Second order sliding mode and adaptive observer for synchronization of a chaotic system: a comparative study

Francisco J. Bejarano, Malek Ghanes, Jean-Pierre Barbot, Leonid Fridman

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Francisco J. Bejarano * **Malek Ghanes** * **Jean-Pierre Barbot** *
Leonid Fridman **

* *Equipe Commande des Systèmes (ECS), ENSEA 6 Avenue du Ponceau, 95014 Cergy-Pontoise, France, (e-mail: {Francisco.Bejarano, Malek.Ghanes, barbot}@ensea.fr)*
** *National Autonomous University of Mexico, Engineering Faculty, Division of Electrical Engineering, UNAM 04510, México, D.F., (e-mail: lfridman@servidor.unam.mx)*

Abstract:

1. INTRODUCTION

2. PROBLEM STATEMENT

Consider the following chaotic system

$$\left\{\begin{array}{c} \dot{x}_1(t) = a(x_2(t) - x_1(t)) + x_2(t)\,x_3(t) + m_1(t) \\ \dot{x}_2(t) = b(x_1(t) + x_2(t)) - x_1(t)\,x_3(t) \\ \dot{x}_3(t) = -cx_3(t) - ex_4(t) + x_1(t)\,x_2(t) + m_1(t) \\ \dot{x}_4(t) = fx_3(t) - dx_4(t) + x_1(t)\,x_3(t) + m_2(t) \\ y_1(t) = x_1(t) \\ y_2(t) = x_2(t) \end{array}\right. \tag{1}$$

where $x_i \in \mathbb{R}$ $(i = 1, 2, 3, 4)$ are the states of the system; m_1 and m_2 represent the information to be transmitted, which for the observation problem are considered as the unknown inputs; y_1 and y_2 are the outputs of the system, these are the signals to be sent by a public channel.

Assumption. 2.1. The signals m_1 and m_2 are assumed to be differentiable, bounded and with the derivative bounded.

The goal is to reconstruct the states x_3 and x_4 which allows to reconstruct the messages m_1 and m_2. For achieving such a goal two approaches are tested, namely, an observer based on the super-twisting algorithm that basically allows to obtain information from the derivatives of the output and consequently to reconstruct the states and the messages (unknown inputs), and an adaptive observer. A comparison between these two methods will be discussed.

It should be notice that a singularity appears in the point $x_1 = 0$, that is, from the outputs x_1 and x_2 is not possible to know the states x_3 and x_4 and consequently neither the messages m_1 and m_2.

3. SUPER-TWISTING OBSERVER

First in order to generate a new output, namely the variable x_3, we use the super-twisting algorithm for designing the observer for $\hat{x}_3$ in the following way:

$$\left\{\begin{array}{c} \dot{x}_{a,1} = b(x_1 + x_2) + v_1 \\ v_1 = \hat{x}_{1,3} + \lambda_1 |s_1|^{1/2} \operatorname{sign} s \\ \dot{\hat{x}}_{1,3} = \alpha_1 \operatorname{sign} s \\ s_1 = x_2 - x_{a,1} \end{array}\right. \tag{2}$$

In this way, the derivative of s_1 takes the form

$$\dot{s}_1 = -x_1 x_3 - v^1 \tag{3}$$

Choosing the gains $\lambda_1 \geq \frac{(\alpha_1 + M_1)(1+\theta)}{1-\theta}\sqrt{\frac{2}{\alpha_1 + M_1}}$ and $\alpha_1 > M_1 \geq \left|\frac{d}{dt}(x_1 x_3)\right|$ $(0 < \theta < 1)$, we get, according to (Levant, 93), (Levant, 98), and (Davila, 05), the second order sliding motion, that is, $s(t) = 0$, $\dot{s}(t) = 0$ after some finite time T_1. Notice that for $s = 0$, $v^1 = \hat{x}_3$; therefore, from (3), we get

$$\hat{x}_{1,3}(t) \equiv -x_1(t)\,x_3(t) \tag{4}$$

From (4) we can reconstruct $x_3(t)$ provided $x_1(t) \neq 0$. Thus, the observer for x_3 is designed in the form

