Shape and topological sensitivity analysis in domains with cracks

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SHAPE AND TOPOLOGICAL SENSITIVITY ANALYSIS IN DOMAINS WITH CRACKS*

A. KHLUDNEV†, J. SOKOLOWSKI‡, AND K. SZULC§

Abstract. Framework for shape and topology sensitivity analysis in geometrical domains with cracks is established for elastic bodies in two spatial dimensions. Equilibrium problem for elastic body with cracks is considered. Inequality type boundary conditions are prescribed at the crack faces providing a non-penetration between the crack faces. Modelling of such problems in two spatial dimensions is presented with all necessary details for further applications in shape optimization in structural mechanics. In the paper, general results on the shape and topology sensitivity analysis of this problem are provided. The results are interesting on its own. In particular, the existence of the shape and topological derivatives of the energy functional is obtained. It is shown, in fact, that the level set type method [4] can be applied to shape and topology opimization of the related variational inequalities for elasticity problems in domains with cracks, with the nonpenetration condition prescribed on the crack faces. The results presented in the paper can be used for numerical solution of shape optimization and inverse problems in structural mechanics.

Key words. Crack with non-penetration, shape sensitivity, derivative of energy functional, topological derivative

AMS subject classifications. Primary 35J85, 74K20 Secondary 35J25, 74M15

1. Introduction.

Shape optimization requires few mathematical results, in the framework of modelling and numerical solution, for any specific class of problems governed by partial differential equations of mathematical physics. Usually, we need to show the well posedness of the specific problem, and also we can propose a numerical method for the effective solution procedure. Hence, in order to solve a shape optimization problem we are obliged to have the results on

- the existence and continuous dependence with respect to the shape of solutions to the model, which may result in the existence of optimal shapes,
- the differentiability of solutions with respect to the boundary variations, which imply the existence of shape gradients and leads to some necessary conditions for optimality, of the first order and possibly of the second order which leads to the Newton method of shape optimization,
- and in addition, perform the asymptotic analysis of the related boundary value problem in singularly perturbed geometrical domains and derive the form of the topological derivative for the shape functional of interest, which allows for the topology changes in the process of numerical optimization, if necessary,
- and finally, we may device a numerical method and show its efficiency in numerical examples, and its convergence form the mathematical point of view.

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One of the most important applications of shape optimization with long tradition is structural mechanics. From practical point of view, it is useful for applications, that the analysis of a specific shape optimization problem is performed taking into account possible presence of the cracks, in particular in order to avoid the damage of the structure under considerations. Unfortunately, the analysis of elastic bodies with cracks is quite complex, and a little is known about mathematical modelling of cracks, however the presence of cracks is evident, so the subject is important and the research well developed in experimental branch of mechanics. In the present paper we consider cracks in two spatial dimensions, and we also prescribe the so-called nonpenetration conditions on the crack faces in the framework of linear elasticity. For such models we provide results on the existence of solutions, variational formulations, shape sensitivity analysis, and asymptotic analysis with respect to the singular perturbations of geometrical domains. We select the results in such a way, that we construct the theoretical background for possible application of the level set method of shape optimization for the related problems. Most of the results presented here are obtained recently, and are results of the long term collaboration between Nancy and Novosibirsk.

The aim of this paper is twofold, modelling of elastic bodies with cracks and sensitivity analysis of the energy functionals with respect to the boundary variations and singular perturbations of geometrical domains. The results presented in our paper can be implemented in the framework of the so-called level set method of shape optimization, we refer the reader to [4] for the numerical results obtained for the Signorini problem.

Thus, we provide some of the new results obtained last years and related to the crack theory in elasticity with possible contact between crack faces, which are then required for the sensitivity analysis. The energy functional of an elastic body in two spatial dimensions is a representative example of possible shape functional which can be minimized or maximized over a class of admissible domains.

First, for the modelling issue, we discuss problem formulations, peculiarities of the problems and possible relations between topics under investigation. It is well known that classical crack theory in elasticity is characterized by linear boundary conditions which leads to linear boundary value problems. This approach has a clear shortcoming from mechanical standpoint since opposite crack faces can penetrate each other. We consider nonlinear boundary conditions on crack faces, the so-called non-penetration conditions, written in terms of inequalities. From the standpoint of applications these boundary conditions are preferable since they provide a mutual non-penetration between crack faces. As a result a free boundary problem is obtained which means that a concrete boundary condition at a given point can be found provided that we have a solution of the problem.

The main attention in this paper is paid to dependence of solutions of the problem on domain perturbations, and in particular, on the crack shape. The technique of boundary variations [27] is used in section 5 in order to obtain the shape gradient of the energy functional. On the other hand, asymptotic analysis in singularly perturbed domains [21] is performed in section 6 in order to obtain the topological derivative [28] of the same energy functional. In this way we have all tools nececessary in the framework of the level set method of shape optimization, which is the subject of the subsequent publication.

The outline of the paper can be described as follows. We start with the strong formulation (1.1)-(1.5) of the elliptic free boundary problem. We have some inequality

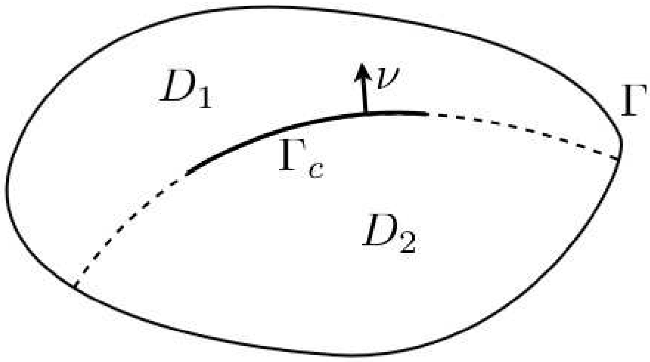

FIG. 1.1. *Domain* Ω_c

type condtions in (1.4)-(1.5) which leads to the free boundary and to the unknown coincidence set, which should be determined for the solution of the problem. In section 2 the existence of weak solutions for variational formulations of problem (1.1)-(1.5) is presented. In particular, the smooth domain formulation introduced by the authors is given. In section 3 the fictitious domain method is described in details for the crack problem. In section 4, the case of a crack on the boundary of rigid inclusion is analysed, this topic is new in the field of mathematical crack modelling to the best of our knowledge. In section 5 the shape sensitivity analysis is performed for the singular parts of the boundary, i.e. the perturbations of the crack tips. Since the singularities of the displacement field are not explicitely known for our model, the treatment of the tips requires appropriate technique proposed by the authors, we present in details the construction which allows us to derive the shape derivative of the energy functional for the perturbations of the crack tips. In particular, our results can be used in the Griffith criteria for the crack propagation. The results of section 6 are new, and constitutes the topological sensitivity analysis part of the paper. The form of the topological derivative of the energy functional is obtained here for the first time, and it is exactly of the same form, as in the case of linear problem, however the proof is not the same, we use the method proposed for the Signorini problem in [30]. In section 7 the shape sensitivity analysis is applied to the modelling of the kinking crack. Finally, in section 8 some open problems, important for a progress in the field, are formulated.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ, and $\Gamma_c \subset \Omega$ be a smooth curve without self-intersections, $\Omega_c = \Omega \setminus \overline{\Gamma}_c$ (see Fig. 1.1).

It is assumed that Γ_c can be extended in such a way that this extension crosses Γ at two points, and Ω_c is divided into two subdomains D_1 and D_2 with Lipschitz boundaries ∂D_1, ∂D_2, $meas(\Gamma \cap \partial D_i) > 0$, $i = 1, 2$. Denote by $\nu = (\nu_1, \nu_2)$ a unit normal vector to Γ_c. We assume that Γ_c does not contain its tip points, i.e. $\Gamma_c = \overline{\Gamma}_c \setminus \partial\Gamma_c$.

Equilibrium problem for a linear elastic body occupying Ω_c is as follows. In the domain Ω_c we have to find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i,j = 1,2$, such that

$$-\text{div}\sigma = f \quad \text{in} \quad \Omega_c, \tag{1.1}$$

$$\sigma = A\varepsilon(u) \quad \text{in} \quad \Omega_c, \tag{1.2}$$

$$u = 0 \quad \text{on} \quad \Gamma, \tag{1.3}$$

$$[u]\nu \geq 0, \quad [\sigma_\nu] = 0, \quad \sigma_\nu \cdot [u]\nu = 0 \quad \text{on} \quad \Gamma_c, \tag{1.4}$$

$$\sigma_\nu \leq 0, \quad \sigma_\tau = 0 \quad \text{on} \quad \Gamma_c^\pm. \tag{1.5}$$

Here $[v] = v^+ - v^-$ is a jump of v on Γ_c, and signs $\pm$ correspond to positive and negative crack faces with respect to ν, $f = (f_1, f_2) \in L^2(\Omega_c)$ is a given function,

$$\sigma_\nu = \sigma_{ij}\nu_j\nu_i, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \cdot \nu, \quad \sigma_\tau = (\sigma_\tau^1, \sigma_\tau^2),$$
$$\sigma\nu = (\sigma_{1j}\nu_j, \sigma_{2j}\nu_j),$$

the strain tensor components are denoted by $\varepsilon_{ij}(u)$,

$$\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \varepsilon(u) = \{\varepsilon_{ij}(u)\}, \quad i,j = 1,2.$$

Elasticity tensor $A = \{a_{ijkl}\}$, $i,j,k,l = 1,2$, is given and satisfies the usual properties of symmetry and positive definiteness

$$a_{ijkl}\xi_{kl}\xi_{ij} \geq c_0|\xi|^2, \quad \forall\, \xi_{ij},\ \xi_{ij} = \xi_{ji}, \quad c_0 = const,$$

$a_{ijkl} = a_{klij} = a_{jikl}$, $a_{ijkl} \in L^\infty(\Omega)$.

Relations (1.1) are equilibrium equations, and (1.2) is the Hooke's law, $u_{i,j} = \frac{\partial u_i}{\partial u_j}$, $(x_1, x_2) \in \Omega_c$. All functions with two below indices are symmetric in those indices, i.e. $\sigma_{ij} = \sigma_{ji}$ etc. Summation convention is assumed over repeated indices throughout the paper.

The first condition in (1.4) is called the non-penetration condition. It provides a mutual non-penetration between the crack faces $\Gamma_c^\pm$. The second condition of (1.5) provides zero friction on Γ_c. For simplicity we assume a clamping condition (1.3) at the external boundary Γ.

Note that a priori we do not know points on Γ_c where strict inequalities in (1.4), (1.5) are fulfilled. Due to this, the problem (1.1)-(1.5) is a free boundary value problem. If we have $\sigma_\nu = 0$ then, together with $\sigma_\tau = 0$, the classical boundary condition $\sigma\nu = 0$ follows which is used in the linear crack theory. On the other hand, due to (1.4), the condition $\sigma_\nu < 0$ implies $[u]\nu = 0$, i.e. we have a contact between the crack faces at a given point. The strict inequality $[u]\nu > 0$ at a given point means that we have no contact between the crack faces.

Hence, the first difficulty in studying the problem (1.1)-(1.5) is concerned with boundary conditions (1.4)-(1.5). The second one is related to the general crack problem difficulty - a presence of non-smooth boundaries.

2. The existence of weak solutions. We show that the analysed problem is well posed. Therefore, there is a unique weak solution to the associated variational inequality. We introduce also the so-called smooth domain formulation [15] which have some implications in numerical analysis, for the related results in the case of a scalar problem of an elastic membrane with a cut we refere the reader to [1]. The smooth domain formulation allows obtain variational solutions to the crack problem in the geometrical domain without any cut, the crack is *present* only in the subset of admissible functions for the variational solution, i.e., some inequality constraints are imposed for the admissible functions over the crack Γ_c.

