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Abstract: This paper addresses the design of a controller family that, given a continuous-time linear plant sampled at a varying rate, asymptotically stabilizes the closed loop. Under the assumption of having a finite and known set of allowable sampling intervals, the problem is formulated as that of stabilizing a discrete-time switched system (DTSS). The solution approach consists in choosing the controller parameters in order for the Lie algebra generated by the closed-loop DTSS-matrices to be solvable, as this property guarantees the existence of a common Lyapunov function for the control system. The results can be applied to design digital controllers sharing resources, as it is the case of networked control systems, where the need of adapting the rate of task scheduling may originate significant sampling time variations.

1. INTRODUCTION

Point to point communication and dedicated central processing unit (CPU) traditionally used in control systems implementations are being replaced in many applications by distributed communication (via a common network) and/or shared CPUs (at each node or CPU there are several tasks related with the control of different systems).

The use of common shared resources reduces costs and provides better resource's throughput, benefits that are frequently accompanied by noticeable increments in the amount of information exchanged between the system nodes. As a result, the real-time deadlines of measurement and actuation tasks may be violated due to execution delays, which can lead to performance degradation or even instability of a control system.

In order to mitigate these problems, several measures to achieve an efficient use of available resources have been suggested, rate adaptation of task scheduling among them. For a shared CPU, a feedback scheduler that adjusts the sampling rates of the control loops to guarantee schedulability in different resource conditions was used in (Eker et al., 2000) and (Cervin et al, 2002). In a context of networked control systems, (Antunes et al, 2007), (Velazco et al., 2004), and (Felicioni and Junco, 2007), present techniques to adapt the communication rate of the distributed controllers to the available network bandwidth.

Rate adaptation introduces the need to design control laws adapted to these variations and aimed at reducing the performance degradation induced by a varying sampling rate if invariant (non rate-adaptive) controllers were used and, simultaneously, ensuring closed-loop stability.

In order to adapt control laws to rate variations, an optimal design of *state-feedback* linear quadratic controllers (LQ) was proposed in (Cervin et al., 2002). But, as shown in (Schinkel et al., 2002), control systems designed with this optimal-LQ technique, may suffer from instability under certain switching sequences. Due to this undesirable result, (Schinkel et al., 2002) adopts a linear matrix inequalities (LMI) approach to design stable optimal controllers. However, their numerical solutions can be very conservative (see section 4.2).

As a given control system undergoing sampling rate variation can be thought of as a concatenation of systems in time, it can be modeled as a Discrete-Time Switched System (DTSS). This is the mathematical framework exploited in this paper for the control system design problem.

A review of available results to study asymptotic stability of switched systems was presented in (Liberzon and Morse, 1999). In particular, it was established that if a family of systems that constitutes a switched system has a Common Lyapunov Function (CLF), then, asymptotic stability is guaranteed for any switching sequence. Furthermore, global uniform (for all switching signals) exponential stability was proved for a continuous-time switched system described by a compact family of linear systems, under the condition that each matrix in the family is stable and the Lie algebra associated with the family is solvable. This result was demonstrated connecting the Lie algebra solvability with the existence of a Quadratic CLF. A discrete-time counterpart of this result was presented in (Blondel et al., 2004) and (Theys, 2005).

Here, we propose to use Blondel-Theys's result to solve the following problem: given a linear plant sampled at varying rate, design a controller family that asymptotically stabilizes the closed-loop for a given sampling policy.

2. PROBLEM FORMULATION AND BACKGROUND

This section first formalizes the description of a control system with varying sampling rate as a DTSS. It proceeds with the closer examination of related results and mathematical tools used to formulate and solve the problem.

2.1 Switched Linear Systems

Definition (Hespanha and Morse,1999) A *Continuous-Time Switched Linear System* is defined by a parameterised family of realizations P = {(A_i, B_i, C_i, D_i): i ∈ $\underline{p}$={1, 2, …, p}}, together with a family of piecewise constant switching signals S:={σ:[0,∞)→ $\underline{p}$ }, and the (input-state-output) dynamics (1).

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{aligned} \tag{1}$$

Analogously, a *Discrete-Time Switched Linear System* with the discrete time base {t_k : $k \in \mathbb{N}_0$} has the following form:

$$\begin{aligned} x_{k+1} &= A_{\sigma(k)}x_k + B_{\sigma(k)}u_k \\ y_k &= C_{\sigma(k)}x_k + D_{\sigma(k)}u_k \end{aligned} \tag{2}$$

With $u_k = u(t_k)$, $x_k = x(t_k)$, and $y_k = y(t_k)$.

