A necessary condition for dynamic equivalence

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A NECESSARY CONDITION FOR DYNAMIC EQUIVALENCE

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Abstract. If two control systems on manifolds of the same dimension are dynamic equivalent, we prove that either they are static equivalent –*i.e.* equivalent via a classical diffeomorphism– or they are both ruled; for systems of different dimensions, the one of higher dimension must be ruled. A ruled system is one whose equations define at each point in the state manifold, a ruled submanifold of the tangent space. Dynamic equivalence is also known as equivalence by endogenous dynamic feedback, or by a Lie-Bäcklund transformation when control systems are viewed as underdetermined systems of ordinary differential equations; it is very close to absolute equivalence for Pfaffian systems.

It was already known that a differentially flat system must be ruled; this was a particular case of the present result, in which one of the systems was assumed to be "trivial" (or linear controllable).

Key words. Control systems, ordinary differential equations, underdetermined systems, dynamic equivalence, absolute equivalence, ruled submanifolds.

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1. Introduction.

We consider time-invariant control systems, or underdetermined systems of ordinary differential equations (ODEs) where the independent variable is time. *Static equivalence* refers to equivalence via a diffeomorphism in the variables of the equation, or in the state and control variables, with a triangular structure that induces a diffeomorphism (preserving time) in the state variables too. It is also known as "feedback equivalence". *Dynamic equivalence* refers to equivalence via invertible transformations in jet spaces that do not induce any diffeomorphism in a finite number of variables, except when it coincides with static equivalence; these transformations are also known as endogenous dynamic feedback [15, 6], or Lie-Bäcklund transformations [1, 6, 16], although this terminology is more common for systems of partial differential equations (PDEs); dynamic equivalence is also very close to absolute equivalence for Pfaffian systems [4, 18, 19].

The literature on classification and invariants for static equivalence is too large to be quoted here; let us only recall that, as evidenced by all detailed studies and mentioned in [21], each equivalence class (within control systems on the same manifold, or germs of control systems) is very very thin, indeed it has infinite co-dimension except in trivial cases. Since dynamic equivalence is a priori more general, it is natural to ask how more general it is. Systems on manifolds of different dimension may be dynamic equivalent, but not static equivalent. Restricting our attention to systems on the same manifold and considering dynamic equivalence instead of static, how bigger are the equivalence classes ?

The literature on dynamic feedback linearization [11, 5], differential flatness [6, 15], or absolute equivalence [18] tends to describe the classes containing linear controllable systems or "trivial" systems. The authors of [6, 15, 18] made the link with deep differential geometric questions dating back to [9, 4, 10]; see [2] for a recent overview. Despite these efforts, no characterization is available except for systems with one control, *i.e.* whose general solution depends on one function of one variable; there are many systems that one suspects to be non-flat –*i.e.* dynamic equivalent to no trivial system– while no proof is available, see the remark on (3.17) in Section 4.1. There is however one powerful necessary condition [17, 20]: a flat system must be ruled, *i.e.* its

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equations must define a ruled submanifold in each tangent space. As pointed out in [17], this proves that the equivalence class of linear systems for dynamic equivalence, although bigger than for static equivalence, still has infinite co-dimension.

Deciding whether two general systems are dynamic equivalent is at least as difficult. There is no method to prove that two systems are not dynamic equivalent. The contribution of this paper is a necessary condition for two systems to be dynamic equivalent, that generalizes [17, 20]: if they live on manifolds of the same dimension, they must be either both ruled or static equivalent; if not, the one of higher dimension must be ruled. Besides being useful to prove that some pairs of systems are not dynamic equivalent, it also implies that "generic" equivalence classes for dynamic equivalence are the same as for static equivalence.

Outline. Notations on jet bundles and differential operators are recalled in Section 2; the notions of systems, ruled systems, dynamic and static equivalence are precisely defined in Section 3. Our main result is stated and commented in Section 4, and proved in Section 5.

2. Miscellaneous notations. Let M be an n-dimensional manifold, either $\mathbf{C}^\infty$ (infinitely differentiable) or $\mathbf{C}^\omega$ (real analytic).

2.1. Jet bundles. Using the notations and definitions of [8, Chapter II, §2], $J^k(\mathbb{R},M)$ denotes the k^{th} jet bundle of maps $\mathbb{R} \to M$. It is a bundle both over $\mathbb{R}$ and over M. If $(x^1,\ldots,x^n)$ is a system of coordinates on an open subset of M, coordinates on the lift of this open subset are given by $t, x^1,\ldots,x^n,\dot{x}^1,\ldots,\dot{x}^n,\cdots,(x^1)^{(k)},\ldots,(x^n)^{(k)}$ where t is the projection on $\mathbb{R}$.

As an additive group, $\mathbb{R}$ acts on $J^k(\mathbb{R},M)$ by translation of the t-component; the quotient by this action is well defined and we denote it by

$$J^k(M) \;=\; J^k(\mathbb{R},M)\,/\,\mathbb{R}\;. \tag{2.1}$$

Since we only study time-invariant systems, we prefer to work with $J^k(M)$. Quotienting indeed drops the t information: local coordinates on $J^k(M)$ are given by $x^1,\ldots,x^n,\dot{x}^1,\ldots,\dot{x}^n,\cdots,(x^1)^{(k)},\ldots,(x^n)^{(k)}$; for short, we write $x,\dot{x},\ldots,x^{(k)}$. For $\ell<k$, there is a canonical projection

$$\pi_{k,\ell}:\; J^k(M)\to J^\ell(M) \tag{2.2}$$

that makes $J^k(M)$ a bundle over $J^\ell(M)$; in particular it is a bundle over $M=J^0(M)$ and over $\mathrm{T}M=J^1(M)$. In coordinates,

$$\pi_{k,\ell}(x,\dot{x},\ldots,x^{(\ell)},\ldots,x^{(k)}) = (x,\dot{x},\ldots,x^{(\ell)})\;.$$

Notation. To a subset $\Omega\subset J^k(M)$, we associate, for all ℓ, a subset $\Omega_\ell\subset J^\ell(M)$ in the following manner (obviously, $\Omega_k=\Omega$):

$$\Omega_\ell = \begin{cases} \pi_{k,\ell}(\Omega) & \text{if } \ell\le k,\\ {\pi_{\ell,k}}^{-1}(\Omega) & \text{if } \ell\ge k.\end{cases} \tag{2.3}$$

2.2. The k^{th} jet of a smooth ($\mathbf{C}^\infty$) map $x(.) : I \longrightarrow M$. With $I\subset\mathbb{R}$ a time interval, it is a smooth map $j^k(\,x(.)\,) : I\to J^k(M)$ (see again [8]); in coordinates,

$$j^k(\,x(.)\,)\,(t) \;=\; (x(t),\dot{x}(t),\ddot{x}(t),\ldots,x^{(k)}(t))\;.$$

By *a smooth map whose k^{th} jet remains in* Ω, for some $\Omega\subset J^k(M)$, we mean a smooth $x(.) : I\to M$ such that $j^k(\,x(.)\,)(t)\in\Omega$ for all t in I.

2.3. Differential operators. If Ω is an open subset of $J^k(M)$, and M' is a manifold of dimension n', a smooth ($\mathbf{C}^\infty$ or $\mathbf{C}^\omega$) map $\Phi : \Omega \to M'$ defines the smooth differential operator of order[1] k

$$\mathcal{D}^k_\Phi \;=\; \Phi \circ j^k \;. \tag{2.4}$$

Obviously, $\mathcal{D}^k_\Phi$ sends smooth maps $I \to M$ whose k^{th} jet remains in Ω to smooth maps $I \to M'$. In coordinates, the image of $t \mapsto x(t)$ is $t \mapsto \Phi(x(t), \dot{x}(t), \ddot{x}(t), \ldots, x^{(k)}(t))$. Note that we do not require that k be minimal, so Φ *might* not depend on $x^{(k)}$

We call $j^r \circ \mathcal{D}^k_\Phi$ the r^{th} *prolongation of* the differential operator $\mathcal{D}^k_\Phi$; it sends smooth maps $I \to M$ whose k^{th} jet remains in Ω to smooth maps $I \to J^r(M')$; it is indeed the differential operator $\mathcal{D}^{k+r}_{\Phi^{[r]}}$, of order $k+r$, with $\Phi^{[r]}$ the unique smooth map ${\pi_{k+r,k}}^{-1}(\Omega) \to J^r(M')$ such that

$$j^r \circ \Phi \circ j^k \;=\; \Phi^{[r]} \circ j^{k+r} \;. \tag{2.5}$$

We call $\Phi^{[r]}$ the r^{th} prolongation of Φ. One has $\pi_{r,0} \circ \Phi^{[r]} = \Phi \circ \pi_{k+r,k}$ and more generally, for $s < r$,

$$\pi_{r,s} \circ \Phi^{[r]} = \Phi^{[s]} \circ \pi_{k+r,k+s} \;. \tag{2.6}$$

3. Systems and equivalence.

3.1. Systems. Definition 3.1. *A* $\mathbf{C}^\infty$ *or* $\mathbf{C}^\omega$ regular system *with* m *controls on a smooth manifold* M *is a* $\mathbf{C}^\infty$ *or* $\mathbf{C}^\omega$ *sub-bundle* Σ *of the tangent bundle* $\mathrm{T}M$

$$\begin{array}{ccc} \Sigma & \overset{i}{\hookrightarrow} & \mathrm{T}M \\ & \pi\searrow & \downarrow \\ & & M \end{array} \tag{3.1}$$

with fiber Υ*, a* $\mathbf{C}^\infty$ *or* $\mathbf{C}^\omega$ *manifold of dimension* m *(e.g. an open subset of* $\mathbb{R}^m$*). The* velocity set *at a point* $x \in M$ *is the fiber* $\Sigma_x = \pi^{-1}(\{x\})$*, a submanifold of* T_xM *diffeomorphic to* Υ*.*

Definition 3.2 (Solutions of a system). *A solution of system* Σ *on the real interval* I *is a* smooth *(*$\mathbf{C}^\infty$*)* $x(.) : I \to M$ *such that* $j^1(x(.))(t) \in \Sigma$ *for all* $t \in I$*.*

Although a general solution of a system need not be smooth, we only consider *smooth* solutions. They form a rich enough class in the sense that systems are fully characterized by their set of smooth solutions.