$$\hat{x}_3(t) = \left\{\begin{array}{ll} -\dfrac{\hat{x}_{1,3}(t)}{x_1(t)} & \text{if } |x_1(t)| \geq \varepsilon \\ \hat{x}_3(t-\tau) & \text{if } |x_1(t)| < \varepsilon \end{array}\right. \tag{5}$$

where τ and ε are enough small constants [1]. Thus, we get the identity

$$\hat{x}_3(t) \equiv x_3(t) \text{ for } |x_1(t)| \geq \varepsilon.$$

Reconstruction of x_4 Now, defining $\bar{y}_3(t) = x_3(t)$ as a new output, we can rewrite the dynamic equations in (1) as a linear system with output injection and unknown inputs, that is,

[1] The constant τ is chosen enough small but bigger than the sampling time used during the realization of the observer. The constant ε should be chosen sufficiently big to avoid the singularity, but also should be notice that any estimation in this zone can not be considered as an acceptable estimation.

$$\underbrace{\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}}=\underbrace{\begin{bmatrix}\text{-}a & a & 0 & 0\\ b & b & 0 & 0\\ 0 & 0 & \text{-}c & \text{-}e\\ 0 & 0 & f & \text{-}d\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}}_{x}+\underbrace{\begin{bmatrix}y_2\bar{y}_3\\ \text{-}y_1\bar{y}_3\\ y_1y_2\\ y_1\bar{y}_3\end{bmatrix}}_{\phi(y_1,y_2,\bar{y}_3)}+\underbrace{\begin{bmatrix}1 & 0\\ 0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}m_1\\ m_2\end{bmatrix}}_{w}$$

$$\underbrace{\begin{bmatrix}y_1\\ y_2\\ \bar{y}_3\end{bmatrix}}_{\bar{y}}=\underbrace{\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}}_{C}x \tag{6}$$

Now, let z be defined by the solution of the following differential equation

$$\dot{z} = Az + \phi(y_1, y_2, \bar{y}_3)$$

Thus, defining $e_z = x - z$ we obtain the dynamic equation for e_z

$$\begin{array}{c}\dot{e_z}(t) = Ae_z(t) + Dw(t)\\ y_z = Ce_z\end{array} \tag{7}$$

Hence, the system (7) is basically a linear system with unknown inputs, just the sort of systems considered in (Bejarano, 07). Then, for the reconstruction of x_4 we follow in essence, but with a little modification, the algorithm proposed in (Bejarano, 07). That is, the basic idea proposed in (Bejarano, 07) is to obtain an algebraic expression of e_z in terms of the output y_z and its derivatives. Thus, we get the equalities [2]

$$y_z = Ce_z$$
$$\frac{d}{dt}(CD)^{\perp} y_z = (CD)^{\perp} CAe_z$$

thus we get [3]

$$e_z(t) = \begin{bmatrix}C\\ (CD)^{\perp} CA\end{bmatrix}^{+}\begin{bmatrix}y_z(t)\\ (CD)^{\perp} y_z(t)\end{bmatrix} \tag{8}$$

Since we already know the first 3 states, we solve (8) for $e_{z,4}$, the fourth state of e_z. Thus,

$$\begin{array}{c}e_{z,4} = \frac{1}{e}\left[\dot{e}_{z,1} - \dot{e}_{z,3} + a(e_{z,1} - e_{z,2})\right.\\ \left.-ce_{z,3} + e_{z,1}e_{z,2} - e_{z,2}e_{z,3}\right]\end{array} \tag{9}$$

3.1 Realization of the observer using super-twisting

To reduce the fast dynamic and have a smaller gains in the super-twisting algorithm, we design the following like-linear estimator $\tilde{x}$ whose dynamics is governed by the following differential equation

$$\dot{\tilde{x}} = \underbrace{\begin{bmatrix}-c & -e\\ f & -d\end{bmatrix}}_{\bar{A}}\tilde{x} + \begin{bmatrix}y_1y_2\\ y_1\bar{y}_3\end{bmatrix} + L(\bar{y}_3 - \tilde{y}_3)$$
$$\tilde{y}_3 = \underbrace{[\,1\ 0\,]}_{\bar{C}}\tilde{x} \tag{10}$$