Due to the switching of the matrices inside the family P, both systems, (1) and (2), are time-varying.

2.2 Varying-rate Sampled-data Control Systems

The discretization with a sampling time $h_i = t_{k+1} - t_k$ of the *open-loop continuous-time plant* described by the linear model (3) yields the *discrete-time linear system* (4), whose matrices are given in (5), if a zero-order hold is considered.

$$\begin{aligned} \dot{x} &= A\,x + B\,u \\ y &= C\,x + D\,u \end{aligned} \tag{3}$$

$$\begin{aligned} x_{k+1} &= \Phi(h_i)x_k + \Gamma(h_i)u_k \\ y_k &= C_d(h_i)x_k + D_d(h_i)u_k \end{aligned} \tag{4}$$

$$\begin{aligned} \Phi(h_i) &= e^{Ah_i} & C_d(h_i) &= C \\ \Gamma(h_i) &= \int_0^{h_i} e^{As}B\,ds & D_d(h_i) &= D \end{aligned} \tag{5}$$

For varying sampling time h_i, system (4) can be modeled as (2), with σ(k) = i ∈ $\underline{p}$, which yields the equivalence of matrices (6).

$$\begin{aligned} A_i &= \Phi(h_i) \\ B_i &= \Gamma(h_i) \\ C_i &= C(h_i) \\ D_i &= D(h_i) \end{aligned} \tag{6}$$

A *Closed-Loop Discrete-Time System* results from the interconnection of the sampled plant with the linear discrete-time dynamic controller (7). Note that the controller parameters adapt themselves to the sampling period.

$$\begin{aligned} v_{k+1} &= A_{C,i}\,v_k + B_{C,i}\,C_d(h_i)\,x_k \\ u_k &= C_{C,i}\,v_k + D_{C,i}\,C_d(h_i)\,x_k \end{aligned} \tag{7}$$

Defining the augmented state $(x_k \;\; v_k)^T$, then the closed-loop dynamics is described by (8)

$$\begin{aligned} \begin{pmatrix} x_{k+1} \\ \nu_{k+1} \end{pmatrix} &= \begin{pmatrix} \Phi(h_i) + \Gamma(h_i)\cdot D_{C,i}\cdot C(h_i) & \Gamma(h_i)\cdot C_{C,i} \\ B_{C,i}\cdot C(h_i) & A_{C,i} \end{pmatrix} \cdot \begin{pmatrix} x_k \\ \nu_k \end{pmatrix} \\ &= \Phi_i^{CL} \cdot \begin{pmatrix} x_k \\ \nu_k \end{pmatrix} \end{aligned} \tag{8}$$

Particularly, for a static state feedback controller $u_k = L_i\,x_k$, the closed-loop matrix of equation (8) reduces to

$$\Phi_i^{CL} = \Phi(h_i) + \Gamma(h_i)L_i = \Phi_i + \Gamma_i L_i \tag{9}$$

2.3 Asymptotic Stability in Closed-loop.

The problem to solve in this paper is the following:

Given the family describing the open-loop systems (4), associated with a finite number of sampling periods {h_i, i ∈ $\underline{p}$ } , find a family of controllers ($A_{C,i}$, $B_{C,i}$, $C_{C,i}$, $D_{C,i}$; i ∈ $\underline{p}$) such that the DTSS closed-loop system (9) is asymptotically stable.

Variations of this controller design problem have been approached in (Sala, 2005) and (Schinkel et al., 2002) adopting a LMI framework to synthesize optimal controllers that guarantee closed-loop stability under any sampling sequence. More specifically, Sala solves the problem with a unique common controller, while Schinkel obtains a family of controllers associated to the sampling rates.

The solution provided by this work consists of a family of stabilizing controllers with the property that the resulting closed-loop descriptions have a Quadratic CLF (QCLF). It is assumed that the switching signal (which generates the varying parameter h_i in our system (8)) is determined by a scheduler policy yielding a finite number of known sampling periods.

As reported in (Liberzon and Morse, 1999), the existence of a QCLF for a family of (autonomous) systems guarantees asymptotic stability for arbitrary switching signals. Blondel and Theys, approaching DTSS in a similar way as (Liberzon and Morse, 1999) did for continuous-time systems, proved that if the Lie algebra associated to a family of stable matrices is solvable, then, there exists a QCLF. This result is quoted in the theorem below.