First of all we note that problem (1.1)-(1.5) admits several equivalent formulations. In particular, it corresponds to minimization of the energy functional. To check this, introduce Sobolev space

$$H^1_\Gamma(\Omega_c) = \{v = (v_1, v_2) \mid v_i \in H^1(\Omega_c),\ v_i = 0 \text{ on } \Gamma,\ i = 1, 2\}$$

and closed convex set of admissible displacements

$$K = \{v \in H^1_\Gamma(\Omega_c) \mid [v]\nu \geq 0 \quad \text{a.e. on} \quad \Gamma_c\}. \tag{2.1}$$

In this case, due to the Weierstrass theorem, the problem

$$\min_{v\in K} \left\{ \frac{1}{2} \int\limits_{\Omega_c} \sigma_{ij}(v)\varepsilon_{ij}(v) - \int\limits_{\Omega_c} f_i v_i \right\}$$

has (a unique) solution u satisfying the variational inequality

$$u \in K, \tag{2.2}$$

$$\int\limits_{\Omega_c} \sigma_{ij}(u)\varepsilon_{ij}(v-u) \geq \int\limits_{\Omega_c} f_i(v_i - u_i), \quad \forall v \in K, \tag{2.3}$$

where $\sigma_{ij}(u) = \sigma_{ij}$ are defined from (1.2).

Problem formulations (1.1)-(1.5) and (2.2)-(2.3) are equivalent. Any smooth solution of (1.1)-(1.5) satisfies (2.2)-(2.3) and conversely, from (2.2)-(2.3) it follows (1.1)-(1.5).

Below we provide two more equivalent formulations for the problem (1.1)-(1.5), the so-called mixed and smooth domain formulations. To this end, we first discuss in what sense boundary conditions (1.4)-(1.5) are fulfilled. Denote by Σ a closed curve without self-intersections of the class $C^{1,1}$, which is an extension of Γ_c such that $\Sigma \subset \Omega$, and the domain Ω is divided into two subdomains Ω_1 and Ω_2 (see Fig. 2.1). In this case Σ is the boundary of the domain Ω_1, and the boundary of Ω_2 is $\Sigma \cup \Gamma$.

Introduce the space $H^{\frac{1}{2}}(\Sigma)$ with the norm

$$\|v\|^2_{H^{\frac{1}{2}}(\Sigma)} = \|v\|^2_{L^2(\Sigma)} + \int\limits_\Sigma \int\limits_\Sigma \frac{|v(x)-v(y)|^2}{|x-y|^2} dxdy \tag{2.4}$$

and denote by $H^{-\frac{1}{2}}(\Sigma)$ a space dual of $H^{\frac{1}{2}}(\Sigma)$. Also, consider the space

$$H^{1/2}_{00}(\Gamma_c) = \left\{ v \in H^{\frac{1}{2}}(\Gamma_c) \mid \frac{v}{\sqrt{\rho}} \in L^2(\Gamma_c) \right\}$$

with the norm

$$\|v\|^2_{1/2,00} = \|v\|^2_{1/2} + \int\limits_{\Gamma_c} \rho^{-1} v^2,$$

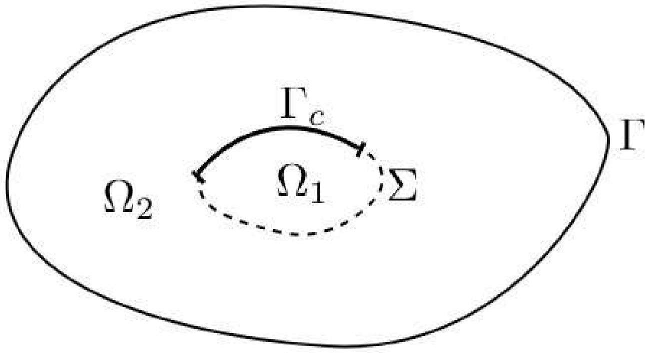

FIG. 2.1. *Extension of* Γ_c *to* Σ.

where $\rho(x) = dist(x; \partial\Gamma_c)$ and $\|v\|_{1/2}$ is the norm in the space $H^{1/2}(\Gamma_c)$. It is known that functions from $H^{1/2}_{00}(\Gamma_c)$ can be extended to Σ by zero values, and moreover this extention belongs to $H^{1/2}(\Sigma)$. More precisely, let v be defined at Γ_c, and $\overline{v}$ be the extension of v by zero, i.e.

$$\overline{v}(x) = \begin{cases} v(x), & x \in \Gamma_c \\ 0, & x \in \Sigma \setminus \Gamma_c. \end{cases}$$

Then (see [9])

$$v \in H^{1/2}_{00}(\Gamma_c) \quad \text{if and only if} \quad \overline{v} \in H^{1/2}(\Sigma).$$

With the above notations, it is possible to describe in what sense boundary conditions (1.4)-(1.5) are fulfilled. Namely, the condition $\sigma_\nu \leq 0$ in (1.5) means that

$$\langle \sigma_\nu, \phi \rangle_{1/2,00} \leq 0, \quad \forall\, \phi \in H^{1/2}_{00}(\Gamma_c), \quad \phi \geq 0 \text{ a.e. on } \Gamma_c,$$

where $\langle \cdot, \cdot \rangle_{1/2,00}$ is a duality pairing between $H^{-1/2}_{00}(\Gamma_c)$ and $H^{1/2}_{00}(\Gamma_c)$. The condition $\sigma_\tau = 0$ in (1.5) means that

$$\langle \sigma^i_\tau, \phi \rangle_{1/2,00} = 0, \quad \forall\, \phi = (\phi_1, \phi_2) \in H^{1/2}_{00}(\Gamma_c), \quad i = 1, 2.$$

The last condition of (1.4) holds in the following sense

$$\langle \sigma_\nu, [u]\nu \rangle_{1/2,00} = 0.$$

2.1. Mixed formulation of the problem. Now we are interested to give a mixed formulation of the problem (1.1)-(1.5). Introduce the space for stresses

$$H(\mathrm{div}) = \left\{ \sigma = \{\sigma_{ij}\} \mid \sigma \in L^2(\Omega_c), \mathrm{div}\sigma \in L^2(\Omega_c) \right\}$$

with the norm

$$\|\sigma\|^2_{H(\mathrm{div})} = \|\sigma\|^2_{L^2(\Omega_c)} + \|\mathrm{div}\sigma\|^2_{L^2(\Omega_c)}$$

and the set of admissible stresses

$$H(\mathrm{div}; \Gamma_c) = \left\{ \sigma \in H(\mathrm{div}) \mid [\sigma\nu] = 0 \text{ on } \Gamma_c; \quad \sigma_\nu \leq 0, \quad \sigma_\tau = 0 \text{ on } \Gamma^\pm_c \right\}.$$

We should note at this step that for $\sigma \in H(\mathrm{div})$ the traces $(\sigma\nu)^{\pm}$ are correctly defined on $\Sigma^{\pm}$ as elements of $H^{-1/2}(\Sigma)$. The first condition in the definition of $H(\mathrm{div};\Gamma_c)$ is fulfilled in the following sense

$$(\sigma\nu)^{+} = (\sigma\nu)^{-} \text{ on } \Sigma$$

for any curve Σ with the prescribed properties (see [9]). Relations $\sigma_\nu \leq 0$, $\sigma_\tau = 0$ on $\Gamma_c^{\pm}$ also make sense. Values σ_ν, σ_τ are defined as elements of the space $H_{00}^{-1/2}(\Gamma_c)$.

The mixed formulation of the problem (1.1)-(1.5) is as follows. We have to find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$u \in L^2(\Omega_c), \quad \sigma \in H(\mathrm{div};\Gamma_c), \tag{2.5}$$

$$-\mathrm{div}\sigma = f \quad \text{in} \quad \Omega_c, \tag{2.6}$$

$$\int\limits_{\Omega_c} C\sigma(\overline{\sigma} - \sigma) + \int\limits_{\Omega_c} u(\mathrm{div}\overline{\sigma} - \mathrm{div}\sigma) \geq 0 \quad \forall \overline{\sigma} \in H(\mathrm{div};\Gamma_c). \tag{2.7}$$

The tensor C is obtained by inverting the Hooke's law (1.2), i.e.

$$C\sigma = \varepsilon(u).$$

It is possible to prove a solution existence to the problem (2.5)-(2.7) and check that (2.5)-(2.7) is formally equivalent to (1.1)-(1.5) (see [13]). Solution existence to (2.5)-(2.7) can be proved independently of (1.1)-(1.5). On the other hand, the solution exists due to the equivalence, and we already have the solution to the problem (1.1)-(1.5).

2.2. Smooth domain formulation. Along with the mixed formulation (2.5)-(2.7) the so-called smooth domain formulation of the problem (1.1)-(1.5) can be provided. In this case the solution of the problem is defined in the smooth domain Ω. To do this, we should notice that the solution of the problem (1.1)-(1.5) satisfies (2.2)-(2.3), thus the condition

$$[\sigma\nu] = 0 \quad \text{on} \quad \Gamma_c$$

holds, and therefore it can be proved that in the distributional sense

$$-\mathrm{div}\sigma = f \quad \text{in} \quad \Omega.$$

Hence, the equilibrium equations (1.1) hold in the smooth domain Ω.

Introduce the space for stresses defined in Ω,

$$\mathcal{H}(\mathrm{div}) = \{\sigma = \{\sigma_{ij}\} \mid \sigma,\ \mathrm{div}\sigma \in L^2(\Omega)\}$$

and the set of admissible stresses

$$\mathcal{H}(\mathrm{div};\Gamma_c) = \{\sigma \in \mathcal{H}(\mathrm{div}) \mid \sigma_\tau = 0, \quad \sigma_\nu \leq 0 \text{ on } \Gamma_c\}.$$

The norm in the space $\mathcal{H}(\mathrm{div})$ is defined as follows

$$\|\sigma\|^2_{\mathcal{H}(\mathrm{div})} = \|\sigma\|^2_{L^2(\Omega)} + \|\mathrm{div}\sigma\|^2_{L^2(\Omega)}.$$

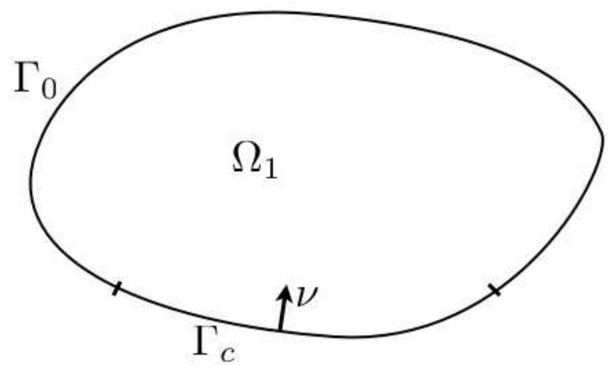

FIG. 3.1. *Signorini problem.*

We see that for $\sigma \in \mathcal{H}(\text{div})$, the boundary condition $\sigma_\tau = 0$, $\sigma_\nu \le 0$ on Γ_c are correctly defined in the sense $H_{00}^{-1/2}(\Gamma_c)$. Thus, we can provide the smooth domain formulation for the problem (1.1)-(1.5). It is necessary to find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$u \in L^2(\Omega), \quad \sigma \in \mathcal{H}(\text{div}; \Gamma_c), \tag{2.8}$$

$$-\text{div}\sigma = f \quad \text{in} \quad \Omega, \tag{2.9}$$

$$\int\limits_\Omega C\sigma(\overline{\sigma} - \sigma) + \int\limits_\Omega u(\text{div}\overline{\sigma} - \text{div}\sigma) \ge 0 \quad \forall \overline{\sigma} \in \mathcal{H}(\text{div}; \Gamma_c). \tag{2.10}$$

It is possible to prove a solution existence to the problem (2.8)-(2.10) (see [15]). Moreover, any smooth solution of (1.1)-(1.5) satisfies (2.8)-(2.10) and conversely, from (2.8)-(2.10) it follows (1.1)-(1.5). Advantage of the formulation (2.8)-(2.10) is that it is given in the smooth domain. This formulation reminds contact problems with thin obstacle when restrictions are imposed on sets of small dimensions (see [12]).

Numerical aspects for the problems like (1.1)-(1.5) can be found, for example, in [1], [17]. In particular, in [1] the convergence of the finite element approximation is proved for a scalar problem, and some error estimates are derived.

3. Fictitious domain method. This type of modelling is also interesting from numerical point of view, since theoretically allows for numerical computations in a fixed domain, the shape being defined by some additional constraints involving the Lagrangian multipliers. We discuss here in details only one aspect of this technique which can be useful for numerical methods of shape optimization for frictionless contact problems.