The matrix L is chosen so that the eigenvalues of the matrix $(\bar{A} - L\bar{C})$ have negative real part. In this way we have that the dynamic equations for $\bar{e} = \tilde{x} - \begin{bmatrix}x_3\\ x_4\end{bmatrix}$ are

$$\dot{\bar{e}}(t) = (\bar{A} - LC)\,\bar{e}(t) + w(t)$$

where w is defined in (6). Hence we get some upper bounds for the norm of $\bar{e}$ and for the norm of $\dot{\bar{e}}$ that is

$$\|\bar{e}(t)\| \le \gamma \exp(-\lambda t)\,\|\bar{e}(0)\| + \mu\,\|w(t)\|$$
$$\gamma,\ \lambda,\ \mu \text{ are positive constants}$$

That is, with the estimator (10), $\bar{e}$ is constrained to stayed in a zone depending on the amplitude of m_1 and m_2; and this zone can be made smaller by moving the eigenvalues of $(\bar{A} - LC)$ more to the left in the left half-plane.

Then, following the algebraic expression obtained in (9) we design the observer for x_4 using the super-twisting algorithm as follows

$$\begin{cases}\dot{x}_{a,2} = \frac{1}{e}\left[a(x_2 - x_1) + c\hat{x}_3 - x_1x_2 + x_2\hat{x}_3\right] + \tilde{x}_4 + v_2\\ v_2 = v_{2,1} + \lambda_2 |s_2| \operatorname{sign} s_2,\ \dot{v}_{2,1} = \alpha \operatorname{sign} s_2\\ s_2 = \frac{1}{e}(x_1 - x_3) - x_{a,2}\\ \hat{x}_4(t) = \begin{cases}\tilde{x}_4 + v_{2,1} & \text{if } |x_1(t)| \ge \varepsilon\\ \hat{x}_4(t - \tau) & \text{if } |x_1(t)| < \varepsilon\end{cases}\end{cases}$$

where $\tilde{x}_4$ is the second component of the vector $\tilde{x}$ defined in (10). Then, the time derivative of s_2 is

$$\dot{s}_2 = x_4 - \tilde{x}_4 + v_2$$

Thus, for $\lambda_2 \ge \frac{(\alpha_2+M_2)(1+\theta)}{1-\theta}\sqrt{\frac{2}{\alpha_2+M_2}}$ and $\alpha_2 > M_2 \ge \left|\frac{d}{dt}(x_4 - \tilde{x}_4)\right|$ $(0 < \theta < 1)$, after some finite time T_2, we get $s_2 = 0$ and $\dot{s}_2 = 0$; therefore,

$$\hat{x}_4 \equiv x_4 \text{ for } |x_1(t)| \ge \varepsilon.$$

3.2 Messages reconstruction

The reconstruction of m_1 is made in the following way

$$\begin{cases}\dot{x}_{\mathrm{a},3} = a(x_2 - x_1) + x_2\hat{x}_3 + v_3\\ v_3 = v_{3,1} + \lambda_3 |s_3| \operatorname{sign} s_3\\ \dot{v}_{3,1} = \alpha_3 \operatorname{sign} s_3\\ \hat{m}_1 = \begin{cases}v_{3,1} & \text{if } |x_1(t)| \ge \varepsilon\\ \hat{m}_1(t - \tau) & \text{if } |x_1(t)| < \varepsilon\end{cases}\\ s_3 = x_1 - x_a^3\end{cases}$$