Theorem [Blondel] [Theys]. Let P be the family of matrices {M_1, $M_2$$\cdots$, M_p} and consider the switched linear system $x_{k+1} = M_i\,x_k$, M_i ∈ P. If all matrices in P have spectral radius ρ(M_i) less than 1 (largest modulus of their eigenvalues) and the Lie algebra associated to P is solvable, then the system has a QCLF.

An equivalent result was first established in (Gurvits, 1995) for the particular case where each M_i in the family takes the form e^{Ai}.

In this paper, the matrices M_i are the closed loop matrices Φ_i^{CL}, i.e., the approach to solve the stability problem consists in imposing the conditions of the precedent theorem to the family of matrices $P=\left\{\Phi_i^{CL}; i\in \underline{p}\right\}$ (8), in order to design the family of controllers ($A_{C,i}$, $B_{C,i}$, $C_{C,i}$, $D_{C,i}$; $i \in \underline{p}$) looked for.

2.4 Mathematical Background

The *matrix Lie algebra* g:={M_i : $i \in \underline{p}$} generated by the family of square matrices P ={M_1, $M_2$$\cdots$, M_p} wrt the standard Lie bracket $[M_\nu, M_\gamma] = M_\nu M_\gamma - M_\gamma M_\nu$, is the linear space spanned by iterated Lie brackets of matrices of the family (Liberzon et al.,1999), (Theys, 2005).

For the Lie algebra g, the sequence $g^{(m)}$ is defined as $g^{(0)}$:=g, $g^{(m+1)}:=[g^{(m)}, g^{(m)}]\subset g^{(m)}$.

If $g^{(m)}$ =0 for some m, then g is said to be *solvable*.

Given the Lie algebra g, the sequence g^m is defined as g^0:=g , $g^{m+1}:=[g, g^m]\subset g^m$.

If g^m =0 for some m, then g is called *nilpotent*.

The nilpotency of a Lie algebra is a sufficient condition for its solvability.

Also, the Lie algebra g:={M_i: $i \in \underline{p}$} is solvable *if and only* if the Lie algebra $\tilde{g}$:= {$\tilde{M}_i$: $i \in \underline{p}$} is solvable, where $\tilde{M}_i = R^{-1}M_i R$, with R the matrix of a similarity transformation.

3. CONTROLLER DESIGN

As already pointed-out, we propose here to choose the controller parameters in order for the Lie algebra generated by the closed-loop DTSS-matrices (8) to be solvable. More explicitly, as the closed-loop model (8) is associated to each sampling time, a family of closed-loop matrices is recovered, each of them being parameterized by the corresponding controller set of parameters, which have to be determined in such a way that the sequence $g^{(m)}$ generated by the closed-loop family is forced to be zero for some m.

Three cases are presented in the sequel, and the conditions to obtain solvable algebras associated with a family of DTSS matrices are given. These methods are illustrated by some examples in the next section.

Case 1: If $g^{(1)}$=0, the algebra is solvable, the matrices are pairwise commutative (*abelian* case), i.e. all the brackets between matrices in the set P={$\Phi_1^{CL},\Phi_2^{CL},\cdots,\Phi_P^{CL}$} are zero. This is an application of a particular case of Blondel's Theorem, which was first proved in (Kumpati et al., 1994).

For each pair of matrices in the family, this condition requires that $\left[\Phi_1^{CL},\Phi_2^{CL}\right]=\Phi_1^{CL}\,\Phi_2^{CL}-\Phi_2^{CL}\,\Phi_1^{CL}=0$. To do that, the controller parameters must to satisfy the following design matrix equation

$$\Phi_1^{CL}\,\Phi_2^{CL}=\Phi_2^{CL}\,\Phi_1^{CL} \tag{10}$$

For the controller structure in (9), due to $\Phi_1\,\Phi_2=e^{Ah_1}e^{Ah_2}=\Phi_2\,\Phi_1$, (10) reduces to

$$\Phi_1\,\Gamma_2L_2+\Gamma_1L_1\,\Gamma_2L_2=\Phi_2\,\Gamma_1L_1+\Gamma_2L_2\,\Gamma_1L_1 \tag{11}$$

For a closed-loop matrix of order n x n the matrix equation (11) implies n^2 equations to satisfy. The possibility of satisfying these constraints depends on the number of open-loop inputs and outputs and the chosen controller structure.

