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Abstract: Recently, a framework for the approximation of the entire set of ϵ-efficient solutions (denote by E_ϵ) of a multi-objective optimization problem with stochastic search algorithms has been proposed. It was proven that such an algorithm produces – under mild assumptions on the process to generate new candidate solutions –a sequence of archives which converges to E_ϵ in the limit and in the probabilistic sense. The result, though satisfactory for most discrete MOPs, is at least from the practical viewpoint not sufficient for continuous models: in this case, the set of approximate solutions typically forms an n-dimensional object, where n denotes the dimension of the parameter space, and thus, it may come to perfomance problems since in practise one has to cope with a finite archive.
Here we focus on obtaining finite and tight approximations of E_ϵ, the latter measured by the Hausdorff distance. We propose and investigate a novel archiving strategy theoretically and empirically. For this, we analyze the convergence behavior of the algorithm, yielding bounds on the obtained approximation quality as well as on the cardinality of the resulting approximation, and present some numerical results.

Key-words: multi-objective optimization, convergence, ϵ-efficient solutions, approximate solutions, stochastic search algorithms.

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Résumé : Dans des travaux précédent, nous avons proposé un environnement ("framework") pour l'approximation de l'intégralité de l'ensemble des solutions ϵ-efficaces (noté E_ϵ) d'un problème d'optimisation multi-objectifs à l'aide d'une recherche stochastique. Il a été prouvé que suivant certaines hypothèses relatives au processus de génération de nouvelles solutions candidates, un tel algorithme produit une séquence d'archives qui converge asymptotiquement vers E_ϵ, au sens probabiliste du terme. Le résultat, s'il est satisfaisant pour la plupart des MOP discrets, ne l'est pas d'un point de vue pratique pour les problèmes continus. Dans ce dernier cas, l'ensemble des solutions approximées forme un objet à n dimentions, où n est la dimension de l'espace des paramètres. Ceci peut amener à des problèmes de performances puisqu'en pratique la taille de l'archive est finie.

Dans le travail présenté, nous nous concentrons sur l'obtention d'approximations finies et précises de E_ϵ qui est mesuré par la distance de Hausdorff. Nous proposons et nous étudions une nouvelle stratégie d'archivage des points de vue théorique et pratique. Pour ce faire, nous analysons le comportement asymptotique de l'algorithme, en fournissant les limites de qualité de l'approximation obtenue, aussi bien que la cardinalité de l'approximation et nous présentons également quelques résultats numériques.

Mots-clés : optimisation multi-objectif, convergence, solutions ϵ-efficaces, solutions approwimées, algorithmes de recherche stochastique.

1 Introduction

Since the notion of ϵ-efficiency for multi-objective optimization problems (MOPs) has been introduced more than two decades ago ([8]), this concept has been studied and used by many researchers, e.g. to allow (or tolerate) nearly optimal solutions ([8], [18]), to approximate the set of optimal solutions ([11]), or in order to discretize this set ([7], [14]). ϵ-efficient solutions or approximate solutions have also been used to tackle a variety of real world problems including portfolio selection problems ([19]), a location problem ([2]), or a minimal cost flow problem ([11]).
As an illustrative example where it could make sense from the practical point of view to consider in addition to the exact solutions also approximate ones we consider a plane truss design problem, where the volume of the truss as well as the displacement of the joint to a given position have to be minimized (see also Section 6.2). Since the designs of this problem – as basically in all other engineering problems – have to obey certain physical contraints such as in this case the weight and stability of the structural element, the objective values of all feasible solutions are located within a relatively tight and a priori appreciable range. Hence, the maximal tolerable loss of a design compared to an 'optimal' one with respect to the objective values can easily be determined quantitatively and qualitatively by the decision maker (DM) before the optimization process. The resulting set of exact *and* approximate (but physically relevant) solutions obtained by the optimization algorithm[1] leads in general to a larger variety of possibilities to the DM than 'just' the set of exact solutions: this is due to the fact that solutions which are 'near' in objective space can differ significantly in design space (e.g., when the model contains symmetries, or see Section 6.3 for another example).
The computation of such approximate solutions has been addressed in several studies. In most of them, scalarization methods have been empoyed (e.g., [18], [2], [4]). By their nature, such algorithms can deliver only single solutions by one single execution. The only work so far which deals with the approximation of the entire set of approximate solutions (denote by E_ϵ) is [13], where an archiving strategy for stochastic search algorithms is proposed for this task. Such a sequence of archives obtained by this algorithm provably converges – under mild assumptions on the process to generate new candidate solutions – to E_ϵ in the limit and in the probabilistic sense. This result, though satisfactory for most discrete MOPs, is at least from the practical viewpoint not sufficient for continuous models (i.e., continous objectives defined on a continuous domain): in this case, the set of approximate solutions typically forms an n-dimensional object, where n denotes the dimension of the parameter space (see below). Thus, it may come to performance problems since it can easily happen that a given threshold on the magnitude of the archives is exceeded before a 'sufficient' approximation of the set of interest in terms of diversity and/or convergence is obtained. An analogue statement holds for the approximation of the Pareto front, which is 'only' $(k-1)$-dimensional for MOPs with k objectives, and suitable discretizations have been subject of research since several years (e.g., [7], [6], [14]).
The scope of this paper is to develop a framework for finite size representations of the set E_ϵ

[1] Here we assume an idealized algorithm, since in practise every solution is an approximate one.

with stochastic search algorithms such as evolutionary multi-objective (EMO) algorithms. This will call for the design of a novel archiving strategy to store the 'required' solutions found by the stochastic search process. We will further analyze the convergence behavior of this method, yielding bounds on the approximation quality as well as on the cardinality of the resulting approximations. Finally, we will demonstrate the practicability of the novel approach by several examples.

The remainder of this paper is organized as follows: in Section 2, we state the required background including the set of interest $P_{Q,\epsilon}$. In Section 3, we propose a novel archiving strategy for the approximation of $P_{Q,\epsilon}$ and state a convergence result, and give further on an upper bound on the resulting archive sizes in Section 4. In Section 5, we present some numerical results, and make finally some conclusions in Section 6.

2 Background

In the following we consider continuous multi-objective optimization problems

$$\min_{x\in Q}\{F(x)\}, \qquad \text{(MOP)}$$

where $Q \subset \mathbb{R}^n$ and F is defined as the vector of the objective functions $F : Q \to \mathbb{R}^k$, $F(x) = (f_1(x), \dots, f_k(x))$, and where each $f_i : Q \to \mathbb{R}$ is continuously differentiable. Later we will restrict the search to a compact set Q, the reader may think of an n-dimensional box.

Definition 2.1 *(a) Let $v, w \in Q$. Then the vector v is* less than *w ($v <_p w$), if $v_i < w_i$ for all $i \in \{1, \dots, k\}$. The relation $\le_p$ is defined analogously.*

(b) $y \in \mathbb{R}^n$ is dominated *by a point $x \in Q$ ($x \prec y$) with respect to (MOP) if $F(x) \le_p F(y)$ and $F(x) \ne F(y)$, else y is called nondominated by x.*

(c) $x \in Q$ is called a Pareto point *if there is no $y \in Q$ which dominates x.*

The set of all Pareto optimal solutions is called the *Pareto set* (denote by P_Q). This set typically – i.e., under mild regularity assumptions – forms a $(k-1)$-dimensional object. The image of the Pareto set is called the *Pareto front*.
We now define another notion of dominance, which is the basis of the approximation concept used in this study.

Definition 2.2 *Let $\epsilon = (\epsilon_1, \dots, \epsilon_k) \in \mathbb{R}^k_+$ and $x, y \in Q$.*

(a) x is said to ϵ-dominate y ($x \prec_\epsilon y$) with respect to (MOP) if $F(x) - \epsilon \le_p F(y)$ and $F(x) - \epsilon \ne F(y)$.

(b) *x is said to $-\epsilon$-dominate y ($x \prec_{-\epsilon} y$) with respect to (MOP) if $F(x) + \epsilon \leq_p F(y)$ and $F(x) + \epsilon \neq F(y)$.*

The notion of $-\epsilon$-dominance is of course analogous to the 'classical' ϵ-dominance relation but with a value $\tilde{\epsilon} \in \mathbb{R}^k_-$. However, we highlight it here since it will be used frequently in this work. While the ϵ-dominance is a weaker concept of dominance, $-\epsilon$-dominance is a stronger one.
We now define the set which we want to approximate in the sequel.