Locally, one may write "explicit" equations of Σ in the following form. Of course there are many choices of coordinates and the map f depends on this choice.

Proposition 3.3. *For each* $\xi \in \Sigma$*, with* $\Sigma \hookrightarrow \mathrm{T}M$ *a regular system (3.1), there is*

- *an open neighborhood* $\mathcal{U}$ *of* ξ *in* $\mathrm{T}M$*,* $\mathcal{U}_0$ *its projection on* M*,*
- *a system of local coordinates* $(x_{\mathrm{I}}, x_{\mathrm{II}})$ *on* $\mathcal{U}_0$*, with* x_{I} *a block of dimension* $n-m$ *and* x_{II} *of dimension* m*,*
- *an open subset* U *of* $\mathbb{R}^{n+m}$ *and a smooth (*$\mathbf{C}^\infty$ *or* $\mathbf{C}^\omega$*) map* $f : U \to \mathbb{R}^{n-m}$*,*

such that the equation of $\Sigma \cap \mathcal{U}$ *in these coordinates is*

$$\dot{x}_{\mathrm{I}} = f(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}})\,, \quad (x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}) \in U \;. \tag{3.2}$$

Proof. Consequence of the implicit function theorem. ∎

[1] "Of order no larger than k" would be more accurate: if Φ does not depend on k^{th} derivatives, the order in the usual sense would be smaller than k. See for instance Ψ in example (3.16).

Control systems. A more usual representation of a system with m controls is

$$\dot{x} = F(x,u)\ , \quad x \in M\ ,\ u \in \mathcal{B}\ , \tag{3.3}$$

with $\mathcal{B}$ an open subset of $I\!R^m$ and $F : M \times \mathcal{B} \to \mathrm{T}M$ smooth enough. It can be brought locally, in block coordinates $(x_{\mathrm{I}}, x_{\mathrm{II}})$, to the form

$$\dot{x}_{\mathrm{I}} = f(x_{\mathrm{I}}, x_{\mathrm{II}}, u)\ , \quad \dot{x}_{\mathrm{II}} = u \tag{3.4}$$

modulo a static feedback on u, at least around nonsingular points (x,u) where

$$\operatorname{rank} \frac{\partial F}{\partial u}(x,u) = m\ . \tag{3.5}$$

Equation (3.2) can be obtained by eliminating the control u in (3.4).

If (3.5) holds, (3.3) defines a system in the sense of Definition 3.1. All results on systems in that sense may easily be translated to control systems (3.3).

Implicit systems of ODEs. A smooth *system of* $n-m$ *ODEs* on M: $R(x,\dot{x}) = 0$ with $R : \mathrm{T}M \to I\!R^{n-m}$ also defines a system in the sense of Definition 3.1 if it is nonsingular, *i.e.* rank $\frac{\partial R}{\partial \dot{x}}(x,\dot{x}) = n-m$.

Singularities. With the above rank assumptions, or the one that Σ is a sub-bundle in Definition 3.1, we carefully avoid singular systems. This paper does not apply to singular control systems or singular implicit systems of ODEs.

Prolongations of Σ. For integers $k \geq 1$, we denote by Σ_k the prolongation of the system Σ to k^{th} order; it is *the* subbundle $\Sigma_k \hookrightarrow J^k(M)$ with the following property: for any smooth map $x(.) : I \to M$, with $j^k(x(.))$ defined in section 2.2,

$$j^1(x(.))(t) \in \Sigma\ ,\ t \in I \quad \Leftrightarrow \quad j^k(x(.))(t) \in \Sigma_k\ ,\ t \in I\ . \tag{3.6}$$

The left-hand side means that $x(.)$ is a *solution* of Σ according to Definition 3.2. Obviously, $\Sigma_1 = \Sigma$. We may describe Σ_k in coordinates.

Proposition 3.4. *Let K be a positive integer. There is a unique sub-bundle $\Sigma_K \hookrightarrow J^K(M)$ such that:*

$$\text{\textit{a smooth map } } x(.) : I \to M \text{ \textit{ is a solution of system} } \Sigma \text{ \textit{ on the real interval} } I \text{ \textit{ if and only if} } j^K(x(.))(t) \in \Sigma_K \text{ \textit{ for all} } t \in I. \tag{3.7}$$

For all $\xi \in \Sigma_K$, its projection $\xi_1 = \pi_{K,1}(\xi)$ is in Σ and, with $\mathcal{U}$ the neighborhood of ξ_1, $(x_{\mathrm{I}}, x_{\mathrm{II}})$ the coordinates on $\mathcal{U}_0$, U the open subset $I\!R^{n+m}$ and $f : U \to I\!R^m$ the map given by Proposition 3.3, the equations of $\mathcal{U}_K \cap \Sigma_K$ in $J^K(M)$ are, in the coordinates $(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{I}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{I}}^{(K)}, x_{\mathrm{II}}^{(K)})$ induced on $\mathcal{U}$ by $(x_{\mathrm{I}}, x_{\mathrm{II}})$,

$$\begin{aligned} &x_{\mathrm{I}}^{(i)} = f^{(i-1)}(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(i)})\ , \quad 1 \leq i \leq K\ , \\ &(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(K)}) \ \in\ U \times I\!R^{(K-1)m}\ , \end{aligned} \tag{3.8}$$

where, for a smooth map $f : U \to I\!R^{n-m}$, and $\ell \geq 0$, $f^{(\ell)}$ is the smooth map $U \times I\!R^{Km} \to I\!R^{n-m}$ defined by $f^{(0)} = f$ and, for $i \geq 1$,

$$f^{(i)}(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(i+1)}) = \frac{\partial f^{(i-1)}}{\partial x_{\mathrm{I}}} f(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}) + \sum_{i=0}^{i} \frac{\partial f^{(i-1)}}{\partial x_{\mathrm{II}}^{(i)}} x_{\mathrm{II}}^{(i+1)}\ . \tag{3.9}$$

Proof. This is classical, and obvious in coordinates. □

REMARK 3.5. Each Σ_{k+1} $(k \geq 1)$ is an affine bundle over Σ_k, and may be viewed as an affine sub-bundle of $\mathrm{T}\Sigma_k$, *i.e.* it is a system in the sense of Section 3.1 on the manifold Σ_k instead of M.

In particular $\Sigma_2 \hookrightarrow \mathrm{T}\Sigma$ is the system obtained by "adding an integrator in each control" of the system $\Sigma \hookrightarrow \mathrm{T}M$. It is an affine system (i.e. affine sub-bundle) even when Σ is not.

3.2. Ruled systems. Recall that a smooth submanifold of an affine space is ruled if and only if it is a union of straight lines, i.e. if through each point of the submanifold passes a straight line contained in the submanifold. Such a manifold must be unbounded; since we want to consider the intersection of a submanifold with an arbitrary open set and allow this patch to be "ruled", we use the same slightly abusive notion as [14]: a submanifold N is ruled if and only if, through each point of it, passes a straight line which is contained in N "until it reaches the boundary of N". Here, the boundary of the submanifold N is $\partial N = \overline{N} \setminus N$.

A system will be called ruled if and only if Σ_x is, for all x, a ruled submanifold of $\mathrm{T}_x M$. This is formalized below in a self-contained manner.

DEFINITION 3.6. *Let $\mathcal{O}$ be an open subset of* $\mathrm{T}M$. *System Σ (see (3.1)) is* ruled *in $\mathcal{O}$ if and only if, for all $(x, \dot{x}) \in (\mathcal{O} \cap \Sigma)$, there is a nonzero vector $w \in \mathrm{T}_x M \setminus \{0\}$ and two possibly infinite numbers $\lambda^- \in [-\infty, 0)$ and $\lambda^+ \in (0, +\infty]$ such that $(x, \dot{x} + \lambda w) \in \mathcal{O} \cap \Sigma$ for all λ, $\lambda^- < \lambda < \lambda^+$ and*

$$\begin{aligned} \lambda^- > -\infty &\Rightarrow (x, \dot{x} + \lambda^- w) \in \partial\,(\mathcal{O} \cap \Sigma)\ , \\ \lambda^+ < +\infty &\Rightarrow (x, \dot{x} + \lambda^+ w) \in \partial\,(\mathcal{O} \cap \Sigma)\ . \end{aligned} \tag{3.10}$$

Recall that, by definition, $\partial\,(\mathcal{O} \cap \Sigma) = \overline{\mathcal{O} \cap \Sigma} \setminus (\mathcal{O} \cap \Sigma)$.

We shall need the following characterisation.

PROPOSITION 3.7 ([14]). *Let $\mathcal{O}$ be an open subset of* $\mathrm{T}M$. *Σ is ruled in $\mathcal{O}$ if and only if, for all $\xi = (x, \dot{x})$ in $\Sigma \cap \mathcal{O}$, there is a straight line in $\mathrm{T}_x M$ passing through $\dot{x}$ that has contact of infinite order with Σ_x at $\dot{x}$.*

Proof. From [14, Theorem 1], a "patch of" submanifold of dimension m in a manifold of dimension n is ruled if and only if there is, through each point, a straight line that has contact of order $n + 1$. This is of course implied by infinite order. □

3.3. Dynamic equivalence. The following notion is usually called dynamic equivalence, or equivalence by (endogenous) dynamic feedback transformations in control theory, see [15, 7, 12, 16]. It is in fact also the notion of Lie-Bäcklund transformation, limited to ordinary differential equation, as noted in [7] or [16].

