of $\alpha = \sin^6(\omega t/2)$ on the interval $(0, T)$, with the a priori assumption $T = 5 >> \tau$. Moreover, and in order to avoid unavailable measurements, the starting phase algorithm (i.e. in the vicinity of $t = 0$) uses an increasing window size, from 0 to T.

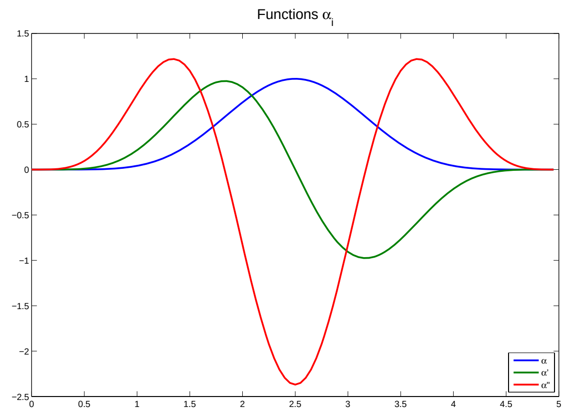

Fig. 2. Candidate functions $\alpha^{(p)}$, $p = 0, 1, 2$ for Eq. (25).

A simulation result based on the same configurations as those of section 2.2 is shown in Figure 3, where the sliding window size T has been fixed to $5\,s$. Note that a good estimation is obtained for $t < T$. The small and local deviations one can observe result from the specific trajectories for which the eigenvalue problem (29) becomes nearly singular.

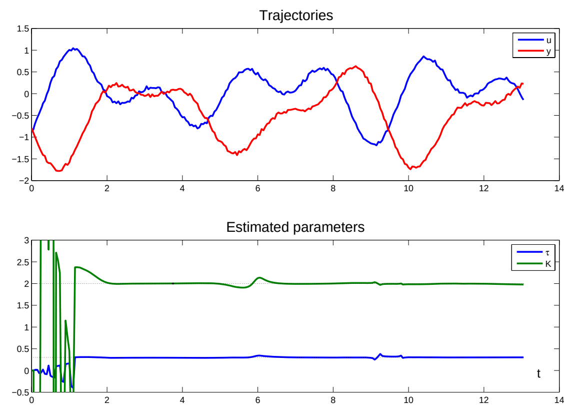

Fig. 3. Trajectories and estimated parameters k, τ of Eq. (21).

In this new context, the ability to identify from bounded sets $(t, t-T)$ of measurements make it possible to extend the parameters estimations to non stationary cases with slowly time varying coefficients. In Figure 4, these estimations are shown for a delayed integrator with slowly time varying gain $k(t) = 2(1 + .2\sin(.03t))$ and delay $\tau(t) = 0.3(1 - 0.8\sin(.06t)\cos(.01t))$.

4. CONCLUSION

This note has presented a new method for the identification of delay systems based on arbitrary input-output trajectories. The ability of identification on bounded sets of measurements allows us to extend the estimation problem to slowly time varying parameters and delay. Extensions to multivariable and multidelay cases, rigorous proofs for non stationary processes, a well as a deeper study of the eigenvalue problem singularities are under investigation.

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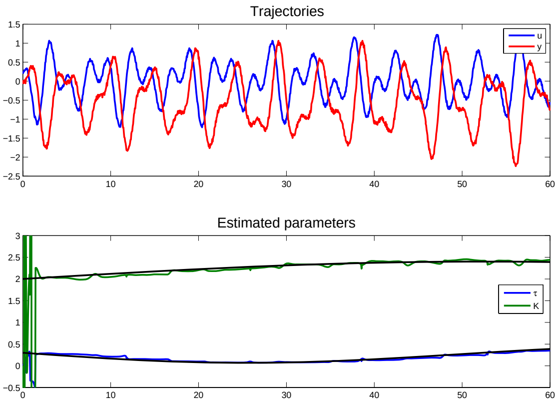

Fig. 4. Trajectories and estimated parameters $k(t), \tau(t)$ of Eq. (21) in case of slowly time varying coefficients.

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