In this section we provide a connection between the problem (1.1)-(1.5) and the Signorini contact problem. It is turned out that the Signorini problem is a limit problem for a family of problems like (1.1)-(1.5). First we give a formulation of the Signorini problem. Let $\Omega_1 \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ_1, $\Gamma_1 = \Gamma_c \cup \Gamma_0$, $\Gamma_c \cap \Gamma_0 = \emptyset$, $meas\Gamma_0 > 0$ (see Fig. 3.1).

For simplicity, we assume that Γ_c is a smooth curve (without its tip points). Denote by $\nu = (\nu_1, \nu_2)$ a unit normal inward vector to Γ_c. We have to find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\mathrm{div}\sigma = f \quad \text{in} \quad \Omega_1, \tag{3.1}$$

$$\sigma = A\varepsilon(u) \quad \text{in} \quad \Omega_1, \tag{3.2}$$

$$u = 0 \quad \text{on} \quad \Gamma_0, \tag{3.3}$$

$$u\nu \geq 0,\ \sigma_\nu \leq 0,\ \sigma_\tau = 0,\ u\nu \cdot \sigma_\nu = 0 \quad \text{on} \quad \Gamma_c. \tag{3.4}$$

Here $f = (f_1, f_2) \in L^2_{loc}(\mathbb{R}^2)$ is a given function, $A = \{a_{ijkl}\}$, $i, j, k, l = 1, 2$ is a given elasticity tensor, $a_{ijkl} \in L^\infty_{loc}(\mathbb{R}^2)$, with the usual properties of symmetry and positive definiteness.

It is well known that the problem (3.1)-(3.4) has a variational formulation providing a solution existence. Namely, denote

$$H^1_{\Gamma_0}(\Omega_1) = \{v = (v_1, v_2) \in H^1(\Omega_1) \mid v_i = 0 \text{ on } \Gamma_0, \quad i = 1, 2\}$$

and introduce the set of admissible displacements

$$K_c = \{v = (v_1, v_2) \in H^1_{\Gamma_0}(\Omega_1) \mid v\nu \geq 0 \text{ a.e. on } \Gamma_c\}.$$

In this case the problem (3.1)-(3.4) is equivalent to minimization of the functional

$$\frac{1}{2}\int\limits_{\Omega_1} \sigma_{ij}(v)\varepsilon_{ij}(v) - \int\limits_{\Omega_1} f_i v_i$$

over the set K_c and can be written in the form of variational inequality

$$u \in K_c, \tag{3.5}$$

$$\int\limits_{\Omega_1} \sigma_{ij}(u)\varepsilon_{ij}(v - u) \geq \int\limits_{\Omega_1} f_i(v_i - u_i) \quad \forall v \in K_c. \tag{3.6}$$

Here $\sigma_{ij}(u) = \sigma_{ij}$ are defined from the Hooke's law (3.2). Variational inequality (3.5)-(3.6) is equivalent to (3.1)-(3.4) and conversely, i.e., any smooth solution of (3.1)-(3.4) satisfies (3.5)-(3.6) and from (3.5)-(3.6) it follows (3.1)-(3.4). Along with variational formulation (3.5)-(3.6) the problem (3.1)-(3.4) admits a mixed formulation which is omitted here.

The aim of this section is to prove that the problem (3.1)-(3.4) is a limit problem for a family of problems like (1.1)-(1.5). In what follows we provide explanation of this statement.

First of all we extend the domain Ω_1 by adding a domain Ω_2 with smooth boundary Γ_2. An extended domain is denoted by Ω_c, and it has a crack (cut) Γ_c. Boundary of Ω_c is $\Gamma \cup \Gamma_c^\pm$ (see Fig. 3.2). Denote $\Sigma_0 = \Gamma_1 \cap \Gamma_2$, $\Sigma = \Sigma_0 \setminus \Gamma$, thus Σ does not contain its tip points.

We introduce a family of elasticity tensors with a positive parameter λ,

$$a^\lambda_{ijkl} = \begin{cases} a_{ijkl} & \text{in} \quad \Omega_1 \\ \lambda^{-1} a_{ijkl} & \text{in} \quad \Omega_2. \end{cases}$$

Denote $A^\lambda = \{a^\lambda_{ijkl}\}$, and in the extended domain Ω_c, consider a family of the crack problems. Find a displacement field $u^\lambda = (u^\lambda_1, u^\lambda_2)$, and stress tensor components

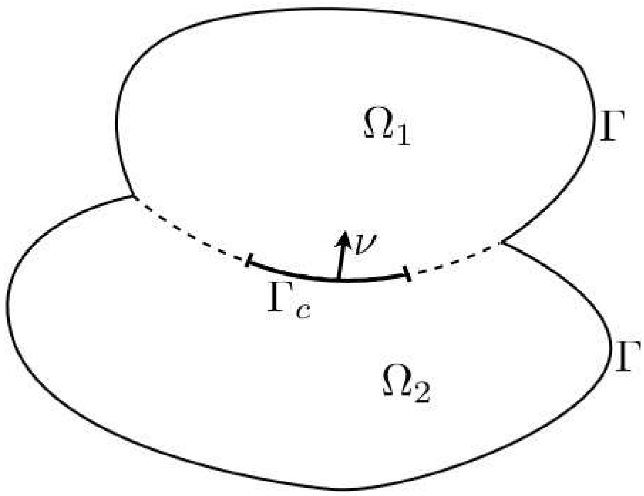

FIG. 3.2. *Extended domain* Ω_c.

$\sigma^\lambda = \{\sigma^\lambda_{ij}\}$, $i,j = 1,2$, such that

$$-\mathrm{div}\sigma^\lambda = f \quad \text{in} \quad \Omega_c, \tag{3.7}$$

$$\sigma^\lambda = A^\lambda \varepsilon(u^\lambda) \quad \text{in} \quad \Omega_c, \tag{3.8}$$

$$u^\lambda = 0 \quad \text{on} \quad \Gamma, \tag{3.9}$$

$$[u^\lambda]\nu \geq 0,\ [\sigma^\lambda_\nu] = 0,\ \sigma^\lambda_\nu \cdot [u]\nu = 0 \quad \text{on} \quad \Gamma_c, \tag{3.10}$$

$$\sigma^\lambda_\nu \leq 0,\ \sigma^\lambda_\tau = 0 \quad \text{on} \quad \Gamma^\pm_c. \tag{3.11}$$

As before, $[v] = v^+ - v^-$ is a jump of v through Γ_c, where $\pm$ fit positive and negative crack faces $\Gamma^\pm_c$. All the rest notations correspond to those of Section 1. We see that for any fixed $\lambda > 0$ the problem (3.7)-(3.11) describes an equilibrium state of linear elastic body with the crack Γ_c where non-penetration conditions are prescribed. Hence, the problem (3.7)-(3.11) is exactly the problem like (1.1)-(1.5), and we are interested in passage to the limit as $\lambda \to 0$. In particular, the problem (3.7)-(3.11) admits a variational formulation. Boundary conditions (3.10)-(3.11) are fulfilled in the form as it is explained in Section 2. It can be proved (see [7]) that the following convergence takes place as $\lambda \to 0$

$$u^\lambda \to u^0 \quad \text{strongly in} \quad H^1_\Gamma(\Omega_c), \tag{3.12}$$

$$\frac{u^\lambda}{\sqrt{\lambda}} \to 0 \quad \text{strongly in} \quad H^1(\Omega_2), \tag{3.13}$$

where $u^0 = u$ on Ω_1, i.e. a restriction of the limit function from (3.12) to Ω_1 coincides with the unique solution of the Signorini problem (3.1)-(3.4). From (3.12)-(3.13) it is seen that the limit function u^0 is zero in Ω_2. On the other hand, there is no limit σ^λ in Ω_2 as $\lambda \to 0$. Thus, the domain Ω_2 can be understood as undeformable body. This means that the Signorini problem is, in fact, a crack problem with non-penetration condition between crack faces, where the crack Γ_c is located between the elastic body

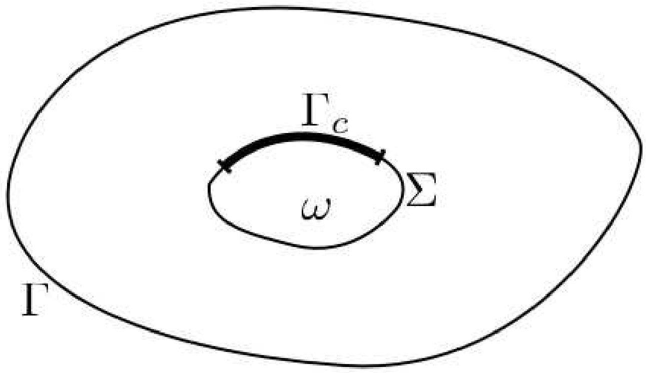

FIG. 4.1. *Rigid inclusion ω in elastic body.*

Ω_1 and nondeformable (rigid) body Ω_2. It is worth noting that, in fact, we can write the problem (3.7)-(3.11) in the equivalent form in the smooth domain $\Omega_c \cup \overline{\Gamma}_c$ by using the smooth domain formulation (Section 2.2). Some additional details of fictitious domain method in the crack theory can be found in [7].

4. Crack on the boundary of rigid inclusion. The inclusions in elastic bodies are also important for applications, both in design procedures and in numerical solution of some inverse problems. We restrict ourselves to the limit case of a rigid inclusion, with a crack at the interface. This seems to be a new class of problems, both for the analysis and for the shape optimization. One can also attemp to find the shape derivative of the elastic energy with respect to the perturbations of the crack tip, some results in this direction are given with all details in section 5.

We consider a rigid inclusion inside of the rigid body. This section is concerned with a crack situated on the boundary of the rigid inclusion.

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ, and $\omega \subset \Omega$ be a subdomain with smooth boundary Σ, $\overline{\omega} \subset \Omega$. Assume that Σ is composed of two parts: $\Sigma = \Gamma_c \cup (\Sigma \setminus \Gamma_c)$, $meas(\Sigma \setminus \Gamma_c) > 0$, see Fig. 4.1. Denote $\Omega_c = \Omega \setminus \overline{\Gamma}_c$. As before, by $A = \{a_{ijkl}\}$ we denote an elasticity tensor with the usual symmetry and positive definiteness properties, $a_{ijkl} \in L^\infty_{loc}(\mathbb{R}^2)$. For a positive parameter $\lambda > 0$, introduce the following elasticity tensor

$$a^\lambda_{ijkl} = \begin{cases} a_{ijkl} & \text{in} \quad \Omega \setminus \overline{\omega}, \\ \lambda^{-1} a_{ijkl} & \text{in} \quad \omega, \end{cases} \qquad i,j,k,l = 1,2,$$

and consider a boundary value problem for finding a displacement field $u^\lambda = (u^\lambda_1, u^\lambda_2)$ and stress tensor components $\sigma^\lambda = \{\sigma^\lambda_{ij}\}$, $i,j = 1,2$, such that

$$-\text{div}\sigma^\lambda = f \quad \text{in} \quad \Omega_c, \tag{4.1}$$

$$\sigma^\lambda - A^\lambda \varepsilon(u^\lambda) = 0 \quad \text{in} \quad \Omega_c, \tag{4.2}$$

$$u^\lambda = 0 \quad \text{on} \quad \Gamma, \tag{4.3}$$

$$[u^\lambda]\nu \geq 0, \; [\sigma^\lambda_\nu] = 0, \; \sigma^\lambda_\nu \cdot [u^\lambda]\nu = 0 \quad \text{on} \quad \Gamma_c, \tag{4.4}$$

$$\sigma^\lambda_\tau = 0, \; \sigma^\lambda_\nu \leq 0 \quad \text{on} \quad \Gamma^\pm_c. \tag{4.5}$$

Here $f = (f_1, f_2) \in L^2(\Omega)$ is a given function. We see that for any $\lambda > 0$ the problem (4.1)-(4.5) is the problem like (1.1)-(1.5) describing an equilibrium state for the elastic body with the crack Γ_c. This problem has the variational formulation, mixed formulation and smooth domain formulation. Our aim is to consider the limit case as $\lambda \to 0$. It can be done by analyzing the variational inequality

$$u^\lambda \in K, \tag{4.6}$$

$$\int\limits_{\Omega_c} \sigma_{ij}^\lambda(u^\lambda)\varepsilon_{ij}(v - u^\lambda) \geq \int\limits_{\Omega_c} f_i(v_i - u_i^\lambda) \quad \forall v \in K. \tag{4.7}$$

Here $\sigma_{ij}^\lambda(u^\lambda) = \sigma_{ij}^\lambda$ are defined from (4.2), and the set K was introduced in (2.1).