Case 2: The algebra generated by a family of upper (or lower) triangular matrices is solvable, with m=n+1. As (8) typically does not have a triangular structure, this proposition requires to put (10), in a triangular form by using a similarity transformation *common* to all.

From digital control theory, the exponential map of the open-loop system $A=R\,T\,R^{-1}$ is

$$e^{At}=e^{RTR^{-1}t}=R\,e^{T\,t}R^{-1} \tag{12}$$

where R is a similarity transformation. If it is a Jordan decomposition, T has an upper (or lower) triangular form (or a diagonal one).

Applying this transformation to the input part in (6)

$$\Gamma(t)=\int_0^t e^{Ax}Bds=\int_0^t R\,e^{Tx}R^{-1}Bds=R\left(\int_0^t e^{Tx}ds\right)R^{-1}B=R\,G_t\,R^{-1}B \tag{13}$$

Here, we consider a state-feedback controller, then by using (13) and (4), (10) becomes

$$x_{k+1}=R\left(e^{Tt}+G_t\left(R^{-1}B\,L_t\,R\right)\right)R^{-1}x_k \tag{14}$$

If $e^{Tt}+G_t\left(R^{-1}B\,L_t\,R\right)$ has an upper (or lower) triangular form for each sampling time t=h_i in the set {h_i, $i \in \underline{p}$}, then the algebra associated to closed-loop family results solvable.

As for a *Jordan* decomposition e^{Tt} has an upper (or lower) triangular form, we propose to design state-feedback controller L_t in such a way that each matrix $G_t\left(R^{-1}B\,L_t\,R\right)$ has an upper (or lower) triangular form.

Depending on the open-loop structure, the satisfaction of this condition requires to choose some controller parameters equal to zero. Thus, in several cases, some degrees of freedom for the election of gains of the state-feedback controllers are lost.

Case 3: Here, we analyse the solvability of the algebra generated by a pair of matrices of order 2.

In order to design the pair of controllers, we compute some brackets and derive two equations that must be satisfied to obtain a solvable algebra.

Consider closed-loop matrices M_1 and M_2, then the sum of diagonal elements of first bracket $[M_1,M_2]=M_3$ is zero, i.e. $M_{3(1,1)}=-M_{3(2,2)}$ (where $M_{i(k,j)}$ refers to the element of column k and row j from matrix M_i). From the computation of next brackets, $M_3=[M_1,M_2]$, $M_4=[M_1,M_3]$, $M_5=[M_2,M_3]$, $M_6=[M_3,M_4]$, $M_7=[M_3,M_5]$, $M_8=[M_6,M_7]$, and so on, it is possible to verify for all i in the set, the same diagonal elements condition, i.e. $M_{i(1,1)}=-M_{i(2,2)}$ and $M_{i(1,1)}=-M_{i(2,2)}$. The generated algebra for this pair is $g^{(0)}=\text{span}\{M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8,\ldots\}$, $g^{(1)}=\text{span}\{M_3, M_4, M_5, M_6, M_7, M_8,\ldots\}$, $g^{(2)}=\text{span}\{M_4, M_5, M_6, M_7, M_8,\ldots\}$, $g^{(3)}=\text{span}\{M_6, M_7, M_8,\ldots\}$, $g^{(4)}=\{M_8,\ldots\}$ and so on.

Consider a pair of indices k and j such that $k\neq j$, $r\leq\{k, j\}\leq r+2$, for some r, and the elements $M_{k(1,1)}$, $M_{k(1,2)}$, $M_{k(2,1)}$, $M_{k(2,2)}$, $M_{j(1,1)}$, $M_{j(1,2)}$, $M_{j(2,1)}$, $M_{j(2,2)}$ are non-zero. To obtain that the next common bracket $[M_k, M_j]$ gives a 0_{2*2} we must to satisfy both following equations:

$$M_{k(1,2)}=\frac{M_{k(1,1)}M_{j(1,2)}}{M_{j(1,1)}}\quad M_{k(2,1)}=\frac{M_{k(1,1)}M_{j(2,1)}}{M_{j(1,1)}} \tag{15}$$

The iterated bracket computation produces non-linear combinations of the controller parameters in the elements in equation (15) (typically they are linear in matrices M_1, M_2). Thus, the solution of these equations becomes more complicated when the value of m is increased. Clearly, non-linear methods must be required to solve this problem.

4. EXAMPLES

This section shows some examples to illustrate the methods mentioned in the last section to satisfy the Lie algebra conditions in each of the three cases.