Definition 2.3 *Denote by $P_{Q,\epsilon}$ the set of points in $Q \subset \mathbb{R}^n$ which are not $-\epsilon$-dominated by any other point in Q, i.e.*

$$P_{Q,\epsilon} := \{x \in Q | \; \not\exists y \in Q : y \prec_{-\epsilon} x\}. \tag{1}$$

To see that $P_{Q,\epsilon}$ typically forms an n-dimensional set let $x_0 \in P_Q$ (such a point, for instance, always exists if Q is compact). That is, there exists no $y \in Q$ such that $y \prec x_0$. Since F is continuous and $\epsilon \in \mathbb{R}^k_+$ there exists a neigborhood U of x_0 such that

$$\not\exists y \in Q : \; y \prec_{-\epsilon} u \quad \forall u \in U \cap Q, \tag{2}$$

and thus, $U \cap Q \subset P_{Q,\epsilon}$, and we are done since U is n-dimensional.
The following result and notions are used for the upcoming proof of convergence.

Theorem 2.4 ([12]) *Let (MOP) be given and $q : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $q(x) = \sum_{i=1}^k \hat{\alpha}_i \nabla f_i(x)$, where $\hat{\alpha}$ is a solution of*

$$\min_{\alpha \in \mathbb{R}^k} \left\{ \left\| \sum_{i=1}^k \alpha_i \nabla f_i(x) \right\|_2^2 ; \alpha_i \geq 0, i = 1, \ldots, k, \sum_{i=1}^k \alpha_i = 1 \right\}.$$

Then either $q(x) = 0$ or $-q(x)$ is a descent direction for all objective functions $f_1, \ldots, f_k$ in x. Hence, each x with $q(x) = 0$ fulfills the first-order necessary condition for Pareto optimality.

Definition 2.5 *Let $u \in \mathbb{R}^n$ and $A, B \subset \mathbb{R}^n$. The semi-distance $dist(\cdot,\cdot)$ and the* Hausdorff distance *$d_H(\cdot,\cdot)$ are defined as follows:*

(a) *$dist(u, A) := \inf_{v \in A} \|u - v\|$*

(b) *$dist(B, A) := \sup_{u \in B} dist(u, A)$*

(c) *$d_H(A, B) := \max \{dist(A, B), dist(B, A)\}$*

Denote by $\overline{A}$ the closure of a set $A \in \mathbb{R}^n$, by $\overset{\circ}{A}$ its interior, and by $\partial A = \overline{A} \backslash \overset{\circ}{A}$ the boundary of A.

Definition 2.6 *(a) A point $x \in Q$ is called a* weak Pareto point *if there exists no point $y \in Q$ such that $F(y) <_p F(x)$.*

(b) A point $x \in Q$ is called $-\epsilon$ weak Pareto point *if there exists no point $y \in Q$ such that $F(y) + \epsilon <_p F(x)$.*

Algorithm 1 gives a framework of a generic stochastic multi-objective optimization algorithm, which will be considered in this work. Here, $Q \subset \mathbb{R}^n$ denotes the domain of the MOP, P_j the candidate set (or population) of the generation process at iteration step j, and A_j the corresponding archive.

Algorithm 1 Generic Stochastic Search Algorithm

$P_0 \subset Q$ drawn at random
$A_0 = ArchiveUpdate(P_0, \emptyset)$
for $j = 0, 1, 2, \dots$ **do**
  $P_{j+1} = Generate(P_j)$
  $A_{j+1} = ArchiveUpdate(P_{j+1}, A_j)$
end for

3 The Algorithm

Here we present and analyze a novel archiving strategy which aims for a finite size representation of $P_{Q,\epsilon}$.
The algorithm which we propose here, $ArchiveUpdateP_{Q,\epsilon}$, is given in Algorithm 2. Denote by $1\Delta := (\Delta, \dots, \Delta) \in \mathbb{R}^k_+$, where $\Delta \in \mathbb{R}_+$ can be viewed as the discretization parameter of the algorithm.

Algorithm 2 $A := ArchiveUpdateP_{Q,\epsilon}\ (P, A_0, \Delta)$

Require: population P, archive A_0, $\Delta \in \mathbb{R}_+$, $\Delta^* \in (0, \Delta)$
Ensure: updated archive A
$A := A_0$
for all $p \in P$ **do**
  if $\nexists a_1 \in A: \ a \prec_{-\epsilon} p$ and $\nexists a_2 \in A: \ d_\infty(F(a), F(p)) \leq \Delta^*$ **then**
    $A := A \cup \{p\}$
    for all $a \in A$ **do**
      if $p \prec_{-(\epsilon+1\Delta)} a$ **then**
        $A := A \backslash \{a\}$
      end if
    end for
  end if
end for

Lemma 3.1 *Let $A_0, P \subset \mathbb{R}^n$ be finite sets, $\epsilon \in \mathbb{R}^k_+$, $\Delta \in \mathbb{R}_+$, and $A :=$ ArchiveUpdateEps1 (P, A_0). Then the following holds:*

$$\forall x \in P \cup A_0 : \quad \exists a \in A : a \prec_{1\Delta} x.$$

Proof: This follows immediately by the construction of the algorithm.

Theorem 3.2 *Let an MOP $F : \mathbb{R}^n \to \mathbb{R}^k$ be given, where F is continuous, let $Q \subset \mathbb{R}^n$ be a compact set and $\epsilon \in \mathbb{R}^k_+$, $\Delta, \Delta^* \in \mathbb{R}_+$ with $\Delta^* < \Delta$. For the generation process we assume*

$$\forall x \in Q \text{ and } \forall \delta > 0 : \quad P\left(\exists l \in \mathbb{N} \ : \ P_l \cap B_\delta(x) \cap Q \neq \emptyset\right) = 1 \tag{3}$$

and for the MOP

(A1) Let there be no weak Pareto point in $Q \backslash P_Q$.

(A2) Let there be no $-\epsilon$ weak Pareto point in $Q \backslash \overline{P_{Q,\epsilon}}$,

(A3) Define $\mathcal{B} := \{x \in Q | \exists y \in P_Q : F(y) + \epsilon = F(x)\}$. Let $\mathcal{B} \subset \overset{\circ}{Q}$ and $q(x) \neq 0$ for all $x \in \mathcal{B}$, where q is as defined in Theorem 2.4.

Then, an application of Algorithm 1, where $AchiveUpdateP_{Q,\epsilon}(P, A, \Delta)$ is used to update the archive, leads to a sequence of archives $A_l, l \in \mathbb{N}$, where the following holds:

(a) For all $l \in \mathbb{N}$ it holds

$$\|F(a_1) - F(a_2)\|_\infty \geq \Delta^* \tag{4}$$

(b) There exists with probability one an $l_0 \in \mathbb{N}$ such that for all $l \geq l_0$:

(b1) $dist(F(P_{Q,\epsilon}), F(A_l)) < \Delta$

(b2) $dist(F(A_l), F(P_{Q,\epsilon})) \leq dist(F(P_{Q,\epsilon+2\Delta}), F(P_{Q,\epsilon}))$

(b3) $d_H(F(P_{Q,\epsilon}), F(A_l)) \leq D$, where $D = max(\Delta, dist(F(P_{Q,\epsilon+2\Delta}), F(P_{Q,\epsilon}))$

Proof: Before we state the proof we have to make some remarks: a point p is discarded from an existing archive A in two cases (see line 3 of Algorithm 2):

$$\begin{array}{ll} (D1) & \exists a_1 \in A : \ a_1 \prec_{-\epsilon} p, \quad \text{or} \\ (D2) & \exists a_2 \in A : \ \|F(a_2) - F(p)\|_\infty \leq \Delta^*. \end{array} \tag{5}$$

Further, we define by

$$B^\infty_\delta(x) := \{y \in \mathbb{R}^k \ : \ \|x - y\|_\infty < \delta\}$$

a k-dimensional open box around $x \in \mathbb{R}^k$. Now we are in the position to state the proof.