We can pass to the limit in (4.6)-(4.7) as $\lambda \to 0$. To this end, we introduce the space of infinitesimal rigid displacements

$$R(\omega) = \{\rho = (\rho_1, \rho_2) \mid \rho(x) = Bx + D,\ x \in \omega\},$$

where

$$B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \quad D = (d^1, d^2); \quad b, d^1, d^2 = const.$$

Consider next the space

$$H_\Gamma^{1,\omega}(\Omega_c) = \{v \in H_\Gamma^1(\Omega_c) \mid v = \rho \text{ on } \omega, \quad \rho \in R(\omega)\}$$

and the set of admissible displacements

$$K_\omega = \{v \in H_\Gamma^{1,\omega}(\Omega_c) \mid (v^+ - \rho)\nu \geq 0 \quad \text{a.e. on} \quad \Gamma_c\}.$$

Here v^+ corresponds to the crack face Γ_c^+. Now we substitute $v = 0$, $v = 2u^\lambda$ as test functions in (4.7). This provides the relation

$$\int\limits_{\Omega_c} \sigma_{ij}^\lambda(u^\lambda)\varepsilon_{ij}(u^\lambda) = \int\limits_{\Omega_c} f_i u_i^\lambda$$

which implies two estimates

$$\|u^\lambda\|_{H_\Gamma^1(\Omega_c)} \leq c_1,$$

$$\frac{1}{\lambda}\int\limits_{\omega} a_{ijkl}\varepsilon_{kl}(u^\lambda)\varepsilon_{ij}(u^\lambda) \leq c_2, \tag{4.8}$$

being uniform in λ, $0 < \lambda < \lambda_0$. Consequently we can assume that as $\lambda \to 0$

$$u^\lambda \to u \quad \text{weakly in} \quad H_\Gamma^1(\Omega_c). \tag{4.9}$$

Moreover, by (4.8),

$$\varepsilon_{ij}(u) = 0 \text{ in } \omega, \quad i, j = 1, 2.$$

This means an existence of a function ρ_0, such that

$$u = \rho_0 \text{ in } \omega, \quad \rho_0 \in R(\omega).$$

Since u^λ converge to u weakly in $H^1_\Gamma(\Omega_c)$ and $u^\lambda \in K$, it follows

$$(u^+ - \rho_0)\nu \geq 0 \text{ on } \Gamma_c.$$

In particular, $u \in K_\omega$. Now we take an arbitrary function $v \in R(\omega)$. In this case, there exists $\rho \in R(\omega)$, such that $v = \rho$ on ω. It is clear that v can be substituted in (4.7) as a test function. Since $A^\lambda = A$ in $\Omega \setminus \overline{\omega}$ we can pass to the limit as $\lambda \to 0$ in (4.6), (4.7) which provides the following variational inequality

$$u \in K_\omega, \tag{4.10}$$

$$\int_{\Omega\setminus\overline{\omega}} \sigma_{ij}(u)\varepsilon_{ij}(v-u) \geq \int_{\Omega_c} f_i(v_i - u_i) \quad \forall v \in K_\omega. \tag{4.11}$$

This problem (4.10)-(4.11) describes an equilibrium state of the body occupying the domain Ω_c which has the crack Γ_c and the rigid inclusion ω. The latter means that any possible displacement in ω has the form $\rho(x)$, $x \in \omega$, where $\rho \in R(\omega)$. The problem (4.10)-(4.11) can be written in the differential form. This problem formulation is as follows. In domain Ω_c, we have to find a displacement field $u = (u_1, u_2)$; $u = \rho_0$ in ω; $\rho_0 \in R(\omega)$; and in domain $\Omega \setminus \overline{\omega}$ we have to find the stress tensor components $\sigma = \{\sigma_{ij}\}$, $i, j = 1, 2$, such that

$$-\mathrm{div}\sigma = f \quad \text{in} \quad \Omega \setminus \overline{\omega}, \tag{4.12}$$

$$\sigma - A\varepsilon(u) = 0 \quad \text{in} \quad \Omega \setminus \overline{\omega}, \tag{4.13}$$

$$u = 0 \quad \text{on} \quad \Gamma, \tag{4.14}$$

$$(u - \rho_0)\nu \geq 0,\ \sigma_\tau = 0,\ \sigma_\nu \leq 0 \quad \text{on} \quad \Gamma_c^+, \tag{4.15}$$

$$\sigma_\nu \cdot (u - \rho_0)\nu = 0 \quad \text{on} \quad \Gamma_c^+, \tag{4.16}$$

$$-\int_\Sigma \sigma\nu \cdot \rho = \int_\omega f_i \rho_i \quad \forall \rho \in R(\omega). \tag{4.17}$$

Problem formulations (4.10)-(4.11) and (4.12)-(4.17) are equivalent. This means that any smooth solution of (4.12)-(4.17) satisfies (4.10)-(4.11) and conversely, from (4.10)-(4.11) it follows (4.12)-(4.17).

Like in the previous sections, it is possible to describe in what sense boundary conditions (4.15)-(4.17) are fulfilled. In particular, the last two conditions of (4.15) are fulfilled in the sense of $H_{00}^{-1/2}(\Gamma_c)$. As for (4.16) it is fulfilled in the form

$$\langle \sigma_\nu^+, (u - \rho_0)\nu \rangle_{1/2,00,\Gamma_c} = 0.$$

Condition (4.17) holds as follows

$$-\langle \sigma\nu, \rho \rangle_{1/2,\Sigma} = \int_\omega f_i \rho_i \quad \forall\ \rho \in R(\omega).$$

To conclude this section, we note that variational inequality (4.10)-(4.11) is equivalent to minimization of the functional

$$\frac{1}{2} \int_{\Omega\setminus\overline{\omega}} \sigma_{ij}(v)\varepsilon_{ij}(v) - \int_{\Omega_c} f_i v_i$$

over the set K_ω.

5. Shape derivatives of energy functionals. This section, and section 6 are the most important contributions from the point of view of the shape optimization. Here, the boundary variation technique is applied in order to derive the form of the shape derivative of the energy functional with respect to the perturbations of the crack tips, we refer to [3], [18] for some results in this direction which apply to the domains with cracks. Such results constitute a complement to the monograph [27], where the shape sensitivity of elliptic boundary value problems in domains with cracks is not performed. On the other hand, in [27] the material derivatives of the solutions to the frictionless contact problem of an elastic body with the rigid foundation are obtained in the framework of the conical differentiability of solutions to variational inequalities.

The difficulty associated with the specific problem analysed in this section is in particular the lack of any information on the form of singularities of the displacement field at the crack tips. Therefore, we provide the precise form of the shape derivative using the path inpendent integrals, which is the standard procedure in the linear fracture mechanics. The structure of shape derivatives for shape differentiable functionals in domains with cracks is given in [18].

In the crack theory, the Griffith criterion is widely used to predict a crack propagation. This criterion says that a crack propagates provided that a derivative of the energy functional with respect to the crack length reaches a critical value. In this section we discuss this question for the model (1.1)-(1.5).

General point of view is that we should consider a perturbed problem with respect to (1.1)-(1.5). In particular, a crack length may be perturbed. Perturbation will be characterised by a small parameter t, and $t = 0$ corresponds to the unperturbed problem, i.e. to the problem (1.1)-(1.5). To describe properly a perturbation of the problem, we should have a perturbation of the domain Ω_c. It will be done via a so-called velocity method (see [27]). This means that we consider a given velocity field V defined in $\mathbb{R}^2$ and describe a perturbation of Ω_c by solving a Cauchy problem for a system of ODE. Namely, let $V \in W^{1,\infty}(\mathbb{R}^2)^2$ be a given field, $V = (V_1, V_2)$. Consider a Cauchy problem for finding a function $\Phi = (\Phi_1, \Phi_2)$,

$$\frac{d\Phi}{dt}(t,\cdot) = V(\Phi(t,\cdot)) \quad \text{for} \quad t \neq 0, \quad \Phi(0,x) = x. \tag{5.1}$$

There exists a unique solution Φ to (5.1) such that

$$\Phi = (\Phi_1, \Phi_2)(t,x) \in C^1([0,t_0]; W^{1,\infty}_{loc}(\mathbb{R}^2)^2), \quad |t_0| > 0. \tag{5.2}$$

Simultaneously, we can find a solution $\Psi = (\Psi_1, \Psi_2)$ to the following Cauchy problem

$$\frac{d\Psi}{dt}(t,\cdot) = -V(\Psi(t,\cdot)) \quad \text{for} \quad t \neq 0, \quad \Psi(0,y) = y \tag{5.3}$$

with the some regularity

$$\Psi = (\Psi_1, \Psi_2)(t,y) \in C^1([0,t_0]; W^{1,\infty}_{loc}(\mathbb{R}^2)^2), \quad |t_0| > 0. \tag{5.4}$$

It can be proved that for any fixed t, the function $\Psi(t,\cdot)$ is inverse with respect to $\Phi(t,\cdot)$ which means the following (see the proof in [10])

$$y = \Phi(t, \Psi(t,y)), \quad x \in \Psi(t, \Phi(t,x)), \quad x, y \in \mathbb{R}^2.$$

Due to this, we have a one-to-one mapping between the domain Ω_c and a perturbed domain Ω_c^t, namely

$$\begin{aligned} y &= \Phi(t,x) : \Omega_c \to \Omega_c^t, \\ x &= \Psi(t,y) : \Omega_c^t \to \Omega_c. \end{aligned}$$

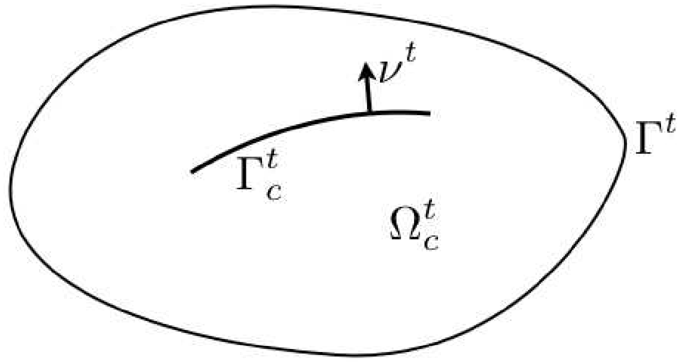

FIG. 5.1. *Perturbed domain* Ω_c^t.

Moreover, by (5.2), (5.4), we have the following asymptotic expansions (I denotes the indentity operator)

$$\Phi(t,x) = x + tV(x) + r_1(t), \tag{5.5}$$
$$\Psi(t,y) = y - tV(y) + r_2(t), \tag{5.6}$$
$$\frac{\partial \Phi(t)}{\partial x} = I + t\frac{\partial V}{\partial x} + r_3(t), \tag{5.7}$$
$$\frac{\partial \Psi(t)}{\partial y} = I - t\frac{\partial V}{\partial y} + r_4(t), \tag{5.8}$$
$$\|r_i(t)\|_{W^{1,\infty}_{loc}(\mathbb{R}^2)^2} = o(t), \quad i = 1,2,$$
$$\|r_i(t)\|_{L^{\infty}_{loc}(\mathbb{R}^2)^{2\times 2}} = o(t), \quad i = 3,4.$$

Hence, in the domain Ω_c^t it is possible to consider the following boundary value problem (perturbed with respect to (1.1)-(1.5)). Find a displacement field $u^t = (u_1^t, u_2^t)$, and stress tensor components $\sigma^t = \{\sigma_{ij}^t\}$, $i,j = 1,2$, such that

$$-\text{div}\sigma^t = f \quad \text{in} \quad \Omega_c^t, \tag{5.9}$$
$$\sigma^t = A\varepsilon(u^t) \quad \text{in} \quad \Omega_c^t, \tag{5.10}$$
$$u^t = 0 \quad \text{on} \quad \Gamma^t, \tag{5.11}$$
$$[u^t]\nu^t \geq 0,\ [\sigma_{\nu^t}^t] = 0,\ \sigma_{\nu^t}^t \cdot [u^t]\nu^t = 0 \quad \text{on} \quad \Gamma_c^t, \tag{5.12}$$
$$\sigma_{\nu^t}^t \leq 0,\ \sigma_{\tau^t}^t = 0 \quad \text{on} \quad \Gamma_c^{t\pm}. \tag{5.13}$$

Here

$$y = \Phi(t,x) : \Gamma \to \Gamma^t, \quad \Gamma_c \to \Gamma_c^t,$$

and we assume in this section that $f = (f_1, f_2) \in C^1(\mathbb{R}^2)$ and that $a_{ijkl} = const$, $i,j,k,l = 1,2$. All the rest notations in (5.9)-(5.13) remind those of (1.1)-(1.5), in particular, $\nu^t = (\nu_1^t, \nu_2^t)$ is a unit normal vector to Γ_c^t.