Finding the solutions requires solving a set of non-linear equations. That can be done in symbolic or numerical forms. For example, in the first case by using the symbolic method *solve* from MAPLE, or in the latter case by using the numerical methods *fsolve* from MATLAB or MAPLE.

4.1 An Introductory Example

In the first order system taken from (Cervin et al., 2002), without noise, controlled by state-feedback, the closed-loop equation for a finite number of sampling periods $\{h_i,\ i\in \underline{p}\}$ is $\Phi_{k,i}^{CL}=e^{ah_i}+\frac{1}{a}\left(e^{ah_i}-1\right)l_i$, i.e. it is a scalar value. Therefore, the algebra generated by family $\{\Phi_1^{CL},\Phi_2^{CL},\cdots,\Phi_P^{CL}\}$ is always solvable ($g^{(1)}=0$). Furthermore, if $\left|e^{ah_i}+\frac{1}{a}\left(e^{ah_i}-1\right)l_i\right|<1$ for all i, both hypothesis of *Theorem* in subsection 2.3 are satisfied, implying the asymptotic stability under any sampling sequence.

The closed-loop for a second sampling time $h_2=\alpha\, h_1$, being α an integer, is $\Phi_2^{CL}=e^{a\alpha h_1}+\frac{1}{a}\left(e^{a\alpha h_1}-1\right)l_2$. Due to the simple structure of this example, it is possible to obtain that $\Phi_2^{CL}=\left(\Phi_1^{CL}\right)^{\alpha}$ in time $t=\alpha h_1+t_0$.

For example if α=2, the satisfaction of the mentioned equation requires to adopt $l_2=\frac{l_1\left(2\,a\,e^{a2h_1}-l_1+e^{ah_1}l_1\right)}{a\left(1+e^{a2h_1}\right)}$. With this adaptation of the control parameter, the plant reaches the same final state value in one step of sampling time h_2 or in two steps of sampling time h_1. Then, this election of l_2 allows us to compensate the augmented sampling interval with no degradation in terms of plant state response.

4.2 Numerical Example

Here, we solve a numerical problem as an example of *Case 3*. Due to the numerical distortion introduced by the discretization (5), it is necessary to consider the number precision as large as possible in order to obtain a good numerical solution.

For the second order plant taken from (Schinkel et al., 2002)

$$\dot{x}=\begin{bmatrix}0 & 1\\ -10000 & -0.1\end{bmatrix}x+\begin{bmatrix}0\\1\end{bmatrix}u$$

with two sampling times (P=2), h_1=0.002 and ε=15 (see note), the family of closed-loop matrices (9) is

$$\left\{\quad \Phi_1^{CL}=\begin{bmatrix}0.9801 & 0.0020\\ -19.8649+0.0020\,l_{1,1} & 0.9799+0.0020\,l_{1,2}\end{bmatrix}\right.,$$

$$\left.\Phi_\varepsilon^{CL}=\begin{bmatrix}-0.9884+0.0002\,l_{2,1} & 0.0014+0.0002\,l_{2,2}\\ -14.0909+0.0014\,l_{2,1} & -0.9886+0.0014\,l_{2,2}\end{bmatrix}\right\}$$

To obtain the controller parameters for this family, we first compute the seven first Lie brackets as exposed in *Case 3*. Assuming k=7 and j=6, we derive the non-linear equations (15) to obtain $M_8=[M_6,M_7]=0_{2*2}$. Then, the algebra g associated with this pair $\{\Phi_{Cl,1}, \Phi_{Cl,\varepsilon}\}$ is solvable being m=4.

The parameters that satisfy equations (15) were obtained by further calculations of *fsolve* command, initialized with a starting guess (parameters designed with LQR in (Schinkel et al., 2002), see also the *Remark* below).

$$L_1=[177.820193769818\ \ -19.683275734678]$$
$$L_\varepsilon=[140.481223090893\ \ -0.113548333882]$$

The spectral values for closed loop matrices are $\rho(\Phi_1^{CL})$= 0.97997188< 1 and $\rho(\Phi_\varepsilon^{CL})$= 0.984352978< 1 and $g^{(4)}=0$, then the hypothesis of *Theorem* in Section 2 are satisfied and it guarantees asymptotic stability for arbitrary switching.