Claim (a): follows immediately by construction of the algorithm and by an inductive argument.
Claim (b1): By *(a) it follows that for an element a from a given archive A it holds*

$$F(\tilde{a}) \notin B^{\infty}_{\Delta^*}(F(a)), \quad \forall \tilde{a} \in A \backslash \{a\}, \tag{6}$$

Since further Q is compact and F is continuous it follows that F(Q) is bounded, and thus, there exits an upper bound for the number of entries in the archive for a given MOP, denote by $n_0 = n_0(\Delta^*, F(Q))$ *(see also Section 4).*
Since $\overline{P_{Q,\epsilon}}$ *is compact and*
$dist(F(P_{Q,\epsilon}), F(A_l)) = dist(F(\bar{P_{Q,\epsilon}}), F(A_l))$*, and since* $A_l, l \in \mathbb{N}$*, is finite it follows that*

$$dist(F(P_{Q,\epsilon}), F(A_l)) = \max_{p \in P_{Q,\epsilon}} \min_{a \in A_l} \|F(p) - F(a)\|_\infty$$

That is, the claim is right for an archive $A_l, l \in \mathbb{N}$*, if for every* $p \in P_{Q,\epsilon}$ *there exists an element* $a \in A_l$ *such that* $\|F(p) - F(a)\|_\infty < \Delta$*. Thus,* $F(P_{Q,\epsilon})$ *must be contained in* $C_{A_l,\Delta}$*, where*

$$C_{A,\Delta} := \bigcup_{a \in A} B^{\infty}_{\Delta}(F(a)).$$

First we show that if there exists an $l_0 \in \mathbb{N}$ *with*
$dist(F(P_{Q,\epsilon}), F(A_l)) < \Delta$*, this property holds for all* $l \geq l_0$*. Assume that such an* l_0 *is given. Define*

$$\tilde{A} := \{a \in A_{l_0} | \exists p \in P_{Q,\epsilon} : \|F(p) - F(a)\|_\infty < \Delta\} \tag{7}$$

Since it holds that

$$p \in P_{Q,\epsilon} \text{ and } a \in Q : \|F(p) - F(a)\| \leq \Delta \;\Rightarrow a \in P_{Q,\epsilon+1\Delta}$$

it follows that $\tilde{A} \subset P_{Q,\epsilon+1\Delta}$*, and thus, no element* $a \in \tilde{A}$ *will be discarded further on due to the construction of* $ArchiveUpdateP_{Q,\epsilon}$*. Since* $dist(F(P_{Q,\epsilon}), F(A_l)) < \Delta$ *it follows that for all* $p \in P_{Q,\epsilon}$ *there exists an element* $a \in \tilde{A}$ *such that* $\|F(p) - F(a)\|_\infty < \Delta$*. By the above discussion this holds for all* $l \geq l_0$*, and since no element* $a \in \tilde{A}$ *is discarded during the run of the algorithm, and the claim follows.*
It remains to show the existence of such an integer l_0*, which we will do by contradiction: first we show that by using* $ArchiveUpdateP_{Q,\epsilon}$ *and under the assumptions made above only finitely many replacements can be done during the run of the algorithm. Then we construct a contradiction by showing that under the asssumptions made above infinitely many replacements have to be done during the run of the algorithm with the given setting.*
Let a finite archive A_0 *be given. If a point* $p \in \mathbb{R}^n$ *replaces a point* $a \in A_0$ *(see lines 4 and 7 of Algorithm 2) it follows by construction of* $ArchiveUpdateP_{Q,\epsilon}$ *that*

$$F(p) <_p F(a) - \Delta \tag{8}$$

Since the relation '$\prec$*' is transitive, there exists for every* $a \in A$ *a 'history' of replaced points* $a_i \in A_{l_i}$ *where equation (8) holds for* a_i *and* a_{i-1}*. Since* $F(Q)$ *is bounded there exist*

$l_i, u_i \in \mathbb{R}, i = 1, \ldots, k$, such that $F(Q) \subset [l_1, u_1] \times \ldots \times [l_k, u_k]$. After r replacements there exists at least one $a \in A_{l(r)}$ such that the length h of the history of a is at least $h \geq \lceil r/n_0 \rceil$, where n_0 is the maximal number of entries in the archive (see above). Denote by $a_0 \in A_0$ the root of the history of a. For a, a_0 it follows that

$$F(a) \leq F(a_0) - h\Delta$$

For $\tilde{h} > h_{max} := \Delta^{-1} \max_{i=1,\ldots,k} u_i - l_i$ we obtain a contradiction since in that case there exists $i \in \{1, \ldots, n\}$ with $f_i(a) < l_i$ and thus $F(a) \notin F(Q)$. Hence it follows that there can be done only finitely many such replacements during the run of an algorithm.

Assume that such an integer l_0 as claimed above does not exist, that is, that $F(P_{Q,\epsilon}) \not\subset C_{A_l,\Delta}$ for all $l \in \mathbb{N}$. Hence there exists a sequence of points

$$p_i \in P_{Q,\epsilon}: \quad y_i = F(p_i) \in F(P_{Q,\epsilon}) \backslash C_{A_i,\Delta} \quad \forall i \in \mathbb{N}. \tag{9}$$

Since $P_{Q,\epsilon} \subset Q$ and Q is compact there exists an accumulation point $p^* \in \overline{P_{Q,\epsilon}}$, that is, there exists a subsequence $\{i_j\}_{j \in \mathbb{N}}$ with

$$p_{i_j} \to p^* \in \overline{P_{Q,\epsilon}} \text{ for } j \to \infty. \tag{10}$$

In [13] it was shown that under the assumptions (A1)–(A3) it follows that

$$\overline{\overset{\circ}{P}_{Q,\epsilon}} = \overline{P_{Q,\epsilon}}, \tag{11}$$

i.e., that $P_{Q,\epsilon}$ is not 'flat' anywhere. Hence, the set

$$\tilde{U}_1 := B^{\infty}_{(\Delta - \tilde{\Delta})/2}(y^*) \cap \overset{\circ}{P}_{Q,\epsilon}, \tag{12}$$

where $y^* := F(p^*)$, is not empty. By (3) it follows that there exists with probability one an $l_1 \in \mathbb{N}$ and an element $\tilde{x}_1 \in P_{l_0+l_1}$ generated by Generate() *with $\tilde{y}_1 = F(\tilde{x}_1) \in \tilde{U}_1$. There are two cases for the archive $A_{l_0+l_1}$: (a) x_1 can be discarded from the archive, or (b) x_1 is added to it. Assume first that x_1 is discarded. Since $x_1 \in P_{Q,\epsilon}$ there exists no $\bar{x} \in Q$ such that $\bar{x}$ $-\epsilon$-dominates x_1. Hence, (D1) can not occur (see (5)), and thus, there must exist an $a_2 \in A_{l_0+l_1}$ such that $\|F(a_2) - F(x_1)\|_\infty \leq \Delta^*$ (see (D2)). Thus, whether x_1 is added to the archive or not there exists an $\tilde{a}_1 \in A_{l_0+l_1}$ such that $\|F(\tilde{a}_1 - y^*\|_\infty \leq \Delta$ (since in case x_1 is added to the archive $\tilde{a}_1 = x_1$ can be chosen), and we obtain*

$$\|F(\tilde{a}_1) - \tilde{y}\|_\infty \leq \|F(\tilde{a}_1) - F(x_1)\|_\infty + \|F(x_1) - \tilde{y}\|_\infty < \Delta \quad \forall \tilde{y} \in U_1 \tag{13}$$

By (9) and (10) there exist integers $j_1, \tilde{l}_1 \in \mathbb{N}$ with

$$y_{i_{j_1}} \in \tilde{U}_1 \backslash C_{l_0+l_1+\tilde{l}_1,\Delta}. \tag{14}$$

Since by (13) it holds that $\|y_{i_{j_1}} - F(a_1)\|_\infty < \Delta$ it follows that $a_1 \notin A_{l_0+l_1+\tilde{l}_1}$, which is only possible via a replacement in Algorithm 2 (lines 4 and 7).