We can provide a variational formulation of the problem (5.9)-(5.13). Indeed, introduce the Sobolev space

$$H^1_{\Gamma^t}(\Omega_c^t) = \{v = (v_1, v_2) \mid v_i \in H^1(\Omega_c^t),\ v_i = 0 \text{ on } \Gamma^t, \quad i = 1,2\}$$

and the set of admissible displacements

$$K^t = \{v \in H^1_{\Gamma^t}(\Omega^t_c) \mid [v]\nu^t \geq 0 \quad \text{a.e. on} \quad \Gamma^t_c\}.$$

Consider the functional

$$\Pi(\Omega^t_c; v) = \frac{1}{2}\int\limits_{\Omega^t_c} \sigma^t_{ij}(v)\varepsilon_{ij}(v) - \int\limits_{\Omega^t_c} f_i v_i$$

and the minimization problem

$$\min_{v\in K^t} \Pi(\Omega^t_c; v). \tag{5.14}$$

Here $\sigma^t_{ij}(v)$ are defined from Hooke's law similar to (5.10). Solution of the problem (5.14) exists and it satisfies the variational inequality

$$u^t \in K^t, \tag{5.15}$$

$$\int\limits_{\Omega^t_c} \sigma^t_{ij}(u^t)\varepsilon_{ij}(v - u^t) \geq \int\limits_{\Omega^t_c} f_i(v_i - u^t_i) \quad \forall v \in K^t. \tag{5.16}$$

Having found a solution of the problem (5.15)-(5.16) we can define the energy functional

$$\Pi(\Omega^t_c; u^t) = \frac{1}{2}\int\limits_{\Omega^t_c} \sigma^t_{ij}(u^t)\varepsilon_{ij}(u^t) - \int\limits_{\Omega^t_c} f_i u^t_i.$$

Note that for $t = 0$, we have $\Omega^0_c = \Omega_c$ and $u^0 = u$, where u is the solution of the unperturbed problem (2.2), (2.3). The question whether it is possible to differentiate the functional $\Pi(\Omega^t_c; u^t)$ with respect to t? We have in mind an existence of the following derivative

$$\frac{d}{dt}\Pi(\Omega^t_c; u^t)|_{t=0} = \lim_{t\to 0} \frac{\Pi(\Omega^t_c; u^t) - \Pi(\Omega_c; u)}{t}.$$

The answer is positive in many practical situations. We consier two cases, where the derivative

$$I = \frac{d}{dt}\Pi(\Omega^t_c; u^t)|_{t=0} \tag{5.17}$$

exists.

a) Assume that the normal vector ν to Γ_c keeps its value under the mapping $x \to \Phi(t, x)$, i.e. $\nu^t = \nu$. In this case, it is proved that the formula for I can be obtained, namely, (see [11], [16], [19])

$$I = \frac{1}{2}\int\limits_{\Omega_c} \{\text{div}V \cdot \varepsilon_{ij}(u) - 2E_{ij}(V; u)\}\, \sigma_{ij}(u) - \int\limits_{\Omega_c} \text{div}(V f_i)u_i, \tag{5.18}$$

where

$$E_{ij}(U; v) = \frac{1}{2}(v_{i,k}U_{k,j} + v_{j,k}U_{k,i}), \quad U = \{U_{ij}\}, \quad i, j = 1, 2;$$

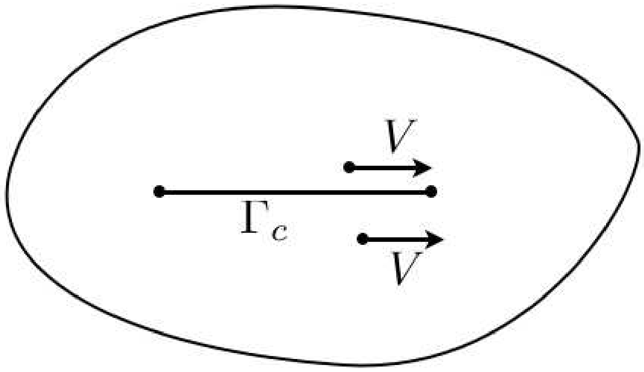

FIG. 5.2. *Rectilinear crack* Γ_c *and tangential field* V.

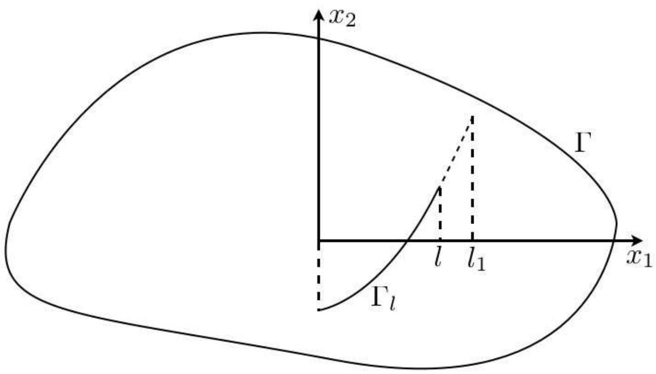

FIG. 5.3. *Domain* Ω_l *with a crack* Γ_l.

Note that the assumption concerning the normal vector ν takes place for rectilinear cracks Γ_c and vector fields V tangential to Γ_c (see Fig. 5.2). In this situation, (5.18) can provide a formula for the derivative of the energy functional with respect to the crack length what is practically needed for using the Griffith criterion. It will be the case when $V = 1$ in a vicinity of the right crack tip and *suppV* belongs to a small neighborhood of this tip (see Fig. 5.2).

b) Formula for the derivative (5.17) can be derived and for curvilinear cracks when the above assumption on the normal vector ν is not fulfilled. We provide here the formula (5.17) when the crack Γ_c is described as a graph of a smooth function.

Let $\psi \in H^3(0, l_1)$ be a given function, $l_1 > 0$, and

$$\Sigma = \{(x_1, x_2) \mid x_2 = \psi(x_1), \quad 0 < x_1 < l_1\}.$$

Consider a crack Γ_l, $\Gamma_l \subset \Sigma$, as a graph of the function ψ, see Fig. 5.3

$$\Gamma_l = \{(x_1, x_2) \mid x_2 = \psi(x_1), \quad 0 < x_1 < l\}, \quad 0 < l < l_1.$$

Here l is a parameter that characterizes the length of the projection of the crack Γ_l onto x_1 axis. Consider a smooth cut-off function θ with a support in a vicinity of the crack tip $(l, \psi(l))$, moreover we assume that $\theta = 1$ in a small neighborhood of $(l, \psi(l))$. We can consider a perturbation of the crack Γ_l along Σ via a small parameter t. Denote $\Omega_l = \Omega \setminus \overline{\Gamma}_l$. Perturbed crack Γ_l^t has a tip $(l+t, \psi(l+t))$, and we consider a perturbed domain $\Omega_l^t = \Omega \setminus \overline{\Gamma}_l^t$. It is possible to establish a one-to-one correspondence between Ω_l and Ω_l^t by formulas

$$\begin{aligned} y_1 &= x_1 + t\theta(x), \\ y_2 &= x_2 + \psi(x_1 + t\theta(x)) - \psi(x_1), \end{aligned} \qquad (x_1, x_2) \in \Omega_l,\ (y_1, y_2) \in \Omega_l^t. \tag{5.19}$$

Transformation (5.19) is equivalent to the following (cf. (5.5))

$$y = x + tV(x) + r(t, x)$$

with the velocity field

$$V(x) = (\theta(x), \psi'(x_1)\theta(x)). \tag{5.20}$$

In the domain Ω_l^t, we can consider a perturbed problem formulation. Namely, it is necessary to find a displacement field $u^t = (u_1^t, u_2^t)$ and the stress tensor components $\sigma^t = \{\sigma_{ij}^t\}$, $i, j = 1, 2$, such that

$$-\mathrm{div}\sigma^t = f \quad \text{in} \quad \Omega_l^t, \tag{5.21}$$
$$\sigma^t = A\varepsilon(u^t) \quad \text{in} \quad \Omega_l^t, \tag{5.22}$$
$$u^t = 0 \quad \text{on} \quad \Gamma, \tag{5.23}$$
$$[u^t]\nu^t \geq 0,\ [\sigma_{\nu^t}^t] = 0,\ \sigma_{\nu^t}^t \cdot [u^t]\nu^t = 0 \quad \text{on} \quad \Gamma_l^t, \tag{5.24}$$
$$\sigma_{\nu^t}^t \leq 0,\ \sigma_{\tau^t}^t = 0 \quad \text{on} \quad \Gamma_l^{t\pm}. \tag{5.25}$$

Here $\nu^t = (\nu_1^t, \nu_2^t)$ is a unit normal vector to Γ_l^t. For a solution u^t of (5.21)-(5.25) it is possible to define the energy functional

$$\Pi(\Omega_l^t; u^t) = \frac{1}{2} \int\limits_{\Omega_l^t} \sigma_{ij}^t(u^t)\varepsilon_{ij}(u^t) - \int\limits_{\Omega_l^t} f_i u_i^t$$

and to find the derivative

$$\Pi'(l) = \frac{d\Pi(\Omega_l^t; u^t)}{dt}\Big|_{t=0}$$

with the formula (see [24])

$$\begin{aligned} \Pi'(l) \quad &= \quad \frac{1}{2} \int\limits_{\Omega_l} \{\mathrm{div}V \cdot \varepsilon_{ij}(u) - 2E_{ij}(V; u)\}\sigma_{ij}(u) \\ &- \int\limits_{\Omega_l} \mathrm{div}(Vf_i)u_i + \int\limits_{\Omega_l} \sigma_{ij}(u)\varepsilon_{ij}(w) - \int\limits_{\Omega_l} f_i w_i, \end{aligned} \tag{5.26}$$

where the vector field V is defined in (5.20) and $w = (0, \theta\psi'' u_1)$ is a given function. Note that the formula (5.26) contains the function θ, but in fact there is no dependence of the right-hand side of (5.26) on θ. In particular, if $\psi'' = 0$, the formula (5.26)

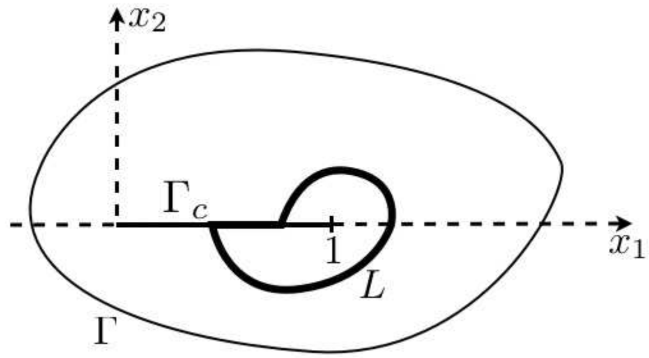

FIG. 5.4. *Curve L surrounding a crack tip.*

reduces to (5.18) with $\Omega_c = \Omega_l$. In this case we have a rectilinear crack and $\nu^t = \nu$. Formula (5.26) defines a derivative of the energy functional with respect to the length of the projection of the crack Γ_l onto the x_1 axis. Hence, the derivative of the energy functional with repsect to the length of the curvilinear crack is as follows

$$\Pi'(s) = \Pi'(l)(\psi'(l)^2 + 1)^{-1/2},$$

where

$$s = \int_0^l \sqrt{\psi'(t)^2 + 1}$$

is the length of the crack Γ_l.