The difference between the final value of L_1 and the starting one, $L_{1,0}$=[195.401 –19.4121] designed with LQR, is rather small compared with the difference between $L_{1,0}$ and $L_{1,f}$=[0.5784 –0.0507] obtained via *LMI* by Schinkel et al..

Remark: From digital control theory the sampling time must be chosen under the limit established by *Shannon* theorem ($h<\pi/\omega_0$) In (Schinkel et al., 2002) that limit is not respected by the second sampling time, i.e. 0.094. For that reason, we select h_2 =0.03 (ε=15) ($h<\pi/99.999$).

The controller parameters for this sampling time h_2, L2=[-130.29664515561 1.123104069925], were designed as in (Schinkel et al., 2002). For example, if the switching signal is $\sigma=\{1,2\}$, the dynamics switches between Φ_1^{CL} and Φ_ε^{CL} and always in that sequence, the augmented sampling time system is a periodic one, described by $\Phi_{1+e}^{CL}=\Phi_1^{CL} * \Phi_\varepsilon^{CL}$. The spectral radius is $\rho(\Phi_{1+e}^{CL})>1$ shows that, even each matrix is stable, a sequence of them it is not.

4.3 Symbolic Example – Commuting matrices

Here, we solve a symbolic problem in the frame of *Case 1* for the linear second order model of the cart system of Fig 1.

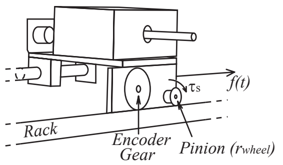

Figure 1: Cart System

The closed-loop system with a state-feedback controller, for $h_i=\alpha\, h_1$ with $\alpha\geq 1$, is

$$\begin{bmatrix} \frac{k_2}{k_1}\left(\alpha h_1-\frac{\left(1-e^{-k_1\alpha h_1}\right)}{k_1}\right)l_{\alpha,1} & \frac{1-e^{-k_1\alpha h_1}}{k_1}-\frac{k_2}{k_1}\left(\alpha h_1-\frac{\left(1-e^{-k_1\alpha h_1}\right)}{k_1}\right)l_{\alpha,1} \\ \frac{k_2}{k_1}\left(1-e^{-k_1\alpha h_1}\right)l_{\alpha,2} & e^{-k_1\alpha h_1}-\frac{k_2}{k_1}\left(1-e^{-k_1\alpha h_1}\right)l_{\alpha,2} \end{bmatrix}$$

Symbolic solution of (11) for the controller L_α is presented in Appendix I (equations (17) and (18)). This controller can be calculated as a non-linear function depending on plant parameters k_1 and k_2, the sampling times h_1 and $\alpha\, h_1$, and the controller parameters L_1 designed via traditional methods (pole-placement, LQR,..).

For example, consider the plant k_1=12.65, k_2=1.92, the sampling period h_1=0.02 and the controller L_1=[-121 –6.5] designed via LQR for Q=[1 0;0 0] R=0.0006. The controller parameters L_α are continuous functions of α, as we can see in Fig. 2 a) and b).

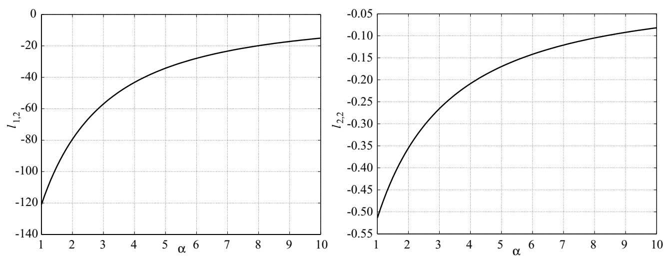

Figure 2: Controller parameters as function of α.

4.4 Upper diagonal matrices

Here, we solve a numerical problem using the *Case 2*. A paper machine head box model taken from (Franklin et al., 1997) is as follows:

$$\dot{x}=\begin{bmatrix} -0.2 & 0.1 & 1 \\ -0.05 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}x+\begin{bmatrix} 0 & 1 \\ 0 & 0.7 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_1 & u_2 \end{bmatrix}$$

This matrix can be brought to diagonal form (different eigenvalues) and is also stable. The similarity transformation (12) R is

$$R=\begin{bmatrix} -1.24223 & -.02133 & 1.4556 \\ -0.06211 & -0.3642 & 0.4263 \\ 1 & 0 & 0 \end{bmatrix}, e^{Tt}=\begin{bmatrix} e^{-0.17071t} & 0 & 0 \\ 0 & e^{-0.0292t} & 0 \\ 0 & 0 & e^{-t} \end{bmatrix}$$

The gain matrix of the state-feedback controller is

$$L_t=\begin{bmatrix} l_{t,1} & l_{t,2} & l_{t,3} \\ l_{t,4} & l_{t,5} & l_{t,6} \end{bmatrix}$$

The transformation of matrix (R^{-1} B L_t R) into an upper diagonal matrix requires to adopt the solution $l_{t,1}=0, l_{t,2}=0, l_{t,5}=-3.4142\, l_{t,4}$.