Thus, for α_3 and λ_3 satisfying $\lambda_3 \ge \frac{(\alpha_3+M_3)(1+\theta)}{1-\theta}\sqrt{\frac{2}{\alpha_3+M_3}}$ and $\alpha_3 > M_3 \ge |\dot{m}_1|$ $(0 < \theta < 1)$, after some finite time, we get the equalities $s_3 = 0$, $\dot{s}_3 = 0$. Thus, we get the equality

$$\hat{m}_1 \equiv m_1, \text{ for } |x_1(t)| \ge \varepsilon$$

The reconstruction of m_2 is made in a similar way, that is,

$$\begin{cases}\dot{x}_{a,4} = b(x_1 + x_2) + fx_3 - dx_4 + v_4\\ v_4 = v_{4,1} + \lambda_4 |s_4| \operatorname{sign} s_4\\ \dot{v}_{4,1} = \alpha_4 \operatorname{sign} s_4\\ \hat{m}_2 = \begin{cases}v_{4,1} & \text{if } |x_1(t)| \ge \varepsilon\\ \hat{m}_2(t - \tau) & \text{if } |x_1(t)| < \varepsilon\end{cases}\\ s_4 = x_2 + x_4 - x_{a,4}\end{cases}$$

Thus, taking into account (1) and the derivative of $x_{\mathrm{a},4}$, with the choosing of α_4 and λ_4 satisfying the inequalities $\lambda_4 \ge \frac{(\alpha_4+M_4)(1+\theta)}{1-\theta}\sqrt{\frac{2}{\alpha_4+M_4}}$ and $\alpha_4 > M_4 \ge |\dot{m}_2|$ $(0 < \theta < 1)$ (see, (Levant, 93), (Levant, 98)), we get, after some finite time,

$$\hat{m}_2 \equiv m_2 \text{ for } |x_1(t)| \ge \varepsilon$$

[2] $Y^{\perp}$ is a full row rank matrix such that $Y^{\perp}Y = 0$.

[3] The matrix X^{+} is defined to be the pseudo-inverse of X. The matrix considered in (8) belongs to sort of matrices of full column rank. Hence, in such a case, $X^{+} = (X^TX)^{-1}X^T$.

Remark. 3.1. At a glance, it seems that, during the estimation of the state x_3, it is sufficient with to use $x_1(t) \neq 0$ instead of $|x_1(t)| \geq \varepsilon$, but the justification is given in the following lines. It is known that during the realization of the super-twisting algorithm s_1 and $\dot{s}_1$ are not exactly zero, hence, instead of having (4) we really have the equality $\hat{x}_{1,3}(t) = -x_1(t)x_3(t) + \Delta(t)$, where Δ represents the estimation error and which does not tends to zero. Then, after dividing over x_1, it yields the equality $\bar{x}_3 := -\frac{\hat{x}_{1,3}}{x_1} = x_3 - \frac{\Delta}{x_1}$, which means that, when x_1 is very close to zero, the error between $\bar{x}_3$ and x_3 is equal to $O(1/x_1)$. Therefore, in a small neighborhood of $x_1 = 0$, the error between $\bar{x}_3$ and x_3 is extremely big. This justify the structure of $\hat{x}_3$.

4. ADAPTIVE OBSERVER

The model of a chaotic system (1) can be rewritten in the following interconnected compact form

$$\begin{cases} \dot{X}_1 = A_1(y_2)X_1 + g_1(y, X_1, X_2) + \phi_1(t) \\ y_1 = C_1X_1 \end{cases} \quad (11)$$

$$\begin{cases} \dot{X}_2 = A_2(y_1)X_2 + g_2(y, X_2, X_1) + \phi_2(t) \\ y_2 = C_2X_2 \end{cases} \quad (12)$$

where $X_1 = (x_1, x_3, x_4, x_5)^T$ is the state of the first subsystem with $x_5 := m_2$, $X_2 = (x_2, x_3, x_6)^T$ is the state of the second subsystem with $x_6 := m_1$. $y = [x_1, x_2]^T$ are the output of the whole system, and

$$A_1(y_2) = \begin{pmatrix} 0 & y_2 & 0 & 0 \\ 0 & 0 & -c & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \; A_2(y_1) = \begin{pmatrix} 0 & -y_1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$g_1(y, X_2, X_1) = \begin{pmatrix} a(y_1 - y_2) + x_6 \\ -ex_4 + y_1y_2 + x_6 \\ fx_3 - dx_4 + y_1x_3 \\ 0 \end{pmatrix}$$