In an analogous way a sequence $\{a_i\}_{i\in\mathbb{N}}$ of elements can be constructed which have to be replaced by other elements. Since this leads to a sequence of infinitely many replacements. This is a contradiction to the assumption, and the proof is complete.
Claim (b2): Let $\tilde{A}$ and l_0 as above, and let $l \geq l_0$. Further, let $x \in Q\backslash P_{Q,\epsilon+2\Delta}$, that is, there exists a $p \in P_{Q,\epsilon}$ such that $p \prec_{-(\epsilon+2\Delta)} x$. Since $l \geq l_0$ there exists an $a \in \tilde{A} \subset A_l$ such that $\|F(p) - F(a)\|_\infty < \Delta$. Combining both facts we see that $a \prec_{-(\epsilon+1\Delta)} x$. Thus, no element $x \in Q\backslash P_{Q,\epsilon+2\Delta}$ is contained in $A_l, l \geq l_0$, or will ever be added to the archive further on. The claim follows since the archive can only contain elements in $P_{Q,\epsilon+2\Delta}$ (see also Examples 3.4 and 3.5).
Claim (b3): follows immediately by (b1) and (b2).

Remarks 3.3 *(a) For $\Delta = \Delta^* = 0$ the archiver coincides with the one proposed in [13], which reads as*

$$UpdateP_{Q,\epsilon}(P, A) := \{x \in P \cup A : y \not\prec_{-\epsilon} x \;\forall y \in P \cup A\}. \tag{15}$$

(b) The convergence result holds for a scalar $\Delta_0 \in \mathbb{R}_+$ which is used for the discretization of the ϵ-efficient front. However, analogue results can be obtained by using a vector $\Delta \in \mathbb{R}^k_+$. In this case, the exclusion strategy in line 3 of Algorithm 2 has to be replaced by

$$\nexists a_2 \in A : \quad F(p) \in B(F(a_2), \Delta), \tag{16}$$

where

$$B(y, \Delta) := \{x \in \mathbb{R}^k : |x_i - y_i| \leq \Delta_i, \; i = 1, .., k\}.$$

Further, elements a have to be discarded from the archive if they are $-(\epsilon+\Delta)$ dominated by p (lines 6-8).

(c) In the algorithm the discretization is done in the image space (line 3 of Algorithm 2). By replacing this exclusion strategy by

$$\nexists a_2 \in A : \; d_\infty(a, p) \leq \Delta^*, \tag{17}$$

an analogue result with discretization in parameter space can be obtained. This will lead on one hand to approximations which could be more 'complete', but on the other hand certainly to archives with much larger magnitudes since $P_{Q,\epsilon}$ is n-dimensional (see also the discussion in Section 4).

(d) The parameter $\Delta^* \in \mathbb{R}_+$ with $\Delta^* < \Delta$ is used for theoretical purposes. In practise, $\Delta^* = \Delta$ can be chosen.

(e) Note that the convergence result also holds for discrete models. In that case, assumption (3) can be modified using Markov chains such that it can easier be verified (see e.g., [9]).

The next two examples show that with using $ArchiveUpdateP_{Q,\epsilon}$ one cannot prevent to maintain points $x \in P_{Q,\epsilon+2\Delta} \backslash P_{Q,\epsilon}$ in the limit archive, and that the distance between $F(P_{Q,\epsilon+1\Delta})$ (respectively $F(P_{Q,\epsilon+2\Delta})$) and $F(P_{Q,\epsilon})$ can get large in some (pathological) examples.

Example 3.4 *Consider the following MOP:*

$$F : \mathbb{R} \to \mathbb{R}, \qquad F(x) = x \tag{18}$$

Let $Q = [0, 5]$, $\epsilon = 1, \Delta = 0.1$, and let $\Delta^* = \Delta$. Thus, we have $P_{Q,\epsilon} = [0, 1]$. Assume that $A = \{a_1\}$ with $a_1 = 1.2$. If next $a_2 = 0.1$ is considered, it will be inserted into the archive since $d_\infty(F(a_1), F(a_2)) > \Delta$ and since $a_2 \in P_{Q,\epsilon}$ is not $-\epsilon$-dominated by a_1 nor by any other point $x \in Q$, and will thus remain in the archive further on. Since a_2 is not $-(\epsilon + \Delta)$-dominating a_1 we have for the updated archive $A = \{a_1, a_2\}$. Hence, no element $a \in [0, \Delta]$ will be taken to the archive since for these points it holds $d_\infty(F(a), F(a_2)) \leq \Delta^*$, and thus, $a_2 \in P_{Q,\epsilon+2\Delta} \backslash P_{Q,\epsilon}$ will not be discarded from the archive during the run of the algorithm.
When on the other side $a_1 = 0$ is taken to the archive, no element $a \in Q \backslash P_{Q,\epsilon}$ will ever be accepted further on.

Example 3.5 *Let the MOP be given by $F : \mathbb{R} \to \mathbb{R}^2$, where*

$$f_1(x) = |x + 1|, \qquad f_2(x) = \begin{cases} |x - 1| & \text{for } x \leq 1 \\ \alpha|x - 1| & \text{for } x > 1 \end{cases}, \tag{19}$$

where $\alpha \in (0, 1)$ (see also Figure 1). For simplicity we assume that $\epsilon = (\bar{\epsilon}, \bar{\epsilon}) \in \mathbb{R}^2_+$. It is $P_Q = [-1, 1]$ with

$$F(-1) = (0, 2), \qquad F(1) = (2, 0) \tag{20}$$

Further, it is

$$F(-1 - \bar{\epsilon}) = (\bar{\epsilon}, 2 + \bar{\epsilon}), \qquad F(1 + \frac{\bar{\epsilon}}{\alpha}) = (2 + \frac{\bar{\epsilon}}{\alpha}, \bar{\epsilon}) \tag{21}$$

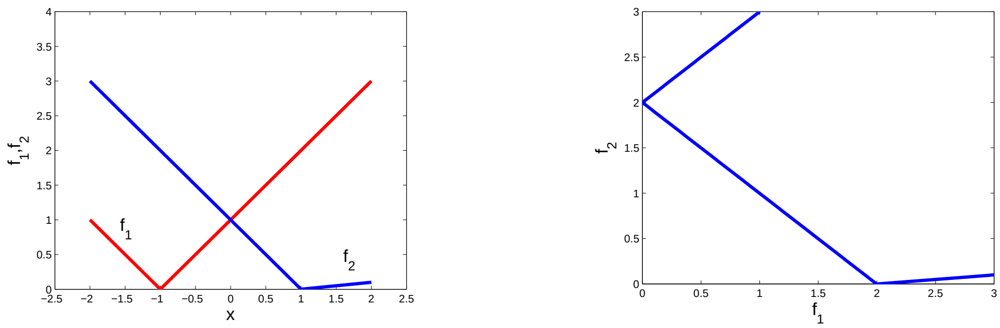
Using this and some monoticity arguments on f_1 and f_2 we see that

$$P_{Q,\epsilon} = \Big(-1 - \bar{\epsilon}, 1 + \frac{\bar{\epsilon}}{\alpha}\Big] \tag{22}$$

Since $F(1 + \frac{\bar{\epsilon}+\Delta}{\alpha}) = (2 + \frac{\bar{\epsilon}+\Delta}{\alpha}, \bar{\epsilon} + \Delta)$ it follows that

$$dist(F(P_{Q,\epsilon+1\Delta}), F(P_{Q,\epsilon})) = \frac{\Delta}{\alpha}, \tag{23}$$

which can get large for small values of α.



(a) Parameter space (b) Image space

Figure 1: Example of MOP (19) for $\alpha = 0.1$.

4 Bounds on the Archive Sizes

Here we give an upper bound U on the size of the limit archive obtained by the novel strategy, and discuss that the order of U is already optimal.