DEFINITION 3.8. *Let $\Sigma \hookrightarrow \mathrm{T}M$ and $\Sigma' \hookrightarrow \mathrm{T}M'$ be $\mathbf{C}^\infty$ (resp. $\mathbf{C}^\omega$) regular systems (see (3.1)) on two manifolds M and M' of dimension n and n', K, K' two integers, $\Omega \subset J^K(M)$ and $\Omega' \subset J^{K'}(M')$ two open subsets.*

Systems Σ and Σ' are dynamic equivalent over Ω and Ω' *if and only if there exists two mappings of class $\mathbf{C}^\infty$ (resp. $\mathbf{C}^\omega$) :*

$$\Phi : \Omega \to M'\,, \qquad \Psi : \Omega' \to M \tag{3.11}$$

inducing differential operators $\mathcal{D}_\Phi^K$ and $\mathcal{D}_\Psi^{K'}$ –see (2.4)– such that, for any interval I,

- *for any solution $x(.) : I \to M$ of Σ whose K^{th} jet remains inside Ω, $\mathcal{D}_\Phi^K(\,x(.)\,)$ is a solution of Σ' whose K'^{th} jet remains inside Ω' and $\mathcal{D}_\Psi^{K'}(\,\mathcal{D}_\Phi^K(\,x(.)\,)\,) \;=\; x(.)$,*

- *for any solution $z(.) : I \to M'$ of Σ' whose K'^{th} jet remains inside Ω', $\mathcal{D}_{\Psi}^{K'}(\,z(.)\,)$ is a solution of Σ whose K^{th} jet remains inside Ω and $\mathcal{D}_{\Phi}^{K}(\,\mathcal{D}_{\Psi}^{K'}(\,z(.)\,)\,) \;=\; z(.)$.*

REMARK 3.9. Since all properties are tested on *solutions*, only the restriction of Φ and Ψ to Σ_K and $\Sigma_{K'}$ (see Proposition 3.4) matter; for instance, Φ can be arbitrarily modified away from Σ_K without changing any conclusions. Borrowing this language from the literature on Lie-Bäcklund transformations, Φ and Ψ above are "external" correspondences.

In [7] or in [16], the "internal" point of view prevails: for instance Φ and Ψ are replaced, in [7], by diffeomorphisms between diffieties. This is more intrinsic because maps are defined only where they are to be used. However the definitions are equivalent because these internal maps admit infinitely many "external" prolongations.

Here, this external point of view is adopted because it makes the statement of the main result less technical. Note however that, as a preliminary to the proofs, an "internal" translation is given in section 5.1.

REMARK 3.10. In the theorems, we shall require that Ω and Ω' satisfy

$$\Omega_1 \cap \Sigma \;\subset\; (\Omega \cap \Sigma_K)_1 \quad \text{and} \quad \Omega'_1 \cap \Sigma' \;\subset\; (\Omega' \cap {\Sigma'}_{K'})_1 \;, \tag{3.12}$$

i.e. any (jet of) solution whose first jet is in Ω_1 lifts to at least one (jet of) solution whose K^{th} jet is in Ω. Note the following facts about this requirement.
- These inclusions are equalities for the reverse inclusions always hold.
- Replacing the original Ω with $\Omega \setminus \Big(\overline{(\Omega_1 \cap \Sigma) \setminus (\Omega \cap \Sigma_K)_1}\Big)_K$ and Ω' accordingly forces (3.12); alternatively, keeping arbitrary open sets, Theorem 4.2 and Theorem 4.1 would hold with Ω_1 replaced with $\Omega_1 \setminus \overline{(\Omega_1 \cap \Sigma) \setminus (\Omega \cap \Sigma_K)_1}$.
- When $\Sigma' = \mathrm{T}M'$ is the trivial system (see section 3.5), any open Ω' satisfies (3.12).

3.4. Static equivalence. DEFINITION 3.11. *Let $\mathcal{O} \subset \mathrm{T}M$ and $\mathcal{O}' \subset \mathrm{T}M'$ be open subsets. Systems Σ and Σ' are* static equivalent *over $\mathcal{O}$ and $\mathcal{O}'$ if and only if there is a smooth diffeomorphism $\Phi : \mathcal{O}_0 \to \mathcal{O}'_0$ such that the following holds:*

$$\left.\begin{array}{l}\textit{a smooth map } t \mapsto x(t) \textit{ is a solution of } \Sigma \textit{ whose first jet remains in } \mathcal{O} \\ \textit{if and only if } t \mapsto \Phi(x(t)) \textit{ is a solution of } \Sigma' \textit{ whose first jet remains in } \mathcal{O}'.\end{array}\right\} \tag{3.13}$$

DEFINITION 3.12 (Local static equivalence). *Let $\mathcal{O} \subset \mathrm{T}M$ and $\mathcal{O}' \subset \mathrm{T}M'$ be open subsets. Systems Σ and Σ' are* locally *static equivalent over $\mathcal{O}$ and $\mathcal{O}'$ if and only if there are coverings of $\mathcal{O} \cap \Sigma$ and $\mathcal{O}' \cap \Sigma'$:*

$$\Sigma \cap \mathcal{O} \;\subset\; \Sigma \cap \bigcup_{\alpha \in A} \mathcal{O}^{\alpha}\,, \quad \Sigma' \cap \mathcal{O}' \;\subset\; \Sigma' \cap \bigcup_{\alpha \in A} \mathcal{O}'^{\alpha}$$

where A is a set of indices, $\mathcal{O}^{\alpha}$ and $\mathcal{O}'^{\alpha}$ are open subsets of $\mathcal{O}$ and $\mathcal{O}'$, such that, for all α, systems Σ and Σ' are static equivalent over $\mathcal{O}^{\alpha}$ and $\mathcal{O}'^{\alpha}$.

This definition, stated in terms of solutions, is translated into point (a) below, that only relies on the geometry of Σ and Σ' as submanifolds. Point (b) is used for instance in [13, 22] where "centro-affine" geometry of each Σ_x is studied.

PROPOSITION 3.13. (a) *Systems Σ and Σ' are* static equivalent *over $\mathcal{O} \subset \mathrm{T}M$ and $\mathcal{O}' \subset \mathrm{T}M'$ if and only there is a smooth diffeomorphism $\Phi : \mathcal{O}_0 \to \mathcal{O}'_0$ such that $\Phi_\star$ maps $\mathcal{O} \cap \Sigma$ to $\mathcal{O}' \cap \Sigma'$.*

(b) *If systems Σ and Σ' are static equivalent over $\mathcal{O} \subset \mathrm{T}M$ and $\mathcal{O}' \subset \mathrm{T}M'$, there is, for each $x \in \mathcal{O}_0$ a* linear isomorphism $\mathrm{T}_x M \to \mathrm{T}_{\Phi(x)} M'$ *that maps Σ_x to $\Sigma'_{\Phi(x)}$.*

(c) *Static equivalence preserves ruled systems.*

Proof. (b) and (c) are easy consequences of (a), which in turn is clear by differentiating solutions in Definition 3.2. □

3.5. Examples. *1.* We call *trivial system* on a smooth manifold M the tangent bundle itself $\mathrm{T}M$. Any smooth $x(.) : I \to M$ is a solution of this system; it corresponds to "no equation", or to the control system $\dot{x} = u$, or to the "affine diffieties" in [7]. Following [6, 7], a system $\Sigma \hookrightarrow \mathrm{T}M$ is called *differentially flat* (on $\Omega \subset J^K(M)$) if and only if it is dynamic equivalent (over Ω and Ω') to the trivial system $\mathrm{T}M'$ for some manifold M'.

2. Any system $\Sigma \hookrightarrow \mathrm{T}M$ is dynamic equivalent to the one obtained by "adding integrators". It was described in Remark 3.5 as an affine sub-bundle $\Sigma_2 \hookrightarrow \mathrm{T}\Sigma$; Σ and Σ_2 are equivalent in the sense of Definition 3.8 with $M' = \Sigma$, $K = 1$, $K' = 0$, Ω an open neighborhood of Σ in $J^1(M) = \mathrm{T}M$ such that there is a $\Phi : \Omega \to \Sigma$ that coincides with identity on Σ, $\Omega' = M' = \Sigma$ and $\Psi = \pi$ (see (3.1)).

This may be easier to follow in the coordinates of Proposition 3.3. The prolongation of (3.2) has state $(y_{\mathrm{I}}, y_{\mathrm{II}}) \in U$, with y_{I} a block of dimension n and y_{II} of dimension m, and equation $\dot{y}_{\mathrm{I}} = \big(f(y_{\mathrm{I}}, y_{\mathrm{II}})\,,\, y_{\mathrm{II}}\big)$. In coordinates, the transformations $\Phi : J^1(U_0) \to U$ and $\Psi : U \to U_0$ are given by $(y_{\mathrm{I}}, y_{\mathrm{II}}) = \Phi(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{I}}, \dot{x}_{\mathrm{II}}) = (x, \dot{x}_{\mathrm{II}})$ and $x = \Psi(y) = y_{\mathrm{I}}$.

Static equivalence between these systems of different dimension does not hold.

3. Let us now give, mostly to illustrate the role of the integers K, K' and the open sets Ω and Ω', two more specific examples of systems $\Sigma \hookrightarrow \mathrm{T}\mathbb{R}^3$ and $\Sigma' \hookrightarrow \mathrm{T}\mathbb{R}^3$ with the following equations in $\mathrm{T}\mathbb{R}^3$, with coordinates $(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3)$ or $(y_1, y_2, y_3, \dot{y}_1, \dot{y}_2, \dot{y}_3)$, clearly defining sub-bundles with fiber diffeomorphic to $\mathbb{R}^2$:

$$\Sigma : \ \dot{x}_1 = x_2\ , \qquad \Sigma' : \ \dot{y}_1 = y_2 + (\dot{y}_2 - y_1\dot{y}_3)\,\dot{y}_3\ . \tag{3.14}$$

These equations are even globally in the "explicit" form given by Proposition 3.3.