$$g_2(y, X_2, X_1) = \begin{pmatrix} b(y_1 + y_2) \\ -cx_3 - ex_4 + y_1y_2 \\ 0 \end{pmatrix}$$

$$\phi_i(t) = \begin{bmatrix} 0 \\ 0 \\ \dot{m}_i \end{bmatrix}, \; i = 1, 2$$

$$C_1 = (\,1\;0\;0\;0\,), \; C_2 = (\,1\;0\;0\,).$$

Remark. 4.1. The choice of the variables of each subsystem has been considered in order to separate in one subsystem the message m_1 and in the other one the message m_2. It is clear that other choice can be considered in order to represent these subsystems provided the necessary conditions to design an adaptive are satisfied.

Next, let us introduce an adaptive observer in order to estimate the system's state and the unknown inputs simultaneously. It is based on interconnection between several subsystems which satisfies some required properties, such that the property of inputs persistency ((Hammouri, 90), (Ghanes, 06)).

At first, let us introduce the following assumptions in order to establish the results concerning the adaptive observer design (see more details in (Ghanes, 06)).

Assumption. 4.1.

1. The signals X_1 and X_2 are assumed bounded and to be regularly persistent ((Hammouri, 90), (Ghanes, 06)) in order to guarantee the observability property of subsystems (11) and (12), respectively.
2. $A_1(y_2)$ and $A_2(y_1)$ are uniformly bounded.
3. $g_1(y, X_1, X_2)$ is globally Lipschitz with respect to X_2 and uniformly with respect to (y, X_1).
4. $g_2(y, X_2, X_1)$ is globally Lipschitz with respect X_1 and uniformly with respect to (y, X_2).
5. The unknown functions $\dot{m}_i(t)$ $(i = 1, 2)$ are assumed to be bounded.

Then, an adaptive observer for interconnected subsystems (11) and (12) estimating the state and unknown parameters is given by

$$\begin{cases} \dot{Z}_1 = A_1(y_2)Z_1 + g_1(y, Z_1, Z_2) + S_1^{-1}C_1^T(y_1 - \hat{y}_1) \\ \dot{S}_1 = -\theta_1 S_1 - A_1^T(y_2)S_1 - S_1A_1(y_2) + C_1^TC_1 \\ \hat{y}_1 = C_1Z_1 \end{cases} \quad (13)$$

$$\begin{cases} \dot{Z}_2 = A_2(y_1)Z_2 + g_2(y, Z_2, Z_1) + S_2^{-1}C_2^T(y_2 - \hat{y}_2) \\ \dot{S}_2 = -\theta_2 S_2 - A_2^T(y_1)S_2 - S_2A_2(y_1) + C_2^TC_2 \\ \hat{y}_2 = C_2Z_2 \end{cases} \quad (14)$$

where $Z_1 = (\hat{x}_1, \hat{x}_3, \hat{x}_4, \hat{x}_5)^T$; $Z_2 = (\hat{x}_2, \hat{x}_3, \hat{x}_6)^T$ $S_i = S_i^T > 0$, $i = 1, 2$. Note that $S_1^{-1}C_1^T$ and $S_2^{-1}C_2^T$ are the gains of the observers (13) and (14), respectively.

Remark. 4.2. It is worth noticing that $\|S_1\|$ and $\|S_2\|$ are bounded for θ_1 and θ_2 large enough due to the persistency of input considered in assumption 4.1.