Theorem 4.1 *Let $\epsilon \in \mathbb{R}^k_+$, $\Delta^*, \Delta \in \mathbb{R}_+$ with $\Delta^* < \Delta$ be given. Further let $m_i = \min_{x\in Q} f_i(x)$ and $M_i = \max_{x\in Q} f_i(x)$, $1 \leq i \leq k$, and l_0 as in Theorem 3.2. Then, when using ArchiveUpdate$P_{Q,\epsilon}$, the archive size maintained in Algorithm 1 for all $l \geq l_0$ is bounded as*

$$|A_l| \leq \left(\frac{1}{\Delta^*}\right)^k \sum_{i=1}^{k} (\epsilon_i + 2\Delta + \Delta^*) \prod_{\substack{j=1\\ j\neq i}}^{k} (M_j - m_j + \Delta^*). \tag{24}$$

Proof: Let $l \geq l_0$ and the archive A_l be given. Since $A_l \subset P_{Q,\epsilon+2\Delta}$ (see Theorem 3.2) we are interested in an upper bound on the volume of $F(P_{Q,\epsilon+2\Delta})$. For this, we consider first the $(k-1)$-dimensional volume of the Pareto front $F(P_Q)$. Due to the nature of nondominance we can assume that $F(P_Q)$ is located in the graph of a map

$$\begin{aligned} &\Phi_f : K \to \mathbb{R}^k \\ &\Phi_f(u_1, \ldots, u_{k-1}) = (u_1, \ldots, u_{k-1}, f(u_1, \ldots, u_{k-1})), \end{aligned} \tag{25}$$

where $K := [m_1, M_1] \times \ldots \times [m_{k-1}, M_{k-1}]$ and $f : K \to [m_k, M_k]$ which satisfies the monotonicity conditions

$$\begin{aligned} f(u_1, \ldots, u_{i-1}, x_i, u_{i+1}, \ldots, u_{k-1}) \leq f(u_1, \ldots, u_{i-1}, y_i, u_{i+1}, \ldots, u_{k-1}) \\ \forall i = 1, \ldots, k-1, \\ u_j \in [m_j, M_j], j = 1, \ldots, i-1, i+1, \ldots, k-1, \\ x_i, y_i \in [m_i, M_i], x_i \leq y_i. \end{aligned} \tag{26}$$

Further, we can assume that f is sufficiently smooth. If not, we can replace f by a smooth function $\tilde{f}$ such that the volume of $\Phi_{\tilde{f}}$ is larger than the volume of Φ_f as the following discussion shows:
if $f(m_i, \ldots, m_{k-1}) = m_k$, the Pareto front consists of one point, $F(P_Q) = \{(m_1, \ldots, m_k)\}$, and has minimal volume 0. Since we are interested in upper bounds on the volume we can omit this case. Doing so, a smooth function $\tilde{f} : K \to [m_k, M_k]$ exists with $\tilde{f}(x) \leq f(x)$, $\forall x \in K$ that fulfills the monotonicity conditions (26). The $(k-1)$-dimensional volume of $\Phi_{\tilde{f}}$ is obviously larger than the volume of Φ_f. Under the smoothness assumption we can replace condition (26) by

$$\frac{\partial f}{\partial u_i} u \leq 0, \quad \forall u \in K, \ \forall i = 1, \ldots, k-1. \tag{27}$$

The $(k-1)$-dimensional volume of Φ_f with parameter range K is given by (see [5]):

$$Vol_{k-1}(\Phi_f) = \int_K \sqrt{||\nabla f||^2 + 1}\, du, \tag{28}$$

where ∇f denotes the gradient of f. Analogue to [14] (see also Appendix 1) the volume can be estimated by using partial integration and the monotonicity conditions (27) by:

$$Vol_{k-1}(\Phi_f) \leq \sum_{i=1}^{k} \prod_{\substack{j=1 \\ j \neq i}}^{k} (M_j - m_j). \tag{29}$$

Considering this and the nature of $-\epsilon$-dominance we can bound the k-dimensional volume of $F(P_{Q,\epsilon+2\Delta})$ by:

$$Vol_k(F(P_{Q,\epsilon+2\Delta})) \leq \sum_{i=1}^{k} (\epsilon_i + 2\Delta) \prod_{\substack{j=1 \\ j \neq i}}^{k} (M_j - m_j), \tag{30}$$

Since $\|F(a_1) - F(a_2)\| \geq \Delta^*$ for all $a_1, a_2 \in A_l$ it follows that the boxes

$$B^{\infty}_{\frac{1}{2}\Delta^*}(F(a)), \quad a \in A_l, \tag{31}$$

are mutually nonoverlapping. Further, if $F(a) \in F(P_{Q,\epsilon+2\Delta})$, then $B^{\infty}_{\frac{1}{2}\Delta^*}(F(a))$ is included in a $\Delta^*/2$-neighborhood $\tilde{F}$ of $F(P_{Q,\epsilon+2\Delta})$ with

$$Vol_k(\tilde{F}) \leq \sum_{i=1}^{k}(\epsilon_i + 2\Delta + \Delta^*) \prod_{\substack{j=1\\ j\neq i}}^{k}(M_j - m_j + \Delta^*). \tag{32}$$

The maximal number of entries in A_l can now be estimated by

$$|A_l| \leq \frac{Vol_k(\tilde{F})}{Vol_k(B^{\infty}_{\frac{1}{2}\Delta^*}(F(a)))}, \tag{33}$$

and the claim follows since the volume of $B^{\infty}_{\frac{1}{2}\Delta^*}(F(a))$ is obviously given by $(\Delta^*)^k$.

In particular interesting is certainly the growth of the magnitudes of the (limit) archives for vanishing discretization parameter Δ. Since for every meaningful computation the value Δ will be smaller than every entry of ϵ, we can assume $\epsilon_i = c_i\Delta$ with $c_i > 1$. Using (24) and for simlicity $\Delta = \Delta^*$ we see that

$$|A_l| \leq \left(\frac{1}{\Delta}\right)^{k-1} \sum_{i=1}^{k}(c_i + 3) \prod_{\substack{j=1\\ j\neq i}}^{k}(M_j - m_j + \Delta^*). \tag{34}$$

Thus, the growth of the magnitudes is of order $\mathcal{O}\left(\left(\frac{1}{\Delta}\right)^{k-1}\right)$ for $\Delta \to 0$. Regarding the fact that P_Q, which is contained in $P_{Q,\epsilon}$ for all values of $\epsilon \in \mathbb{R}^k_+$, typically forms a $(k-1)$-dimensional object, we see that the order of the magnitude of the archive with respect to Δ is already optimal. This is due to the fact that the discretization (line 3 of Algorithm 2) is realized in image space. An analogue result for a discretization in parameter space, however, can not hold since $P_{Q,\epsilon}$ is n-dimensional.

Remark 4.2 *In case the algorithm is modified as described in Remark 3.3 (c), the upper bound for the magnitude of the archive is given by*

$$|A_l^x| \leq \left(\frac{1}{\Delta^*} + 1\right)^n \prod_{j=1}^{n}(b_i - a_i), \tag{35}$$

where $Q \subset [a_1, b_1] \times \ldots \times [a_n, b_n]$ *(*$P_{Q,\epsilon+2\Delta}$ *is certainly included in* $[a_1, b_1] \times \ldots \times [a_n, b_n]$*, and maximal* $1/\Delta^* + 1$ *elements can be placed in each coordinate direction). To see that this bound is tight we consider the example*

$$F : [0,1]^n \to \mathbb{R}^k, \quad F(x) \equiv c_0 \in \mathbb{R}^k, \tag{36}$$

and let $\Delta = 1/s,\ s \in \mathbb{N}$*. Define* $x_{i_1,\ldots,i_n} = (i_1\Delta, \ldots, i_n\Delta)$ *and*

$$\mathcal{D} := \{x_{i_1,\ldots,i_n} | 0 \leq i_1, \ldots, i_n \leq s\}. \tag{37}$$

Table 1: Comparison of the magnitudes of the final archive ($|A_{final}|$, rounded) and the corresponding update times (T, in seconds) for MOP (38) and for different values of Δ. We have taken the average result of 100 test runs.

Δ	$\|A_{final}\|$	T
0	3836	32.98
0.01	827	6.22
0.05	68	1.80

Since $\mathcal{D} \subset [0,1]^n$ and $d_\infty(z_1, z_2) \geq \Delta > \Delta^*$ for all $z_1, z_2 \in \mathcal{D}$, $z_1 \neq z_2$, all points in $\mathcal{D}$ will be accepted by the archiver (assuming that only points $z \in \mathcal{D}$ are inserted) leading to an archive A with $|A| = |\mathcal{D}| = (s+1)^n$.
Since $P_{Q,\epsilon}$ is n-dimensional, the growth of the magnitudes of the archives is also beyond this constructed example of order $\mathcal{O}\left(\left(\frac{1}{\Delta}\right)^n\right)$ for $\Delta \to 0$. This makes a huge difference to the other archiver since for general MOPs we have $n \gg k$.

5 Numerical Results

Here we demonstrate the practicability of the novel archiver on five examples. For this, we run and compare *ArchiveUpdate*$P_{Q,\epsilon}$ for different values of Δ including $\Delta_0 = 0$, which is the archiver which accepts all test points which are not $-\epsilon$ dominated by any other test point (see Remark 4.3 (a)). To obtain a fair comparison we have decided to take a random search operator for the generation process (the same sequence of points for all settings). An implementation of the archiver including these examples can be found in [1].