First of all, Σ is dynamic equivalent to the trivial system $\Sigma'' = \mathrm{T}\mathbb{R}^2$, with $\Phi : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $\Phi(x_1, x_2, x_3) = (x_1, x_3)$ and $\Psi : J^1(\mathbb{R}^2) \to \mathbb{R}^3$ given by $\Psi(z_1, z_2, \dot{z}_1, \dot{z}_2) = (z_1, \dot{z}_1, z_2)$. Here $K = 0, K' = 1, \Omega = \mathbb{R}^2, \Omega' = J^1(\mathbb{R}^2)$.

Also, with $K = 1$ and $K' = 2$, systems Σ *and* Σ' *are dynamic equivalent* over $\Omega \subset J^1(\mathbb{R}^3)$ and $\Omega' \subset J^2(\mathbb{R}^3)$ defined by

$$\begin{aligned} \Omega &= \{(x_1, x_2, x_3,\ \dot{x}_1, \dot{x}_2, \dot{x}_3),\ 1 - \dot{x}_2 - {x_2}^3 \neq 0\}\,, \\ \Omega' &= \{(y_1, y_2, y_3,\ \dot{y}_1, \dot{y}_2, \dot{y}_3,\ \ddot{y}_1, \ddot{y}_2, \ddot{y}_3),\ 1 - \ddot{y}_3 - \dot{y}_3^3 \neq 0\}\,. \end{aligned}$$

The maps $\Phi : \Omega \to \mathbb{R}^3$ and $\Psi : \Omega' \to \mathbb{R}^3$ are given by

$$\Phi(x_1, x_2, x_3,\ \dot{x}_1, \dot{x}_2, \dot{x}_3) = \Big(\, \frac{(1 - \dot{x}_2)x_3 + x_2\,\dot{x}_3}{1 - \dot{x}_2 - {x_2}^3}\,,\ \frac{{x_2}^2\,x_3 + \dot{x}_3}{1 - \dot{x}_2 - {x_2}^3}\,,\ x_1 \,\Big)\,, \tag{3.15}$$

$$\Psi(y_1, y_2, y_3,\ \dot{y}_1, \dot{y}_2, \dot{y}_3,\ \ddot{y}_1, \ddot{y}_2, \ddot{y}_3) = (\, y_3\,,\ \dot{y}_3\,,\ y_1 - \dot{y}_3\,y_2\,)\,. \tag{3.16}$$

REMARK 3.14. Since Ψ does not depend on second derivatives, $K' = 2$ is not the order of the differential operator $\mathcal{D}_\Psi^{K'}$ in the usual sense; this illustrates the footnote after (2.4); it is however necessary to go to second jets to describe the domain Ω' where the restriction to solutions of Σ' of this first order operator can be inverted.

Finally, note that systems Σ *and* Σ' *are not static equivalent* because, from Proposition 3.13-(b), this would imply that each Σ_x is sent to some Σ'_y by a linear isomorphism $\mathrm{T}_xM \to \mathrm{T}_yM'$, which is not possible because each Σ_x is an affine subspace of T_xM and Σ'_y a non degenerate quadric of T_yM'.

4. Consider two more systems, $\Sigma \hookrightarrow \mathrm{T}I\!\!R^3$ and $\Sigma' \hookrightarrow \mathrm{T}I\!\!R^3$ described as in (3.14):

$$\Sigma: \ \dot{x}_1 = x_2 + (\dot{x}_2 - x_1\dot{x}_3)^2\,\dot{x}_3^{\,2}\ , \qquad \Sigma': \ \dot{y}_1 = y_2 + (\dot{y}_2 - y_1\dot{y}_3)^2\,\dot{y}_3\ . \tag{3.17}$$

System Σ is ruled –each Σ_y is the union of lines $\dot{y}_2 - y_1\dot{y}_3 = \lambda$, $\dot{y}_1 = y_2 + \lambda^2\,\dot{y}_3$ for λ in $I\!\!R$– while Σ' is not. Hence, from point (c) of Proposition 3.13, Σ *and* Σ' *are not static equivalent.* We shall come back to these two systems from the point of view of flatness and dynamic equivalence in sections 4.1 and 4.3.

4. Necessary conditions.

4.1. The case of flatness. It has been known since [17, 20] that a system which is dynamic equivalent to a *trivial system* –see the beginning of section 3.5; such a system is called differentially flat– must be ruled; of course, at least in the smooth case, this is true only on the domain where equivalence is assumed.

THEOREM 4.1 ([17, 20]). *If Σ is dynamic equivalent to the trivial system $\Sigma' = \mathrm{T}M'$ over $\Omega \subset J^K(M)$ and $\Omega' \subset J^{K'}(M')$ satisfying (3.12), then Σ is ruled in Ω_1.*

Application. Since Σ in (3.17) is not ruled, this theorem implies that it is not flat, i.e. not dynamic equivalent to the trivial system $\mathrm{T}I\!\!R^2$. On the contrary, Σ' in (3.17) is ruled, hence the result does not help deciding it being flat or not; in fact, one conjectures that this system is not flat, but no proof is available; see [3].

4.2. Main idea of the proofs. Our main result, stated in next section, studies what remains of Theorem 4.1 when Σ' is not the trivial system. Due to many technicalities concerning regularity conditions, the main ideas may be difficult to grasp in the proof given in section 5.2. In order to enlighten these ideas, and even the result itself, let us first sketch the proof of the above theorem, following the line of [17] (itself inspired from [10]), but *without assuming a priori that Σ' is trivial.*

Take two arbitrary systems Σ and Σ', and assume that they are dynamic equivalent. From Proposition 3.3, one may use locally the explicit forms

$$\Sigma: \ \dot{x}_{\mathrm{I}} = f(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}})\ , \quad \Sigma': \ \dot{z}_{\mathrm{I}} = g(z_{\mathrm{I}}, z_{\mathrm{II}}, \dot{z}_{\mathrm{II}})\ .$$

Recall that n and n' denote the dimensions of x and z; assume $n \le n'$. Since we work only on *solutions* (see Remark 3.9 and also Section 5.1) and the above equations allow one to express each time-derivative $x_{\mathrm{I}}^{(j)}$, $j \ge 1$, as a function of $x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(j)}$, we may work with the variables $x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ddot{x}_{\mathrm{II}}, x_{\mathrm{II}}^{(3)}, \ldots$ and $z_{\mathrm{I}}, z_{\mathrm{II}}, \dot{z}_{\mathrm{II}}, \ddot{z}_{\mathrm{II}}, z_{\mathrm{II}}^{(3)}, \ldots$ only. The map Φ of Definition 3.8 translates, in these coordinates, into a correspondence $z_{\mathrm{I}} = \phi_{\mathrm{I}}(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(K)})$, $z_{\mathrm{II}} = \phi_{\mathrm{II}}(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(K)})$; here the number K is chosen such that *the dependence of ϕ versus $x_{\mathrm{II}}^{(K)}$ is effective.*

If $K = 0$, this reads $z = \phi(x)$, and $n < n'$ is absurd because it would imply (around points where the rank of ϕ is constant) some nontrivial relations $R(z) = 0$. Hence $n = n'$, ϕ is a local diffeomorphism and static equivalence holds locally.

If $K \ge 1$, note that Φ mapping solutions of Σ to solution of Σ' implies (plug the expression of z given by ϕ into state equations of Σ') the following identity, valid for all $x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(K+1)}$:

$$\begin{aligned}
&\frac{\partial\phi_{\mathrm{I}}}{\partial x_{\mathrm{I}}} f(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}) + \frac{\partial\phi_{\mathrm{I}}}{\partial x_{\mathrm{II}}}\dot{x}_{\mathrm{II}} + \frac{\partial\phi_{\mathrm{I}}}{\partial \dot{x}_{\mathrm{II}}}\ddot{x}_{\mathrm{II}} + \cdots + \frac{\partial\phi_{\mathrm{I}}}{\partial x_{\mathrm{II}}^{(K)}} x_{\mathrm{II}}^{(K+1)} \\
&\quad = \ g\left(\phi_{\mathrm{I}}, \phi_{\mathrm{II}},\ \frac{\partial\phi_{\mathrm{II}}}{\partial x_{\mathrm{I}}} f(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}) + \frac{\partial\phi_{\mathrm{II}}}{\partial x_{\mathrm{II}}}\dot{x}_{\mathrm{II}} + \frac{\partial\phi_{\mathrm{II}}}{\partial x_{\mathrm{II}}}\ddot{x}_{\mathrm{II}} + \cdots + \frac{\partial\phi_{\mathrm{II}}}{\partial x_{\mathrm{II}}^{(K)}} x_{\mathrm{II}}^{(K+1)}\right)
\end{aligned}$$

where $\phi_{\rm I}$ and $\phi_{\rm II}$ depend on $x_{\rm I}, x_{\rm II}, \dot{x}_{\rm II}, \ldots, x_{\rm II}^{(K)}$ only and, at least at generic points, $\left(\frac{\partial \phi_{\rm I}}{\partial x_{\rm II}^{(K)}}, \frac{\partial \phi_{\rm II}}{\partial x_{\rm II}^{(K)}} \right) \neq (0,0)$. Fixing such $x_{\rm I}, x_{\rm II}, \dot{x}_{\rm II}, \ldots, x_{\rm II}^{(K)}$ and consequently $z = \phi(x_{\rm I}, x_{\rm II}, \dot{x}_{\rm II}, \ldots, x_{\rm II}^{(K)})$, and examining Σ'_z as a submanifold of $T_z M'$ with equation $\dot{z}_{\rm I} = g(z, \dot{z}_{\rm II})$, it is clear that moving $x_{\rm II}^{(K+1)}$ in a direction which is not in the kernel of $\frac{\partial \phi_{\rm II}}{\partial x_{\rm II}^{(K)}}(x_{\rm I}, x_{\rm II}, \dot{x}_{\rm II}, \ldots, x_{\rm II}^{(K)})$ provides a straight line of $T_z M'$ contained in Σ'_z and, since this covers all points of Σ'_z, proves that the latter is a ruled submanifold of $T_z M'$ and finally that system Σ' is ruled. We only examined regular points; see Section 5.2 for a proper proof.