To conclude this section we shortly discuss an existence of so-called invariant integrals in the crack theory analyzed. It is turned out that the formula (5.18) for the derivative of the energy functional can be rewritten as an integral over closed curve surrounding the crack tip.

Consider the most simple case of a rectilinear crack $\Gamma_c = (0,1) \times \{0\}$ assuming that $\overline{\Gamma}_c \subset \Omega$, see Fig. 5.4. Let θ be a smooth cut-off function equal to 1 near the point $(1,0)$, and *supp*θ belong to a small neighborhood of the point $(1,0)$. Then we can take the vector field

$$V = (\theta, 0)$$

in (5.1), (5.3) which, according to (5.5), corresponds to the following change of independent variables

$$\begin{aligned} y_1 &= x_1 + t\theta(x) + r_{11}(t), \\ y_2 &= x_2. \end{aligned}$$

In this case the formula (5.18) (or the formula (5.26) in a particular case $\psi = 0$) provides a derivative of the energy functional with respect to the crack length. This formula can be rewritten as an integral over curve L surrounding the crack tip $(1,0)$ (see Fig. 5.4 solid line). Namely, the following formula is valid (see [14], [16])

$$I = \int_L \{\frac{1}{2}\nu_1 \sigma_{ij}(u)\varepsilon_{ij}(u) - \sigma_{ij}(u)u_{i,1}\nu_j\} \tag{5.27}$$

provided that f is equal to zero in a neighborhood of the point $(1,0)$. We should underline two important points. First, the formula (5.27) is independent of L, and second, the right-hand side of (5.27) is equal to the derivative of the energy functional with respect to the crack length.

In fact, invariant integrals like (5.27) can be obtained in more complex situations. For example, we can assume that the crack Γ_c is situated on the interface between two media which means that the elasticity tensor $A = \{a_{ijkl}\}$ is as follows (see Fig. 5.4)

$$a_{ijkl} = \begin{cases} a^1_{ijkl} & \text{for} \quad x_2 > 0 \\ a^2_{ijkl} & \text{for} \quad x_2 < 0. \end{cases}$$

Here $a^1_{ijkl} = const$, $a^2_{ijkl} = const$, $i,j,k,l = 1,2$, and $\{a^1_{ijkl}\}$, $\{a^2_{ijkl}\}$ satysfy the usual properties of symmetry and positive definiteness. In this case, formula (5.18) for the derivative of the energy functional holds true provided that V is tangential to Γ_c. This formula provides an existence of invariant integral of the form (5.27). We should remark at this point that while the integral (5.27) is calculated, the values $\sigma_{ij}(u)u_{i,1}\nu_j$ can be taken at Γ_c^+ or at Γ_c^-. It gives the same value of the integral (5.27). This statement takes place due to the equality (see [8])

$$[\sigma_{ij}(u)u_{i,1}\nu_j] = 0 \text{ on } \Gamma_c.$$

On the other hand, we can analyze the case when a rigidity of the elastic body part $\Omega_c \cap \{x_2 < 0\}$ goes to infinity. Indeed, consider the following elasticity tensor for a positive parameter $\lambda > 0$,

$$a^\lambda_{ijkl} = \begin{cases} a^1_{ijkl} & \text{for} \quad x_2 > 0 \\ \lambda^{-1} a^2_{ijkl} & \text{for} \quad x_2 < 0. \end{cases}$$

Then for any fixed $\lambda > 0$, the solution of the equilibrium problem like (1.1)-(1.5) exists, and a passage to the limit as $\lambda \to 0$ can be fulfilled. As we already noted in Section 3, in the limit the following contact Signorini problem is obtained. Find a displacement field $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i,j = 1,2$, such that

$$-\text{div}\sigma = f \quad \text{in} \quad \Omega_c \cap \{x_2 > 0\}, \tag{5.28}$$

$$\sigma = A\varepsilon(u) \quad \text{in} \quad \Omega_c \cap \{x_2 > 0\}, \tag{5.29}$$

$$u = 0 \quad \text{on} \quad \partial(\Omega_c \cap \{x_2 > 0\}) \setminus \Gamma_c, \tag{5.30}$$

$$u\nu \geq 0, \ \sigma_\nu \leq 0, \ \sigma_\tau = 0, \ \sigma_\nu \cdot u\nu = 0 \quad \text{on} \quad \Gamma_c. \tag{5.31}$$

For the problem (5.28)-(5.31) it is possible to differentiate the energy functional in the direction of the vector field $V = (\theta, 0)$, where the properties of θ are described above. The formula for the derivative has the following form (cf. (5.18))

$$I = \frac{1}{2}\int\limits_{\Omega_1} \{\text{div}V \cdot \sigma_{ij}(u) - 2E_{ij}(V,u)\}\sigma_{ij}(u) - \int\limits_{\Omega_1} \text{div}(Vf_i)u_i. \tag{5.32}$$

Assume that $f = 0$ in a neighborhood of the point $(1,0)$. In this case, formula (5.32) can be rewritten in the form of invariant integral

$$I = \int\limits_{L_1} \{\frac{1}{2}\nu_1\sigma_{ij}(u)\varepsilon_{ij}(u) - \sigma_{ij}(u)u_{i,1}\nu_j\}, \tag{5.33}$$

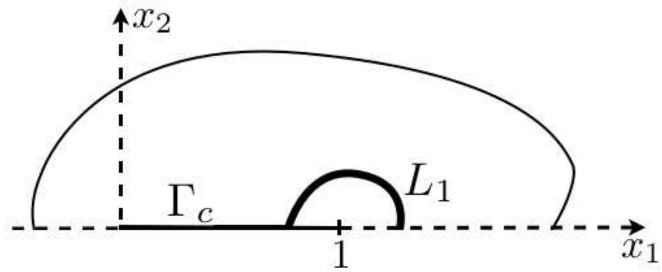

FIG. 5.5. *Curve L_1 "covering" a tip of contact set.*

where L_1 is a smooth curve "covering" the point $(1, 0)$ (see Fig. 5.5, solid line). Like for invariant integrals in the crack problems, formula (5.33) is independent of a choice of L_1.

6. Singular domain perturbations, topological derivatives. Boundary variations technique applied in section 5 to prove the shape differentiability of the elastic energy functional in domains with crack should be be complemented by the asymptotic analysis of the functional [21] in singularly perturbed domains. Such an analysis is particularly related to the topology optimization, and the knowledge of the so-called *topological derivative* of the shape functional provides the information [28] whenever a small hole can be created in the process of numerical solution of some shape optimization problem e.g., in the framework of the so-called *level set* method for variational inequalities [4].

In this section the topological derivative of the energy functional for the elasticity boundary value problems in domains with cracks is obtained. To this end the domain decomposition technique is used in the same way as it is proposed in [30] for the Signorini problem, and used in [4] for the purposes of numerical methods of shape optimization. Therefore, the results given here can be applied in numerical solution of shape optimization in domains with cracks.

We briefly explain, what we mean by the topological derivative of a shape functional. This notion of the topological derivative is new, the results are obtained in the framework of asymptotic analysis of elliptic boundary value problems in singularly perturbed geometrical domains in the spirit of [22], full mathematical framework for linear elasticity boundary value problems can be found in [21].

First, let us precise, what is the meaning of singularly perturbed geometrical domain for an elastic body with cracks. We introduce a small parameter $\rho > 0$ which describes the singular perturbations of the elastic body under considerations. We divide the elastic body $\mathcal{D}$ into two parts denoted by Ω_0 and Ω_c, respectively, and denote by $B_\rho(x)$ the hole which be located in Ω_0, the domain with the hole is denoted by $\Omega_\rho = \Omega_0 \setminus \overline{B_\rho(x)}$, the boundary Σ of Ω_0 is fixed and independent of the small parameter $\rho > 0$, see Fig. 6.1. It means that for $\rho > 0$ we consider the geometrical domain $\mathcal{D}_\rho = \Omega_\rho \cup \Sigma \cup \Omega_c$, the crack being located in Ω_c, and the hole $B_\rho(x)$ being located in Ω_ρ. The domain Ω_ρ with the boundary $\Sigma \cup \partial B_\rho(x)$ includes the hole $B_\rho(x)$, see Fig. 6.1. For the purposes of asymptotic analysis with respect to small parameter ρ, we assume that the domain Ω_ρ is located far from the outer boundary Γ, and far from the crack Γ_c. We assume also that in the domain Ω_ρ the elastic body is isotropic

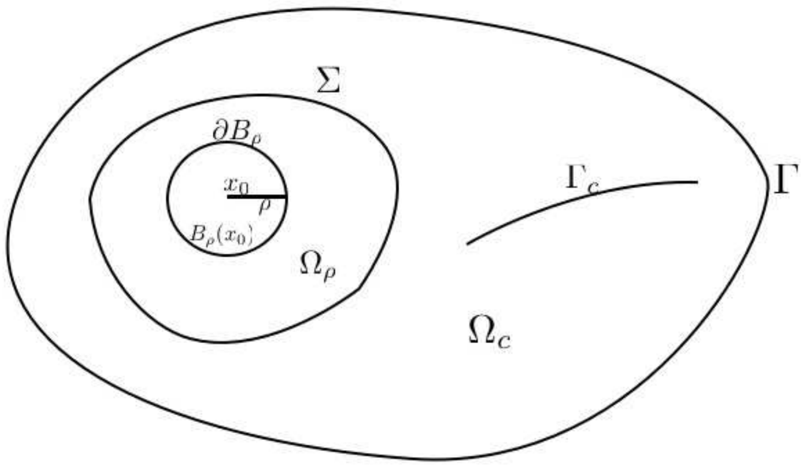

Fig. 6.1.

and homogeneous, so we can perform the asymptotic analysis of the Steklov-Poincaré operator associated to the domain Ω_ρ with respect to the small parameter $\rho \to 0$ along the lines of [30]. We refer to [31] for all details of such an analysis in the framework of exact solutions to elasticity boundary value problems by means of the elastic potentials [23]. For the convenience of the reader we recall here some facts on the topological derivatives of the shape functionals for linear elliptic boundary value problems, all proofs are given e.g., in [28].

The topological derivative $\mathcal{T}_\Omega$ of a shape functional $\mathcal{J}(\Omega)$ is introduced in [28] in order to characterize the variation of $\mathcal{J}(\Omega)$ with respect to the infinitesimal variation of the topology of the domain Ω. In our context the notion of the *topological* derivative (TD) has the following meaning. Assume that $\Omega \subset \mathbb{R}^2$ is an open set and that there is given a shape functional

$$\mathcal{J} \ : \ \Omega \setminus D \to \mathbb{R}$$

for any compact subset $D \subset \overline{\Omega}$. We denote by $B_\rho(x)$, $x \in \Omega$, the ball of radius $\rho > 0$, $B_\rho(x) = \{y \in \mathbb{R}^2 \mid \|y - x\| < \rho\}$, $\overline{B_\rho(x)}$ is the closure of $B_\rho(x)$, and assume that there exists the following limit

$$\mathfrak{T}(x) = \lim_{\rho \downarrow 0} \frac{\mathcal{J}(\Omega \setminus \overline{B_\rho(x)}) - \mathcal{J}(\Omega)}{|\overline{B_\rho(x)}|}.$$

The function $\mathfrak{T}(x)$, $x \in \Omega$, is called the topological derivative of $\mathcal{J}(\Omega)$, and provides the information on the infinitesimal variation of the shape functional $\mathcal{J}$ if a small hole is created at $x \in \Omega$. This definition is suitable for traction free boundary ∂B_ρ of the hole $B_\rho(x)$.

In several cases this characterization is constructive [5, 20, 6, 21, 29, 30, 31], i.e. TD can be evaluated for shape functionals depending on solutions of elliptic partial differential equations defined in the domain Ω.