5.1 Example 1

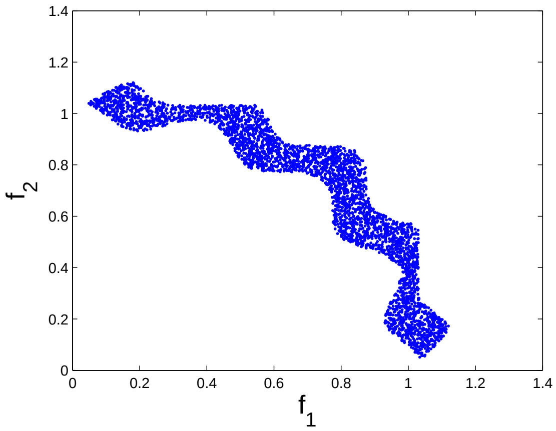
First we consider the MOP suggested by Tanaka ([16]):

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \quad F(x_1, x_2) = (x_1, x_2) \tag{38}$$

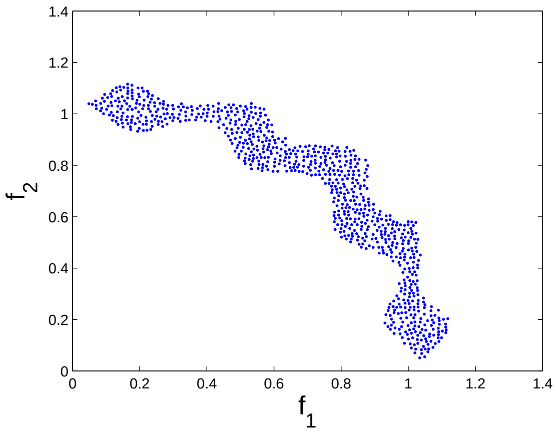
where

$$C_1(x) = x_1^2 + x_2^2 - 1 - 0.1\cos(16\arctan(x_1/x_2)) \geq 0$$
$$C_2(x) = (x_1 - 0.5)^2 + (x_2 - 0.5)^2 \leq 0.5$$

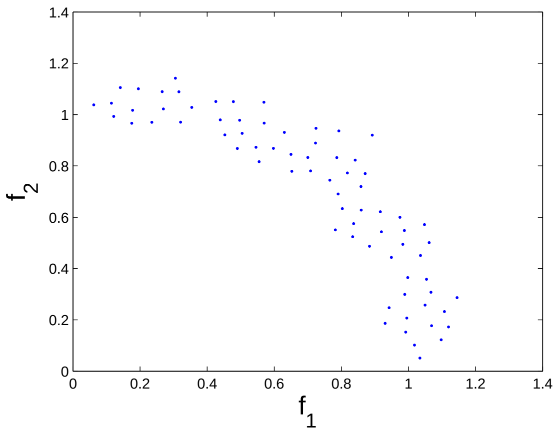
Figure 2 shows a numerical result for $N = 200,000$ randomly chosen points within $Q = [0,\pi]^2$ and for three different values of the discretization parameter: $\Delta_0 = 0$, $\Delta_1 = 0.01$ and $\Delta_2 = 0.05$. As anticipated, the granularity of the resulting archive increases with the value of Δ. Thus, the approximation quality decreases, but also the running time of the algorithm (see Table 1).



(a) $\Delta_0 = 0, \quad |A_{final}| = 3824$



(b) $Delta_1 = 0.01, \quad |A_{final}| = 834$



(c) $Delta_2 = 0.05, \quad |A_{final}| = 73$

Figure 2: Results for MOP (38) for different values of Δ leading to different granularities of the approximation.

5.2 Example 2

Next, we consider the following MOP proposed in [10]:

$$F : \mathbb{R}^2 \to \mathbb{R}^2$$
$$F(x_1, x_2) = \begin{pmatrix} (x_1 - t_1(c + 2a) + a)^2 + (x_2 - t_2 b)^2 \\ (x_1 - t_1(c + 2a) - a)^2 + (x_2 - t_2 b)^2 \end{pmatrix}, \quad (39)$$

where

$$t_1 = \text{sgn}(x_1) \min\left(\left\lceil \frac{|x_1| - a - c/2}{2a + c} \right\rceil, 1 \right), t_2 = \text{sgn}(x_2) \min\left(\left\lceil \frac{|x_2| - b/2}{b} \right\rceil, 1 \right).$$

The Pareto set consists of nine connected components of length a with identical images. We have chosen the values $a = 0.5$, $b = c = 5$, $\epsilon = (0.1, 0.1)$, and the domain as $Q = [-20, 20]^2$. Figure 3 shows some numerical results for $N = 100,000$ randomly chosen points within Q and for the three variants of the archiver $ArchiveUpdateP_{Q,\epsilon}$: (a) $\Delta = (0, 0)$, i.e., the archiver which aims to store the entire set $P_{Q,\epsilon}$, (b) a discretization (in image space) using $\Delta = (0.02, 0.02)$, and (c) the variant which is described in Remark 3.3 (c) using $\Delta_x = (0.1, 0.1)$ for a discretization of the parameter space. As anticipated, the solution in (b) is more uniform in image space compared to the solution in (c), which is, in turn, more uniform in parameter space. Note that the solution in (b), i.e., where the discretization has been done in image space, already 'detects' all nine connected components in parameter space. This is certainly due to the fact that the search has been done by selecting random points wich are uniformly distributed within Q. One interesting point for future studies would be to investigate if this capability still holds when other search strategies are chosen (which include, e.g., local search strategies).

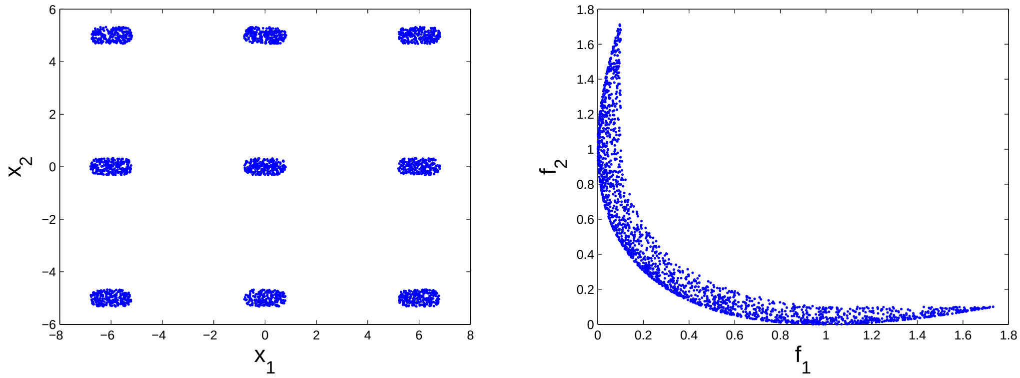
5.3 Example 3

Finally, we consider the production model proposed in [12]:
$f_1, f_2 : \mathbb{R}^n \to \mathbb{R}$,

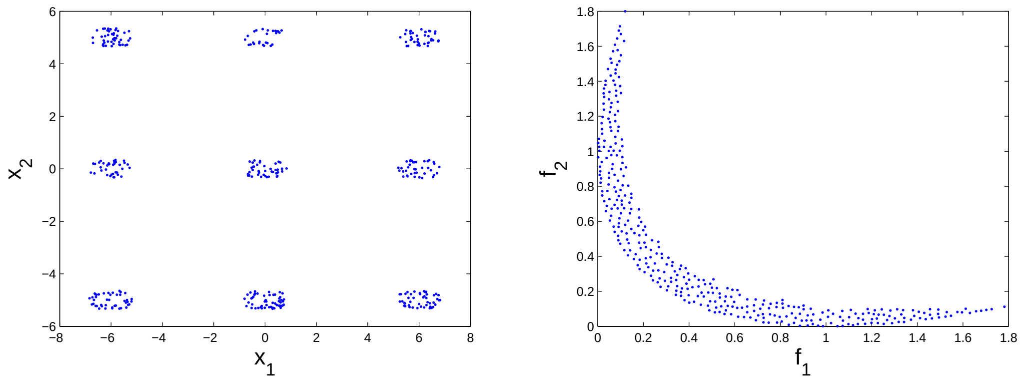
$$f_1(x) = \sum_{j=1}^{n} x_j,$$
$$f_2(x) = 1 - \prod_{j=1}^{n} (1 - w_j(x_j)), \quad (40)$$