Now, in order to guarantee the convergence of the proposed observer, sufficient conditions are established in the following result. Denote the estimation errors:

$$\epsilon_1 = X_1 - Z_1 \text{ and } \epsilon_2 = X_2 - Z_2$$

whose dynamics are given by

$$\begin{cases} \dot{\epsilon}_1 = [A_1(y_2) - S_1^{-1}C_1^TC_1]\epsilon_1 \\ +g_1(y, X_1, X_2) - g_1(y, Z_1, Z_2) + \phi_1(t) \end{cases} \quad (15)$$

$$\begin{cases} \dot{\epsilon}_2 = [A_2(y_1) - S_2^{-1}C_2^TC_2]\epsilon_2 \\ +g_2(y, X_2, X_1) - g_2(y, Z_2, Z_1) + \phi_2(t). \end{cases} \quad (16)$$

The values of θ_1 and θ_2 are chosen to satisfy the inequalities

$$\delta_1 = (\theta_1 - \Gamma\eta) > 0, \; \delta_2 = (\theta_2 - \frac{\Gamma}{\eta}) > 0 \quad (17)$$

where $\Gamma = \tilde{\mu}_1 + \tilde{\mu}_2$, with $\tilde{\mu}_i = \frac{\mu_i}{\sqrt{\lambda_{min}(S_1)}\sqrt{\lambda_{min}(S_2)}}$, $i = 1, 2$; and $\mu_1 = k_1k_2$, $\mu_2 = k_3k_4$, $\eta \in]0, 1[$. The parameters k_1, k_2, k_3, k_4 are positive constants and $\lambda_{min}(S_1), \lambda_{min}(S_2)$ are the minimal eigenvalues of S_1 and S_2 respectively.

Lemma 4.1. Consider the system (13)-(14) and that assumption 4.1 holds. Then, the system (13)-(14) is a practical exponential observer for system (11)-(12) for θ_1 and θ_2 satisfying the inequalities (17). Furthermore, the observer converges arbitrarily fast with a convergence rate fixed by a parameter δ, $\delta = min(\delta_1, \delta_2)$.

Sketch of proof 4.1. Consider the following Lyapunov function candidate:

$$V_o = V_1 + V_2$$

where $V_1 = \epsilon_1^T S_1 \epsilon_1$ and $V_2 = \epsilon_2^T S_2 \epsilon_2$.

From assumption 4.1, the following inequalities hold

$\|S_1\| \leq k_1$;
$\|\{g_1(y, X_1, X_2) - g_1(y, Z_1, Z_2)\}\| \leq k_2 \|\epsilon_2\|$;
$\|S_2\| \leq k_3$;
$\|\{g_2(y, X_2, X_1) - g_2(y, Z_2, Z_1)\}\| \leq k_4 \|\epsilon_1\|$.
$\|\phi_1\| \leq k_5$.
$\|\phi_2\| \leq k_6$.

Computing the time derivative of V_o, , by using the above inequalities it follows that

$$\dot{V}_o \leq -\theta_1 \epsilon_1^T S_1 \epsilon_1 + 2\mu_1 \|\epsilon_1\| \|\epsilon_2\| + \|\phi_1\| \qquad (18)$$
$$- \theta_2 \epsilon_2^T S_2 \epsilon_2 + 2\mu_2 \|\epsilon_2\| \|\epsilon_1\| + \|\phi_2\|$$

Now, consider that the following inequalities are satisfied $\lambda \min(S_i) \|\epsilon_i\|^2 \leq \|\epsilon_i\|^2_{S_i} \leq \lambda \max(S_i) \|\epsilon_i\|^2, i = 1, 2$.
By writing (18) in terms of functions V_1 and V_2, it follows that

$$\dot{V}_o \leq -\theta_1 V_1 - \theta_2 V_2 + 2(\tilde{\mu}_1 + \tilde{\mu}_2)\sqrt{V_1}\sqrt{V_2} + k_5 + k_6$$

where the parameters $\tilde{\mu}_1, \tilde{\mu}_2$, are defined just before lemma 4.1.
Next, by using the following inequality $\sqrt{V_1}\sqrt{V_2} \leq \frac{\upsilon}{2} V_1 + \frac{1}{2\upsilon} V_2$, $\forall \upsilon \in]0, 1[$, one get

$$\dot{V}_o \leq -(\theta_1 - \Gamma)V_1 - (\theta_2 - \frac{\Gamma}{\upsilon})V_2 + k_5 + k_6.$$

where Γ is defined just before lemma 4.1.