For h=0.2 results

$$(R^{-1}\text{ B } L_1\text{ R})=\begin{bmatrix} 0 & 0.504\, l_{t,4} & -0.504\, l_{t,4}+l_{t,3}+0.489\, l_{t,6} \\ 0 & -1.39\, l_{t,4} & 1.39\, l_{t,4}+l_{t,3}-1.35\, l_{t,6} \\ 0 & 0 & l_{t,3} \end{bmatrix}$$

We can see that the first column elements cannot be modified by changing the controller parameters. Thus, $\Phi_{t,(1,1)}^{CL}$ remains unchanged (equals its original value).

Using this controller, the closed-loop matrix (for h_1=0.2) results

$$\Phi_{h_1}^{CL}=\begin{bmatrix} 0.966 & 0.099\, l_{1,4} & -0.099\, l_{1,4}+0.196\, l_{1,3}+0.096\, l_{1,6} \\ 0 & 0.994-0.277\, l_{1,4} & 0.277\, l_{1,4}+0.194\, l_{1,3}-0.269\, l_{1,6} \\ 0 & 0 & 0.8187+0.181\, l_{1,3} \end{bmatrix}$$

Its eigenvalues are $\{0.966, 0.994-0.277\, l_{1,4}, 0.8187+0.181\, l_{1,3}\}$

Thus, we have a fixed eigenvalue at 0.966 and the others two can be placed by using $l_{1,3}$ and $l_{1,4}$.

In the same way for h_2=0.4

$$\Phi_{h2}^{CL}=\begin{bmatrix} 0.934 & 0.195\, l_{2,4} & -0.195\, l_{1,4}+0.386\, l_{2,3}+0.189\, l_{2,6} \\ 0 & 0.988-0.552\, l_{2,4} & 0.553\, l_{2,4}+0.397\, l_{2,3}-0.536\, l_{2,6} \\ 0 & 0 & 0.6703+0.3296\, l_{2,3} \end{bmatrix}$$

Its eigenvalues are $\{0.934, 0.988-0.552\, l_{2,4}, 0.6703+0.3296\, l_{2,3}\}$

Thus, we have a fixed eigenvalue at 0.934 and the others two can be placed by using $l_{2,4}$ y $f_{2,3}$.

In the same way we can proceed for any sampling period.

As each closed-loop matrix has an upper diagonal form, the generated algebra associated with a family of them is ***solvable***. The initial introduction of the similarity matrix transformation (R and R^{-1}) is a key feature that makes this controller design possible.

5. CONCLUSIONS

Rate adaptation of task execution is increasingly used in order to optimize allocation and throughput of shared resources in communication and computing systems. This practice may provoke significant variations in the sampling time of digital controllers immersed in networked control systems, with negative impact in closed-loop stability and performance. Motivated by these problems, this paper addressed the problem of designing a controller family that, given a continuous-time linear plant sampled at a varying rate, asymptotically stabilizes the closed loop.

The solvability of the Lie algebra generated by a family of stable matrices, as a sufficient condition for the asymptotic stability of the discrete-time switching system represented by them, is a previous result being used to solve the problem. This condition has been imposed to the closed-loop sampled system in order to obtain the parameterisation of a given state-feedback controller structure that solves the problem. Conditions to find the solutions have been provided for three cases, which have been illustrated with examples.

In a paper following this one, this work has been complemented with some results on closed-loop performance and robustness, that we have obtained via the calculation of the joint spectral radius of the set of matrices defining the discrete-time switching system.

In what concerns future work, we will explore other Lie-algebra solvability conditions, essentially in order to be able to take into account different types of controller structures, beyond the state-feedback controllers considered in this research.