where

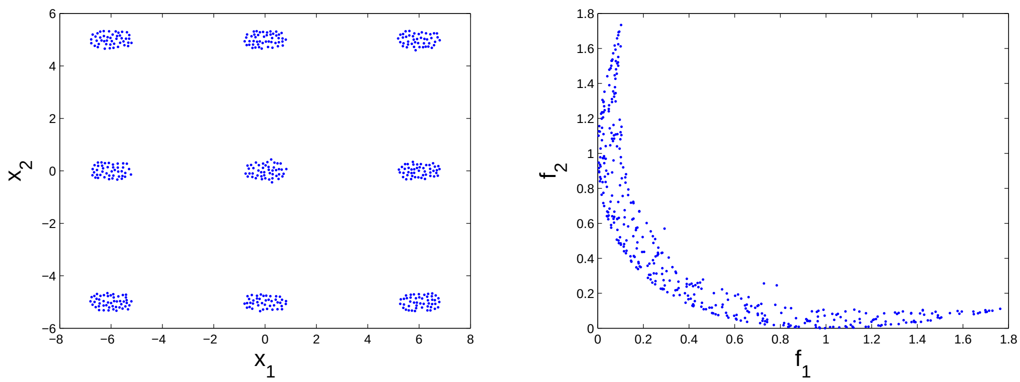
$$w_j(z) = \begin{cases} 0.01 \cdot \exp(-(\frac{z}{20})^{2.5}) & \text{for} \quad j = 1, 2 \\ 0.01 \cdot \exp(-\frac{z}{15}) & \text{for} \quad 3 \leq j \leq n \end{cases}$$



(a) $\Delta = (0, 0)$, $|A_{final}| = 2150$



(b) $\Delta = (0.02, 0.02)$, $|A_{final}| = 365$



(c) $\Delta_x = (0.1, 0.1)$, $|A_{final}| = 410$

Figure 3: Numerical results for MOP (39) for the three different variants of $ArchiveUpdateP_{Q,\epsilon}$.

The two objective functions have to be interpreted as follows. f_1 represents the sum of the additional cost necessary for a more reliable production of n items. These items are needed for the composition of a certain product. The function f_2 describes the total failure rate for the production of this composed product.
Here we have chosen $n = 5$, $Q = [0, 40]^n$, and $\epsilon = (0.1, 0.001)$ which corresponds to 10 percent of one cost unit for one item (ϵ_1), and to 0.1 percent of the total failure rate (ϵ_2). Figure 4 shows numerical results for (a) $\Delta = (0, 0)$ and (b) for $\Delta = \epsilon/3$. Note the symmetries in the model: it is e.g., $F(x_1, x_2, \ldots) = F(x_2, x_1, \ldots)$ by which is follows that the two connected components at $x_1 = 0$ and $x_2 = 0$ (see Figure 4 (a)) have the same image. Also in this case the archiver detects both components though the discretization has been done in image space (Figure 4 (b)).

5.4 Example 4

Next, we consider a real-life engineering problem, namely the design of a four-bar plane truss ([15]):

$$\begin{aligned} F &: \mathbb{R}^4 \to \mathbb{R}^2 \\ f_1(x) &= L(2x_1 + \sqrt{2}x_2 + \sqrt{2}x_3 + x_4) \\ f_2(x) &= \frac{FL}{E}\left(\frac{2}{x_1} + \frac{2\sqrt{2}}{x_2} - \frac{2\sqrt{2}}{x_3} + \frac{1}{x_4}\right) \end{aligned} \tag{41}$$

f_1 models the volume of the truss, and f_2 the displacement of the joint. The model constants are the length L of each bar ($L = 200$ cm), the elasticity constants E and σ of the materials involved ($E = 2 \times 10^5$ kN/cm^3, $\sigma = 10$ kN/cm^2), and the force F which causes the stress of the truss ($F = 10$ kN). The parameters x_i represent the cross sectional areas of the four bars of the truss. The physical restrictions are given by

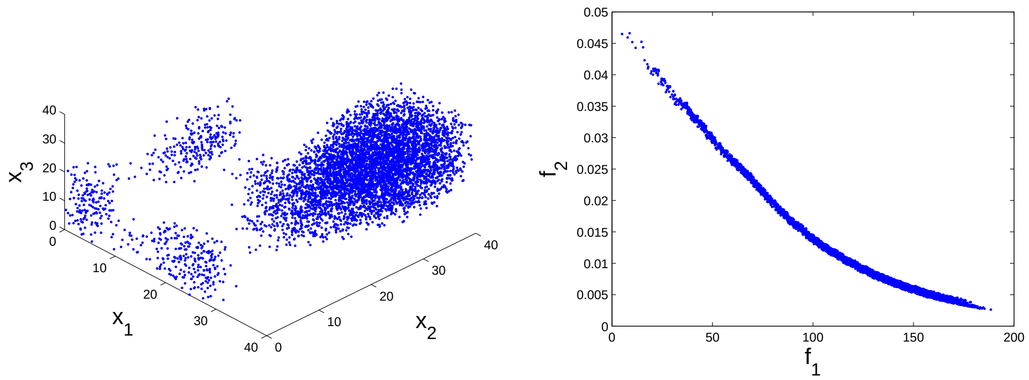
$$Q = [F/\sigma, 3F/\sigma] \times [\sqrt{2}F/\sigma, 3F/\sigma]^2 \times [F/\sigma, 3F/\sigma] \tag{42}$$

For the allowed tolerances we follow the suggestion made in [4] and set $\epsilon_1 = 50$ cm^3 and $\epsilon_2 = 0.0005$ cm. Figure 5 shows a result for $N = 500,000$ randomly chosen points within Q and for $\Delta = (10, 0.0001)$, i.e., $\Delta_i = \epsilon_i/5$ (see Remark 4.3 (b)). The final archive consists of 78 elements, and the computational time was 5.5 seconds. In contrast, a run of the same algorithm with the same setting but with $\Delta = 0$ took 4 minutes and 21 seconds leading to an archive with 8377 elements.

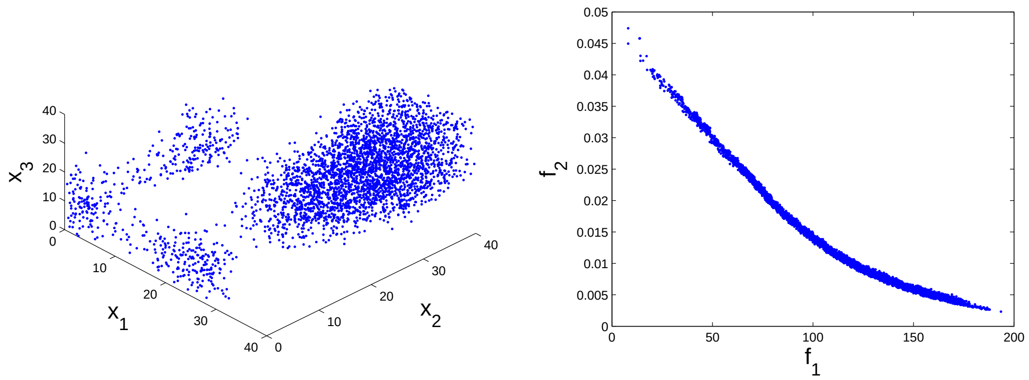
5.5 Example 5

Finally, we consider a bi-objective {0,1}-knapsack problem which should demonstrate that the additional consideration of approximate solutions can be beneficial for the decision maker.

$$f_1, f_2 : \{0,1\}^n \to \mathbb{R},\ f_1(x) = \sum_{j=1}^n c_j^1 x_j,\ f_2(x) = \sum_{j=1}^n c_j^2 x_j \tag{43}$$



(a) $\Delta = (0,0)$, $|A_{final}| = 5939$



(b) $\Delta = (0.033, 0.00033)$, $|A_{final}| = 3544$

Figure 4: Numerical result for MOP (40): projections to the coordinates x_1, x_2, x_3 of the final archive (left) and their images (right).