By taking δ and r such that $\delta = min(\delta_1, \delta_2)$ and $r = k_5 + k_6$ one has

$$\dot{V}_o \leq -\delta V_o + r. \qquad (19)$$

Finally, by choosing θ_1 and θ_2 such that the inequalities (17) are satisfied and sufficiently large, the inequality (19) shows that arbitrarily bounded perturbation will not result in large error estimation deviations. This ends the proof.

5. NUMERICAL EXAMPLE AND DISCUSSIONS

For the system (1) we use the parameters $a = 42.5$, $b = 24$, $c = 13$, $d = 20$, $e = 50$, $f = 40$. The parameters used for the super-twisting observer are $\alpha_1 = 7 \times 10^7$, $\lambda_1 = 7 \times 10^3$, $\alpha_2 = 300$, $\lambda_2 = 100$, $\alpha_3 = 1000$, $\lambda_3 = 200$, $\alpha_4 = 600$, $\lambda_4 = 300$. For the adaptive observer the parameters used are $\theta_1 = \theta_2 = 400$.

Figures 1 and 2 show the trajectories of the states x_3 and x_4 as well as the ones of $\hat{x}_3$ and $\hat{x}_4$ for both the super-twisting and the adaptive observers. We can see in the figures that the trajectories of the super-twisting observer converge much faster than the ones of the adaptive observer.

Figures 3 and 4 show the messages m_1 and m_2 together with their estimations $\hat{m}_1$ and $\hat{m}_2$, respectively. In this figures we note that the singularity in the point $x_1 = 0$ affects more the estimation of the messages made with the super-twisting than the one made with the adaptive observer. This is clear due to the explanation of Remark 3.1 and the fact that, in a ball of radio ε and center in $x_1 = 0$, it is not done a estimation neither of the states nor of the messages.

In order to test both observers with respect to parameters uncertainties, we introduce a variation of 1% in the nominal parameters. Figures 5 and 6 show how the parameters

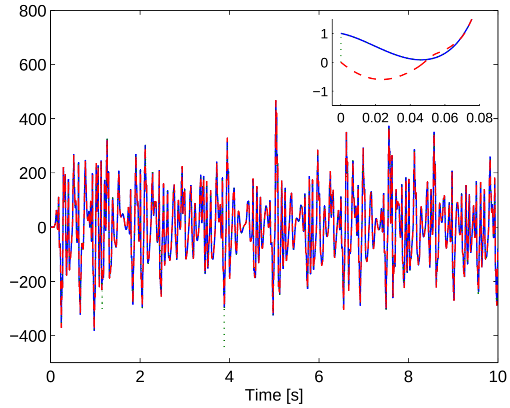

Fig. 1. x_3 (solid line) and its estimation $\hat{x}_3$ using the super-twisting (dot line) and adaptive (dash line) observers

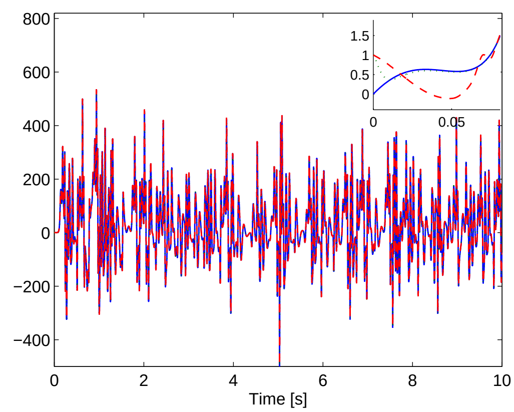

Fig. 2. x_4 (solid line) and its estimation $\hat{x}_4$ using the super-twisting (dot line) and adaptive (dash line) observers

uncertainties affects the behavior of the observers. In the case of the adaptive observer the parameter uncertainty destroy completely the estimation of the messages. Nevertheless, no matter what observe is used, in this case, the estimation of the messages can be consider unacceptable.