6.1. Problem setting for elasticity systems. We introduce elasticity system in the form convenient for the evaluation of topological derivatives. Let us consider the elasticity boundary value problem for isotropic and homogeneous elastic body $\Omega_0 \subset \mathbb{R}^2$, with the boundary $\Gamma_D \cup \Gamma_N$,

$$\operatorname{div}\sigma(u) = 0 \quad \text{in} \quad \Omega_0, \tag{6.1}$$
$$u = g \quad \text{on} \quad \Gamma_D, \tag{6.2}$$
$$\sigma(u)n = T \quad \text{on} \quad \Gamma_N, \tag{6.3}$$

and the same elasticity boundary value problem in the domain $\Omega_\rho = \Omega_0 \setminus \overline{B_\rho(x_0)}$ with the spherical cavity $B_\rho(x_0) \subset \Omega_0$ centered at $x_0 \in \Omega_0$,

$$\operatorname{div}\sigma_\rho(u_\rho) = 0 \quad \text{in} \quad \Omega_\rho, \tag{6.4}$$
$$u_\rho = g \quad \text{on} \quad \Gamma_D, \tag{6.5}$$
$$\sigma_\rho(u_\rho)n = T \quad \text{on} \quad \Gamma_N, \tag{6.6}$$
$$\sigma_\rho(u_\rho)n = 0 \quad \text{on} \quad \partial B_\rho(x_0), \tag{6.7}$$

where n is the unit outward normal vector on $\partial\Omega_\rho = \partial\Omega_0 \cup \partial B_\rho(x_0)$. In addition, g, T must be compatible with $u \in H^1(\Omega_0)$. Assuming that $0 \in \Omega_0$, we can consider the case $x_0 = 0$. Here u and u_ρ denote the displacement vectors fields, g is a given displacement on the fixed part Γ_D of the boundary, T is a traction prescribed on the loaded part Γ_N of the boundary. In addition, σ is the Cauchy stress tensor given, for $\xi = u$ (6.1)-(6.3) or $\xi = u_\rho$ (6.4)-(6.7), by

$$\sigma(\xi) = A\varepsilon(\xi), \tag{6.8}$$

where $\varepsilon(\xi)$ is the strain tensor $\varepsilon(\xi) = \{\varepsilon_{ij}(\xi)\}$, $i, j = 1, 2$, and A is the elasticity tensor,

$$A = 2\mu \mathbb{I} + \lambda\,(I \otimes I)\,, \tag{6.9}$$

with

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{\nu E}{1-\nu^2} \tag{6.10}$$

being E the Young's modulus, ν the Poisson's ratio. In addition, I and $\mathbb{I}$ respectively are the second and fourth order identity tensors. Thus, the inverse of A is

$$A^{-1} = \frac{1}{2\mu}\left[\mathbb{I} - \frac{\lambda}{2\mu + N\lambda}\,(I \otimes I)\right].$$

The first shape functional under consideration depends on the displacement field, for our purposes it is sufficient to consider the linear form,

$$J_1(\rho) = \int_{\Omega_\rho} F u_\rho, \tag{6.11}$$

where F is a given function, in particular $F = f$, where f stands for the right-hand side in (1.1) is a possible choice. It is also useful to introduce the functional of the form

$$J_2(\rho) = \int_{\Omega_\rho} S\sigma(u_\rho) \cdot \sigma(u_\rho), \tag{6.12}$$

where S is an isotropic fourth-order tensor. Isotropicity means here, that S may be expressed as follows

$$S = 2m\mathbb{I} + l\,(I \otimes I)\,,$$

where l, m are real constants. Their values may vary for specific cases, in particular $S = A^{-1}$ can be selected for our purposes. The following assumption assures, that J_1, J_2 are well defined for solutions of the elasticity boundary value problems in Ω_0.

For simplicity the following notation is used for functional spaces,

$$H^1_g(\Omega_\rho) = \{v \in [H^1(\Omega_\rho)]^2 \mid v = g \text{ on } \Gamma_D\},$$

$$H^1_{\Gamma_D}(\Omega_\rho) = \{v \in [H^1(\Omega_\rho)]^2 \mid v = 0 \text{ on } \Gamma_D\},$$

$$H^1_{\Gamma_D}(\Omega_0) = \{v \in [H^1(\Omega_0)]^2 \mid v = 0 \text{ on } \Gamma_D\}\ .$$

The weak solutions to the elasticity systems are defined in the standard way. Find $u_\rho \in H^1_g(\Omega_\rho)$ such that, for every $\phi \in H^1_{\Gamma_D}(\Omega_\rho)$,

$$\int_{\Omega_\rho} A\varepsilon(u_\rho)\cdot\varepsilon(\phi) = \int_{\Gamma_N} T\cdot\phi. \tag{6.13}$$

The solution u_ρ for $\rho = 0$ is denoted by u.
We introduce the adjoint state equations in order to simplify the form of shape derivatives of functionals J_1, J_2. For the functional J_1 the equation takes on the form:
Find $w_\rho \in H^1_{\Gamma_D}(\Omega_\rho)$ such that, for every $\phi \in H^1_{\Gamma_D}(\Omega_\rho)$,

$$\int_{\Omega_\rho} A\varepsilon(w_\rho)\cdot\varepsilon(\phi) = -\int_{\Omega_\rho} F\cdot\phi, \tag{6.14}$$

whose Euler-Lagrange equation reads

$$\operatorname{div}\sigma_\rho(w_\rho) = F \quad \text{in} \quad \Omega_\rho, \tag{6.15}$$
$$w_\rho = 0 \quad \text{on} \quad \Gamma_D, \tag{6.16}$$
$$\sigma_\rho(w_\rho)n = 0 \quad \text{on} \quad \Gamma_N, \tag{6.17}$$
$$\sigma_\rho(w_\rho)n = 0 \quad \text{on} \quad \partial B_\rho(x_0), \tag{6.18}$$

while $v_\rho \in H^1_{\Gamma_D}(\Omega_\rho)$ is the adjoint state for J_2 and satisfies for all test functions $\phi \in H^1_{\Gamma_D}(\Omega)$ the following integral identity:

$$\int_{\Omega_\rho} A\varepsilon(v_\rho)\cdot\varepsilon(\phi) = -2\int_{\Omega_\rho} AS\sigma(u_\rho)\cdot\varepsilon(\phi)\ , \tag{6.19}$$

which associated Euler-Lagrange equation becomes

$$\operatorname{div}\sigma_\rho(v_\rho) = -2\operatorname{div}\left(AS\sigma_\rho(u_\rho)\right) \quad \text{in} \quad \Omega_\rho, \tag{6.20}$$
$$v_\rho = 0 \quad \text{on} \quad \Gamma_D, \tag{6.21}$$
$$\sigma_\rho(v_\rho)n = -2AS\sigma_\rho(u_\rho)n \quad \text{on} \quad \Gamma_N, \tag{6.22}$$
$$\sigma_\rho(v_\rho)n = -2AS\sigma_\rho(u_\rho)n \quad \text{on} \quad \partial B_\rho(x_0).. \tag{6.23}$$

We denote the adjoint states for $\rho = 0$ by $w = w_0$, $v = v_0$, respectively.

Remark 1. *We observe that AS can be written as*

$$AS = 4\mu m \mathbb{I} + \gamma\,(I \otimes I)\,, \tag{6.24}$$

where

$$\gamma = 2\lambda l + 2\,(\lambda m + \mu l)\ . \tag{6.25}$$

Thus, when $\gamma = 0$, the boundary condition on $\partial B_\rho(x_0)$ in (6.20)-(6.23) becomes homogeneous and the tensor S must satisfy the constraint

$$\frac{m}{l} = -\left(\frac{\mu}{\lambda} + 1\right), \tag{6.26}$$

which is naturally satisfied for the energy shape functional, for instance. In fact, in this particular case, tensor S is given by

$$S = \frac{1}{2}A^{-1} \quad \Rightarrow \quad \gamma = 0 \quad \text{and} \quad 2m + l = \frac{1}{2E}, \tag{6.27}$$

which implies that the adjoint solution associated to J_2 can be explicitly obtained. Finally we describe the construction of the Steklov-Poincaré operator $A_\rho\ :\ H^{1/2}(\Sigma) \to H^{-1/2}(\Sigma)$ defined for the domain Ω_ρ in the following way.

Given the solution z_ρ to the boundary value problem

$$\operatorname{div}\sigma_\rho(z_\rho) = 0 \quad \text{in} \quad \Omega_\rho, \tag{6.28}$$
$$z_\rho = g \quad \text{on} \quad \Sigma, \tag{6.29}$$
$$\sigma_\rho(z_\rho)n = 0 \quad \text{on} \quad \partial B_\rho(x_0), \tag{6.30}$$

we define the traction on Σ as the value of the operator,

$$A_\rho(g) = \sigma_\rho(z_\rho)n.$$

6.2. Topological derivatives. The topological derivatives of shape functionals in elasticity in two spatial dimensions are obtained in [28]. In three spatial dimensions the results are less explicite, and can be found e.g., in [21], [6]. The principal stresses associated with the displacement field u are denoted by $\sigma_I(u)$, $\sigma_{II}(u)$, the trace of the stress tensor $\sigma(u)$ is denoted by $\operatorname{tr}\sigma(u) = \sigma_I(u) + \sigma_{II}(u)$.

Theorem 6.1. *The expressions for the topological derivatives of the functionals J_1, J_2 have the form*

$$\begin{aligned} \mathcal{T}J_1(x_0) &= -\left[F(u) + \frac{1}{E}\left(a_u a_w + 2b_u b_w \cos 2\delta\right)\right]_{x=x_0} \\ &= -\left[F(u) + \frac{1}{E}\left(4\sigma(u)\cdot\sigma(w) - \operatorname{tr}\sigma(u)\operatorname{tr}\sigma(w)\right)\right]_{x=x_0}, \end{aligned} \tag{6.31}$$

and

$$\begin{aligned} \mathcal{T}J_2(x_0) &= -\left[(\alpha+\beta)a_u^2 + 2(\alpha-\beta)b_u^2 + \frac{1}{E}\left(a_u a_v + 2b_u b_v \cos 2\delta\right)\right]_{x=x_0} \\ &= -\Big[4(\alpha-\beta)\sigma(u)\cdot\sigma(u) - (\alpha-3\beta)(\operatorname{tr}\sigma(u))^2 \\ &+ \frac{1}{E}\left(4\sigma(u)\cdot\sigma(v) - \operatorname{tr}\sigma(u)\operatorname{tr}\sigma(v)\right)\Big]_{x=x_0}. \end{aligned} \tag{6.32}$$

Some of the terms in (6.31), (6.32) require explanation.

According to (6.25) constants α and β are given by

$$\alpha = l + 2\left(m + \gamma\frac{\nu}{E}\right) \qquad \text{and} \qquad \beta = 2\frac{\gamma}{E}. \tag{6.33}$$

Furthermore, we denote

$$\begin{array}{ll} a_u = \sigma_I(u) + \sigma_{II}(u), & b_u = \sigma_I(u) - \sigma_{II}(u), \\ a_w = \sigma_I(w) + \sigma_{II}(w), & b_w = \sigma_I(w) - \sigma_{II}(w), \\ a_v = \sigma_I(v) + \sigma_{II}(v), & b_v = \sigma_I(v) - \sigma_{II}(v). \end{array} \tag{6.34}$$

Finally, the angle δ denotes the angle between principal stress directions for displacement fields u and w in (6.31), and for displacement fields u and v in (6.32).

Remark 2. *For the energy stored in a 2D elastic body, tensor S is given by (6.27), $\gamma = 0$, $\alpha = 1/(2E)$ and $\beta = 0$. Thus, we obtain the following well-known result*

$$\mathcal{T}J_2(x_0) = \frac{1}{2E}\left[4\sigma(u)\cdot\sigma(u) - (\mathrm{tr}\sigma(u))^2\right]_{x=x_0} \tag{6.35}$$

which we use below for derivation of the topological derivatives of the energy functional for the domains with cracks.