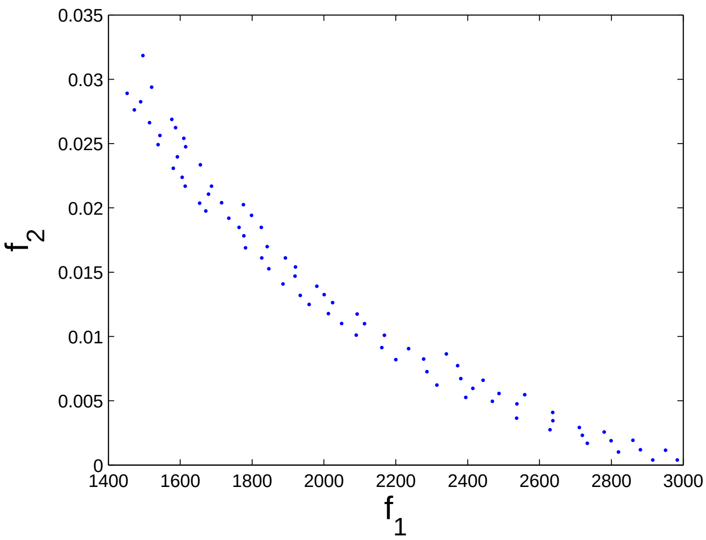


Figure 5: Numerical result for MOP (41). Here, we have chosen $\epsilon = (50, 0.0005)$ and $\Delta = (10, 0.0001)$.

s.t.

$$\sum_{j=1}^{n} w_j x_j \leq W,\ x_j \in \{0,1\},\ j = 1,\ldots,n,$$

where c_j^i represents the value of item j on criterion $i, i = 1,2$; $x_j = 1$, $j = 1,\ldots,n$, if item j is included in the knapsack, else $x_j = 0$. w_j is the weight of item j, and W the overall knapsack capacity. Figure 6 shows one numerical result obtained by an evolutionary strategy[2] for an instance with $n = 30$ items and with randomly chosen values $c_j^i \in [8,12]$, and capacity $W = 15$ (note that we are faced with a maximization problem). For $\epsilon = (2,2)$ and $\Delta = 0.1$ a total of 182 elements forms the final archive, and only six of them are nondominated. When taking, for instance, x_0 as reference (assuming, e.g., that this point has been selected by the DM out of the nondominated points) and assuming a tolerance of 1 which represents a possible loss of 0.6% compared to x_0 for each objective value, the resulting region of interest includes seven approximate solutions (see Figure 6). These solutions, though similar in objective space, differ significantly in parameter space: two solutions differ compared to x_0 in 8 items, one in 10, and 4 solutions differ even in 12 items. Thus, in this case it is obvious that by tolerating approximate solutions – where the loss of them can be determined a priori – a larger variety of possibilities is offered to the DM.

6 Conclusion and Future Work

We have proposed and investigated a novel archiving strategy for stochastic search algorithms which allows – under mild assumptions on the generation process – for a finite size approximation of the set $P_{Q,\epsilon}$ which contains all ϵ-efficient solutions of an MOP within a compact domain Q. We have proven convergence of the algorithm toward a finite size representation of the set of interest in the probabilistic sense, yielding bounds on the approximation quality and the cardinality of the archives. Finally, we have presented some numerical results indicating the usefulness of the approach.

The consideration of approximate solutions certainly leads to a larger variety of possible options to the DM, but, in turn, also to a higher demand on the related decision making process. Thus, the support for this problem could be one focus of future research. Further, it could be interesting to integrate the archiving strategy directly into the stoachastic search process (as e.g. done in [3] for an EMO algorithm) in order to obtain a fast and reliable search procedure.

Acknoledgements

The second author gratefully acknowledges support from CONACyT project no. 45683-Y.

[2]A modification of the algorithm presented in [17] which uses the novel archiver.

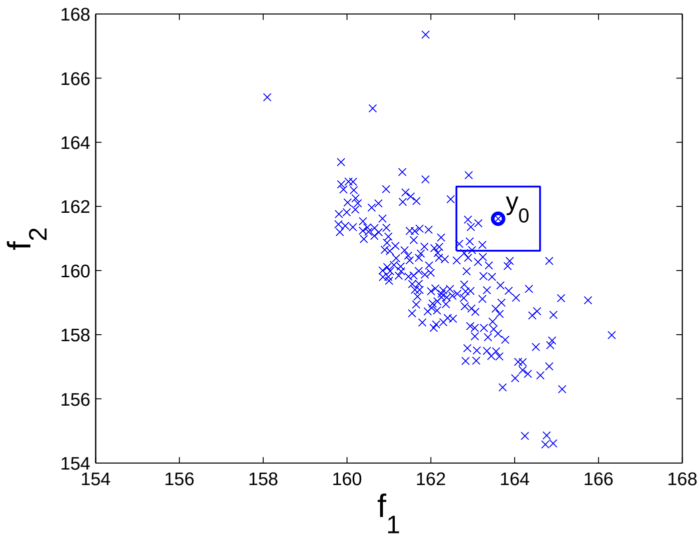


Figure 6: Numerical result for MOP (41) with $\epsilon = (2, 2)$ and $\Delta = 0.1$. The rectangle defines one possible region of interest around $y_0 = F(x_0)$ including seven approximate solutions (see text).

Table 2: ..

c^1	c^2	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7
		1	1	1	1	1	1	0	0
		0	0	1	1	0	1	1	0
		1	1	0	1	0	1	1	1
		1	1	1	1	1	1	0	1
		0	1	0	1	1	1	1	1
		0	0	1	1	0	0	0	0
		0	1	1	0	1	0	1	1
		1	0	0	0	1	0	0	0
		1	1	1	1	0	1	1	1
		0	0	1	0	0	1	1	0
		1	0	0	1	1	1	1	1
		0	0	0	0	0	0	0	0
		1	1	1	1	1	0	0	0
		0	0	0	0	0	0	1	0
		1	1	0	1	1	0	1	0
		1	1	1	1	1	1	1	1
		0	1	1	0	0	1	0	1
		0	0	0	0	0	0	0	0
		1	1	0	1	0	0	1	1
		0	0	0	0	0	1	0	1
		0	0	0	0	1	0	0	1
		1	1	1	0	1	1	1	1
		1	0	1	0	1	0	0	0
		0	1	1	1	1	1	1	1
		1	0	1	0	0	1	1	0
		0	0	0	0	0	0	0	0
		0	1	0	0	0	0	0	0
		1	1	0	1	1	1	0	1
		0	0	0	0	0	0	0	0
		1	0	1	1	1	0	1	1

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7 Appendix 1

Here we show the inequality in (29), which is taken from [14].
Define

$$\begin{aligned} K &:= [m_1, M_1] \times \ldots \times [m_{k-1}, M_{k-1}], \\ K_{(i)} &:= [m_1, M_1] \times \ldots \times [m_{i-1}, M_{i-1}] \times [m_{i+1}, M_{i+1}] \times \ldots \times [m_{k-1}, M_{k-1}], \\ u_{(i)} &:= (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_{k-1}),\ i = 1, \ldots, k-1. \end{aligned} \tag{44}$$

Then, the $(k-1)$-dimensional volume of Φ_f with parameter range K can be bounded as follows:

$$\begin{aligned} Vol_{k-1}(\Phi_f) &= \int_K \sqrt{||\nabla f||^2 + 1}du = \int_K \sqrt{\left(\frac{\partial f}{\partial u_1}\right)^2 + \ldots + \left(\frac{\partial f}{\partial u_{k-1}}\right)^2 + 1}du \\ &\leq \int_K \left|\frac{\partial f}{\partial u_1}\right| du + \ldots + \int_K \left|\frac{\partial f}{\partial u_{k-1}}\right| du + \int_K 1du \\ &= \sum_{i=1}^{k-1} \left(\int_{K_{(i)}} \left(\int_{m_i}^{M_i} \left|\frac{\partial f}{\partial u_i}\right| du_i \right) du_{(i)} \right) + \int_K 1du \\ &= \sum_{i=1}^{k-1} \left(\int_{K_{(i)}} \left(-\int_{m_i}^{M_i} \frac{\partial f}{\partial u_i} du_i \right) du_{(i)} \right) + \int_K 1du \\ &\leq \sum_{i=1}^{k} \prod_{\substack{j=1 \\ j \neq i}}^{k} (M_j - m_j). \end{aligned} \tag{45}$$

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