Collecting the two cases, we have proved that, if $n \leq n'$, either Σ' is ruled or $n = n'$ and Σ' is static equivalent to Σ. This is stated formally in Theorem 4.2.

4.3. The result for general systems. The contribution of this paper is the following strong necessary condition for dynamic equivalence between two general systems. Ω_1 and Ω'_1 are defined by (2.3).

THEOREM 4.2. *Let Σ and Σ' be systems on manifolds of dimension n and n', K, K' two integers and $\Omega \subset J^K(M)$, $\Omega' \subset J^{K'}(M')$ two open subsets satisfying (3.12).*

If Σ and Σ' are dynamic equivalent over Ω and Ω', then

if $n > n'$, *system Σ is ruled in Ω_1,*

if $n < n'$, *system Σ' is ruled in Ω'_1,*

if $n = n'$, *then (see Definition 3.12 for "locally static equivalent")*

- in the real analytic case*, and if $\Omega_1 \cap \Sigma$ and $\Omega'_1 \cap \Sigma'$ are connected,*
  either systems Σ and Σ' are ruled in Ω_1 and Ω'_1 respectively,
  or they are locally static equivalent over Ω_1 and Ω'_1,
- in the smooth ($\mathbf{C}^\infty$) case*, there are open subsets $\mathcal{R}, \mathcal{S}$ of Ω_1 and $\mathcal{R}', \mathcal{S}'$ of Ω'_1 such that Ω_1 and Ω'_1 are covered as*

$$\Omega_1 = \overline{\mathcal{R}} \cup \mathcal{S} = \mathcal{R} \cup \overline{\mathcal{S}}\,, \quad \Omega'_1 = \overline{\mathcal{R}'} \cup \mathcal{S}' = \mathcal{R}' \cup \overline{\mathcal{S}'} \tag{4.1}$$

  and the systems have the following properties on these sets:
  1. *Σ and Σ' are ruled in $\mathcal{R}$ and $\mathcal{R}'$ respectively,*
  2. *Σ and Σ' are* locally *static equivalent over $\mathcal{S}$ and $\mathcal{S}'$.*

Proof. See Section 5.2. ☐

A few remarks are in order:

1. Theorem 4.1 is a consequence. Indeed, $n' = m'$ because Σ' is trivial, dynamic equivalence implies $m' = m$ (this is common knowledge; see [4], [7] or [16, Theorem 1]), and $n \geq m$ for any system; hence $n \geq n'$ and Theorem 4.2 directly implies that Σ is ruled except if the systems are static equivalent, but this also implies that Σ is ruled from point (c) of Proposition 3.13 and the fact that the trivial system Σ' is ruled.

Static equivalence still appears explicitly in Theorem 4.2 because two general systems can be static equivalent without being ruled.

2. The part "$n > n'$ or $n < n'$" can be rephrased as follows: if a system is not ruled, it cannot be dynamic equivalent to any system of smaller dimension. No necessary condition is given on the system of lower dimension; indeed *any* system is dynamic equivalent to at least its first prolongation, see Example 2 in Section 3.5.

3. The case $n = n'$ states that dynamic equivalence, except when it reduces to static equivalence, forces both systems to be ruled (in the real analytic case, the added rigidity prevents the two situations from occurring simultaneously).

In other words, if two systems are not static equivalent and at least one of them is not ruled, they are not dynamic equivalent. Since the two conditions can be checked rather systematically, this yields a new and powerful method for proving that two systems are *not* dynamic equivalent, a difficult task in general because very few invariants of dynamic equivalence are known.

For instance, to the best of our knowledge, the state of the art does not allow one to decide whether Σ and Σ' in (3.17) are dynamic equivalent or not. In section 3.5, it was noted that they are not static equivalent and Σ' is not ruled. This implies:

COROLLARY 4.3. *Σ and Σ' in (3.17) are not dynamic equivalent over any domains.*

4. Since being ruled is non-generic [17], we have the following general consequence (in terms of germs of systems because the conclusion in the theorem is only local).

COROLLARY 4.4. *Generic static equivalence classes for germs of systems of the same dimension at a point are also dynamic equivalence classes.*

Note that this is in the mathematical sense of "generic": this does not prevent many interesting systems from being dynamic equivalent without being static equivalent... it might even be that "most interesting systems" fall in this case !

5. Proofs. Recall that subscripts always refer to the order of the jet space. The notation (2.3) is constantly used.

5.1. Preliminaries: a re-formulation of dynamic and static equivalence. The maps Φ and Ψ are always applied to jets of solutions, and, according to (3.6), the K^{th} jets of solutions of Σ remain in Σ_K; hence the only information to retain about Φ and Ψ is their restriction to, respectively,

$$\widetilde{\Omega} = \Omega \cap \Sigma_K \quad \text{and} \quad \widetilde{\Omega}' = \Omega' \cap {\Sigma'}_{K'} \,. \tag{5.1}$$

We need one more piece of notation: according to Section 2.3, the ℓ^{th} prolongation of a smooth map $\widetilde{\Phi} : \widetilde{\Omega} \to M'$, is a map $\pi^{-1}_{K+\ell,\ell}(\widetilde{\Omega}) \to J^\ell M'$; again, only its restriction to $\widetilde{\Omega}_{K+\ell}$ will matter; for this reason, the notations $\widetilde{\Phi}^{[\ell]}$ and $\widetilde{\Psi}^{[\ell]}$ will not stand for the prolongations as defined earlier, but rather these restrictions:

$$\widetilde{\Phi}^{[\ell]} : \widetilde{\Omega}_{K+\ell} \to J^\ell(M') \,, \quad \widetilde{\Psi}^{[\ell]} : \widetilde{\Omega'}_{K'+\ell} \to J^\ell(M) \,, \tag{5.2}$$

$$\text{with} \quad \widetilde{\Omega}_{K+\ell} = \Omega_{K+\ell} \cap \Sigma_{K+\ell} \,, \quad \widetilde{\Omega}'_{K'+\ell} = {\Omega'}_{K'+\ell} \cap {\Sigma'}_{K'+\ell} \,. \tag{5.3}$$

We may now state the following proposition. Smooth ($\mathbf{C}^\infty$ or $\mathbf{C}^\omega$) maps on $\widetilde{\Omega}_{K+\ell}$ or $\widetilde{\Omega'}_{K'+\ell}$ can be defined in a standard way because, from Proposition 3.3, these are smooth embedded submanifolds.

PROPOSITION 5.1 (Dynamic Equivalence). *Let K, K' be integers, $\Omega \subset J^K(M)$ and $\Omega' \subset J^{K'}(M')$ two open subsets. Systems Σ and Σ' are* dynamic equivalent over Ω and Ω' *if and only if, with $\widetilde{\Omega}, \widetilde{\Omega}'$ defined in (5.1), there exist two smooth (real analytic, in the real analytic case) mappings*

$$\widetilde{\Phi} : \widetilde{\Omega} \to M' \quad \textit{and} \quad \widetilde{\Psi} : \widetilde{\Omega}' \to M \,,$$

such that

$$\widetilde{\Phi}^{[1]}(\widetilde{\Omega}_{K+1}) \subset \Sigma' \,, \quad \widetilde{\Psi}^{[1]}(\widetilde{\Omega'}_{K'+1}) \subset \Sigma \,, \tag{5.4}$$

and, with $\widetilde{\Phi}^{[K]}$ and $\widetilde{\Psi}^{[K]}$ defined by (5.2),

$$\widetilde{\Phi}^{[K']}(\widetilde{\Omega}_{K+K'}) \subset \Omega' \,, \quad \widetilde{\Psi}^{[K]}(\widetilde{\Omega}'_{K+K'}) \subset \Omega \,, \tag{5.5}$$

$$\widetilde{\Psi} \circ \widetilde{\Phi}^{[K']} = \pi_{K+K',0}\Big|_{\widetilde{\Omega}_{K+K'}} \,, \quad \widetilde{\Phi} \circ \widetilde{\Psi}^{[K]} = \pi_{K+K',0}\Big|_{\widetilde{\Omega}'_{K+K'}} \,. \tag{5.6}$$

Proof. If the above conditions on Φ and Ψ are satisfied, and $x(.) : I \to M$ is a solution of Σ whose K^{th} jet remains inside Ω, then the first part of (5.4) implies that $\mathcal{D}_\Phi^K(\,x(.)\,)$ is a solution of Σ', the first part of (5.5) implies that its K^{th} jet remains inside Ω', and the first part of (5.6) implies that $\mathcal{D}_\Psi^{K'}(\,\mathcal{D}_\Phi^K(\,x(.)\,)\,) = x(.)$. This proves the first item of Definition 3.8; the second item follows in the same way from the second part of (5.4), (5.5) and (5.6).