Now we consider the domain $\mathcal{D}_\rho = \Omega_\rho \cup \Sigma \cup \Omega_c$, see Fig. 6.1. The convex set K is defined by the same formula (2.1), with the only difference that in the present stuation the boundary $\partial\Omega_c = \Sigma \cup \Gamma \cup \Gamma_c^\pm$, and there is no condition prescribed on Σ, hence

$$K = \{v \in H^1_\Gamma(\Omega_c) \mid [v]\nu \geq 0 \quad \text{a.e. on} \quad \Gamma_c\}.$$

The energy in $\mathcal{D}_\rho$ is given by the functional depending on the size of the cavity

$$\begin{aligned} j(\rho) &= \min_{v\in K}\left\{\frac{1}{2}\int_{\mathcal{D}_\rho}\sigma_{ij}(v)\varepsilon_{ij}(v) - \int_{\mathcal{D}_\rho} f_i v_i\right\} \\ &= \min_{v\in K}\left\{\frac{1}{2}\int_{\Omega_c}\sigma_{ij}(v)\varepsilon_{ij}(v) - \int_{\Omega_c} f_i v_i + \langle A_\rho(v), v\rangle_{1/2,\Sigma}\right\}, \end{aligned}$$

where the expression for the energy in the domain Ω_ρ with the hole $B_\rho(x_0)$ uses the Steklov-Poincaré operator of the specific annulus domain Ω_ρ, we refer the reader to [31] for the derivation of asymptotics of arbitrary order for the operator. Thus, the argument of [30] apply, and in view of (6.35) we have

$$j''(0^+) = -\frac{1}{4E}\left[4\sigma(u)\cdot\sigma(u) - (\mathrm{tr}\sigma(u))^2\right]_{x=x_0}$$

which gives the expression for the topological derivative of the energy functional at the point x_0.

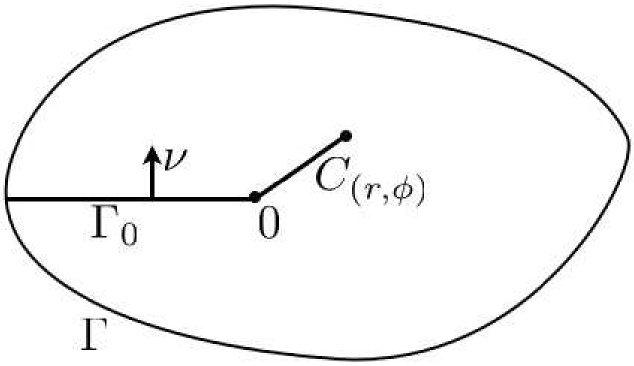

FIG. 7.1. *Kinking crack.*

7. Evolution of a kinking crack. The problem of kink is of special interest, because it represents a change of topology from smooth crack to the non-smooth one. The topology change is the main difficulty of mathematical analysis of cracks with a kink. In this section we apply the shape optimization approach to a two-parameter problem for kinking crack. Namely, we fix a point of kink and find unknown shape parameters of the kink angle and the crack length, which minimize the total potential energy due to the Griffith approach. This nonlinear minimization problem describes evolution of the kinking crack with respect to time-like loading parameter. In the linear crack theory, the optimization Griffith approach was used in [2].

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ. Assuming that the origin $\mathcal{O} = (0,0)$ belongs to $\overline{\Omega}$, we consider a given crack $\Gamma_0 \subset \Omega$ with tips at Γ and at the origin, and unknown part $C_{(r,\phi)}$ of the crack, which tip is described in polar coordinates as

$$(r\cos\phi, r\sin\phi), \quad (r,\phi) \in \overline{\omega},$$

where ω is the set of admissible parameters

$$\omega = \{(r,\phi) \mid 0 < r < R(\phi) \text{ for } \phi \in (\phi_0,\phi_1)\}, \quad [\phi_0,\phi_1] \subset (-\pi,\pi),$$

with a given periodic function $R \in W^{2,\infty}(-\pi,\pi)$.

Admissible kinking cracks are defined as a union $\Gamma_{(r,\phi)} = \Gamma_0 \cup C_{(r,\phi)}$. Denote by $\Omega_{(r,\phi)}$ a domain with a crack $\Gamma_{(r,\phi)}$, i.e. $\Omega_{(r,\phi)} = \Omega \setminus \overline{\Gamma}_{(r,\phi)}$, see Fig. 7.1. In the domain $\Omega_{(r,\phi)}$ we can consider an equilibrium problem like (1.1)-(1.5). Namely, let ν be a normal vector to $\Gamma_{(r,\phi)}$ and $f = (f_1, f_2) \in C^1(\overline{\Omega})$ be a given function. Problem formulation is as follows. In the domain $\Omega_{(r,\phi)}$ we have to find a displacement vector $u = (u_1, u_2)$ and stress tensor components $\sigma = \{\sigma_{ij}\}$, $i,j = 1,2$, such that

$$-\text{div}\sigma = f \quad \text{in} \quad \Omega_{(r,\phi)}, \tag{7.1}$$

$$\sigma = A\varepsilon(u) \quad \text{in} \quad \Omega_{(r,\phi)}, \tag{7.2}$$

$$u = 0 \quad \text{on} \quad \Gamma, \tag{7.3}$$

$$[u]\nu \geq 0,\ [\sigma_\nu] = 0,\ \sigma_\nu \cdot [u]\nu = 0 \quad \text{on} \quad \Gamma_{(r,\phi)}, \tag{7.4}$$

$$\sigma_\nu \leq 0,\ \sigma_\tau = 0 \quad \text{on} \quad \Gamma^{\pm}_{(r,\phi)}. \tag{7.5}$$

For any given $(r,\phi) \in \overline{\omega}$, solution of the problem (7.1)-(7.5) exists in the Sobolev space $H^1_\Gamma(\Omega_{(r,\phi)})$. Hence, for any $(r,\phi) \in \overline{\omega}$ we can define a solution $u^{(r,\phi)}$ and the energy functional

$$\Pi(\Omega_{(r,\phi)}; u^{(r,\phi)}) = \frac{1}{2} \int\limits_{\Omega_{(r,\phi)}} \sigma_{ij}(u^{(r,\phi)})\varepsilon_{ij}(u^{(r,\phi)}) - \int\limits_{\Omega_{(r,\phi)}} f_i u_i^{(r,\phi)},$$

where $\sigma_{ij}(u^{(r,\phi)}) = \sigma_{ij}$ are found from (7.2). Thus, a differentiability of the energy functional with respect to (r,ϕ) can be analyzed. These results can be found in [10]. The main difficulty in study of differentiability is the following. Considering perturbations of the problem (7.1)-(7.5), we have no a one-to-one correspondence between sets of admissible displacements for perturbed and unperturbed problems. This requires additional considerations to prove a differentiability of $\Pi(\Omega_{(r,\phi)}; u^{(r,\phi)})$ with respect to r, ϕ.

In what follows, we formulate an evolution problem for a kinking crack. Denote

$$P(r,\phi) = \Pi(\Omega_{(r,\phi)}; u^{(r,\phi)}).$$

For a time-like loading parameter $t \geq 0$ we consider a family of forces tf in (7.1). Let the length of the crack Γ_0 be equal to $l_0 \geq 0$. Note that if the solution $u^{(r,\phi)}$ corresponds to the force f in (7.1), we obtain a solution $tu^{(r,\phi)}$ for the force tf due to a homogeneity property for the problem (7.1)-(7.5). Let the initial crack (at $t = 0$) be given as Γ_0. For the loading tf, we look for a propagating crack $\Gamma_{(r(t),\phi^*)} \subset \Omega$ with the kink at the origin $\mathcal{O}$ and unknown shape parameters of the crack length $l_0 + r(t)$ and the kink angle $\phi^* \in [\phi_0, \phi_1]$. To this end, we use a shape optimization approach, which is based on the Griffith hypothesis. Following this hypothesis, we define a function of total potential energy

$$T(r,\phi)(t) = 2\gamma(l_0 + r) + t^2 P(r,\phi), \quad (r,\phi) \in \overline{\omega}. \tag{7.6}$$

The first term in (7.6) represents the surface energy distributed uniformly at two crack faces with a constant density $\gamma > 0$ (the given material parameter). The second term in (7.6) represents the potential energy which is quadratic in t,

$$P(r,\phi)(t) = \Pi(\Omega_{(r,\phi)}; tu^{(r,\phi)}) = t^2 P(r,\phi).$$

Thus we arrive at the problem formulation of the evolution of kinking crack:

$$r(0) = 0; \tag{7.7}$$

for $t > 0$, find parameters $(r(t), \phi(t)) \in \overline{\omega}$ that

$$\text{minimize} \quad T(r,\phi)(t) \quad \text{over} \quad (r,\phi) \in \overline{\omega}, \tag{7.8}$$

$$\text{subject to} \quad \phi \in \bigcap_{s<t} \{\phi(s)\}. \tag{7.9}$$

The constraint (7.9) allows us to preserve the shape of kinking crack during its evolution. This means that if kinking angle ϕ^* is found, its value is preserved during the evolution. Problem (7.7)-(7.9) has a solution (see [10]). It is turned out that the radius $r(t)$ during the evolution may be multi-value, i.e. $r(t) \in [r^-(t), r^+(t)]$, which means a nonstable crack evolution.

8. 3D problems and open questions. The most problems discussed in the paper can be solved in 3D case when a crack is presented as 2D smooth surface. For example, the crack can be described as

$$x_i = x_i(y_1, y_2), \quad i = 1, 2, 3,$$

where $(y_1, y_2) \in D$, $D \subset \mathbb{R}^2$ is a bounded domain with smooth boundary, and the mapping $y \to x$ is non-degenerating.

All formulas and statements of Sections 1-5 hold true with suitable specifications of the situation. In particular, by discussing a fulfillment of the boundary conditions (1.4)-(1.5) we should introduce the Hilbert space $H^{\frac{1}{2}}(\Sigma)$, where Σ is an extension of Γ_c to a closed 2D smooth surface. The norm in $H^{\frac{1}{2}}(\Sigma)$ in this case is defined as follows (cf. (2.4))

$$\|v\|^2_{H^{\frac{1}{2}}(\Sigma)} = \|v\|^2_{L^2(\Sigma)} + \int\limits_{\Sigma}\int\limits_{\Sigma} \frac{|v(x) - v(y)|^2}{|x-y|^3} dxdy.$$

Mixed and smooth domain formulations in 3D case hold true as well as the fictitious domain method.

Also, we can consider a crack located on the boundary of a rigid inclusion for a 3D elastic body and prove all statements of Section 5. Notice that in 3D case the space of infinitesimal rigid inclusions is defined as follows

$$R(\omega) = \{\rho = (\rho_1, \rho_2, \rho_3) \mid \rho(x) = Bx + D, \quad x \in \omega\},$$

where

$$B = \begin{pmatrix} 0 & b_{12} & b_{13} \\ -b_{21} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{pmatrix}, \quad D = (d^1,\ d^2,\ d^3),$$

$$b_{ij}, d^i = const,\ i, j = 1, 2, 3.$$

As for the differentiation of energy functionals with respect to a perturbed parameter (Section 5), we have a big variety of perturbations in 3D case. The most simple ones provide a perturbation of the crack front. For example, let Γ_c be chosen in the form

$$\Gamma_c = \{(x_1, x_2, 0) \mid 0 \le x_1 \le \phi(x_2),\ x_2 \in [-1, 1],\ \phi(x_2) > 0\}$$

with a given smooth function ϕ. In this case, the 3D vector field can be taken as follows

$$V(x) = (\theta(x), 0, 0),$$

where θ is a given smooth function with a support in a vicinity of the crack front

$$\{(x_1, x_2, x_3) \mid x_1 = \phi(x_2),\ x_3 = 0,\ x_2 \in [-1, 1]\}.$$

This allows us to differentiate the energy functional in the direction of the field V which implies the formula (5.17) with $i, j = 1, 2, 3$; see [11], [16].

Like in Section 5, in 3D case we can consider curvilinear cracks described as a graph of a function

$$x_3 = \psi(x_1, x_2),\ (x_1, x_2) \in D,$$

where $D \subset \mathbb{R}^2$ is a bounded domain with a smooth boundary. The needed formulas for derivatives of the energy functional in this case can be found in [25].

As for invariant integrals, in 3D case we should integrate over closed 2D surfaces surrounding a crack front, see [8], [14].

To conclude the paper, we formulate some open questions.

- For a crack Γ_c which crosses the external boundary Γ with a zero angle, there is no solvability of problem (1.1)-(1.5) in the general case since Korn's inequality is non valid. Is it possible to overcome this difficulty?
- There is the uniqueness of solutions to problem (7.7)-(7.9) ?
- Find the form of the shape derivative for the energy functional with respect to the perturbations of the crack tip in the case of the crack at the interface beween an elastic body and a rigid inclusion.

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