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Appendix A. FIRST APPENDIX

In order to verify conditions (11) for the plant (16), the controller parameters $L_\alpha = \left[l_{1,\alpha} \quad l_{2,\alpha} \right]$ for the sampling period αh were obtained in a symbolic form by using MAPLE, as follow:

$$
\begin{aligned}
l_{\alpha,1} = &\left(-l_{1,1} k_1^2 \alpha\, e^{-k_1 \alpha h_1} + e^{-k_1 h_1} - 1 - e^{-k_1 h_1 (\alpha+1)}\right) / \left(-h_1 k_1 k_2 \alpha\, e^{-k_1 \alpha h_1} l_{1,1}\right. \\
&+ h_1 k_1 k_2 \alpha\, l_{1,1} - h_1 k_1 k_2 \alpha^2 l_{1,1} + h_1 k_1 k_2 \alpha^2 e^{-k_1 h_1} l_{1,1} + 2k_2 e^{-k_1 \alpha h_1} l_{1,1} \\
&- k_2 e^{-2k_1 \alpha h_1} l_{1,1} - k_2 l_{1,1} + \alpha k_2 l_{1,1} k_2 \alpha\, e^{-k_1 \alpha h_1} l_{1,1} + k_2 \alpha\, e^{-k_1 h_1 (\alpha+1)} l_{1,1} \\
&- k_2 \alpha\, e^{-k_1 h_1} l_{1,1} + k_1 k_2 e^{-2k_1 \alpha h_1} l_{1,2} + k_1 k_2 l_{1,2} + k_1 k_2 \alpha\, e^{-k_1 \alpha h_1} l_{1,2} \\
&- k_1 k_2 \alpha\, e^{-k_1 h_1 (\alpha+1)} l_{1,2} - 2k_1 k_2 \alpha\, e^{-k_1 \alpha h_1} l_{1,2} + k_1 k_2 \alpha\, e^{-k_1 h_1} l_{1,2} \\
&\left. - k_1 k_2 \alpha\, l_{1,2} - k_1^2 \alpha\, e^{-k_1 \alpha h_1} + k_1^2 \alpha - k_1^2 \alpha\, e^{-k_1 h_1} + k_1^2 \alpha\, e^{-k_1 (\alpha+1) h_1} \right)
\end{aligned}
\tag{17}
$$

$$
\begin{aligned}
l_{\alpha,2} = &-k_1 \left(e^{-2k_1 \alpha h_1} l_{1,1} - k_1 e^{-2k_1 \alpha h_1} l_{1,2} - \alpha e^{-k_1 (\alpha+1) h_1} l_{1,1} + \alpha e^{-k_1 \alpha h_1} l_{1,1} \right. \\
&\left. - 2e^{-k_1 \alpha h_1} l_{1,1} + 2k_1 e^{-k_1 \alpha h_1} l_{1,2} + \alpha e^{-k_1 h_1} l_{1,1} - k_1 l_{1,2} - \alpha l_{1,1} + l_{1,1} \right) / \\
&\left(-k_1 k_2 h_1 \alpha e^{-k_1 \alpha h_1} l_{1,1} + k_1 k_2 h_1 \alpha l_{1,1} - k_1 k_2 h_1 \alpha^2 l_{1,1} + k_1 k_2 h_1 \alpha^2 e^{-k_1 h_1} l_{1,1} \right. \\
&+ 2k_2 e^{-k_1 \alpha h_1} l_{1,1} + k_2 e^{-k_1 \alpha h_1} l_{1,1} - k_2 l_{1,1} + k_2 \alpha l_{1,1} - k_2 \alpha e^{-k_1 \alpha h_1} l_{1,1} \\
&+ k_2 \alpha e^{-k_1 (\alpha+1) h_1} l_{1,1} - k_2 \alpha\, e^{-k_1 h_1} l_{1,1} + k_1 k_2 e^{-k_1 \alpha h_1} l_{1,2} + k_1 k_2 l_{1,2} \\
&+ k_1 k_2 \alpha\, e^{-k_1 \alpha h_1} l_{1,2} - k_1 k_2 \alpha e^{-k_1 (\alpha+1) h_1} l_{1,2} - 2k_1 k_2 e^{-k_1 \alpha h_1} l_{1,2} + k_1 k_2 e^{-k_1 h_1} l_{1,2} \\
&\left. - k_1 k_2 l_{1,2} - k_1^2 \alpha\, e^{-k_1 \alpha h_1} + k_1^2 \alpha - k_1^2 \alpha\, e^{-k_1 h_1} + k_1^2 \alpha e^{-k_1 (\alpha+1) h_1} \right)
\end{aligned}
\tag{18}
$$