Conversely, if Φ and Ψ satisfy the properties of Definition 3.8, their restrictions $\widetilde{\Phi}$ and $\widetilde{\Psi}$ to $\widetilde{\Omega}$ and $\widetilde{\Omega}'$ respectively satisfy the above relations because through each point in $\widetilde{\Omega}_{K+1}$, $\widetilde{\Omega'}_{K'+1}$, $\widetilde{\Omega}_{K+K'}$ or $\widetilde{\Omega}_{K+K'}$ passes a jet of order $K+1$, $K'+1$ or $K+K'$ of a solution of Σ or Σ'; differentiating yields the required relations. □

Proposition 5.2 (Static Equivalence). *With $\Omega_1 \subset J^1(M) = \mathrm{T}M$ and $\Omega'_1 \subset J^1(M') = \mathrm{T}M$ two open subsets, systems Σ and Σ' are* static equivalent over Ω_1 and Ω'_1 *if and only if, with $\widetilde{\Omega}_1, \widetilde{\Omega}'_1$ defined in (5.1), there exist a smooth diffeomorphism $\Phi_0 : \widetilde{\Omega}_0 \to \widetilde{\Omega}'_0$, and its inverse Ψ_0 such that $\widetilde{\Phi}_0^{[1]}(\widetilde{\Omega}_1) = \widetilde{\Omega}'_1$ (and $\widetilde{\Psi}_0^{[1]}(\widetilde{\Omega}'_1) = \widetilde{\Omega}_1$).*

Proof. This is a re-phrasing of point (a) of Proposition 3.13. □

5.2. Proof of Theorem 4.2. Assume that Σ and Σ' are dynamic equivalent over the open sets $\Omega \subset J^K(M)$ and $\Omega' \subset J^{K'}(M')$; let $\widetilde{\Phi} : \widetilde{\Omega} \to M'$ and $\widetilde{\Psi} : \widetilde{\Omega}' \to M$ be the smooth maps given by Proposition 5.1 (recall that $\widetilde{\Omega}$ and $\widetilde{\Omega}'$ are open subsets of Σ_K and $\Sigma'_{K'}$). We define open subsets $\widetilde{\Omega}^S \subset \widetilde{\Omega}$ and $\widetilde{\Omega}'^S \subset \widetilde{\Omega}'$ and state four lemmas concerning these :

$$\xi \in \widetilde{\Omega}^S \quad \Leftrightarrow \quad \begin{array}{l}\text{There is a neighborhood } V \text{ of } \xi \text{ in } \widetilde{\Omega} \text{ and a smooth map} \\ \widetilde{\Phi}_0 : V_0 \to M' \text{ such that } \widetilde{\Phi}\Big|_V = \widetilde{\Phi}_0 \circ \pi_{K,0}\,,\end{array} \tag{5.7}$$

$$\xi' \in \widetilde{\Omega}'^S \quad \Leftrightarrow \quad \begin{array}{l}\text{There is a neighborhood } V' \text{ of } \xi' \text{ in } \widetilde{\Omega}' \text{ and a smooth map} \\ \widetilde{\Psi}_0 : V'_0 \to M \text{ such that } \widetilde{\Psi}\Big|_{V'} = \widetilde{\Psi}_0 \circ \pi_{K,0}\,.\end{array} \tag{5.8}$$

Lemma 5.3. *In the analytic case, and if $\widetilde{\Omega} = \Omega \cap \Sigma$ and $\widetilde{\Omega}' = \Omega' \cap \Sigma'$ are connected, one has either $\widetilde{\Omega}^S = \widetilde{\Omega}$ or $\widetilde{\Omega}^S = \varnothing$, and either $\widetilde{\Omega}'^S = \widetilde{\Omega}'$ or $\widetilde{\Omega}'^S = \varnothing$.*

Lemma 5.4. *One has the following identities, where the two first ones hold for any subsets $S \subset \widetilde{\Omega}$, $S' \subset \widetilde{\Omega}'$ and any integer ℓ, $0 \le \ell \le K+K'$,*

$$\pi_{K+K',\ell}\left(\widetilde{\Phi}^{[K']^{-1}}(S')\right) = \widetilde{\Psi}^{[\ell]}\left(S'_{K'+\ell}\right),\quad \pi_{K+K',\ell}\left(\widetilde{\Psi}^{[K]^{-1}}(S)\right) = \widetilde{\Psi}^{[\ell]}\left(S_{K+\ell}\right), \tag{5.9}$$

$$\widetilde{\Phi}^{[1]}(\widetilde{\Omega}_{K+1}) = \widetilde{\Omega}'_1\,, \qquad \widetilde{\Psi}^{[1]}(\widetilde{\Omega}'_{K'+1}) = \widetilde{\Omega}_1\,. \tag{5.10}$$

Lemma 5.5. ***If $n < n'$,** then $\widetilde{\Omega}^S = \varnothing$. **If $n > n'$,** then $\widetilde{\Omega}'^S = \varnothing$.*

***If $n = n'$,** there is, for all $\xi_K \in \widetilde{\Omega}^S$, a neighborhood $\mathcal{V}_1$ of $\xi_1 = \pi_{K,1}(\xi_K)$ in Ω_1 and an open subset $\mathcal{V}'_1$ of Ω'_1 such that systems Σ and Σ' are static equivalent over $\mathcal{V}_1$ and $\mathcal{V}'_1$. There is also, for all $\xi'_{K'} \in \widetilde{\Omega}'^S$, a neighborhood $\mathcal{W}'$ of $\xi'_1 = \pi_{K',1}(\xi'_{K'})$ in Ω'_1 and an open subset $\mathcal{W}_1$ of Ω_1 such that systems Σ and Σ' are static equivalent over $\mathcal{W}_1$ and $\mathcal{W}'_1$. Finally,*

$$\pi_{K+K',K'}\left(\widetilde{\Psi}^{[K]^{-1}}\left(\widetilde{\Omega}^S\right)\right) = \widetilde{\Phi}^{[K']}\left(\widetilde{\Omega}^S_{K+K'}\right) \;=\; \widetilde{\Omega}'^S\,, \tag{5.11}$$

$$\pi_{K+K',K}\left(\widetilde{\Phi}^{[K']^{-1}}\left(\widetilde{\Omega}'^S\right)\right) = \widetilde{\Psi}^{[K]}\left(\widetilde{\Omega}'^S_{K'+K}\right) \;=\; \widetilde{\Omega}^S\,. \tag{5.12}$$

LEMMA 5.6. *For all $\xi_{K+1} \in \widetilde{\Omega}_{K+1}$ such that $\xi_K = \pi_{K+1,K}(\xi_{K+1}) \in \widetilde{\Omega} \setminus \widetilde{\Omega}^S$, there is a straight line in $\mathrm{T}_{\widetilde{\Phi}(\xi_K)}M'$ that has contact of infinite order with Σ' at $\widetilde{\Phi}^{[1]}(\xi_{K+1})$.*

These lemmas will be proved later. Let us finish the proof of the Theorem.

If $n < n'$, (5.10) implies existence, for each $\xi' \in \widetilde{\Omega}'_1 = \Omega_1 \cap \Sigma'$, of some $\xi_{K+1} \in \widetilde{\Omega}_{K+1}$ such that $\widetilde{\Phi}^{[1]}(\xi_{K+1}) = \xi'$ and finally, since $\widetilde{\Omega}^S$ is empty according to Lemma 5.5, Lemma 5.6 yields a straight line in $\mathrm{T}_{\xi'_0}M'$ that has contact of infinite order with Σ' at ξ'; from Proposition 3.7, this implies that system Σ' is ruled over Ω_1. If $n > n'$, one concludes in the same way.

Now assume $\boldsymbol{n = n'}$. For all ξ' in $\widetilde{\Phi}^{[1]}\left((\widetilde{\Omega} \setminus \widetilde{\Omega}^S)_{K+1}\right)$, there is, according to Lemma 5.6, a straight line in $\mathrm{T}_{\xi'_0}M'$ that has contact of infinite order with Σ' at ξ'. By continuity, this is also true for all ξ' in the topological closure

$$\widetilde{R}' \;=\; \overline{\widetilde{\Phi}^{[1]}\left((\widetilde{\Omega} \setminus \widetilde{\Omega}^S)_{K+1}\right)} = \overline{\pi_{K+K',1}\left({\widetilde{\Psi}^{[K]}}^{-1}\left(\widetilde{\Omega} \setminus \widetilde{\Omega}^S\right)\right)}, \tag{5.13}$$

where the second equality come from (5.9). Let $i(\widetilde{R}')$ be the interior of $\widetilde{R}'$ for the induced topology on Σ'; since $\widetilde{R}' = \overline{i(\widetilde{R}')}$, there is an open subset $\mathcal{R}'$ of $\Omega'_1 \subset \mathrm{T}M'$, enjoying the property that it is the interior of its topological closure, and such that $\mathcal{R}' \cap \Sigma' = i(\widetilde{R}')$ and $\overline{\mathcal{R}'} \cap \Sigma' = \widetilde{R}'$. From Proposition 3.7, Σ' is ruled over $\mathcal{R}'$. Setting $\mathcal{S}' = \Omega'_1 \setminus \overline{\mathcal{R}'}$, one has $\Omega'_1 = \overline{\mathcal{R}'} \cup \mathcal{S}' = \mathcal{R}' \cup \overline{\mathcal{S}'}$. Along the same lines, Σ is ruled over $\mathcal{R}$, open subset of $\Omega_1 \subset \mathrm{T}M$ such that $\mathcal{R} \cap \Sigma$ is the relative interior of

$$\widetilde{R} \;=\; \overline{\widetilde{\Psi}^{[1]}\left((\widetilde{\Omega}' \setminus \widetilde{\Omega}'^S)_{K'+1}\right)} = \overline{\pi_{K+K',1}\left({\widetilde{\Phi}^{[K']}}^{-1}\left(\widetilde{\Omega}' \setminus \widetilde{\Omega}'^S\right)\right)}, \tag{5.14}$$

and such that $\Omega_1 = \overline{\mathcal{R}} \cup \mathcal{S} = \mathcal{R} \cup \overline{\mathcal{S}}$ with $\mathcal{S} = \Omega_1 \setminus \mathcal{R}$.

We have proved (4.1) and point 1; let us prove point 2. Obviously,

$$\mathcal{S} \cap \Sigma \subset \pi_{K+K',1}\left({\widetilde{\Phi}^{[K']}}^{-1}(\widetilde{\Omega}'^S)\right) \quad \text{and} \quad \mathcal{S}' \cap \Sigma' \subset \pi_{K+K',1}\left({\widetilde{\Psi}^{[K]}}^{-1}(\widetilde{\Omega}^S)\right) .$$

Using identities (5.11) and (5.12), this implies

$$\mathcal{S} \cap \Sigma \subset \pi_{K,1}\big(\widetilde{\Omega}^S\big) \quad \text{and} \quad \mathcal{S}' \cap \Sigma' \subset \pi_{K',1}\big(\widetilde{\Omega}'^S\big) . \tag{5.15}$$

For all ξ in $\mathcal{S} \cap \Sigma$, there is one $\xi_K \in \widetilde{\Omega}^S$ such that $\xi = \pi_{K,1}(\xi_K)$ and, from Lemma 5.5, a neighborhood $\mathcal{V}_1^\xi$ of ξ in Ω_1 and an open subset ${\mathcal{V}'_1}^\xi$ of Ω'_1 such that systems Σ and Σ' are static equivalent over $\mathcal{V}_1^\xi$ and ${\mathcal{V}'_1}^\xi$. For all ξ' in $\mathcal{S}' \cap \Sigma'$, there is one $\xi'_{K'} \in \widetilde{\Omega}'^S$ such that $\xi' = \pi_{K',1}(\xi'_{K'})$ and, from Lemma 5.5, a neighborhood ${\mathcal{W}'}^{\xi'}$ of $\xi'_1 = \pi_{K',1}(\xi'_{K'})$ in Ω'_1 and an open subset $\mathcal{W}_1^{\xi'}$ of Ω_1 such that systems Σ and Σ' are static equivalent over $\mathcal{W}_1^{\xi'}$ and ${\mathcal{W}'_1}^\xi$.