Remark. 5.1. For the super-twisting observer we do some simulations with $\tilde{x} = 0$, that is without using a linear estimator. In the simulations, not shown here, we obtained a bigger error in the estimation of x_4 which had a big effect in the error of the estimation of m_2. It was due to the fact that using $\tilde{x} = 0$, the variable to be estimated with the super-twisting algorithm became much faster and for having an acceptable estimation the sampling step must be reduce considerably.

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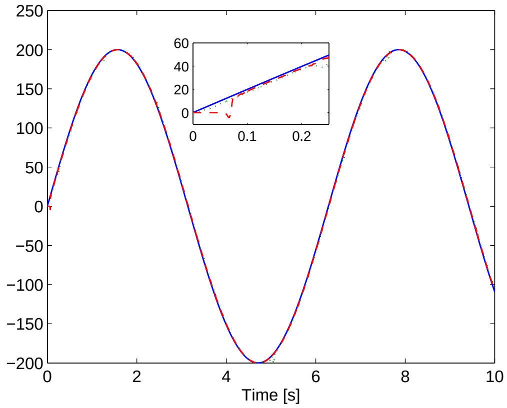

Fig. 3. Message m_1 (solid line) and its estimation $\hat{m}_1$ using the super-twisting (dot line) and adaptive (dash line) observers

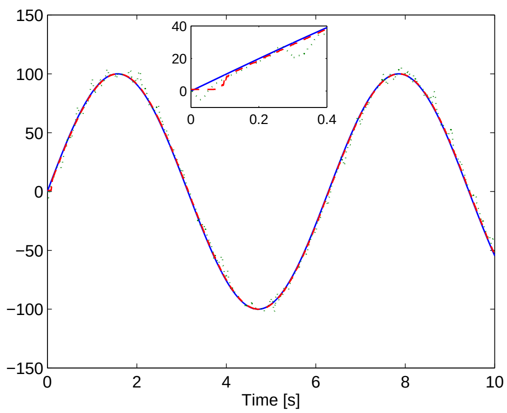

Fig. 4. Message m_2 (solid line) and its estimation $\hat{m}_2$ using the super-twisting (dot line) and adaptive (dash line) observers

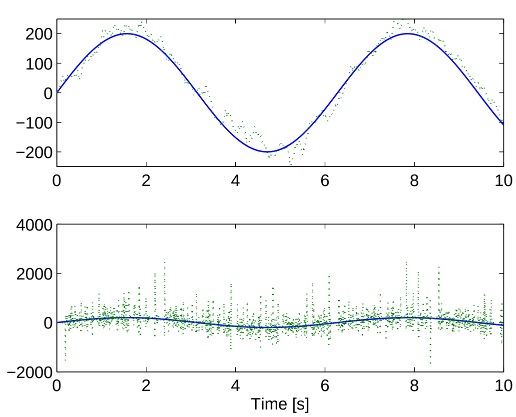

Fig. 5. Message m_1 (solid line) and its estimation $\hat{m}_1$ (dot line) for the system with 1% of uncertainty in the parameters. Above with the super-twisting observer, below with the adaptive observer

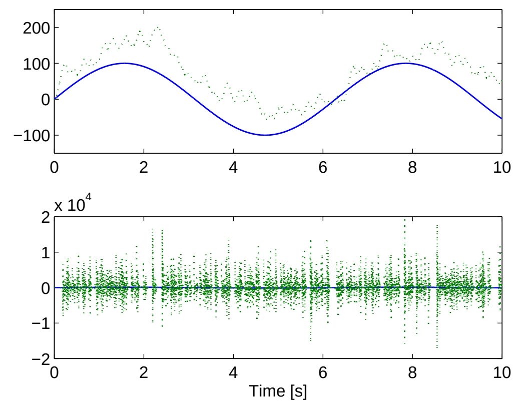

Fig. 6. Message m_2 (solid line) and its estimation $\hat{m}_2$ (dot line) for the system with 1% of uncertainty in the parameters. Above with the super-twisting observer, below with the adaptive observer

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