Now, $(\mathcal{V}_1^\xi)_{\xi \in \mathcal{S}\cap\Sigma}$ is an open covering of $\mathcal{S} \cap \Sigma$ and $({\mathcal{W}'_1}^{\xi'})_{\xi' \in \mathcal{S}'\cap\Sigma'}$ is an open covering of $\mathcal{S}' \cap \Sigma'$. Take for $(\widetilde{\mathcal{S}}^\alpha)_{\alpha \in A}$ the union of $(\mathcal{V}_1^\xi)_{\xi \in \mathcal{S}\cap\Sigma}$ and $(\mathcal{W}_1^{\xi'})_{\xi' \in \mathcal{S}'\cap\Sigma'}$; take for $(\widetilde{\mathcal{S}}'^\alpha)_{\alpha \in A}$ the union of $({\mathcal{V}'_1}^\xi)_{\xi \in \mathcal{S}\cap\Sigma}$ and $({\mathcal{W}'_1}^{\xi'})_{\xi' \in \mathcal{S}'\cap\Sigma'}$.

This proves the smooth case, and obviously implies the real analytic one from Lemma 5.3. ◻

Let us now prove the four lemmas used in the above proof.

Proof of Lemma 5.3. If $\widetilde{\Omega}^S \neq \varnothing$, then there is at least an open set in $\widetilde{\Omega}$ derivatives of $\widetilde{\Phi}$ along any vertical vector field (preserving fibers of $\Sigma_K \to M$) are identically zero; since these are real analytic they must be zero all over $\widetilde{\Omega}$, assumed connected, hence $\widetilde{\Omega}^S = \widetilde{\Omega}$. The proof is similar in $\widetilde{\Omega}'$. ∎

Proof of Lemma 5.4. The first relation in (5.9) is a consequence of the two identities

$$\pi_{K'+\ell,K'} \circ \widetilde{\Phi}^{[K'+\ell]} = \widetilde{\Phi}^{[K']} \circ \pi_{K+K'+\ell,K+K'} \;\text{ and }\; \widetilde{\Psi}^{[\ell]} \circ \widetilde{\Phi}^{[K'+\ell]} = \pi_{K+K'+\ell,\ell}\,, \quad (5.16)$$

respectively (2.6) with $(r, s) = (K' + \ell, K')$ and the ℓ^{th} prolongation of (5.6). The second relation follows from interchanging K, Φ, S with K', Ψ, S'.

From equations (5.4) and (5.5), one has, for any positive integer ℓ,

$$\widetilde{\Phi}^{[\ell]}(\widetilde{\Omega}_{K+\ell}) \subset \widetilde{\Omega}'_\ell \;\text{ and }\; \widetilde{\Psi}^{[\ell]}(\widetilde{\Omega}'_{K'+\ell}) \subset \widetilde{\Omega}_\ell \quad (5.17)$$

(for instance, (5.4) implies $\widetilde{\Phi}^{[\ell]}(\widetilde{\Omega}_{K+\ell}) \subset \Sigma'_\ell$, (5.5) implies $\widetilde{\Phi}^{[\ell]}(\widetilde{\Omega}_{K+\ell}) \subset \Omega'_\ell$, hence the first relation above because $\widetilde{\Omega}'_\ell = \Omega'_\ell \cap \Sigma'_\ell$). We only need to prove the reverse inclusions for $\ell = 1$. Let us do it for the second one. The second relation in (5.16) for $\ell = 1$ implies $\widetilde{\Omega}_1 = \widetilde{\Psi}^{[1]}\left(\widetilde{\Phi}^{[K'+1]}(\widetilde{\Omega}_{K+K'+1})\right)$, and finally $\widetilde{\Omega}_1 \subset \widetilde{\Psi}^{[1]}(\widetilde{\Omega}'_{K'+1})$ from the first relation in (5.16) with $\ell = K' + 1$. ∎

Proof of Lemma 5.5. Assume for instance that $\widetilde{\Omega}^S$ is non-empty; then it contains an open subset V and there is a smooth $\widetilde{\Phi}_0 : V_0 \to M'$ such that, in restriction to V, $\widetilde{\Phi} = \widetilde{\Phi}_0 \circ \pi_{K,0}$. Hence (5.6) implies, on the open subset $V' = \left(\widetilde{\Psi}^{[K]}\right)^{-1}(V)$ of ${\Sigma'}_{K+K'}$,

$$\widetilde{\Phi}_0 \circ \pi_{K,0} \circ \widetilde{\Psi}^{[K]} = \pi_{K+K',0}|_{V'}\,. \quad (5.18)$$

The rank of the map on the left-hand side is n' while the rank of the right-hand side is no larger than n (rank of $\pi_{K,0}$), hence $\widetilde{\Omega}^S \neq \varnothing$ implies $n' \leq n$. By interchanging the two systems, this proves the fist sentence of the Lemma.

Let us now turn to the case where $n = n'$. Consider ξ_K in $\widetilde{\Omega}^S$. By definition of $\widetilde{\Omega}^S$, there is a neighborhood V and a smooth (real analytic in the real analytic case) map $\widetilde{\Phi}_0 : V_0 \to M'$ such that $\widetilde{\Phi} = \widetilde{\Phi}_0 \circ \pi_{K,0}$ on V. Let V' be defined from V as

$$V' = \pi_{K+K',K'}\left({\widetilde{\Psi}^{[K]}}^{-1}(V)\right) = \widetilde{\Phi}^{[K']}(V_{K+K'})\,, \quad (5.19)$$

where the second equality comes from (5.9). Applying $\widetilde{\Psi}$ and $\widetilde{\Psi}^{[1]}$ to both sides of the first equality in (2.6) and using (5.19) with $(r, s) = (K, 0)$ and $(r, s) = (K, 1)$ yields

$$\widetilde{\Psi}(V') = V_0\,, \qquad \widetilde{\Psi}^{[1]}(V'_{K'+1}) = V_1\,. \quad (5.20)$$

Substituting $\widetilde{\Phi} = \widetilde{\Phi}_0 \circ \pi_{K,0}$ in (5.6), one has $\widetilde{\Phi}_0 \circ \widetilde{\Psi} \circ \pi_{K+K',K'} = \pi_{K+K',0}$ on ${\widetilde{\Psi}^{[K]}}^{-1}(V)$, and finally

$$\widetilde{\Phi}_0 \circ \widetilde{\Psi} = \pi_{K',0} \text{ on } V'\,; \quad (5.21)$$

in a similar way, substituting $\widetilde{\Phi}^{[1]} = \widetilde{\Phi}_0^{[1]} \circ \pi_{K+1,1}$ in the first prolongation of (5.6),

$$\widetilde{\Phi}_0^{[1]} \circ \widetilde{\Psi}^{[1]} = \pi_{K'+1,1} \text{ on } V'_{K'+1} \,. \tag{5.22}$$

Applying $\widetilde{\Phi}_0$ to both sides of the first relation and $\widetilde{\Phi}_0^{[1]}$ to both sides of the second relation in (5.20), one has, using (5.21) and (5.22),

$$\widetilde{\Phi}_0(V_0) = V'_0 \,, \quad \widetilde{\Phi}_0^{[1]}(V_1) = V'_1 \,. \tag{5.23}$$

Since the rank of $\pi_{K',0}$ in the right-hand side of (5.21) is $n' = n$ at all points of V', $\widetilde{\Phi}_0$ must be a local diffeomorphism at all point of $\widetilde{\Psi}(V') = V_0$ and in particular at ξ_0: by the inverse function theorem, there is a neighborhood O of $\xi_0 = \pi_{K,0}(\xi)$ in V_0 and a neighborhood O' of $\Phi_0(\xi_0)$ in M' such that Φ_0 defines a diffeomorphism $O \to O'$.

Let us now replace V with $V \cap {\pi_{K,0}}^{-1}(O)$, a smaller neighborhood of ξ_K; V' is still defined by (5.19) from this smaller V, one has $V_0 = O$, the former $\widetilde{\Phi}_0$ is replaced by its restriction to this smaller V_0, and the above relations still hold. In particular, $O' = \widetilde{\Phi}_0(O)$ must be all V'_0 according to (5.23), i.e. $\widetilde{\Phi}_0$ defines a diffeomorphism $V_0 \to V'_0$; let $\widetilde{\Psi}_0$ be its inverse. Composing each side of (5.21) with $\widetilde{\Psi}_0$, one gets $\widetilde{\Psi} = \widetilde{\Psi}_0 \circ \pi_{K',0}$ on V'; hence, by (5.8), one has $V' \subset \widetilde{\Omega}'^S$ and, since this is true for all ξ_K in $\widetilde{\Omega}^S$, one has

$$\pi_{K+K',K'} \left({\widetilde{\Psi}^{[K]}}^{-1}(\widetilde{\Omega}^S) \right) = \widetilde{\Phi}^{[K']}(\widetilde{\Omega}^S_{K+K'}) \subset \widetilde{\Omega}'^S \,. \tag{5.24}$$

Let $\mathcal{V}_1$ and $\mathcal{V}'_1$ and be open subsets of Ω_1 and Ω'_1 such that

$$V_1 = \Sigma \cap \mathcal{V}_1 \,, \quad V'_1 = \Sigma \cap \mathcal{V}'_1 \,. \tag{5.25}$$

From Proposition 5.2, the second relation in (5.23) implies that systems Σ and Σ' are static equivalent over $\mathcal{V}_1$ and $\mathcal{V}'_1$. Interchanging the two systems, one proves that

$$\pi_{K+K',K} \left({\widetilde{\Phi}^{[K']}}^{-1}(\widetilde{\Omega}'^S) \right) = \widetilde{\Psi}^{[K]}(\widetilde{\Omega}'^S_{K+K'}) \subset \widetilde{\Omega}^S \,. \tag{5.26}$$

and that, for all $\xi'_{K'} \in \widetilde{\Omega}'^S$, there are a neighborhood $\mathcal{W}'$ of $\xi'_1 = \pi_{K',1}(\xi'_{K'})$ in Ω'_1 and an open subset $\mathcal{W}_1$ of Ω_1 such that systems Σ and Σ' are static equivalent over $\mathcal{W}_1$ and $\mathcal{W}'_1$.

Now, $\widetilde{\Phi}^{[K']}(\widetilde{\Omega}^S_{K+K'}) \subset \widetilde{\Omega}'^S$ in (5.24) implies $\widetilde{\Omega}^S_{K+K'} \subset {\widetilde{\Phi}^{[K']}}^{-1}(\widetilde{\Omega}'^S)$, and hence $\widetilde{\Omega}^S \subset \pi_{K+K',K} \left({\widetilde{\Phi}^{[K']}}^{-1}(\widetilde{\Omega}'^S) \right)$. Hence (5.24) implies the converse inclusion in (5.26); in a similar way (5.26) implies the converse inclusion in (5.24). This proves (5.11) and (5.12), and ends the proof of Lemma 5.5. □

Proof of Lemma 5.6. Denote by $\bar{\xi}_{K+1}$ the point ξ_{K+1} in the lemma statement and set $\bar{\xi}_K = \pi_{K+1,K}(\bar{\xi}_{K+1}) \in \widetilde{\Omega} \setminus \widetilde{\Omega}^S$, $\bar{\xi}_0 = \pi_{K,0}(\bar{\xi}_{K+1})$, $\bar{\xi}_1 = \pi_{K,1}(\bar{\xi}_{K+1})$. From Proposition 3.4, and after possibly shrinking $\mathcal{U}_K$ so that it is contained in Ω, there exist a neighborhood $\mathcal{U}_K \subset \Omega$ of $\bar{\xi}_K$ in $J^K(M)$, coordinates $(x_{\mathrm{I}}, x_{\mathrm{II}})$ on $\mathcal{U}_0 = \pi_{K,0}(\mathcal{U}_K)$ inducing coordinates $(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{I}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{I}}^{(K)}, x_{\mathrm{II}}^{(K)})$ on $\mathcal{U}_K$, and an open subset $U_K \subset \mathbb{R}^{n+Km}$ such that the equations of $\widetilde{\mathcal{U}}_K = \mathcal{U}_K \cap \Sigma_K$ in $J^K(M)$ in these coordinates are

$$\begin{aligned} &x_{\mathrm{I}}^{(i)} = f^{(i-1)}(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(i)}) \,, \quad 1 \le i \le K \,, \\ &(x_{\mathrm{I}}, x_{\mathrm{II}}, \dot{x}_{\mathrm{II}}, \ldots, x_{\mathrm{II}}^{(K)}) \in U_K \,. \end{aligned} \tag{5.27}$$

By substitution, there is a unique smooth map $\phi_K : U_K \to M'$ such that $\widetilde{\Phi}(\xi) = \phi_K(x_{\rm I}, x_{\rm II}, \dot{x}_{\rm II}, \ldots, x_{\rm II}^{(K)})$ for all ξ in $\widetilde{\mathcal{U}}_K$ with coordinate vector $(x_{\rm I}, x_{\rm II}, \ldots, x_{\rm I}^{(K)}, x_{\rm II}^{(K)})$.

Let $\overline{X}_i = (\overline{x_{\rm I}}, \overline{x_{\rm II}}, \overline{\dot{x}_{\rm I}}, \overline{\dot{x}_{\rm II}}, \ldots, \overline{x_{\rm I}^{(i)}}, \overline{x_{\rm II}^{(i)}})$ be the coordinate vector of $\bar{\xi}_i$ for $i \leq K+1$ and $\bar{\rho}$ the smallest integer such that ϕ_K does not depend on $x_{\rm II}^{(\bar{\rho}+1)}, \ldots, x_{\rm II}^{(K)}$ on at least one neighborhood of $\overline{X}_K$. Shrinking U_K to this neighborhood, and $\widetilde{\mathcal{U}}_K$ accordingly, we may define $\phi : U_{\bar{\rho}} \to M'$, with $U_{\bar{\rho}}$ the projection of U_K on $I\!R^{n+\bar{\rho}\,m}$, such that $\widetilde{\Phi}(\xi) = \phi_K(x_{\rm I}, x_{\rm II}, \dot{x}_{\rm II}, \ldots, x_{\rm II}^{(K)}) = \phi(x_{\rm I}, x_{\rm II}, \dot{x}_{\rm II}, \ldots, x_{\rm II}^{(\bar{\rho})})$. If $\bar{\rho}$ was zero, one would have $\widetilde{\Phi}(\xi) = \phi(x_{\rm I}, x_{\rm II})$, hence the right-hand side of (5.7) would be satisfied for $\xi = \bar{\xi}_K$ with $V = \widetilde{\mathcal{U}}_K$; this is impossible because we assumed $\bar{\xi}_K \in \widetilde{\Omega} \setminus \widetilde{\Omega}^S$. Hence $\bar{\rho} \geq 1$.

For all ξ_{K+1} in $\widetilde{\mathcal{U}}_{K+1}$ with coordinate vector $(x_{\rm I}, x_{\rm II}, \ldots, x_{\rm II}^{(K+1)})$, one has

$$\widetilde{\Phi}^{[1]}(\xi_{K+1}) = \chi(x_{\rm I}, x_{\rm II}, \dot{x}_{\rm II}, \ldots, x_{\rm II}^{(\bar{\rho})}, x_{\rm II}^{(\bar{\rho}+1)}) \tag{5.28}$$

with $\chi : U_{\bar{\rho}+1} \to {\rm T}M'$ the map defined by

$$\chi(x_{\rm I} \ldots x_{\rm II}^{(\bar{\rho}+1)}) = \Big(\phi(x_{\rm I} \ldots x_{\rm II}^{(\bar{\rho})}) \; , \; a(x_{\rm I} \ldots x_{\rm II}^{(\bar{\rho})}) + \frac{\partial \phi}{\partial x_{\rm II}^{(\bar{\rho})}}(x_{\rm I} \ldots x_{\rm II}^{(\bar{\rho})}) \; x_{\rm II}^{(\bar{\rho}+1)} \Big) \tag{5.29}$$

with $a = \frac{\partial \phi}{\partial x_{\rm I}} f + \sum_{i=0}^{\bar{\rho}-1} \frac{\partial \phi}{\partial x_{\rm II}^{(i)}} \, x_{\rm II}^{(i+1)}$. According to (5.5), (5.27) and (5.28), Σ' contains $\chi(U_{\bar{\rho}+1})$. Now, for any $(x_{\rm I}, \ldots, x_{\rm II}^{(\bar{\rho}+1)}) \in U_{\bar{\rho}+1}$ such that the linear map

$$\frac{\partial \phi}{\partial x_{\rm II}^{(\rho)}}(x_{\rm I}, \ldots, x_{\rm II}^{(\bar{\rho})}) : \quad I\!R^m \;\to\; {\rm T}_{\phi(x_{\rm I}, \ldots, x_{\rm II}^{(\bar{\rho})})} M'$$

is nonzero, picking $\underline{w} \neq 0$ in its range, (5.29) implies that the straight line Δ in ${\rm T}_{\phi(x_{\rm I}, \ldots, x_{\rm II}^{(\bar{\rho})})} M'$ passing through $\chi(x_{\rm I} \ldots x_{\rm II}^{(\bar{\rho}+1)})$ with direction $\underline{w}$ has a segment around $\chi(x_{\rm I} \ldots x_{\rm II}^{(\bar{\rho}+1)})$ contained in Σ', hence in particular Δ has contact of infinite order with Σ' at point $\chi(x_{\rm I}, \ldots, x_{\rm II}^{(\bar{\rho}+1)})$. To sum up, we have proved so far that, for all ξ_{K+1} in $\widetilde{\mathcal{U}}_{K+1}$ with coordinate vector $(x_{\rm I}, x_{\rm II}, \ldots, x_{\rm II}^{(K+1)})$ such that $\frac{\partial \phi}{\partial x_{\rm II}^{(\rho)}}(x_{\rm I} \ldots x_{\rm II}^{(\bar{\rho})})$ is nonzero, there is a straight line $\Delta_{\xi_{K+1}}$ in ${\rm T}_{\widetilde{\Phi}(\xi_K)} M'$ passing through $\widetilde{\Phi}^{[1]}(\xi_{K+1})$ that has contact of infinite order with Σ' at $\widetilde{\Phi}^{[1]}(\xi_{K+1})$. The set of such points ξ_{K+1} may not contain $\bar{\xi}_{K+1}$ but its topological closure does, by minimality of $\bar{\rho}$; taking a sequence of points ξ_{K+1} that converges to $\bar{\xi}_{K+1}$, any accumulation point of the compact sequence $\big(\Delta_{\xi_{K+1}}\big)$ is a straight line in ${\rm T}_{\widetilde{\Phi}(\bar{\xi}_K)} M'$ passing through $\widetilde{\Phi}^{[1]}(\bar{\xi}_{K+1})$ that has contact of infinite order with Σ' at $\widetilde{\Phi}^{[1]}(\bar{\xi}_{K+1})$. □

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